Penning trap

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1 Introduction

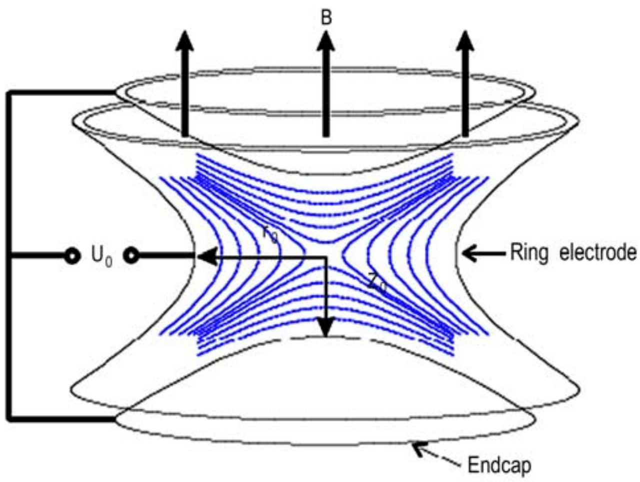

Figure 1: Schematic of the Penning trap

Penning trap is a bound system that can trap a single charges particle by using an external electrostatic quadrupole potential superimposed on a spatially uniform, stable magnetic field. If the particle of charge e and mass m is placed in a uniform magnetic field $\mathbf{B} = B_o\hat{z}$, then the trajectory of the particle will be cyclotron, with the free space cyclotron frequency

$$\omega_c = \frac{|eB|}{m}\hat{z} = \omega_c\hat{z} \tag{1}$$

The formula for the quadrupole is

$$V(\mathbf{r}) = \frac{V_o}{2}\frac{z^2 - \rho^2 2}{d^2} \tag{2}$$

where z and ρ are cylindrical coordinates and d is a characteristic trap dimension. The quadrupole potential has traditionally been produced by placing electrodes along equipotentials of V($\mathbf{r}$). Two 'end caps' follow the hyperbola of revolution

$$z = \pm\sqrt{z_o^2 + \frac{\rho^2}{2}} \tag{3}$$

and one 'ring ' electrode is along the hyperbola of revolution

$$z^2 = \frac{1}{2}(\rho^2 - \rho_o^2) \tag{4}$$

The characteristic trap dimension is defined by

$$d^2 = \frac{1}{2}(z_o^2 + \frac{\rho_o^2}{2}) \tag{5}$$

in terms of the minimum axial and radial distance to the trap electrodes, z_o and ρ_o

The equation of motion result from the Lorentz force on the charged particle

$$\mathbf{F} = -e \bigtriangledown V + e\mathbf{v} \times \mathbf{B} \tag{6}$$

$$\mathbf{a} = -\frac{e}{m}\begin{pmatrix}\frac{\partial V}{\partial x}\\ \frac{\partial V}{\partial x}\\ \frac{\partial V}{\partial x}\end{pmatrix} + \frac{e}{m}\begin{pmatrix}v_x\\ v_y\\ v_z\end{pmatrix}\times\begin{pmatrix}0\\ 0\\ B\end{pmatrix} \tag{7}$$

The axial motion along $\hat{z}$ decouples since $v_z\hat{\mathbf{z}}\times\mathbf{B}$. The resulting equation of motion is that of a simple harmonic oscillator.

$$\ddot{z} + \omega_z^2 z = 0 \tag{8}$$

with angular axial frequency

$$\omega_z^2 = \frac{eV_o}{md^2} \tag{9}$$

The radial equation of motion is

$$m\ddot{\rho} = e\Big[\mathbf{E}_\rho + \dot{\rho}\times\mathbf{B}\Big] \tag{10}$$

where E_ρ is the radial component of the quadrupole electric field which we can express in terms of the axial frequency

$$\mathbf{E}_\rho = -\frac{V}{\rho} = \frac{V_o}{2d^2}\rho = \frac{1}{2}\frac{m}{e}\omega_z^2\rho \tag{11}$$

By substitute Eqn.(10) into Eqn.(9), and express it in Cartesian coordinate, we will get

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \frac{e}{m}\Big[\frac{m\omega_c}{e}\begin{pmatrix}\dot{y}\\ -\dot{x}\end{pmatrix} + \frac{1}{2}\frac{m}{e}\omega_z^2\begin{pmatrix}x\\ y\end{pmatrix}\Big] \tag{12}$$

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \omega_c\begin{pmatrix}\dot{y}\\ -\dot{x}\end{pmatrix} + \frac{1}{2}\omega_z^2\begin{pmatrix}x\\ y\end{pmatrix} \tag{13}$$

By combining both Eqn.(7) and Eqn. (12), we will get

$$\begin{pmatrix}a_x\\ a_y\\ a_z\end{pmatrix} = \begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = \begin{pmatrix}\omega_c\dot{y} + \frac{1}{2}\omega_z^2 x\\ -\omega_c\dot{x} + \frac{1}{2}\omega_z^2 y\\ -\omega_z^2 z\end{pmatrix} \tag{14}$$

By solving Eqn.(14) yield

$$\omega_c' = \omega_c - \frac{\omega_z^2}{2\omega_c'} = \omega_c - \omega_m \tag{15}$$

and

$$\omega_m = \frac{\omega_z^2}{2\omega_c'} \tag{16}$$

The presence of the electric quadrupole field will the changed the cyclotron frequency, $_c$ to ω_c' where it is modified cyclotron frequency. The magnetron frequency ω_m describes the slow circular motion that results from a balance between the radially inward motional electric field and the radially outward electric field. The charges particle will be bound in Penning trap only if the following condition is satisfied.

$$\omega_z \leq \frac{\omega_c}{\sqrt{2}} \tag{17}$$

which required that the inward motional field be larger than the outward motional field. For typical trap sizes and field strengths

$$\omega_m \ll \omega_z \ll \omega_c'$$

By using Eqn.(1), Eqn.(9) and Eqn.(17), relationship between electric quadrupole potential and magnetic field can be determine.

$$\frac{eV_o}{md^2} \leq \frac{e^2B^2}{2m} \tag{18}$$

$$B \geq \frac{2mV_o}{ed^2} \tag{19}$$

Eqn.(18) have to satisfied otherwise the charged particle will not bounded in the Penning trap.

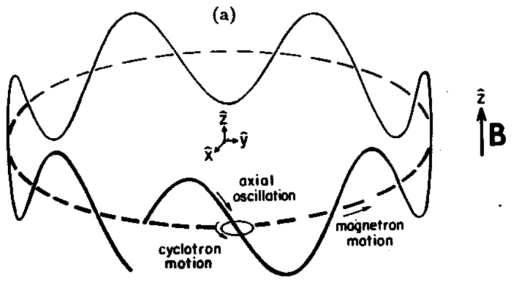

Figure 2: Orbit of a charges particle confined in a Penning trap

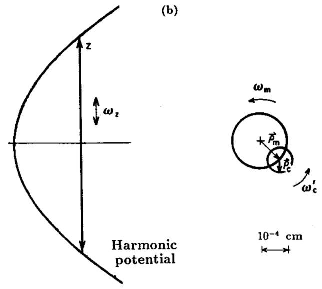

Figure 3: scaled representation of the three oscillatory motions for a confined antiproton

2 Derivation

By using the acceleration in the introduction section, we will get following Ordinary differential equation which described the motion of a particle in panning trap

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} \omega_c \dot{y} + \frac{1}{2}\omega_z^2 x \\ -\omega_c \dot{x} + \frac{1}{2}\omega_z^2 y \\ -\omega_z^2 z \end{pmatrix} \tag{20}$$

Finite different formula for acceleration and velocity are

$$a_i[n] = \frac{i[n+1] - 2i[n] + i[n-1]}{h^2} \tag{21}$$

$$v_i[n] = \frac{i[n+1] - i[n-1]}{h} \tag{22}$$

Substitute each of this finite different formula for acceleration and velocity, we will get

$$\frac{x[n+1] - 2x[n] + x[n-1]}{h^2} = \omega_c \left[\frac{y[n+1] - y[n-1]}{h} \right] + \frac{1}{2}\omega_z^2 x \tag{23}$$

Then after rearrange equation above, we will get

$$x[n+1] = 2x[n] - x[n-1] + k_1 y[n+1] - k_1 y[n-1] + k_2 x[n] \quad , k_1 = \frac{\omega_c h}{2}, k_2 = \frac{\omega_z^2 h^2}{2} \tag{24}$$

Repeat the process for y, we will get

$$y[n+1] = 2y[n] - y[n-1] - k_1 x[n+1] + k_1 x[n-1] + k_2 y[n] \tag{25}$$

By substitute Eqn.(2) into Eqn.(1), we will get a equation that calculate the x position of the particle

$$x[n+1] = k_4[(k_2+2)x[n] + (k_1^2 - 1)x[n-1] + 2k_1 y[n] - 2k_1 y[n-1] + k_1 k_2 y[n]] \tag{26}$$

Do the same thing to z-axis

$$\frac{z[n+1] - 2z[n] + z[n-1]}{h^2} = -\omega_z^2 z[n] \tag{27}$$

Rearrange the equation above we will get

$$z[n+1] = (2 - k_3)z[n] - z[n-1] \quad , k_3 = \omega_z^2 h^2 \tag{28}$$

By using Eqn.(25), Eqn.(26) and Eqn.(28), we can calculate the position of the particle at t[n+1].

3 Numerical analysis of the trajectories

3.1 First two iterations

The initial conditions for the trajectory at n = 1, t[1] = 0
For x- component

$$x[1] = x_o$$
$$v_x[1] = u_x$$
$$a_x[1] = \omega_c u_y + \frac{1}{2}\omega_z^2 x_o$$

For y- component

$$
\begin{aligned}
y[1] &= y_o \\
v_y[1] &= u_y \\
a_y[1] &= -\omega_c u_x + \frac{1}{2}\omega_z^2 y_o
\end{aligned}
$$

For z- component

$$
\begin{aligned}
z[1] &= z_o \\
v_z[1] &= u_z \\
a_z[1] &= -\omega_z^2 z
\end{aligned}
$$

After the first time step the trajectory at n = 2, t[2] = Δ t = h are
For x- component

$$
\begin{aligned}
x[2] &= x_o + v_x[1]h \\
v_x[2] &= v_x[1] + a_x[1]h \\
a_x[2] &= \omega_c v_x[1] + \frac{1}{2}\omega_z^2 x[2]
\end{aligned}
$$

For y- component

$$
\begin{aligned}
y[1] &= y_o + v_y[1]h \\
v_y[2] &= v_y[1] + a_y[1]h \\
a_y[2] &= -\omega_c v_y[1] + \frac{1}{2}\omega_z^2 y[2]
\end{aligned}
$$

For z- component

$$
\begin{aligned}
z[1] &= z_o + v_z[1]h \\
v_z[2] &= v_z[1] + a_z[1]h \\
a_z[2] &= -\omega_z^2 z[2]
\end{aligned}
$$

3.2 Iterations > 2

For time steps when n > 2

$$
\begin{aligned}
x[n+1] &= k_4[(k_2+2)x[n] + (k_1^2-1)x[n-1] + 2k_1 y[n] - 2k_1 y[n-1] + k_1 k_2 y[n]] \\
y[n+1] &= 2y[n] - y[n-1] - k_1 x[n+1] + k_1 x[n-1] + k_2 y[n] \\
z[n+1] &= (2-k_3)z[n] - z[n-1] \quad , k_4 = \omega_z^2 h^2
\end{aligned}
$$

4 Matlab Programming

All the equation that have stated numerical analysis section are used in the Matlab programming.
Matlab input variables:

1. Mass of particle, m
2. Charge on particle, q
3. Voltage, V
4. Initial velocities, (u_x, u_y, u_z)
5. Number of time steps, N

Order of Matlab calculations

1. Input the variables (The variable input need to be satisfied some of the constrains).
2. Calculate the constant k_1, k_2, k_3 and k_4.

3. Initial displacement, velocity and acceleration at time step 1 (t = 0 and n = 1) then at time step 2 (t = h and n =2).
4. Calculate the displacement of the particle by using displacement for loop for n= 3 to n = N.
5. Calculate the velocity of the particle by using velocity for loop for n= 3 to n = N.
6. Calculate the acceleration of the particle by using acceleration for loop for n= 3 to n = N.
7. Output the user input value on the figure 1.
8. Plot figure 8 to 13 for x and y plane of the trajectories of the particle, displacement for x, y and z- axis of the particle, velocity for x, y and z- axis of the particle, acceleration for x, y and z- axis of the particle, and the animation for 3D trajectories of the particle.

4.1 Code detail

There are a five sections in the code which is input parameter, constant, setup, time loops and graphics.

Note: Remember to set the input such that the dimension agreed and also the restriction stated above. Otherwise, the particle will not be bound in the Penning trap.

4.1.1 Input parameter

```
% ======================================================================
% INPUT PARAMETERS
% ======================================================================

    N           = 1000;                 % number of time steps
    q           = 1.602e-19;            % charge of particle
    m           = 9.109384e-31;         % mass of particle
    z_o         = 0.05;                 % minimum axial distance
    z_rho_ratio = 0.05;                 % ratio between axial distance and radial distance
    V           = q*5*10^5;             % Voltage
    V_B_ratio   = 2;                    % Voltage to Magnetic field ratio
    h           = 0.01;                 % time steps

% Initial velocites and displacements
   ux = 1e-5;  x0 = 1e-5;
   uy = 0;     y0 = 0;
   uz = 0;     z0 = 1e-5;
```

Figure 4: Input parameter for penning trap

1. 'h' is the times step for the calculation. This parameter can be change, the smaller the time step, the accurate the graph, but then the animation will be slower. 'N' is the number of times step where it will control how long the particle to travel.
2. 'q' and 'm' is the mass of the charge and the mass of the particle respectively. In this simulation, electron is used.
3. 'z_o' is the minimum axial distance while 'z_rho_ratio' is the ratio of minimum axial distance to the radial distance. This is to control the difference between axial distance and radial distance. Changes of z_o will causes axial distance of the particle to changes.
4. V is the quadrupole potential of the system while 'V_B_ratio' is the ratio of quadrupole potential to magnetic field of the system. This ratio is to restrict the different between the quadrupole potential and the magnetic field. **This constant have to set greater than one in order to satisfied Eqn.(17)**. ω_c will change when magnetic field changed while ω_z will change when quadrupole potential changed.
5. 'x0', 'y0' and 'z0' is the initial position of the particle while 'ux', 'uy' and 'uz' is the initial velocity of the particle. **'uz' cannot be 0, otherwise the particle will not be moving along z- axis.**

4.1.2 Variables and constants

```
% ======================================================================
% Variable and constant
% ======================================================================

% Variable
    rho_o = z_rho_ratio*z_o;                        % rho = minimum radial distance.
    d     = sqrt(1/2*(((z_o)^2+((rho_o^2)/2))));   % characteristic trap dimension
    B     = V_B_ratio* sqrt((2*m*V)/(q*d^2));      % magnetic field
    w_c   = (q*B)/m;                                % free space cyclotron frequency
    w_z   = sqrt((q*V)/(m*(d^2)));                  % angular axial frequency
% Constants
    k1 = (w_c*h)/2;
    k2 = (h^2*w_z^2)/2;
    k3 = w_z^2*h^2; k4 = 1/(1+k1^2);
```

Figure 5: Variables and constant for penning trap

All of the variable will changes according to the changes of the parameter while the constant is defined so that the equation of motion can be simplified.

1. 'rho_o' is the minimum radial distance of the electron. By altering 'z_rho_ratio', the minimum radial distance of the penning trap can be change. Changes of ρ_o can change the radial distance of the electron.
2. 'd' is the characteristic trap dimension. 'd' will change according to ρ_o and z_o.
3. 'B' is the magnetic field applied to the system. This field is applied across z- direction of the system. Magnetic field can change by changing the 'V_B_ratio'. This ratio have to be greater than 1 in order to satisfied Eqn.(19).
4. 'ω_c' and 'ω_z' are free space cyclotron frequency and angular axial frequency of the particle. This two quantity is related by Eqn.(17).
5. k_1, k_2, k_3 and k_4 is the constant defined to simply the derivation.

4.1.3 Setup

```
% ======================================================================
% Setup
% ======================================================================

% Initialize arrays
    t = (1:N).* h;
    x  = zeros(N,1);   y  = zeros(N,1);   z  = zeros(N,1);
    vx = zeros(N,1);   vy = zeros(N,1);   vz = zeros(N,1);
    ax = zeros(N,1);   ay = zeros(N,1);   az = zeros(N,1);

% Time step 1: n = 1
    x(1)  = x0;  y(1)  = y0; z(1)  = z0;
    vx(1) = ux;  vy(1) = uy; vz(1) = uz;

    ax(1) = (w_c)* uy+ 1/2*(w_z^2)*x(1);
    ay(1) = -(w_c)* ux+ 1/2*(w_z^2)*y(1);
    az(1) = -(w_z^2)*z(1);

% Time step 2: n = 2
    x(2)  = x(1) + vx(1)*h;   vx(2) = vx(1) +ax(1)*h;
    y(2) = y(1)  + vy(1)*h;   vy(2) = vy(1) +ay(1)*h;
    z(2) = z(1) + vz(1)*h;    vz(2) = vz(1) +az(1)*h;

    ax(2) =  (w_c)* vy(2)+ 1/2*(w_z^2)*x(2);
    ay(2) = -(w_c)* vx(2)+ 1/2*(w_z^2)*y(2);
    az(2) = -(w_z^2)*z(2);
```

Figure 6: Setup of penning trap

1. The array is initialized to store the displacement, velocity and acceleration at t[n].
2. By using equation in section 3.1, position, velocity and acceleration of the particle for first two iterations are calculated. This will depends on the time step, number of time steps and initial value of position, velocity and acceleration

4.1.4 Time loops

```
% =======================================================================
% TIME LOOPS
% =======================================================================
% for N > 2
% Displacement
   for n = 2 : N-1
       x(n+1) = k4*((2+k2)*x(n) + (k1^2- 1)*x(n-1)+ 2*k1*y(n)- 2*k1*y(n-1)  +k1*k2*y(n));
       y(n+1) = 2*y(n) -y(n-1)- k1*x(n+1) +k1*x(n-1) +k2*y(n);
       z(n+1) = (2 -k3).*z(n) -z(n-1);
   end
 % Velocity  --------------------------------------------------------------
 for n = 2 : N-1
     vx(n) = (x(n+1) - x(n-1))/(2*h);
     vy(n) = (y(n+1) - y(n-1))/(2*h);
     vz(n) = (z(n+1) - z(n-1))/(2*h);

 end
     vx(N) = (x(N)-x(N-1))/h; vy(N) = (y(N)-y(N-1))/h; vz(N) = (z(N)-z(N-1))/h;

 % Acceleration
 for n = 1 : N
      ax(n) =   w_c.* vy(n)+ 1/2.*(w_z^2).*x(n);
      ay(n) = - w_c.* vx(n)+ 1/2.*(w_z^2).*y(n);
      az(n) = -(w_z^2).*z(n);
 end
```

Figure 7: Time loops for penning trap

By using finite different equation and the equation derived in section 3.2, position, velocity and acceleration of the particle can be calculate.

4.1.5 Graphics

There are 6 graph plotted in for the penning trap which are, parameter summary, 2D trajectory of the particle, displacement vs time graph, velocity versus time graph, acceleration vs time graph and 3D trajectory of the particle. This section just used some basic graph plotting technics to plot the graph. The result of the graph will be show under Simulation section.

5 Simulation

The Matlab script Gan_Beng_Yee_matlab_code.m is used for the modelling of a charges particle in a penning trap which have uniform magnetic field and electric quadrupole potential. The direction of the magnetic field is in the z- direction while the electric quadrupole potential is made by two 'end caps' follow the hyperbola of revolution and one 'ring' electrode is along the hyperbola pf revolution.

The graphical output of the mscript Gan_Beng_Yee_matlab_code.m includes a Figure Window which gives a summary of the parameters used in a simulation. Figure (8) gives the parameters used to test the numerical model. Figure (9) to (13) show the 2D trajectory, the displacement, velocity, acceleration and the 3D trajectory of the charged particle respectively.

Number of time steps N = 1000
Charge, q [C] = 1.602e-19
Mass, m [kg] = 9.109e-31
Magnetic field, B [T] = 5.396e-11
Voltage, V [V] = 8.01e-14

Minimum axial distance, z_o [m] = 5.00e-02
Minimum radial distance, ρ_o [m] = 2.50e-03
Characteristic trap dimension, d^2 [m^2] = 1.25e-03

Initial values (t = 0 s) for displacement [m]
x_0 = 0.00 y_0 = 0.00 z_0 = 0.00

Initial values (t = 0 s) for velocity [m/s]
u_x = 1.00e-05 u_y = 0.00e+00 u_z = 0.00e+00

Time step [s] h = 1.00e-02

Figure 8: Parameter summary for the motion of a electron in a penning trap

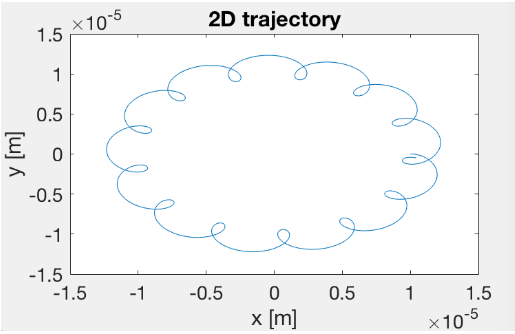

Figure 9: 2D trajectory of the electron in XY plane

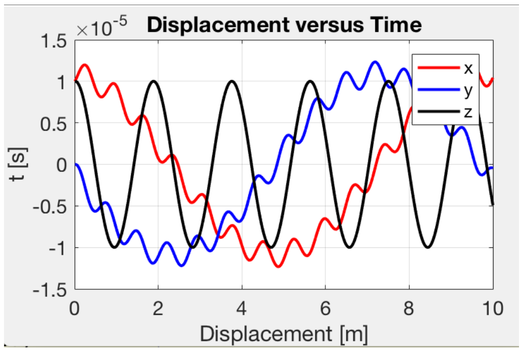

Figure 10: Displacement vs time graph

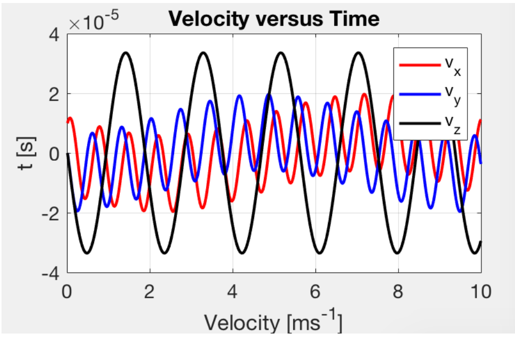

Figure 11: Velocity vs time graph

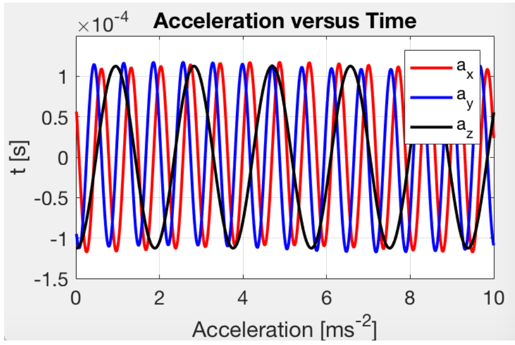

Figure 12: Acceleration vs time graph

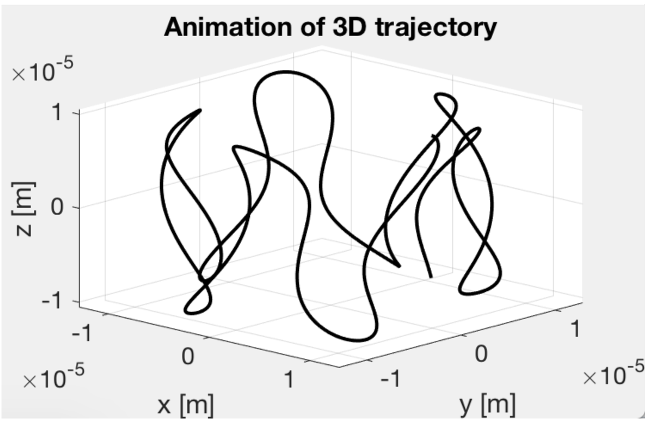

Figure 13: 3D trajectory of the electron

All the figure about just for one set of parameter input. The trajectory will be different if the parameter is different. There are lot of interesting feature of the penning trap. Feel free to change the parameter to explore effect of the parameter on the trajectory of the charged particle.

6 Reference

Note: Introduction part mainly modified from [1] while the code is modified version of [2].

[1] http://gabrielse.physics.harvard.edu/gabrielse/papers/1990/1990_tjoelker/chapter_2.pdf
[2] http://www.physics.usyd.edu.au/teach_res/mp/doc/em_vBE.pdf