A

[illegible]

Linear Algebra Midterm

Dept. of Elec. Eng., National Taiwan University

April 19, 2006

USE OF ALL AUTOMATIC COMPUTING MACHINES IS PROHIBITED

1. Judge if the following statements are true or false. Give a concise proof to each true statement, and a counterexample to each false statement. (25%)
   (a) If $A\mathbf{x} = \mathbf{b}$ is consistent, then the nullity of the augmented matrix $[\ A \ \ \mathbf{b}\ ]$ equals the number of free variables in the general solution to $A\mathbf{x} = \mathbf{b}$.
   (b) If the set S is linearly independent, then no vector in S is a linear combination of the others.
   (c) The determinant of any square matrix equals the product of the diagonal entries of its reduced row echelon form.
   (d) The null space of any matrix equals the null space of its reduced row echelon form.
   (e) For any $m \times n$ matrix A and $n \times m$ matrix B, $\det(AB) = \det(BA)$.

2. For any two subspaces $\mathcal{V}$ and $\mathcal{U}$ of $\mathcal{R}^n$, a set $\mathcal{W} = \{\mathbf{w} \mid \mathbf{w} = \mathbf{v}+\mathbf{u}$, where $\mathbf{v} \in \mathcal{V}$ and $\mathbf{u} \in \mathcal{U}\}$ is defined and denoted as $\mathcal{V}+\mathcal{U}$. In other words, $\mathcal{V}+\mathcal{U}$ is defined as the set of all vectors that are obtained by adding a vector in $\mathcal{V}$ and a vector in $\mathcal{U}$.
   (a) Prove that in general $\mathcal{V}+\mathcal{U}$ is a subspace of $\mathcal{R}^n$. (10%)
   (b) Suppose $\mathcal{V} \cap \mathcal{U} = \{\mathbf{0}\}$, $\mathcal{B}_v$ is a basis of $\mathcal{V}$, and $\mathcal{B}_u$ is a basis of $\mathcal{U}$. Prove that the set $\mathcal{B}_v \cup \mathcal{B}_u$ is a basis of $\mathcal{V}+\mathcal{U}$. (15%)

3. Let R be the reduced row echelon form of an arbitrary $m \times n$ matrix A.
   (a) Find the reduced echelon form of R^T and show that rank R^T = rank A. (15%)
   (b) Use (a) to show that rank A^T = rank A. (10%)

4. Let T be the linear transformation of the orthogonal projection in $\mathcal{R}^3$ onto the plane $\mathcal{P}$. More specifically, for all $\mathbf{v} \in \mathcal{R}^3$, $T(\mathbf{v}) = \mathbf{w}$ as in the following figure, where $\mathbf{w}$ is the perpendicular projection image of $\mathbf{v}$ on the plane $\mathcal{P}$.
   (a) For $\mathcal{P} = \{x_1 - 2x_2 + 3x_3 = 0\}$, select a basis S with two vectors on $\mathcal{P}$ and a vector not on $\mathcal{P}$, and compute $[T]_S$. (15%)
   (b) Prove that for any basis $\mathcal{B}$ of $\mathcal{R}^3$ the value of $\det([T]_{\mathcal{B}})$ is a constant and find the value. (10%)