penning trap simulator

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1. the physics

We jump straight into finding the equations of the fields involved – the magnetic field is simply $B = B_0\hat{z}$, while the E-field can be found via the potential V:

$$\begin{aligned}\vec{E} &= -\nabla V = -\nabla\left[\frac{V_0}{2d^2}\left(z^2 - \frac{x^2+y^2}{2}\right)\right] \\ &= \frac{V_0}{d^2}\left(-z\hat{z} + \frac{x\hat{x}+y\hat{y}}{2}\right)\end{aligned}$$

Now, taking the Lorentz force to be the only force acting on this system, Newton's law of motion states

$$\begin{aligned}\vec{F} &= q\left(\dot{\vec{r}} \times \vec{B} + \vec{E}\right) \\ m\ddot{\vec{r}} &= q\left(\begin{vmatrix}\hat{x} & \hat{y} & \hat{z} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B_0\end{vmatrix} + \frac{V_0}{d^2}\left(-z\hat{z} + \frac{x\hat{x}+y\hat{y}}{2}\right)\right) \\ m\left(\ddot{z}\hat{z} + \ddot{x}\hat{x} + \ddot{y}\hat{y}\right) &= q\left(-\frac{v_0}{d^2}z\hat{z} + \left(B_0\dot{y} + \frac{V_0}{2d^2}x\right)\hat{x} + \left(-B_0\dot{x} + \frac{V_0}{2d^2}y\right)\hat{y}\right) \qquad (1)\end{aligned}$$

Essentially, what we now have is a couple of differential equations. Notice that z only appears in the $\hat{z}$ component, thus we shall examine that first.

1.1 The Vertical Component

Immediately we notice that the $\hat{z}$ component is just a simple harmonic motion equation

$$\ddot{z} = -\underbrace{\frac{qV_0}{md^2}}_{{\omega_z}^2} z$$

With the known solution in terms of initial conditions z_0 and $\dot{z}_0$

$$z(t) = \begin{cases} z_0\cos\left(\omega_z t\right) + \frac{\dot{z}_0}{\omega_z}\sin\left(\omega_z t\right) & \text{for } {\omega_z}^2 > 0 \\ z_0\cosh\left(\left|\omega_z\right| t\right) + \frac{\dot{z}_0}{\left|\omega_z\right|}\sinh\left(\left|\omega_z\right| t\right) & \text{for } {\omega_z}^2 < 0\end{cases} \qquad (2)$$

1.2 The Azimuthal Components

From the $\hat{x}$ and $\hat{y}$ components, we have two coupled differential equations

$$m\ddot{x} = q\left(\frac{V_0}{2d^2}x + B_0\dot{y}\right) \qquad\qquad m\ddot{y} = q\left(\frac{V_0}{2d^2}y - B_0\dot{x}\right)$$

Note that for each component, the reality of the situation means that all the values above has to be real. With that in mind, consider the equation $m\ddot{x} + im\ddot{y}$

$$\begin{aligned} m\left(\ddot{x} + i\ddot{y}\right) &= q\left(\frac{V_0}{2d^2}\left(x+iy\right) + B_0\left(\dot{y} - i\dot{x}\right)\right) \\ \implies 0 &= \left(\ddot{x} + i\ddot{y}\right) - \frac{q}{m}\left(\frac{V_0}{2d^2}\left(x+iy\right) - iB_0\left(\dot{x} + i\dot{y}\right)\right)\end{aligned}$$

Owing to the linearity of the differential equation, we can define a $w \in \mathbb{C}$ such that $w(t) = x(t) + iy(t)$, and

$$\mathbb{Re}\left[w(t)\right] = x(t) \qquad \mathbb{Re}\left[\dot{w}(t)\right] = \dot{x}(t) \qquad \mathbb{Re}\left[\ddot{w}(t)\right] = \ddot{x}(t)$$

$$\mathbb{Im}\left[w(t)\right] = y(t) \qquad \mathbb{Im}\left[\dot{w}(t)\right] = \dot{y}(t) \qquad \mathbb{Im}\left[\ddot{w}(t)\right] = \ddot{y}(t)$$

Using the ansatz $w(t) = e^{-i\omega_0 t}$, the above differential equation can now be rewritten,

$$\begin{aligned}
0 &= \left[\frac{d^2}{dt^2} + i\frac{qB_0}{m}\frac{d}{dt} - \frac{qV_0}{2md^2}\right] w(t) \\
&= \left[(-i\omega_0)^2 + i\frac{qB_0}{m}(-i\omega_0) - \frac{qV_0}{2md^2}\right] w(t) \\
&= -\left[{\omega_0}^2 - \frac{qB_0}{m}\omega_0 + \frac{qV_0}{2md^2}\right] w(t)
\end{aligned}$$

Solving for ω_0,

$$\begin{aligned}
\omega_0 &= \frac{\frac{qB_0}{m} \pm \sqrt{\left(-\frac{qB_0}{m}\right)^2 - 4\left(\frac{qV_0}{2md^2}\right)}}{2} \\
&= \underbrace{\frac{qB_0}{2m}}_{\omega} \pm \underbrace{\frac{qB_0}{2m}\sqrt{1 - \frac{2mV_0}{q{B_0}^2 d^2}}}_{\Omega}
\end{aligned}$$

Let us first consider the case where Ω is real. The general solution is therefore

$$w(t) = e^{-i\omega t}\left(\alpha e^{-i\Omega t} + \beta e^{i\Omega t}\right) \qquad \dot{w}(t) = -ie^{-i\omega t}\left((\omega+\Omega)\,\alpha e^{-i\Omega t} + (\omega-\Omega)\,\beta e^{i\Omega t}\right)$$

By setting $t = 0$, α and β in terms of the initial conditions are

$$w_0 = \alpha + \beta \qquad \dot{w}_0 = -i\left((\omega+\Omega)\,\alpha + (\omega-\Omega)\,\beta\right)$$

Here I utilise the `sympy` package in Python to solve for the coefficients:

```
>>> from sympy import *
>>> var('w_0 alpha beta omega Omega')
(w_0, alpha, beta, omega, Omega)
>>> wp = Symbol("w_0\'")
>>> coef = solve([
...         w_0-alpha-beta,
...         wp+I*((omega+Omega)*alpha + (omega-Omega)*beta)
...         ], [
...         alpha, beta
...         ])
>>> print(latex(coef))
```

$$\left\{\alpha : \frac{1}{2\Omega}\left(w_0\left(\Omega-\omega\right) + i\dot{w}_0\right), \quad \beta : \frac{1}{2\Omega}\left(w_0\left(\Omega+\omega\right) - i\dot{w}_0\right)\right\}$$

Placing the coefficients back into the equation,

$$\begin{aligned}
w(t) &= \frac{e^{-i\omega t}}{2\Omega}\left(\left((\Omega-\omega)\,w_0 + i\dot{w}_0\right)e^{-i\Omega t} + \left((\Omega+\omega)\,w_0 - i\dot{w}_0\right)e^{i\Omega t}\right) \\
&= \frac{e^{-i\omega t}}{2\Omega}\Big(\left((\Omega-\omega)\,(x_0+iy_0) + i\,(\dot{x}_0+i\dot{y}_0)\right)\left(\cos(\Omega t) - i\sin(\Omega t)\right) \\
&\qquad + \left((\Omega+\omega)\,(x_0+iy_0) - i\,(\dot{x}_0+i\dot{y}_0)\right)\left(\cos(\Omega t) + i\sin(\Omega t)\right)\Big) \\
&= \underbrace{\left[\left(x_0\cos(\Omega t) + \frac{\dot{x}_0 - \omega y_0}{\Omega}\sin(\Omega t)\right)\cos(\omega t) + \left(y_0\cos(\Omega t) + \frac{\dot{y}_0 + \omega x_0}{\Omega}\sin(\Omega t)\right)\sin(\omega t)\right]}_{x(t)}
\end{aligned}$$

$$
+ i \underbrace{\left[- \left(x_0 \cos(\Omega t) + \frac{\dot{x}_0 - \omega y_0}{\Omega} \sin(\Omega t) \right) \sin(\omega t) + \left(y_0 \cos(\Omega t) + \frac{\dot{y}_0 + \omega x_0}{\Omega} \sin(\Omega t) \right) \cos(\omega t) \right]}_{y(t)} \tag{3}
$$

$$
\begin{aligned}
x(t) &= \left(x_0 \cos(\Omega t) + \frac{\dot{x}_0 - \omega y_0}{\Omega} \sin(\Omega t) \right) \cos(\omega t) + \left(y_0 \cos(\Omega t) + \frac{\dot{y}_0 + \omega x_0}{\Omega} \sin(\Omega t) \right) \sin(\omega t) \\
&= \frac{1}{2} \sqrt{{x_0}^2 + \left(\frac{\dot{x}_0 - \omega y_0}{\Omega} \right)^2} \Big(\cos\left((\Omega + \omega) t - \operatorname{atan2}(\dot{x}_0 - \omega y_0, \Omega x_0) \right) + \cos\left((\Omega - \omega) t - \operatorname{atan2}(\dot{x}_0 - \omega y_0, \Omega x_0) \right) \Big) \\
&\quad + \frac{1}{2} \sqrt{{y_0}^2 + \left(\frac{\dot{y}_0 + \omega x_0}{\Omega} \right)^2} \Big(\sin\left((\Omega + \omega) t - \operatorname{atan2}(\dot{y}_0 + \omega x_0, \Omega y_0) \right) - \sin\left((\Omega - \omega) t - \operatorname{atan2}(\dot{y}_0 + \omega x_0, \Omega y_0) \right) \Big) \\
y(t) &= - \left(x_0 \cos(\Omega t) + \frac{\dot{x}_0 - \omega y_0}{\Omega} \sin(\Omega t) \right) \sin(\omega t) + \left(y_0 \cos(\Omega t) + \frac{\dot{y}_0 + \omega x_0}{\Omega} \sin(\Omega t) \right) \cos(\omega t) \\
&= \frac{1}{2} \sqrt{{x_0}^2 + \left(\frac{\dot{x}_0 - \omega y_0}{\Omega} \right)^2} \Big(- \sin\left((\Omega + \omega) t - \operatorname{atan2}(\dot{x}_0 - \omega y_0, \Omega x_0) \right) + \sin\left((\Omega - \omega) t - \operatorname{atan2}(\dot{x}_0 - \omega y_0, \Omega x_0) \right) \Big) \\
&\quad + \frac{1}{2} \sqrt{{y_0}^2 + \left(\frac{\dot{y}_0 + \omega x_0}{\Omega} \right)^2} \Big(\cos\left((\Omega + \omega) t - \operatorname{atan2}(\dot{y}_0 + \omega x_0, \Omega y_0) \right) + \cos\left((\Omega - \omega) t - \operatorname{atan2}(\dot{y}_0 + \omega x_0, \Omega y_0) \right) \Big)
\end{aligned}
$$

For the ease of calculation, and in anticipation for purposes of coding, we form the following variables:

$$
\begin{aligned}
A_x &= \frac{1}{2} \sqrt{{x_0}^2 + \left(\frac{\dot{x}_0 - \omega y_0}{\Omega} \right)^2} & O_{\text{sum}} &= \Omega + \omega & P_x &= \operatorname{atan2}(\dot{x}_0 - \omega y_0, \Omega x_0) \\
A_y &= \frac{1}{2} \sqrt{{y_0}^2 + \left(\frac{\dot{y}_0 + \omega x_0}{\Omega} \right)^2} & O_{\text{diff}} &= \Omega - \omega & P_y &= \operatorname{atan2}(\dot{y}_0 + \omega x_0, \Omega y_0)
\end{aligned}
$$

Which means

$$
\begin{aligned}
x(t) &= A_x \left(\cos(O_{\text{sum}} t - P_x) + \cos(O_{\text{diff}} t - P_x) \right) + A_y \left(\sin(O_{\text{sum}} t - P_y) - \sin(O_{\text{diff}} t - P_y) \right) \\
\dot{x}(t) &= -A_x \left(O_{\text{sum}} \sin(O_{\text{sum}} t - P_x) + O_{\text{diff}} \sin(O_{\text{diff}} t - P_x) \right) + A_y \left(O_{\text{sum}} \cos(O_{\text{sum}} t - P_y) - O_{\text{diff}} \cos(O_{\text{diff}} t - P_y) \right) \\
y(t) &= A_x \left(- \sin(O_{\text{sum}} t - P_x) + \sin(O_{\text{diff}} t - P_x) \right) + A_y \left(\cos(O_{\text{sum}} t - P_y) + \cos(O_{\text{diff}} t - P_y) \right) \\
\dot{y}(t) &= A_x \left(-O_{\text{sum}} \cos(O_{\text{sum}} t - P_x) + O_{\text{diff}} \cos(O_{\text{diff}} t - P_x) \right) - A_y \left(O_{\text{sum}} \sin(O_{\text{sum}} t - P_y) + O_{\text{diff}} \sin(O_{\text{diff}} t - P_y) \right)
\end{aligned}
$$

1.3 Non-Oscillatory Cases

There is a possibility that $\frac{2\omega V_0}{q{B_0}^2 d^2} > 1$, which means that $\Omega = i\,|\Omega|$. If we use the trigonometry/hyperbolic identities

$$
\cos(i\,|\Omega|\,t) = \cosh(|\Omega|\,t) \qquad\qquad \sin(i\,|\Omega|\,t) = i \sinh(|\Omega|\,t)
$$

Then Equation 3 becomes

$$
\begin{aligned}
w(t) &= \left[\left(x_0 \cos(\Omega t) + \frac{\dot{x}_0 - \omega y_0}{\Omega} \sin(\Omega t) \right) \cos(\omega t) + \left(y_0 \cos(\Omega t) + \frac{\dot{y}_0 + \omega x_0}{\Omega} \right) \sin(\omega t) \right] \\
&\quad + i \left[- \left(x_0 \cos(\Omega t) + \frac{\dot{x}_0 - \omega y_0}{\Omega} \sin(\Omega t) \right) \sin(\omega t) + \left(y_0 \cos(\Omega t) + \frac{\dot{y}_0 + \omega x_0}{\Omega} \right) \cos(\omega t) \right] \\
&= \left[\left(x_0 \cos(i\,|\Omega|\,t) + \frac{\dot{x}_0 - \omega y_0}{i\,|\Omega|} \sin(i\,|\Omega|\,t) \right) \cos(\omega t) + \left(y_0 \cos(i\,|\Omega|\,t) + \frac{\dot{y}_0 + \omega x_0}{i\,|\Omega|} \sin(\Omega t) \right) \sin(\omega t) \right] \\
&\quad + i \left[- \left(x_0 \cos(i\,|\Omega|\,t) + \frac{\dot{x}_0 - \omega y_0}{i\,|\Omega|} \sin(i\,|\Omega|\,t) \right) \sin(\omega t) + \left(y_0 \cos(i\,|\Omega|\,t) + \frac{\dot{y}_0 + \omega x_0}{i\,|\Omega|} \sin(\Omega t) \right) \cos(\omega t) \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\left(x_0 \cosh\left(|\Omega| t\right) + \frac{\dot{x}_0 - \omega y_0}{i|\Omega|} i \sinh\left(|\Omega| t\right)\right) \cos\left(\omega t\right) + \left(y_0 \cosh\left(|\Omega| t\right) + \frac{\dot{y}_0 + \omega x_0}{i|\Omega|} i \sin\left(|\Omega| t\right)\right) \sin\left(\omega t\right)\right] \\
&\quad + i\left[-\left(x_0 \cosh\left(|\Omega| t\right) + \frac{\dot{x}_0 - \omega y_0}{i|\Omega|} i \sinh\left(|\Omega| t\right)\right) \sin\left(\omega t\right) + \left(y_0 \cosh\left(|\Omega| t\right) + \frac{\dot{y}_0 + \omega x_0}{i|\Omega|} i \sin\left(|\Omega| t\right)\right) \cos\left(\omega t\right)\right] \\
&= \left[\left(x_0 \cosh\left(|\Omega| t\right) + \frac{\dot{x}_0 - \omega y_0}{|\Omega|} \sinh\left(|\Omega| t\right)\right) \cos\left(\omega t\right) + \left(y_0 \cosh\left(|\Omega| t\right) + \frac{\dot{y}_0 + \omega x_0}{|\Omega|} \sinh\left(|\Omega| t\right)\right) \sin\left(\omega t\right)\right] \\
&\quad + i\left[-\left(x_0 \cosh\left(|\Omega| t\right) + \frac{\dot{x}_0 - \omega y_0}{|\Omega|} \sinh\left(|\Omega| t\right)\right) \sin\left(\omega t\right) + \left(y_0 \cosh\left(|\Omega| t\right) + \frac{\dot{y}_0 + \omega x_0}{|\Omega|} \sinh\left(|\Omega| t\right)\right) \cos\left(\omega t\right)\right]
\end{aligned}
$$

What we notice here, and also with Equation 2, is that in the case that either ω_z or Ω is imaginary, then we merely take its absolute value and change the trigonometric function into its corresponding hyperbolic one (not generally true, but it is in the case of $x(t)$, $y(t)$, and $z(t)$).

1.4 The Forces

From Equation 1, we find that the forces are given by

$$
\vec{F}_B = qB_0\left(\dot{y}\hat{x} - \dot{x}\hat{y}\right) \qquad\qquad \vec{F}_E = \frac{qV_0}{d^2}\left(-z\hat{z} + \frac{1}{2}\left(x\hat{x} + y\hat{y}\right)\right)
$$

which we can find now that we have the solved equations of motion.

2. the code

With $\vec{r}(t)$ found in explicit form, there's not much to talk about for the coding part (the specifics are better understood alongside the code with the comments). I've used `three.js` to create my animation. Just briefly,

0. penning.html
   This HTML file forms the plaform of the programme, which doesn't actually contain much but merely exists for the scripts to act upon.

1. penning.js
   This file is just the code version of the above equations we've found, with helper functions (`setB0`, `setV0`, etc.) to make changes to the environmental variables easier.

   Also, it has to check whether or not the solution is the oscillatory one, and to use the appropriate equation when calculating position.

2. animation.js
   This all of the animations. Essentially, with $\vec{r}(t)$ known, all we have to do is to tell `three.js` to create a sphere of a certain size, then to move it every few milliseconds to the position $\vec{r}(t_1 - t_0)$.

   There are other bonus parts, of course, but most of it is just things which look nice (or helps with spatially placing the sphere)

3. gui.js
   This handles the user interaction, where `dat.gui.js` have been used. It just attaches itself to the objects that has been already created, and updates the values as the user does.