Department of Electrical Engineering, National Taiwan University

Engineering Mathematics-Differential Equation, 2010, Fall

Final Examination

2011/1/12 Wednesday, 10:20-12:10

1. (14 scores) Find the general solutions of

(a)
$$\begin{cases} 2\frac{d}{dt}x(t)-\frac{d}{dt}y(t)=0 \\ \frac{d}{dt}x(t)+\frac{d}{dt}y(t)=3\left[-x(t)+y(t)\right] \end{cases}$$
(7 scores)

(b)
$$\begin{cases} \frac{d}{dt}x_1(t)-x_1(t)=x_2(t) \\ \frac{d}{dt}x_2(t)=x_2(t) \\ \frac{d}{dt}x_3(t)=x_3(t) \\ \frac{d}{dt}x_4(t)-x_4(t)=x_5(t) \\ \frac{d}{dt}x_5(t)=x_5(t) \end{cases}$$
(7 scores)

2. (16 scores)

(a) Find power series solutions about $x=0$. **Also specify the region of convergence.**

$$(x^2+1)y''(x)-2y(x)=0$$

(8 scores)

(b) Find power series solutions about $x=0$.

$$x^2y''(x)+xy'(x)+\left(x^2-\frac{4}{9}\right)y(x)=0$$

(8 scores)

3. (10 scores) Legendre polynomials $P_n(x)$ are orthogonal on the interval$[-1,1]$. The first 3 Legendre polynomials are

$$P_0(x)=1$$
$$P_1(x)=x$$
$$P_2(x)=\frac{3}{2}x^2-\frac{1}{2}$$

(a) Verify $\{P_0(x),P_1(x),P_2(x)\}$ is an orthogonal set. (5 scores)

(b) Find the first three coefficients, a_0, a_1 and a_2 in the orthogonal series expansion of $f(x)$.

$$f(x)=a_0P_0(x)+a_1P_1(x)+a_2P_2(x)+\cdots$$

where $f(x)=\begin{cases} -1, & -1<x<0 \\ 1, & 0<x<1 \end{cases}$ (5 scores)

4. **(10 scores) Let $\Phi_n(x) \triangleq e^{i\frac{n\pi}{p}x}$, where $i = \sqrt{-1}$, n are integers.**

Define the set S as

$$S \triangleq \{\ldots \Phi_{-n}(x), \ldots \Phi_{-2}(x), \Phi_{-1}(x), \Phi_0(x), \Phi_1(x), \Phi_2(x), \ldots \Phi_n(x), \ldots\}$$

The inner product of $\Phi_m(x)$ and $\Phi_l(x)$ over $[-p, p]$ is defined as

$$(\Phi_m(x), \Phi_l(x)) = \int_{-p}^{p} \Phi_m^*(x)\Phi_l(x)dx$$

where $\Phi_m^*(x)$ denotes the complex conjugate of $\Phi_m(x)$.

(a) Show that S is an orthogonal set.

(5 scores)

(b) Assume $f(x)$ is a piecewise continuous real function over $[-p, p]$. The complex form Fourier series of $f(x)$ is

$$\sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi}{p}x}.$$

Show that

$$C_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-i\frac{n\pi}{p}x}dx.$$

(5 scores)

5. **(15 scores) Solve $16\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$ for $\pi > x > 0,\ t > 0$ with**

$$\begin{cases} u(0,t) = 0, u(\pi,t) = 0 \\ u(x,0) = \sin x + 2\sin 2x \\ \dfrac{\partial u(x,t)}{\partial t}\Big|_{t=0} = 0 \end{cases}$$

(a) Find $u(x,t)$ by the method of separation of variables.

(10 scores)

(b) Show that $u(x,t)$ can be expressed as $\frac{1}{2}[f(x+at) + f(x-at)]$.

(5 scores)

6. **(5 scores)**

(a) Expand $f(x) = e^x, 0 < x < 1$ in a cosine series.

(3 scores)

(b) Find the value of $f(999)$ in Problem6(a)

(2 scores)

7. (10 scores) Using Fourier series to solve the following DE,

$$x'' + 10x = f(t),\ x(0) = 0,\ x'(0) = 0$$

$$f(t) = \begin{cases} 5, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases};\ f(t+2\pi) = f(t)$$

8. (10 scores)

(a) Find the corresponding time-domain waveform $f(t)$ of the S-domain function $\frac{e^{-s}}{s}(\frac{1}{1-e^{-s}})$, and graph it.

(2 scores)

(b) Find the corresponding time-domain waveform $f(t)$ of the S-domain function $\frac{e^{-s}}{s}(\frac{1}{1+e^{-s}})$, and graph it.

(2 scores)

(c) Use the $f(t)$ of Problem8(b) and Laplace Transform to solve the DE:

$$x'' + 2x' + x = 5f(t), x(0) = x'(0) = 0$$

(4 scores)

(d) Discuss the differences of responses from the Fourier Transform in **Problem 7** and the Laplace Transform in Problem8(c).

(2 scores)

9. (10 scores) Please use the following initial value problem

$$\boldsymbol{t^2y'' + ty' + t^2y = 0, y(0) = 1, y'(0) = 0}$$

to show the inverse Laplace transform of $\frac{1}{\sqrt{s^2+1}}$ is Bessel Function:

$$\boldsymbol{\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2+1}}\right\} = J_0(t)}$$

- 期末考成績預定公告時間：1/17（一）中午 12:00 公告於電機系助教公布欄
- 期末考預定看考卷時間：1/17（一）晚上 18:00-19:30 在博理 113
- 如有更改，將另行公告於電機系二館助教公布欄與 ptt 電機系功課板，請各位同學密切注意。
- 祝各位學弟妹們期末考順利，新年快樂！

國立台灣大學電機工程學系

網路與多媒體實驗 兼任工程數學-微分方程/機率與統計

專任助教

趙式隆 敬上