Signals and Systems, Final Exam

Solutions (Draft)

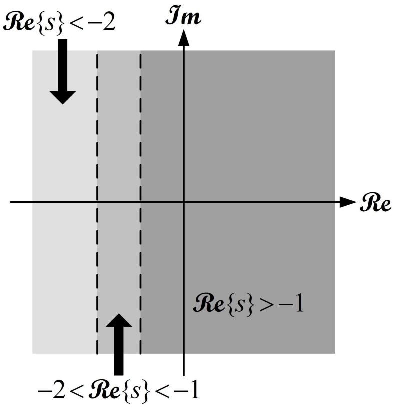
Spring 2009, Edited by bypeng

1. **[16]** Consider that a continuous-time signal $x(t)$ with its Laplace transform given by

$$X(s) = \frac{(s+3)}{(s+1)(s+2)}$$

   (a) Find all possible inverse bilateral Laplace transforms and justify your answer.
   (b) Sketch the region of convergence (ROC) in each case.
   (c) Determine each time function as causal, noncausal, or two-sided and justify your answer.
   (d) Determine each time function as bound-input and bound-output (BIBO) stable or not BIBO stable and justify your answer.

Solution:



We may find that

$$X(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{1}{s+2}$$

The possible ROCs are $\mathrm{Re}\{s\} > -1$, $-2 < \mathrm{Re}\{s\} < -1$, and $\mathrm{Re}\{s\} < -2$.

For the case $\mathrm{Re}\{s\} > -1$, $x(t) = 2e^{-t}u(t) - e^{-2t}u(t)$,
which is causal and BIBO stable.

For the case $-2 < \mathrm{Re}\{s\} < -1$, $x(t) = -2e^{-t}u(-t) - e^{-2t}u(t)$,
which is two-sided and not BIBO stable.

For the case $\mathrm{Re}\{s\} < -2$, $x(t) = -2e^{-t}u(-t) + e^{-2t}u(-t)$,
which is anti-causal and not BIBO stable.

2. **[16]** Consider that a linear time-invariant (LTI) system $h(t)$ has its transfer function $H(z)$ given by

$$H(z) = \frac{\sum_{i=0}^{M} b_i z^{-i}}{\sum_{i=0}^{N} a_i z^{-i}}$$

   Let $H_i(z)$ be the inverse system of $H(z)$.
   (a) Determine the constraint on the orders M and N if the LTI system $H(z)$ is causal and justify your answer.
   (b) Determine the constraint on the orders M and N if the inverse system is also causal and justify your answer.
   (c) Determine the constraint on the poles of $H(z)$ if $H(z)$ is stable and justify your answer.
   (d) Determine the constraint on the poles and zeros of $H(z)$ if $H_i(z)$ is also stable and justify your answer.

Solution:

(a) Letting $K = \max\{M, N\}$, it is shown

$$H(z) = \frac{\sum_{i=0}^{M} b_i z^{-i}}{\sum_{i=0}^{N} a_i z^{-i}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} = \frac{b_0 z^K + b_1 z^{K-1} + b_2 z^{K-2} + \cdots + b_M z^{K-M}}{a_0 z^K + a_1 z^{K-1} + a_2 z^{K-2} + \cdots + a_N z^{K-N}}$$

which shows $H(z)$ as a ratio of polynomials in z. We now that a discrete-time LTI system with rational system function $H(z)$ is causal if and only if (a) the ROC is exterior of a circle outside the outermost pole, and (b) with $H(z)$ expressed as a ratio of polynomial in z, the order of the numerator cannot be greater than the order of the denominator. For any (M,N), this derivation satisfies the above condition of causality, since both the order of the numerator and that of the denominator are K. We claim that there are no constraints on the orders M and N to make $H(z)$ causal.

(b) It is well known that

$$H_i(z) = H^{-1}(z) = \frac{\sum_{i=0}^{N} a_i z^{-i}}{\sum_{i=0}^{M} b_i z^{-i}}$$

Similar derivation in (a) may be applied on $H_i(z)$. We claim that there are no constraints on the orders M and N to make $H_i(z)$ causal.

(c) An LTI system is stable if and only if the ROC of its system function $H(z)$ includes the unit circle. Since $H(z)$ is rational, any of the possible ROCs of $H(z)$ is in one of the following form: $|z|<\alpha$, $\alpha<|z|<\beta$, or $|z|>\beta$. To make the LTI system with system function $H(z)$ stable, we need to select the ROC which includes the unit circle. The only case that we cannot select an ROC which includes the unit circle is that at least one of the poles lies on the unit circle. It is the only constraint that none of the poles lies on the unit circle.

(d) With similar inference, we need neither any of the poles nor any of the zeros lies on the unit circle.

(NOTE: in problem (c) and (d), we do not need the system being causal. So we do not need the poles lying inside the unit circle.)

3. **[8]** Consider a causal LTI system specified by the following linear constant coefficient equation (LCCDE) with input $x(t)$ and output $y(t)$:

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

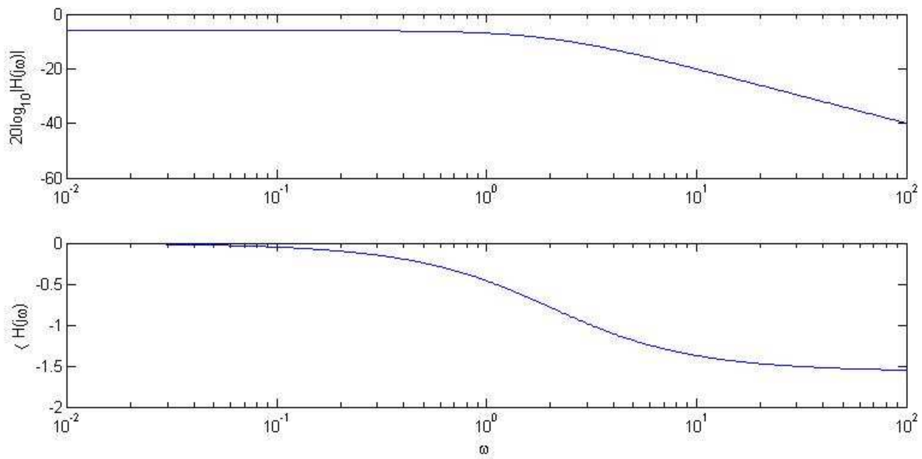
(a) Sketch the Bode plot of the frequency response of the LTI system.

(b) Determine the group delay of the LTI system.

Solution:

$$\frac{dy(t)}{dt} + 2y(t) = x(t),\quad j\omega Y(j\omega) + 2Y(j\omega) = X(j\omega),\quad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{2+j\omega} = \frac{1}{2}\cdot\frac{1}{1+\frac{j\omega}{2}}.$$

The Bode plot is as the following figure.

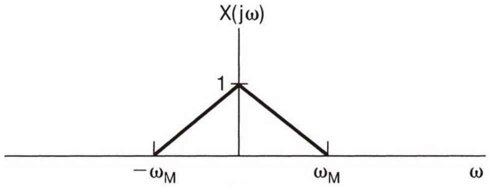


$$\sphericalangle H(j\omega) = -\tan^{-1}\frac{\omega}{2},\quad \tau(\omega) = -\frac{d}{d\omega}\{\sphericalangle H(j\omega)\} = \frac{d}{d\omega}\tan^{-1}\frac{\omega}{2} = \frac{1}{2}\cdot\frac{1}{1+\left(\frac{\omega}{2}\right)^2} = \frac{2}{4+\omega^2}.$$

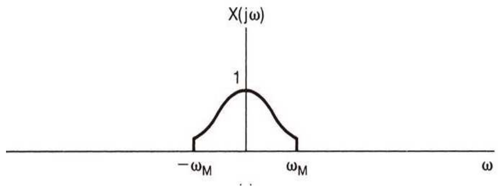
4. **[8]** Consider two message signals $x_1(t)$ and $x_2(t)$ with frequency spectra shown by Figures (a) and (b), respectively. We form the modulated signal $\Phi(t)=x_1(t)\cos\omega_c t+x_2(t)\sin\omega_c t$ for transmission, where $\omega_M \ll \omega_c$. In the receiver, we generate two signals $v_1(t)$ and $v_2(t)$ as

$$v_1(t)=\Phi(t)\cos\omega_c t \quad \text{and} \quad v_2(t)=\Phi(t)\sin\omega_c t\,,$$

respectively. Finally, $v_1(t)$ and $v_2(t)$ are filtered respectively by an ideal low-pass filter with cutoff frequency equal to $2\omega_M$ and unit amplitude to provide the output signals $y_1(t)$ and $y_2(t)$.



(a) $X_1(j\omega)$



(b) $X_2(j\omega)$

(a) Determine $v_1(t)$ and sketch its frequency spectrum.
(b) Determine $v_2(t)$ and sketch its frequency spectrum.
(c) Determine $y_1(t)$ and sketch its frequency spectrum.
(d) Determine $y_2(t)$ and sketch its frequency spectrum.

Solution:

(a) It can be shown

$$v_1(t)=\Phi(t)\cos\omega_c t=\left[x_1(t)\cos\omega_c t+x_2(t)\sin\omega_c t\right]\cdot\cos\omega_c t=x_1(t)\cos^2\omega_c t+x_2(t)\sin\omega_c t\cos\omega_c t$$

$$=x_1(t)\cdot\frac{1+\cos 2\omega_c t}{2}+x_2(t)\cdot\frac{\sin 2\omega_c t}{2}=\frac{1}{2}x_1(t)+\frac{1}{2}\left(x_1(t)\cos 2\omega_c t+x_2(t)\sin 2\omega_c t\right)$$

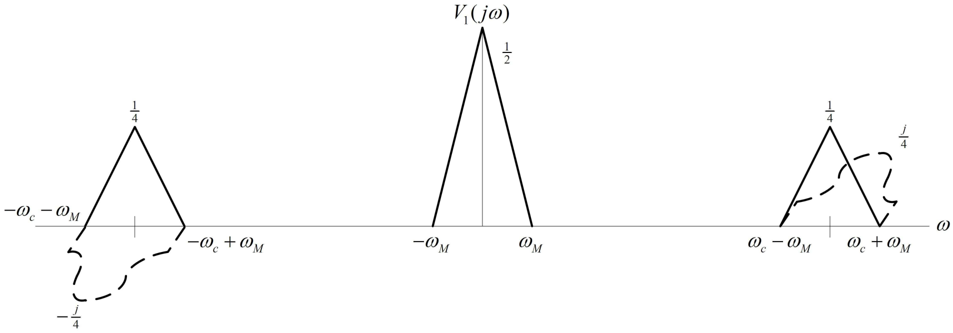
(b) It can be shown

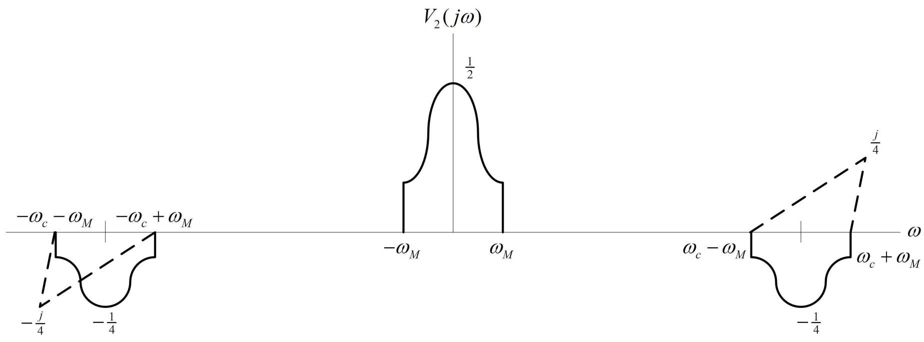
$$v_2(t)=\Phi(t)\sin\omega_c t=\left[x_1(t)\cos\omega_c t+x_2(t)\sin\omega_c t\right]\cdot\sin\omega_c t=x_1(t)\sin\omega_c t\cos\omega_c t+x_2(t)\sin^2\omega_c t$$

$$=x_1(t)\cdot\frac{\sin 2\omega_c t}{2}+x_2(t)\cdot\frac{1-\cos 2\omega_c t}{2}=\frac{1}{2}x_2(t)+\frac{1}{2}\left(x_1(t)\sin 2\omega_c t-x_2(t)\cos 2\omega_c t\right)$$

(c) $y_1(t)=\frac{1}{2}x_1(t)$, and the sketch is the amplitude scaling of $X_1(j\omega)$.

(d) $y_2(t)=\frac{1}{2}x_2(t)$, and the sketch is the amplitude scaling of $X_2(j\omega)$.

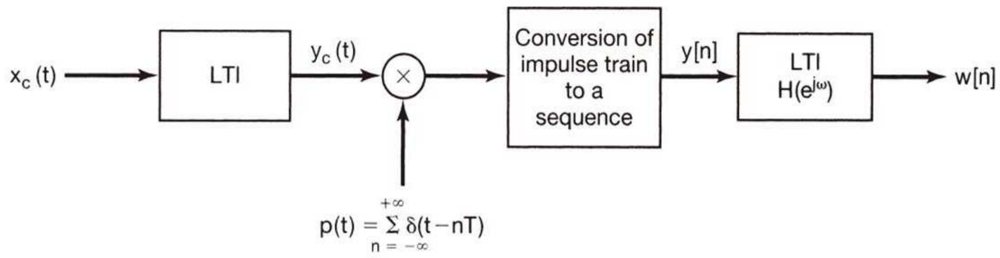




5. **[12]** Consider the system depicted below. Assume that the continuous-time LTI system is causal and specified by the LCCDE

$$\frac{dy_c(t)}{dt} + y_c(t) = x_c(t)$$

Now, we input a unit impulse to the system, i.e., $x_c(t) = \delta(t)$.



(a) Determine the output $y_c(t)$ of the continuous-time LTI system.

(b) Determine the frequency spectrum of $y[n]$.

(c) Determine the unit impulse response $h[n]$ of the discrete-time LTI system $H(e^{j\omega})$ such that $w[n] = \delta[n]$.

Solution:

(a) $\frac{dy_c(t)}{dt} + y_c(t) = x_c(t) \Rightarrow H(j\omega) = \frac{1}{1+j\omega} \Rightarrow h(t) = e^{-t}u(t)$

(b) $y[n] = y_c(nT) = h(nT) = e^{-nT}u[n]$, $Y(e^{j\omega}) = \frac{1}{1-e^{-T}e^{-j\omega}}$

(c) $H(e^{j\omega}) = 1 - e^{-T}e^{-j\omega}$, $h[n] = \delta[n] - e^{-T}\delta[n-1]$

6. **[6]** For a real discrete-time sequence $x[n]$, prove that if $X(z)$, the z-transform of $x[n]$, has a pole (or zero) at $z = z_0$, then it must also have a pole (or zero) at $z = z_0^*$, where z_0^* denotes the complex conjugate of z_0.

Solution:

The conjugation property indicates that $x^*[n] \xleftrightarrow{Z} X^*(z^*)$ with the same ROC. Since $x[n]$ is real, it is shown that $X(z) = X^*(z^*)$, and then $X(z^*) = X^*(z)$. Therefore, if z_0 is a pole of $X(z)$, then z_0 is a pole of $X^*(z) = X(z^*)$, and then z_0^* is a pole of $X(z)$.

7. **[6]** Denote the unilateral z-transform of a discrete-time sequence $x[n]$ by $\mathcal{X}(z)$. Prove that the unilateral z-transform of $x[n+1]$ is $z\mathcal{X}(z) - zx[0]$.

Solution:

$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$, then

$$\sum_{n=0}^{\infty} x[n+1]z^{-n} \overset{m=n+1}{=} \sum_{m=1}^{\infty} x[m]z^{-(m-1)} = z\sum_{m=1}^{\infty} x[m]z^{-m} = z\sum_{m=0}^{\infty} x[m]z^{-m} - zx[0] = z\mathcal{X}(z) - zx[0]$$

8. **[10]** Prove the initial-value theorem of Laplace transform. That is, if a continuous-time signal $x(t) = 0$ for $t < 0$ and contains no impulse or higher order singularities at the origin, then

$$x(0^+) = \lim_{s\to\infty} sX(s).$$

(Hint: Make use of the Taylor series expansion of $x(t)$ at $t = 0^+$ and the Laplace transform pair

$$e^{-at}\left(\frac{t^n}{n!}\right)u(t) \xleftrightarrow{L} \frac{1}{(s+a)^{n+1}},\ \mathrm{Re}\{s\} > -a\ .)$$

Solution:

Since $x(t)=0$ for $t<0$, we may express $x(t)$ as $x(t)=x(t)u(t)$, and expanding $x(t)$ as a Taylor series at $t=0^+$, we obtain

$$x(t)=\left[x(0^+)+x'(0^+)t+x''(0^+)\frac{t^2}{2!}+\ldots+x^{(n)}(0^+)\frac{t^n}{n!}+\ldots\right]u(t)$$

$$=x(0^+)u(t)+x'(0^+)tu(t)+x''(0^+)\frac{t^2}{2!}u(t)+\ldots+x^{(n)}(0^+)\frac{t^n}{n!}u(t)+\ldots$$

Since

$$e^{-at}\left(\frac{t^n}{n!}\right)u(t) \xleftrightarrow{L} \frac{1}{(s+a)^{n+1}},\ \mathrm{Re}\{s\} > -a \text{, implying } \left(\frac{t^n}{n!}\right)u(t) \xleftrightarrow{L} \frac{1}{s^{n+1}},\ \mathrm{Re}\{s\} > 0$$

we obtain

$$X(s)=\frac{x(0^+)}{s}+\frac{x'(0^+)}{s^2}+\frac{x''(0^+)}{s^3}+\ldots+\frac{x^{(n)}(0^+)}{s^{n+1}}+\ldots,\ \mathrm{Re}\{s\}>0$$

and then

$$\lim_{s\to\infty} sX(s)=\lim_{s\to\infty}\left[x(0^+)+\frac{x'(0^+)}{s}+\frac{x''(0^+)}{s^2}+\ldots+\frac{x^{(n)}(0^+)}{s^n}+\ldots\right]=x(0^+)$$

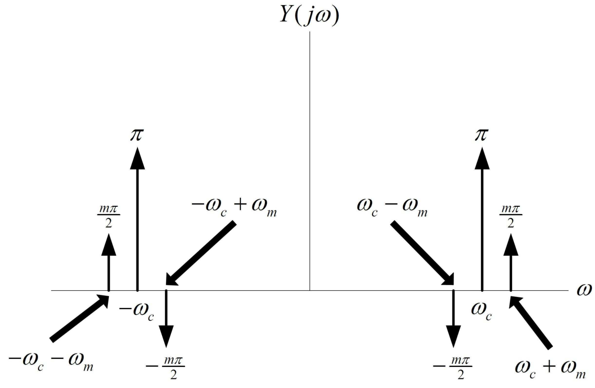
9. **[12]** Consider the case of frequency modulation where the modulated signal is expressed as $y(t)=\cos(\omega_c t+m\sin\omega_m t)=\cos(\omega_c t)\cos(m\sin\omega_m t)-\sin(\omega_c t)\sin(m\sin\omega_m t)$ with ω_c being the frequency of the carrier, ω_m the frequency of the modulating signal, and m the modulation index.
   (a) When m is sufficiently small (say, much smaller than $\pi/2$), the modulated signal can be approximated by $y(t)=\cos(\omega_c t)-m(\sin\omega_m t)(\sin\omega_c t)$. Sketch the spectrum of the approximated signal.
   (b) When m is large, the approximation shown above no longer applies. But it can be noted that the terms $\cos(m\sin\omega_m t)$ and $\sin(m\sin\omega_m t)$ represent periodic signals. What are the fundamental periods of these two periodic signals? Justify your answer.
   (c) Explain how the spectrum of $y(t)$ should look like for the case where m is large.

Solution:

(a) It can be shown that

$$y(t)=\cos(\omega_c t)-m(\sin\omega_m t)(\sin\omega_c t)$$
$$=\cos(\omega_c t)+\tfrac{m}{2}\left[\cos(\omega_c+\omega_m)t-\cos(\omega_c-\omega_m)t\right]$$

The spectrum is as the right figure.



(b) $\sin\omega_m t$ is with frequency ω_m, so $m\sin\omega_m t$ is with frequency ω_m, and then $\cos(m\sin\omega_m t)$ and $\sin(m\sin\omega_m t)$ are with fundamental frequency ω_m, and then they are with fundamental period $\frac{2\pi}{\omega_m}$.

(c) The spectrum of $y(t)$ consists of impulses at frequencies $\pm\omega_c+n\omega_m$, $n=0,\pm1,\pm2,\ldots$, and is not, strictly speaking, band limited around $\pm\omega_c$.

10. **[8]** Suppose $H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\sphericalangle H(e^{j\omega})}$ is the frequency response of a discrete-time system. The response of the system to the input $x[n] = \sin(\omega_0 n + \theta_0)$ would be of this form $y[n] = \left|H(e^{j\omega_0})\right| x[n - n_0]$ if $\sphericalangle H(e^{j\omega_0})$ is related to ω_0 in a certain way. Determine this relation.

Solution:

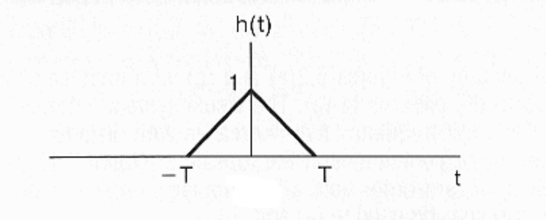
$y[n] = \left|H(e^{j\omega_0})\right| x[n - n_0]$, $Y(e^{j\omega}) = \left|H(e^{j\omega_0})\right| e^{-j\omega n_0} X(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\sphericalangle H(e^{j\omega})} X(e^{j\omega})$.

Since $x[n] = \sin(\omega_0 n + \theta_0) = \sin\omega_0 n \cos\theta_0 + \cos\omega_0 n \sin\theta_0$,

$$X(e^{j\omega}) = \left(\pi\sin\theta_0 + \tfrac{\pi}{j}\cos\theta_0\right)\sum_{l=-\infty}^{\infty}\delta(\omega - \omega_0 - 2\pi l) + \left(\pi\sin\theta_0 - \tfrac{\pi}{j}\cos\theta_0\right)\sum_{l=-\infty}^{\infty}\delta(\omega + \omega_0 - 2\pi l).$$

If $h[n]$ is real, we may claim that $Y(e^{j\omega}) = \left|H(e^{j\omega_0})\right| e^{-j\omega n_0} X(e^{j\omega}) = \left|H(e^{j\omega_0})\right| e^{j\sphericalangle H(e^{j\omega_0})} X(e^{j\omega})$, and then $\sphericalangle H(e^{j\omega_0}) = -n\omega_0 + 2m\pi$ for integer m.

11. **[8]** Show that, in terms of reconstruction of a signal from its samples, the linear interpolation method is equivalent to impulse-train sampling followed by convolution with a triangular impulse response $h(t)$ shown in the following figure:



where T is the sampling period.

Solution:

It is obvious that the linear interpolation method itself is a linear time-invariant system. Therefore all we need to do is to find the impulse response of the system. In the case the output is the impulse response, the input $x_p(t) = \delta(t)$ may be expressed as

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT), \text{ where } x(t) = \begin{cases} 1 & t = 0 \\ 0 & t = nT, n \neq 0 \\ \text{don't-care} & \text{otherwise} \end{cases}$$

<u>Formal approach:</u>

When $nT \le t \le (n+1)T$, the output $y(t)$ is expressed as

$$y(t) = \frac{(n+1)T - t}{T} \cdot x(nT) + \frac{t - nT}{T} \cdot x((n+1)T)$$

For the case $n \le -2$ or $n \ge 1$, $x(nT) = x((n+1)T) = 0$, then $y(t) = 0$.

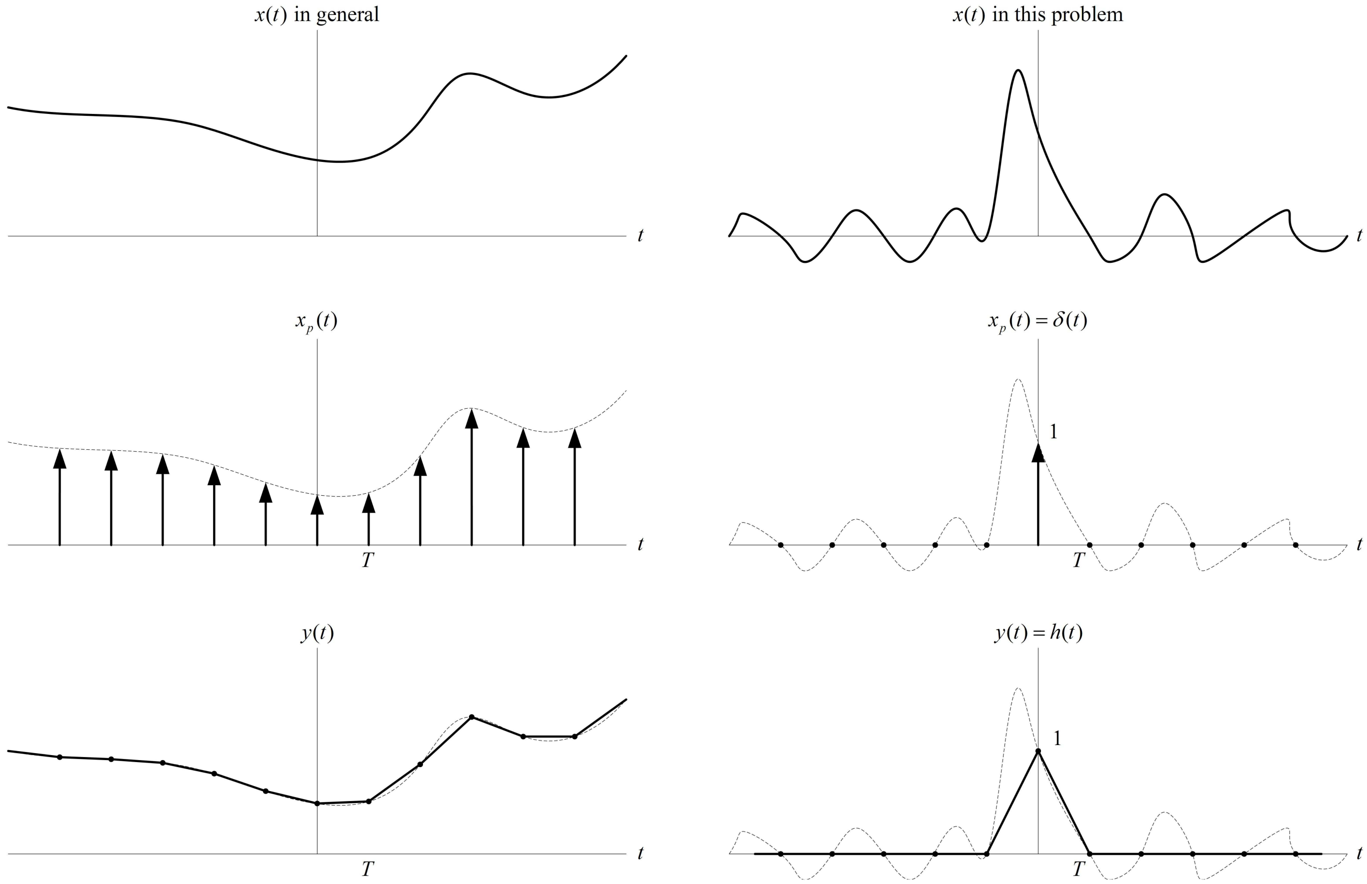
For the case $n = -1$, $x(nT) = 0$ and $x((n+1)T) = 1$, then $y(t) = \frac{t}{T} + 1$.

For the case $n = 0$, $x(nT) = 1$ and $x((n+1)T) = 0$, then $y(t) = -\frac{t}{T} + 1$.

Sketching the curve of $y(t) = h(t)$, it can be shown that $h(t)$ is a triangular pulse shown in the problem.

<u>Figure approach:</u>

To show the process of the linear interpolation, the sketches of $x(t)$, $x_p(t) = \delta(t)$, and $y(t) = h(t)$ are as follows (see next page):



The sketch of $y(t) = h(t)$ matches the figure given in the problem.