Signals and Systems, Final Exam

Solutions (Draft)

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1. **(10)** Consider the linear constant-coefficient second-order differential equation:

$$\frac{d^2}{dt^2}y(t)+2\zeta\omega_n\frac{d}{dt}y(t)+\omega_n{}^2y(t)=\omega_n{}^2x(t) \quad .$$

(a) Find the frequency response $H(j\omega)$ of the system.

(b) For $0<\zeta<\sqrt{2}/2$, what is the frequency ω_m where $\left|H(j\omega_m)\right|$ has a maximum value?

(c) What is the maximum value of $\left|H(j\omega_m)\right|$ at the frequency in (b)?

Solution:

(a) We have

$$(j\omega)^2Y(j\omega)+2\zeta\omega_n(j\omega)Y(j\omega)+\omega_n{}^2Y(j\omega)=\omega_n{}^2X(j\omega)$$

so

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{\omega_n{}^2}{(j\omega)^2+2\zeta\omega_n(j\omega)+\omega_n{}^2}$$

(b) To make $\left|H(j\omega)\right|$ maximum, we need the dominator (or the downstairs)

$$\left|(j\omega)^2+2\zeta\omega_n(j\omega)+\omega_n{}^2\right|^2=\left|\omega_n{}^2-\omega^2+j2\zeta\omega_n\omega\right|^2=\left(\omega_n{}^2-\omega^2\right)^2+4\zeta^2\omega_n{}^2\omega^2$$

to be minimum. So

$$\begin{aligned}\frac{d}{d\omega}\left[\left(\omega_n{}^2-\omega^2\right)^2+4\zeta^2\omega_n{}^2\omega^2\right]&=-4\omega\left(\omega_n{}^2-\omega^2\right)+8\zeta^2\omega_n{}^2\omega\\&=4\omega\left(-\omega_n{}^2+\omega^2+2\zeta^2\omega_n{}^2\right)\\&=4\omega\left(\omega^2-(1-2\zeta^2)\omega_n{}^2\right)\\&=0\end{aligned}$$

which holds when $\omega=0$ or $\omega=\pm\omega_n\sqrt{1-2\zeta^2}$, which is real since $0<\zeta<\sqrt{2}/2$, and of which the positive value is adopted. Considering that

$$\begin{aligned}\frac{d^2}{d\omega^2}\left[\left(\omega_n{}^2-\omega^2\right)^2+4\zeta^2\omega_n{}^2\omega^2\right]&=\frac{d}{d\omega}\left[4\omega\left(\omega^2-(1-2\zeta^2)\omega_n{}^2\right)\right]\\&=4\left(\omega^2-(1-2\zeta^2)\omega_n{}^2\right)+8\omega^2\\&=12\omega^2-4(1-2\zeta^2)\omega_n{}^2\end{aligned}$$

which is positive (minimum occurred) when $\omega=\omega_n\sqrt{1-2\zeta^2}$ and is negative (maximum occurred) when $\omega=0$, we have $\omega_m=\omega_n\sqrt{1-2\zeta^2}$.

(c) We have

$$\begin{aligned}\left|H(j\omega_m)\right|&=\frac{\omega_n{}^2}{\sqrt{\left(\omega_n{}^2-\omega_m{}^2\right)^2+4\zeta^2\omega_n{}^2\omega_m{}^2}}=\frac{\omega_n{}^2}{\sqrt{\left(\omega_n{}^2-\omega_n{}^2(1-2\zeta^2)\right)^2+4\zeta^2\omega_n{}^2\omega_n{}^2(1-2\zeta^2)}}\\&=\frac{\omega_n{}^2}{\sqrt{4\zeta^4\omega_n{}^4+4\zeta^2\omega_n{}^4-8\zeta^4\omega_n{}^4}}\\&=\frac{\omega_n{}^2}{2\zeta\omega_n{}^2\sqrt{1-\zeta^2}}=\frac{1}{2\zeta\sqrt{1-\zeta^2}}\end{aligned}$$

2. **(10)** Consider a continuous-time LTI system with frequency response $H(j\omega)=|H(j\omega)|e^{j\sphericalangle H(j\omega)}$ and real impulse $h(t)$. Suppose that we apply an input $x(t)=\sin(\omega_0 t+\phi_0)$ to the system. The resulting output can be shown to be the form $y(t)=Ax(t-t_0)$, where A is a nonnegative real number representing an amplitude-scaling factor and t_0 is a time delay.

(a) Express A in terms of $|H(j\omega_0)|$.

(b) Express t_0 in terms of $\sphericalangle H(j\omega_0)$.

Solution:

We know that $x(t)=\sin(\omega_0 t+\phi_0)=\frac{1}{2j}\left(e^{j(\omega_0 t+\phi_0)}-e^{-j(\omega_0 t+\phi_0)}\right)=\frac{1}{2j}\left(e^{j\phi_0}e^{j\omega_0 t}-e^{-j\phi_0}e^{-j\omega_0 t}\right)$. Observing that $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ are eigenfunctions with eigenvalues $H(j\omega_0)$ and $H(-j\omega_0)$, respectively, the output $y(t)$ is given by

$$y(t)=\frac{1}{2j}\left(e^{j\phi_0}H(j\omega_0)e^{j\omega_0 t}-e^{-j\phi_0}H(-j\omega_0)e^{-j\omega_0 t}\right)$$

Now $H(j\omega)=|H(j\omega)|e^{j\sphericalangle H(j\omega)}$, and since $h(t)$ is real, we know that $|H(j\omega)|$ is even (implying $|H(j\omega_0)|=|H(-j\omega_0)|$) and $\sphericalangle H(j\omega)$ is odd (implying $\sphericalangle H(j\omega_0)=-\sphericalangle H(-j\omega_0)$), and then we have

$$\begin{aligned}
y(t)&=\frac{1}{2j}\left(e^{j\phi_0}\,|H(j\omega_0)|e^{j\sphericalangle H(j\omega_0)}e^{j\omega_0 t}-e^{-j\phi_0}\,|H(j\omega_0)|e^{-j\sphericalangle H(j\omega_0)}e^{-j\omega_0 t}\right)\\
&=|H(j\omega_0)|\sin\left[\omega_0 t+\phi_0+\sphericalangle H(j\omega_0)\right]\\
&=|H(j\omega_0)|\sin\left[\omega_0\left(t+\frac{\sphericalangle H(j\omega_0)}{\omega_0}\right)+\phi_0\right]
\end{aligned}$$

So $A=|H(j\omega_0)|$, and $t_0=-\dfrac{\sphericalangle H(j\omega_0)}{\omega_0}$

3. **(6)** Compute the group delay of the following frequency response.

$$H(j\omega)=\frac{1}{(j\omega+1)(j\omega+3)}$$

Note that $\frac{d}{dx}\tan^{-1}(x)=\frac{1}{1+x^2}$.

Solution:

$H(j\omega)=\dfrac{1}{j\omega+1}\dfrac{1}{j\omega+3}$, so $\sphericalangle H(j\omega)=-\tan^{-1}\omega-\tan^{-1}\dfrac{\omega}{3}$, and then the group delay is given by

$$\begin{aligned}
\tau(\omega)&=-\frac{d}{d\omega}\sphericalangle H(j\omega)\\
&=\frac{d}{d\omega}\tan^{-1}\omega+\frac{d}{d\omega}\tan^{-1}\frac{\omega}{3}\\
&=\frac{1}{1+\omega^2}+\frac{1}{3}\cdot\frac{1}{1+(\frac{\omega}{3})^2}\\
&=\frac{1}{1+\omega^2}+\frac{3}{9+\omega^2}
\end{aligned}$$

4. **(12)** Let $x(t)=\cos(\omega_0 t)$ and $x(t)$ is sampled and filtered by the following system with a sampling time T, $\omega_s = 2\pi / T$, and $\omega_c = \omega_s / 2$.

(a) If $\omega_0 = 2\omega_s / 6$, identify the signal $x_r(t)$.

(b) If $\omega_0 = 5\omega_s / 6$, identify the signal $x_r(t)$.

(c) Does any aliasing occur in (a) and/or (b)? Justify your answer.

(Figure)

Solution:

We illustrate the result with spectra of $X(j\omega)$, $X_p(j\omega)$, and $X_r(j\omega)$.

(a)

(Figure)

So $x_r(t) = x(t) = \cos\left(\frac{2\omega_s}{6} t\right) = \cos\left(\omega_0 t\right)$.

(b)

(Figure)

So $x_r(t) = \cos\left(\frac{\omega_s}{6} t\right) = \cos\left((\omega_s - \omega_0)t\right) \neq x(t)$.

(c) Aliasing occurs in (b).

5. **(20)** The following figure shows the overall system for filtering a continuous time signal using a discrete-time filter. If $X_c(j\omega)$ and $H(e^{j\omega})$ are as shown in the following figure, with $1/T = 10\text{kHz}$, sketch $X_p(j\omega)$, $X(e^{j\omega})$, $Y(e^{j\omega})$, $Y_p(j\omega)$, and $Y_c(j\omega)$. Note that you need to specify all the critical values on the horizontal and vertical axes of the above plots.

(Figure)

Solution:

We have $\omega_s = 2\pi/T = 2\pi \cdot 10^4\,\text{Hz}$. So the spectra needed are as shown as the following:

(Figure)

6. **(4)** What are amplitude modulation, phase modulation, and frequency modulation? Please describe them in terms of a modulating signal $x(t)$ and a carrier signal $c(t) = A\cos(\omega_c t + \theta_c)$.

Solution:

AM: Modifying the amplitude of $c(t)$ according to $x(t)$. $y(t) = A(t)\cos(\omega_c t + \theta_c)$,

where $A(t) = A_0 + k_a x(t)$ for some constants A_0 and k_a.

FM: Modifying the frequency of $c(t)$ according to $x(t)$. $y(t) = A\cos(\theta(t)) = A\cos(\omega_c(t)\cdot t + \theta_c(t))$,

where $\dfrac{d\theta(t)}{dt} = \omega_c + k_f x(t)$ for some constants ω_c and k_f.

PM: Modifying the phase of $c(t)$ according to $x(t)$. $y(t) = A\cos(\omega_c t + \theta_c(t))$,

where $\theta_c(t) = \theta_0 + k_p x(t)$ for some constants θ_0 And k_p.

7. **(8)** Consider the system for single-sideband amplitude modulation. If $X(j\omega)$ is shown as follows and $\omega_c > 2\omega_M$, please sketch the spectra of $Y_1(j\omega)$, $X_p(j\omega)$, $Y_2(j\omega)$, and $Y(j\omega)$.

(Figure)

Solution:

(Figure)

8. **(8)** If $X(s)$ is the Laplace transform of $x(t)$, that is,

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \quad \text{ROC} = R$$

where R is as shown in the following figure. Show the following time scaling property and plot the new ROC, R_a.

$$x(-0.3t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{10}{3} X(-\frac{10}{3}s), \quad \text{ROC} = R_a$$

(Figure)

Solution:

$$\mathcal{L}\{x(-0.3t)\} = \int_{-\infty}^{\infty} x(-0.3t)e^{-st}dt$$

$$= -\frac{10}{3}\int_{\infty}^{-\infty} x(-0.3t)e^{-\frac{s}{-0.3}(-0.3t)} d(-0.3t)$$

$$= \frac{10}{3}\int_{\infty}^{-\infty} x(\tau)e^{-\frac{10s}{-3}\tau} d\tau \quad (\tau \triangleq -0.3t)$$

$$= \frac{10}{3} X(-\frac{10}{3}s)$$

(Figure)

To hold the last equality, we need $-\frac{10}{3}s \in R = \{s : -2 < \text{Re}[s] < 1\}$.

So $-2 < -\frac{10}{3}\text{Re}[s] < 1$, $-0.3 < \text{Re}[s] < 0.6$. Therefore the ROC is as the above figure and is given by

$$R_a = \{s : -0.3 < \text{Re}[s] < 0.6\}$$

9. **(10)** Given a system function $H(s)$ as follows:

$$H(s) = \frac{-s}{s^2 + 3s + 2}, \quad \text{ROC} = \{-2 < \text{Re}[s] < -1\}$$

(a) Is the system stable? Justify your answer.
(b) Is the system causal, anticausal, or neither? Justify your answer.
(c) If the input to the system is an impulse function, that is, $x(t) = \delta(t)$, find the output $y(t)$.

Solution:

$H(s) = \frac{-s}{s^2 + 3s + 2} = \frac{-s}{(s+1)(s+2)}$, so the poles are $p_1 = -1$, $p_2 = -2$, and the zero is $z = 0$.

(a) Since the ROC doesn't include the $j\omega$-axis, the system is NOT stable.

(b) Since the ROC is neither a right-half plane nor a left-half plane, the system is NEITHER causal NOR anticausal.

(c) $H(s) = \frac{-s}{(s+1)(s+2)} = \frac{-2}{s+2} + \frac{1}{s+1}$, so $h(t) = -2e^{-2t}u(t) - e^{-t}u(-t)$.

10. **(6)** A causal LTI system S with impulse response $h(t)$ has its input $x(t)$ and output $y(t)$ related through a linear constant-coefficient differential equation of the form

$$\frac{d^3y(t)}{dt^3}+(a+1)\frac{d^2y(t)}{dt^2}+a(a+1)\frac{dy(t)}{dt}+a^2y(t)=\frac{dx(t)}{dt}-2x(t)$$

Please use adder, amplifier, and integrator only and as few components as possible to plot the block diagram representation of the system S.

Solution:

An instance:

(Figure)

11. **(10)** Consider a discrete-time signal:

$$x[n]=7\left(\frac{1}{3}\right)^n u[n]-6\left(\frac{1}{2}\right)^n u[-n-1]$$

Find the z-transform.

Solution:

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}=\sum_{n=0}^{\infty}7\left(\frac{1}{3}\right)^n z^{-n}+\sum_{n=-\infty}^{-1}-6\left(\frac{1}{2}\right)^n z^{-n}=7\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n z^{-n}-6\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{-n} z^{n}$$

$$=7\cdot\frac{1}{1-\frac{1}{3}z^{-1}}-6\frac{2z}{1-2z}=7\cdot\frac{1}{1-\frac{1}{3}z^{-1}}+6\frac{1}{1-\frac{1}{2}z^{-1}}$$

Note that we have applied the formula of sum of infinite power series.

So we need the ROC be $\{z:\frac{1}{3}<|z|<\frac{1}{2}\}$.

12. **(8)** If

$$x_1[n]\xleftrightarrow{z}X_1(z),\quad \text{ROC}=R_1$$

and

$$x_2[n]\xleftrightarrow{z}X_2(z),\quad \text{ROC}=R_2$$

show that

$$x_1[n]*x_2[n]\xleftrightarrow{z}X_1(z)X_2(z)\quad \text{with ROC containing } R_1\cap R_2$$

Solution:

We know that $\sum_{n=-\infty}^{\infty}x_1[n]z^{-n}=X_1(z)$ and $\sum_{n=-\infty}^{\infty}x_2[n]z^{-n}=X_2(z)$, so

$$\mathcal{Z}\{x_1[n]*x_2[n]\}=\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}x_1[k]x_2[n-k]\cdot z^{-n}$$

$$=\sum_{k=-\infty}^{\infty}x_1[k]z^{-k}\sum_{n=-\infty}^{\infty}x_2[n-k]z^{-n+k}\overset{m\triangleq n-k}{=}\sum_{k=-\infty}^{\infty}x_1[k]z^{-k}\sum_{m=-\infty}^{\infty}x_2[m]z^{-m}$$

$$=\sum_{k=-\infty}^{\infty}x_1[k]z^{-k}X_2(z)=X_2(z)\sum_{k=-\infty}^{\infty}x_1[k]z^{-k}=X_2(z)X_1(z)=X_1(z)X_2(z)$$

We need the both z-transform relation to hold, so the ROC must contain the intersection of the both ROCs, that is, $R_1\cap R_2$.

13. **(8)** Consider a causal LTI system whose input $x[n]$ and output $y[n]$ are related through the block diagram representation shown in the figure.

(Figure)

(a) Determine a differential equation relating $y[n]$ and $x[n]$.

(b) Is the system stable?

Solution:

(Note: "differential" should be "difference," although it doesn't matter.)

(a) $y[n]-\frac{5}{6}y[n-1]+\frac{1}{6}y[n-2]=x[n]-3x[n-1]+2x[n-2]$

(b) $H(z)=\dfrac{1-3z^{-1}+2z^{-2}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}=\dfrac{1-3z^{-1}+2z^{-2}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$, so the poles are $p_1=\frac{1}{2}$, $p_2=\frac{1}{3}$, which are both inside the unit circle. Since the system is a causal LTI system, the system is stable.