Signals and Systems, Midterm Exam

Solutions

Spring 2008, Edited by bypeng

1. **[10]** Consider a system H to be tested as being **memoryless**, **causal**, **linear**, **time invariant**, and **invertible**. Three signals $x_1(t)$, $x_2(t)$, and $x_3(t)$ are sent to the system, and the corresponding output signals $y_1(t)$, $y_2(t)$, and $y_3(t)$ are obtained as shown in Figure 1.

   Based on the three input-output pairs, is it possible to determine each of the five properties for system H? If yes, what is it? If no, why? Justify your answer.

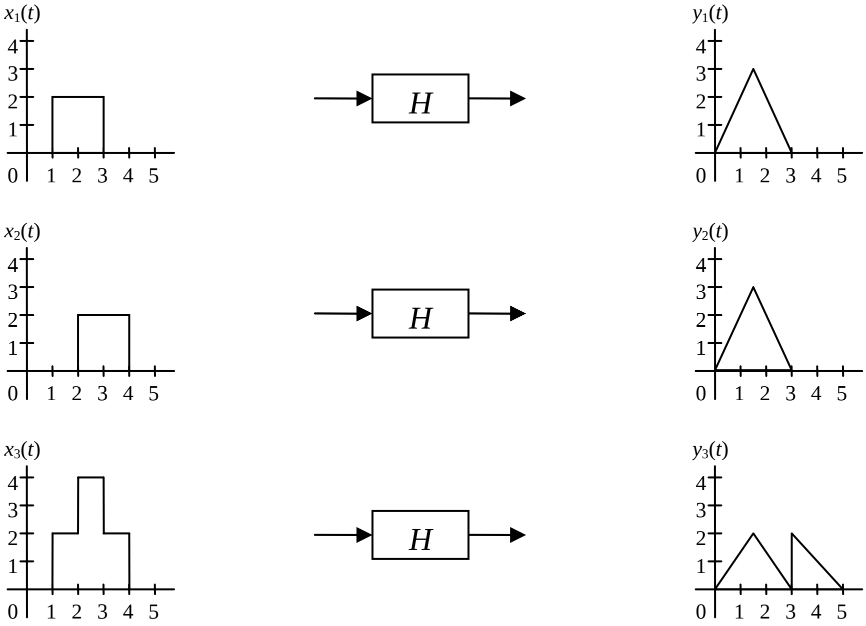


Figure 1

Solution:

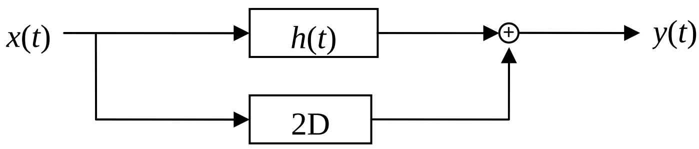
i. NEITHER memoryless NOR causal. Observe that $x_1(t) = x_3(t)$ for $t < 2$, but $y_1(t) \neq y_3(t)$ for $0 < t < 2$.

ii. NOT linear. Observe that $x_3(t) = x_1(t) + x_2(t)$, but $y_3(t) \neq y_1(t) + y_2(t)$.

iii. NOT time invariant. Observe that $x_2(t) = x_1(t-1)$, but $y_2(t) \neq y_1(t-1)$.

iv. NOT invertible. Observe that $x_1(t) \neq x_2(t)$, but $y_1(t) = y_2(t)$.

2. Consider a system as shown in Figure 2, where $h(t)$ is the impulse response of the LTI sub-system in the block, and 2D is the operation of time delay for 2 units.



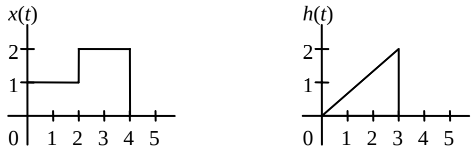


Figure 2

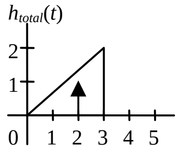
(a) **[4]** Plot the impulse response of the overall system.

(b) **[7]** Plot the output $y(t)$ of the system for input $x(t)$ shown in Figure 2.

(c) **[5]** Repeat $x(t)$ in time with a period of 6, and let $\tilde{x}(t)$ be the corresponding periodic version of $x(t)$. Plot the output $\tilde{y}(t)$ of the system for input $\tilde{x}(t)$.

Solution:

(a) The impulse response of the block 2D is $\delta(t-2)$, so the plot of the impulse response of the overall system is as the right graph.



(b) For the upper part,

$$x(t)*h(t)=\left[u(t)+u(t-2)-2u(t-4)\right]*h(t)=u(t)*h(t)+u(t-2)*h(t)-2u(t-4)*h(t)$$

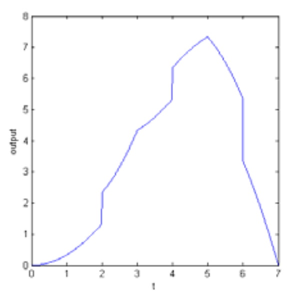
and

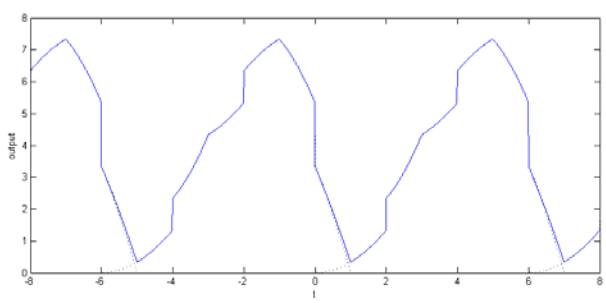
$$u(t)*h(t)=\int_{-\infty}^{t}h(\tau)d\tau=\begin{cases}0 & t<0\\ \frac{1}{3}t^2 & 0\le t<3\\ 3 & t\ge 3\end{cases}$$

So

t interval	$u(t)*h(t)$	$u(t-2)*h(t)$	$2u(t-4)*h(t)$	$x(t)*h(t)$
$t<0$	0	0	0	0
$0\le t<2$	$\frac{1}{3}t^2$	0	0	$\frac{1}{3}t^2$
$2\le t<3$	$\frac{1}{3}t^2$	$\frac{1}{3}(t-2)^2=\frac{1}{3}t^2-\frac{4}{3}t+\frac{4}{3}$	0	$\frac{2}{3}t^2-\frac{4}{3}t+\frac{4}{3}$
$3\le t<4$	3	$\frac{1}{3}t^2-\frac{4}{3}t+\frac{4}{3}$	0	$\frac{1}{3}t^2-\frac{4}{3}t+\frac{13}{3}$
$4\le t<5$	3	$\frac{1}{3}t^2-\frac{4}{3}t+\frac{4}{3}$	$\frac{2}{3}(t-4)^2=\frac{2}{3}t^2-\frac{16}{3}t+\frac{32}{3}$	$-\frac{1}{3}t^2+4t-\frac{19}{3}$
$5\le t<7$	3	3	$\frac{2}{3}t^2-\frac{16}{3}t+\frac{32}{3}$	$-\frac{2}{3}t^2+\frac{16}{3}t-\frac{14}{3}$
$t\ge 7$	3	3	6	0

Adding the lower part, the plot is as following left graph.





(c) Since $\tilde{x}(t)$ is the periodic version of $x(t)$ with the period of 6, $\tilde{y}(t)$ is the periodic version of $y(t)$ with the same period 6. The plot is as the above right graph.

3. **[6]** Let $x[n]$ be a periodic discrete-time sequence with period $N = 8$ and Fourier series coefficients $a_k = -a_{k-4}$. Now generate a sequence

$$y[n] = \left(\frac{1+(-1)^n}{2}\right) x[n-1]$$

with period $N = 8$ based on $x[n]$. Denoting the Fourier series coefficients of $y[n]$ as b_k, find a function $f[k]$ such that $b_k = f[k]\, a_k$.

Solution:

$$y[n] = \left(\frac{1+(-1)^n}{2}\right) x[n-1] = \frac{1}{2} x[n-1] + \frac{1}{2} \cdot (-1)^n x[n-1]$$
$$= \frac{1}{2} x[n-1] + \frac{1}{2} \cdot e^{j\pi n} x[n-1] = \frac{1}{2} x[n-1] + \frac{1}{2} \cdot e^{j4\frac{2\pi}{8}n} x[n-1]$$

So

$$b_k = \frac{1}{2} a_k e^{-jk\frac{2\pi}{8}} + \frac{1}{2} \cdot a_{k-4} e^{-j(k-4)\frac{2\pi}{8}} = \frac{1}{2} e^{-j\frac{\pi}{4}k} a_k + \frac{1}{2} \cdot (-a_k) e^{-j\frac{\pi}{4}k} e^{j\pi} = e^{-j\frac{\pi}{4}k} a_k$$

That is,

$$f[k] = e^{-j\frac{\pi}{4}k}$$

4. Consult tables of Fourier transform pairs and answer the following questions:

(a) **[4]** $x(t) = te^{-3|t-1|}$, what is $X(j\omega)$?

(b) **[4]** $x[n] = \dfrac{\sin(\frac{\pi}{4} n)}{\pi n} * \dfrac{\sin[\frac{\pi}{4}(n-8)]}{\pi(n-8)}$, what is $X(e^{j\omega})$? (Note: * denotes convolution.)

(c) **[6]** $X(j\omega) = \dfrac{d}{d\omega}\left[\dfrac{4\sin(4\omega)\sin(2\omega)}{\omega}\right]$, what is $x(t)$?

(d) **[6]** What is $\displaystyle\int_{-\infty}^{\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt$?

Solution:

Note: The following approaches are not the only approaches.

(a) $$e^{-3|t|} \stackrel{F}{\longleftrightarrow} \frac{6}{9+\omega^2} \quad\Rightarrow\quad e^{-3|t-1|} \stackrel{F}{\longleftrightarrow} \frac{6e^{-j\omega}}{9+\omega^2} \quad\Rightarrow\quad te^{-3|t-1|} \stackrel{F}{\longleftrightarrow} j\frac{d}{d\omega}\frac{6e^{-j\omega}}{9+\omega^2} = \frac{6e^{-j\omega}}{9+\omega^2} - \frac{12 j\omega e^{-j\omega}}{(9+\omega^2)^2}$$

Alternative solution:

$$\begin{aligned} te^{-3|t-1|} &= te^{-3(t-1)}u(t-1) + te^{-3(1-t)}u(1-t) \\ &= (t-1)e^{-3(t-1)}u(t-1) + e^{-3(t-1)}u(t-1) - (1-t)e^{-3(1-t)}u(1-t) + e^{-3(1-t)}u(1-t) \\ &= (t-1)e^{-3(t-1)}u(t-1) + e^{-3(t-1)}u(t-1) + (t-1)e^{3(t-1)}u(-(t-1)) + e^{3(t-1)}u(-(t-1)) \end{aligned}$$

It can be found that

$$e^{-3t}u(t) \stackrel{F}{\longleftrightarrow} \frac{1}{3+j\omega} \quad\Rightarrow\quad e^{-3(t-1)}u(t-1) \stackrel{F}{\longleftrightarrow} \frac{e^{-j\omega}}{3+j\omega} \quad\text{and}\quad e^{3t}u(-t) \stackrel{F}{\longleftrightarrow} \frac{1}{3-j\omega},$$

$$e^{3t}u(-t) \xleftrightarrow{F} \frac{1}{3-j\omega} \;\Rightarrow\; e^{3(t-1)}u(-(t-1)) \xleftrightarrow{F} \frac{e^{-j\omega}}{3-j\omega};$$

$$te^{-3t}u(t) \xleftrightarrow{F} \frac{1}{(3+j\omega)^2} \;\Rightarrow\; (t-1)e^{-3(t-1)}u(t-1) \xleftrightarrow{F} \frac{e^{-j\omega}}{(3+j\omega)^2} \text{ and } te^{3t}u(-t) \xleftrightarrow{F} \frac{-1}{(3-j\omega)^2},$$

$$te^{3t}u(-t) \xleftrightarrow{F} \frac{-1}{(3-j\omega)^2} \;\Rightarrow\; (t-1)e^{3(t-1)}u(-(t-1)) \xleftrightarrow{F} \frac{-e^{-j\omega}}{(3-j\omega)^2}$$

Therefore

$$\begin{aligned} te^{-3|t-1|} &= (t-1)e^{-3(t-1)}u(t-1) + e^{-3(t-1)}u(t-1) + (t-1)e^{3(t-1)}u(-(t-1)) + e^{3(t-1)}u(-(t-1)) \\ &\xleftrightarrow{F} \frac{e^{-j\omega}}{(3+j\omega)^2} + \frac{e^{-j\omega}}{3+j\omega} - \frac{e^{-j\omega}}{(3-j\omega)^2} + \frac{e^{-j\omega}}{3-j\omega} \\ &= \frac{e^{-j\omega}[(3-j\omega)^2-(3+j\omega)^2]}{(9+\omega^2)^2} + \frac{e^{-j\omega}[(3-j\omega)+(3+j\omega)]}{9+\omega^2} \\ &= \frac{e^{-j\omega}(3-j\omega+3+j\omega)(3-j\omega-3-j\omega)}{(9+\omega^2)^2} + \frac{6e^{-j\omega}}{9+\omega^2} \\ &= \frac{-12j\omega e^{-j\omega}}{(9+\omega^2)^2} + \frac{6e^{-j\omega}}{9+\omega^2} \end{aligned}$$

(b) $x[n] = \frac{\sin(\frac{\pi}{4}n)}{\pi n} * \frac{\sin[\frac{\pi}{4}(n-8)]}{\pi(n-8)}$, since $x_1[n]*x_2[n] \xleftrightarrow{F} X_1(e^{j\omega})X_2(e^{j\omega})$ and

$$\frac{\sin(\frac{\pi}{4}n)}{\pi n} \xleftrightarrow{F} \begin{cases} 1 & 0 \le |\omega| \le \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\omega| \le \pi \end{cases} \;\Rightarrow\; \frac{\sin[\frac{\pi}{4}(n-8)]}{\pi(n-8)} \xleftrightarrow{F} \begin{cases} e^{-j8\omega} & 0 \le |\omega| \le \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\omega| \le \pi \end{cases},$$

We conclude that $X(e^{j\omega}) = \begin{cases} e^{-j8\omega} & 0 \le |\omega| \le \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\omega| \le \pi \end{cases}$.

(c) $X(j\omega) = \frac{d}{d\omega}\left[\frac{4\sin(4\omega)\sin(2\omega)}{\omega}\right] = -j\frac{d}{d\omega}\left[j\omega\frac{4\sin(4\omega)\sin(2\omega)}{\omega^2}\right] = -j\frac{d}{d\omega}\left[j\omega\frac{2\sin(4\omega)}{\omega}\frac{2\sin(2\omega)}{\omega}\right]$

Since $x_1(t) = \begin{cases} 1 & |t|<4 \\ 0 & |t|>4 \end{cases} \xleftrightarrow{F} X_1(j\omega) = \frac{2\sin(4\omega)}{\omega}$, $x_2(t) = \begin{cases} 1 & |t|<2 \\ 0 & |t|>2 \end{cases} \xleftrightarrow{F} X_2(j\omega) = \frac{2\sin(2\omega)}{\omega}$,

we conclude that $x(t) = -t\frac{d}{dt}\left[x_1(t)*x_2(t)\right] = \begin{cases} |t| & 2<|t|<6 \\ 0 & |t|<2 \text{ or } |t|>6 \end{cases}$.

(d) Suppose $x(t) = \frac{\sin t}{\pi t}$, then $x(t) \xleftrightarrow{F} X(j\omega) = \begin{cases} 1 & |\omega|<1 \\ 0 & |\omega|>1 \end{cases}$, and

$$\int_{-\infty}^{\infty} t^2\left(\frac{\sin t}{\pi t}\right)^4 dt = \int_{-\infty}^{\infty}\left|t\left(\frac{\sin t}{\pi t}\right)^2\right|^2 dt = \int_{-\infty}^{\infty}\left|tx(t)^2\right|^2 dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|j\frac{d}{d\omega}\frac{1}{2\pi}X(j\omega)*X(j\omega)\right|^2 d\omega = \frac{1}{8\pi^3}\int_{-2}^{2} d\omega = \frac{1}{2\pi^3}$$

5. **[10]** Let $r_{xy}[n] = \sum_{m=-\infty}^{\infty} x[m+n]y^*[m]$ be the cross-correlation of two discrete-time sequences $x[n]$ and $y[n]$, where $y^*[n]$ denotes the complex conjugate of $y[n]$. Let $X(e^{j\omega})$ and $Y(e^{j\omega})$ be the Fourier transform of $x[n]$ and $y[n]$, respectively. Find the Fourier transforms of $r_{xx}[n]$, $r_{xy}[n]$, $r_{yx}[n]$, and $r_{yy}[n]$.

Solution:

$$r_{xy}[n] = \sum_{m=-\infty}^{\infty} x[m+n]y^*[m] = \sum_{m=-\infty}^{\infty} x[n-(-m)]y^*[-(-m)] \overset{\overbrace{m'=-m}}{=} \sum_{m'=-\infty}^{\infty} x[n-m']y^*[-m'] = x[n]*y^*[-n].$$

Since $y^*[-n] \overset{F}{\longleftrightarrow} Y^*(e^{j\omega})$, we have

$$r_{xy}[n] = x[n]*y^*[-n] \overset{F}{\longleftrightarrow} X(e^{j\omega})Y^*(e^{j\omega}),\quad r_{yx}[n] \overset{F}{\longleftrightarrow} Y(e^{j\omega})X^*(e^{j\omega}),$$

$$r_{xx}[n] \overset{F}{\longleftrightarrow} X(e^{j\omega})X^*(e^{j\omega}),\quad r_{yy}[n] \overset{F}{\longleftrightarrow} Y(e^{j\omega})Y^*(e^{j\omega}).$$

6. The continuous-time Fourier transform pair is sometimes defined using the ordinary frequency f instead of the angular frequency ω (that is, $\omega = 2\pi f$) as follows:

$$\mathrm{F}\{x(t)\} = X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt, \quad \text{and}$$

$$\mathrm{F}^{-1}\{X(f)\} = x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df.$$

(a) **[4]** Derive the multiplication property for the new Fourier transform.

(b) **[4]** Derive the duality property for the new Fourier transform:

$$\text{if } x(t) \overset{\mathrm{F}}{\longleftrightarrow} X(f), \text{ then } X(t) \overset{\mathrm{F}}{\longleftrightarrow} \boxed{}\,?$$

(c) **[4]** Let $\mathrm{F}^2\{x(t)\} = \mathrm{F}\left\{\mathrm{F}\{x(t)\}\big|_{f=t}\right\}$, and $\mathrm{F}^n\{x(t)\} = \mathrm{F}\left\{\mathrm{F}^{n-1}\{x(t)\}\big|_{f=t}\right\}$ for $n \in N$ and $n > 2$.

Using the duality property of the Fourier transform, show that

$$\mathrm{F}^2\{x(t)\}\big|_{f=t} = x(-t),\quad \mathrm{F}^3\{x(t)\}\big|_{f=t} = \mathrm{F}^{-1}\{x(f)\},\quad \text{and}\quad \mathrm{F}^4\{x(t)\}\big|_{f=t} = x(t).$$

Solution:

(a) The original multiplication property: $x(t)y(t) \overset{F}{\longleftrightarrow} \frac{1}{2\pi}X(j\omega)*Y(j\omega)$.

So we want to find the new Fourier transform of $x(t)y(t)$.

$$\int_{-\infty}^{\infty} x(t)y(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} x(t)\int_{-\infty}^{\infty} Y(f_0)e^{j2\pi f_0 t}df_0 e^{-j2\pi ft}dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)e^{-j2\pi(f-f_0)t}dt Y(f_0)df_0$$
$$= \int_{-\infty}^{\infty} X(f-f_0)Y(f_0)df_0 = X(f)*Y(f)$$

(b) $\int_{-\infty}^{\infty} X(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} X(t)e^{j2\pi t(-f)}dt = x(-f)$

(c) $\mathrm{F}^2\{x(t)\}\big|_{f=t} = x(-t)$ is exactly problem (b);

$$\mathcal{F}^3\{x(t)\}\Big|_{f=t} = \mathcal{F}\left\{\mathcal{F}^2\{x(t)\}\Big|_{f=t}\right\}\Bigg|_{f=t} = \mathcal{F}\{x(-t)\}\Big|_{f=t} = X(-t) = \mathcal{F}^{-1}\{x(f)\};$$

$$\mathcal{F}^4\{x(t)\}\Big|_{f=t} = \mathcal{F}\left\{\mathcal{F}^3\{x(t)\}\Big|_{f=t}\right\}\Bigg|_{f=t} = \mathcal{F}\left\{\mathcal{F}^{-1}\{x(f)\}\right\}\Big|_{f=t} = x(t)$$

7. **[8]** Consider a discrete-time sequence $x[n]$ and its time-expanded version $x_k[n] = x\left[\lfloor n/k \rfloor\right]$, where $\lfloor z \rfloor$ is the greatest integer less than or equal to z. Let $X(e^{j\omega})$ be the Fourier transform of $x[n]$. Find the Fourier transform $X_k(e^{j\omega})$ for $x_k[n]$.

Solution:

We know that $x_{(k)}[n] = \begin{cases} x[n/k] & k \mid n \\ 0 & k \nmid n \end{cases} \xleftrightarrow{F} X(e^{jk\omega})$, now $x_k[n] = \sum_{l=0}^{k-1} x_{(k)}[n-l]$, so

$$x_k[n] \xleftrightarrow{F} \sum_{l=0}^{k-1} e^{-jl\omega} X(e^{jk\omega}) = \frac{1-e^{-jk\omega}}{1-e^{-j\omega}} X(e^{jk\omega})$$

8. A causal and stable continuous-time LTI system H has the following frequency response

$$H(j\omega) = \frac{j\omega+4}{6-\omega^2+5j\omega}.$$

(a) **[4]** Determine a differential equation relating the input $x(t)$ to the output $y(t)$ of the system.

(b) **[6]** What is the output $y(t)$ when the input is $x(t) = e^{-4t}u(t) - te^{-4t}u(t)$?

Solution:

$$H(j\omega) = \frac{j\omega+4}{6-\omega^2+5j\omega} = \frac{j\omega+4}{6+5j\omega+(j\omega)^2} = \frac{j\omega+4}{(2+j\omega)(3+j\omega)}$$

(a) $\dfrac{d^2}{dt^2}y(t) + 5\dfrac{d}{dt}y(t) + 6y(t) = \dfrac{d}{dt}x(t) + 4x(t)$

(b) $X(j\omega) = \dfrac{1}{4+j\omega} - \dfrac{1}{(4+j\omega)^2} = \dfrac{3+j\omega}{(4+j\omega)^2}$,

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{(2+j\omega)(4+j\omega)} = \frac{\frac{1}{2}}{2+j\omega} - \frac{\frac{1}{2}}{4+j\omega}, \quad y(t) = \tfrac{1}{2}e^{-2t}u(t) - \tfrac{1}{2}e^{-4t}u(t)$$

9. Consider the following transform for a continuous-time signal $x(t)$:

$$\mathcal{H}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)[\cos(\omega t) + \sin(\omega t)]dt.$$

(a) **[6]** Show that $X(j\omega) = X_e(\omega) - jX_o(\omega)$, where $X_e(\omega)$ and $X_o(\omega)$ are the even and odd parts of $X(\omega)$, and $X(j\omega)$ is the continuous-time Fourier transform of $x(t)$.

(b) **[6]** If $x(t)$ is a real-valued function, show that $X(\omega) = \Re e\{X(j\omega)\} - \Im m\{X(j\omega)\}$, where $\Re e\{X(j\omega)\}$ and $\Im m\{X(j\omega)\}$ is the real and imaginary part of $X(j\omega)$, respectively.

(c) **[4]** Evaluate $\mathcal{H}\{t^2 e^{-3t}u(t)\}$.

Solution:

(a) We know that

$$
\begin{aligned}
X_e(\omega) &= \frac{X(\omega)+X(-\omega)}{2} \\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty} x(t)[\cos(\omega t)+\sin(\omega t)]dt + \int_{-\infty}^{\infty} x(t)[\cos(-\omega t)+\sin(-\omega t)]dt\right) \\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty} x(t)[\cos(\omega t)+\sin(\omega t)]dt + \int_{-\infty}^{\infty} x(t)[\cos(\omega t)-\sin(\omega t)]dt\right) \\
&= \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt
\end{aligned}
$$

Similarly, $X_o(\omega)=\int_{-\infty}^{\infty} x(t)\sin(\omega t)dt$, so

$$
\begin{aligned}
X_e(\omega)-jX_o(\omega) &= \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} x(t)\sin(\omega t)dt \\
&= \int_{-\infty}^{\infty} x(t)[\cos(\omega t)-j\sin(\omega t)]dt \\
&= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = X(j\omega)
\end{aligned}
$$

(b) Since $x(t)$ is real, $X(\omega)$ is also real. By (a), we may conclude that $X_e(\omega)=\Re e\{X(j\omega)\}$ and $X_o(\omega)=-\Im m\{X(j\omega)\}$, then $X(\omega)=X_e(\omega)+X_o(\omega)=\Re e\{X(j\omega)\}-\Im m\{X(j\omega)\}$.

(c) $t^2e^{-3t}u(t)=2\cdot\frac{t^2}{2!}e^{-3t}u(t)\xleftrightarrow{F}\frac{2}{(3+j\omega)^3}=\frac{2(3-j\omega)^3}{(9+\omega^2)^3}=\frac{54-18\omega^2}{(9+\omega^2)^3}+j\frac{-54\omega+2\omega^3}{(9+\omega^2)^3}$,

so $\mathrm{H}\{t^2e^{-3t}u(t)\}=\frac{54-18\omega^2}{(9+\omega^2)^3}-\frac{-54\omega+2\omega^3}{(9+\omega^2)^3}=\frac{54-18\omega^2+54\omega-2\omega^3}{(9+\omega^2)^3}$

10. Let $x[n]$ be a discrete-time sequence of finite duration N_1 such that $x[n]=0$ outside the interval $0\le n\le N_1-1$. The N-point discrete Fourier transform (DFT) of $x[n]$ is defined as

$$\tilde{X}[k]=\frac{1}{N}\sum_{n=0}^{N-1}x[n]\,e^{-jk(2\pi/N)n},\qquad k\in Z,$$

where N is an integer larger than N_1.

(a) **[4]** Find the relation between $\tilde{X}[k]$ and $X(e^{j\omega})$, where $X(e^{j\omega})$ is the discrete-time Fourier transform (DTFT) of $x[n]$. Show that $\tilde{X}[k]$ can be considered as samples of $X(e^{j\omega})$ taken at discrete values of ω.

(b) **[4]** If $\tilde{X}[k]$ instead of $X(e^{j\omega})$ is used to recover $x[n]$ using the inverse discrete-time Fourier series (DTFS)

$$\tilde{x}[n]=\sum_{k=0}^{N-1}\tilde{X}[k]\,e^{jk(2\pi/N)n},\qquad n\in Z,$$

describe the difference between the resultant $\tilde{x}[n]$ and the original sequence $x[n]$.

(c) **[4]** Based on the results in (a) and (b), discuss the implication of recovering a finite-duration sequence $x[n]$ using the continuous-time function $X(e^{j\omega})$ and its discrete-time version $\tilde{X}[k]$.

Solution:

(a) $\tilde{X}[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]\, e^{-jk(2\pi/N)n} = \frac{1}{N} X(e^{j\frac{2\pi k}{N}})$

(b) $x[n]$ and $\tilde{x}[n]$ are the same when $0 \le n \le N-1$. $\tilde{x}[n]$ is periodic in time with the period of N.

(c) (Key: Whenever $N < N_1$, $x[n]$ cannot be recovered since some of the samples are dropped.)