Probability and Statistic, Midterm Exam, Spring 2009

1. (10%) Let us deal a bridge hand from a well-shuffled deck of cards (Note that there are 52 cards in a deck).

   (a) (5%) Find the probability of the event that there are two aces and one spade in hand, given that there is no picture (i.e., no kings, queens, nor jakes) in hand.

   (b) (5%) Find the probability of the event that there are less than five spades in hand.

2. (8%) Define a function of two events A and B by

$$g(A; B) \doteq \frac{P[A \cap B]}{P[A \cup B]}$$

   for all events A, B. Now, for any given event B, can $g(A; B)$ be a probability measure? (Hint: A probability measure has to satisfy the Three Axioms of Probability.)

3. (8%) Prove that if events A and B are disjoint, then $P[A] \leq P[B^c]$.

4. (8%) Tom and Mary roll two fair dice alternately starting with Tom. The person that rolls *eleven* wins. Derive the probability p that Tom wins.

5. (8%) Prove that $P[A] = P[A|X \leq x]F_X(x) + P[A|X > x](1 - F_X(x))$.

6. (8%) Let X be a geometric(p) random variable. Prove that $Var[X] = (1 - p)/p^2$.

$f_X(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $\quad Y=aX+b \quad X=\frac{Y-b}{a} \quad X=\frac{Y-b}{a}$

7. (8%) X is Gaussian with mean μ and variance σ. Prove that $Y = aX + b$ is also Gaussian for any $a, b \in \mathbf{R}$.

$F_X(x)=\Phi\left(\frac{X-\mu}{\sigma}\right)$

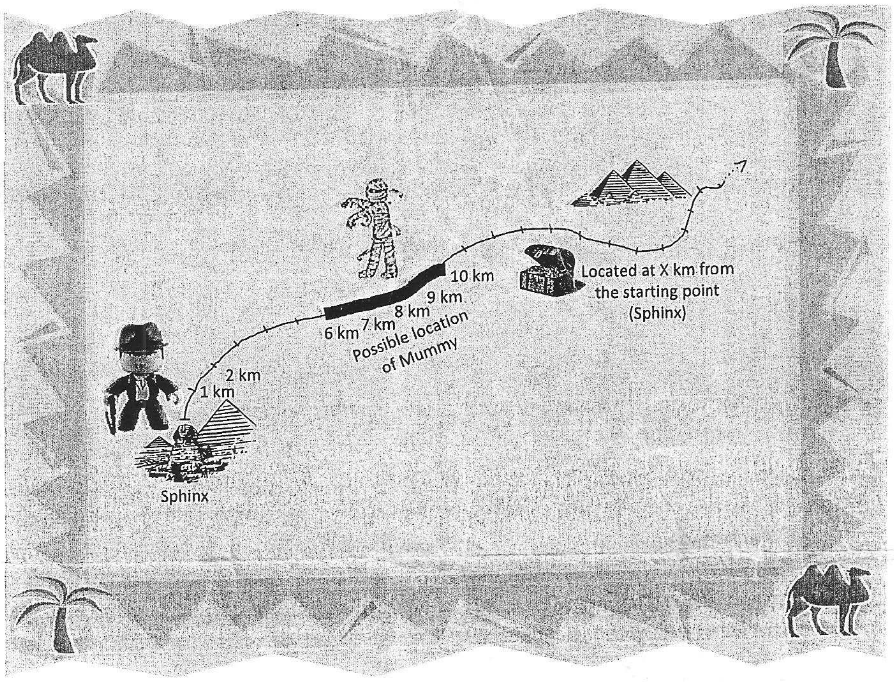
8. Derivation of the third moment.

$\lambda e^{-\lambda x} = e^{-x}$

(a) (5%) X is exponential random variable with $\lambda = 1$. Find the third moment $E[X^3]$.

(b) (5%) X is Gaussian random variable with $\mu = 0$ and $\sigma = 1$. Find the third moment $E[X^3]$.

$P\left[\frac{Y-b}{a}\le x\right]=\Phi\left(\frac{x-\mu}{\sigma}\right)$



$-x^3e^{-x}-3x^2e^{-x}-6xe^{-x}-6e^{-x}$

$-e^{-x}(x^3+3x^2+6x+6)$

$P[Y\le y]$ $\quad P[aX+b\le y]$ $\quad P\left[X\le\frac{y-b}{a}\right]$

Figure 1: Ancient Egyptian Map (translated to English)

9. Dr. Indiana Jones recently discovered an ancient Egyptian manuscript with a map (shown in Fig. 1) which gives hints about the Pharaoh's mysterious treasure. The manuscript mentions that the treasure is located on a secret route started from Sphinx as shown in the map. Due to the magic power of the Pharaoh's priest, the location of the treasure is hard to find and seems to be random. The author mentioned in the manuscript that he gave up searching for the treasure because no matter how far he had traveled on the route, the distribution of the remaining distance to the treasure just seemly remained unchanged due to the magic power of the Pharaoh's priest. The author also mentioned that it is generally believed that the expected distance of the treasure to Sphinx on the secret route is 10 km.

(a) (4%) In the manuscript, it is mentioned that there is a mummy hanging around in the region of 6 km to 10 km on the secret route from Sphinx. This implies that the actual location of the mummy is uniformly distributed in the region. Dr. Jones is really scared of meeting mummy during the treasure hunt (you know...Jones is also a human...). Can you compute the risk, i.e. the probability $g(X)$ of Dr. Jones meeting the mummy during he trip all the way from Sphinx to the treasure location X? (Hint: $g(X) = 0$ if $0 \leq X < 6$. What about the other cases?)

(b) (4%) Though Dr. Jones is well known as a great adventurer and archeologist, he is also an expert in probability and statistics due to a great course he took back in his undergraduate years. From the description of the ancient manuscript, Dr. Jones concludes that the distance X of the treasure from Sphinx on the secret route is exponentially distributed with $\lambda = \frac{1}{10}$. Can you explain why he can reach such conclusion from the manuscript?

(c) (8%) Based on the two sub-questions above, compute the expected risk $E[g(X)]$ of Dr. Jones meeting the mummy before he reaches the treasure location.

(d) (4%) Dr. Jones married couple of years ago and he got several kids. Before he leaves for the treasure hunt, Mrs. Jones forces him to go to the insurance company to buy the Mummy Insurance™ plan, which pays 1 million dollar to Dr. Jones should he meet the mummy during the trip. If you are the manager of the insurance company, what is the minimum price that you will charge Dr. Jones for the insurance plan so that the company is expected not to lose money from the insurance deal?

(e) (6%) During the treasure hunt, Dr. Jones meets the mummy at 6 km on the secret route from Sphinx. Surprisingly, the mummy is quite friendly (mummies are not necessarily all evil!!). The mummy tells Dr. Jones that the remaining distance to the treasure, i.e. $X' = X - 6$ is uniformly distributed in $[0, 5]$. Dr. Jones can not tell whether the mummy is lying or not from the facial expression of the mummy (mummy's face is always covered...). The ancient manuscript mentions that $\frac{1}{3}$ of the mummies like to lie. So there is $\frac{2}{3}$ of the chance that the mummy is telling the truth, and $\frac{1}{3}$ of the chance that Dr. Jones' original exponential assumption of X is correct. Can you find the pdf of the remaining distance $X' = X - 6$ to the treasure?

(f) (6%) Can you calculate the expected value and the variance of the remaining distance $X' = X - 6$ to the treasure from the pdf you derived in the sub-question above?

Eventually Dr. Jones discovers the treasure at $X = 12$ km. He learns a good lesson that friendly mummies are not necessarily all trustworthy. With the treasure found, Dr. Jones lives happily ever after...till his next adventure...