$V\Sigma U^T$ $U\Sigma^{\dagger} V^T$ $U\Sigma V^T$

A

[illegible]

Linear Algebra Final Examination

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USE OF ALL AUTOMATIC COMPUTING MACHINES IS PROHIBITED

1. Judge if the following statements are true or false. Give a concise proof to each true statement, and a counterexample to each false statement. (25%)
   (a) For $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{R}^n$, $\langle \mathbf{u}, \mathbf{v}\rangle = (A\mathbf{u})\bullet\mathbf{v}$ defines an inner product on $\mathcal{R}^n$ with a given $n\times n$ invertible matrix A. $(Au)\cdot u = uA^T u$
   (b) For any two subspaces $\mathcal{V}$ and $\mathcal{W}$ of $\mathcal{R}^n$, $\mathcal{V}\subset\mathcal{W}$ implies $\mathcal{W}^{\perp}\subset\mathcal{V}^{\perp}$.
   (c) For any two subspaces $\mathcal{V}$ and $\mathcal{W}$ of $\mathcal{R}^n$, $P_{\mathcal{V}} + P_{\mathcal{W}}$ is an orthogonal projection matrix, where $P_{\mathcal{V}}$ and $P_{\mathcal{W}}$ are the orthogonal projection matrices for $\mathcal{V}$ and $\mathcal{W}$, respectively.
   (d) If A is a symmetric matrix, then its largest singular value equals its largest eigenvalue.
   (e) For any $n\times n$ invertible matrix A, $A^{\dagger} = A^{-1}$.

   $A = A'$ $A'A = \lambda^2$

2. For $\mathcal{V} = \operatorname{Span}\{e^t, te^t, t^2e^t\}$ and the differential operator D, find all eigenvalues of D and an eigenvector for each eigenvalue. (15%)

3. For any $n\times n$ matrix A, the minimal polynomial of A is defined as the polynomial $p(t) = t^m + p_{m-1}t^{m-1} + \cdots + p_1 t + p_0$ with the lowest degree m such that $p(A) = A^m + p_{m-1}A^{m-1} + \cdots + p_1 A + p_0 I_n = O$. Find the minimal polynomials of the following matrices and justify your answers. $t=1$
   (a) The elementary matrix E obtained from I_n by interchanging its ith and jth rows ($i \neq j$). (10%)
   (b) The $n\times n$ diagonalizable matrix M with k distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$, where $k\le n$ and λ_i has the algebraic multiplicity n_i for $i = 1, 2, \ldots, k$. (10%) $P(D)P^{-1}$

   $(t-1)=0$

4. Find all eigenvalues and their corresponding eigenspaces of the matrix E in Problem 3(a). (10%)

5. Given the SVD of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}\begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{30} & 2/\sqrt{5} & 1/\sqrt{6} \\ 2/\sqrt{30} & -1/\sqrt{5} & 2/\sqrt{6} \\ 5/\sqrt{30} & 0 & -1/\sqrt{6} \end{bmatrix}^T$, plot the image of the unit sphere $\{(x_1\ x_2\ x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1,\ x_i\in\mathcal{R} \text{ for } i = 1,2,3.\}$ in $\mathcal{R}^3$ under the mapping of T_A. (15%)

6. A linear operator T on a finite-dimensional inner product space $\mathcal{V}$ is called an orthogonal operator if $[T]_{\mathcal{B}}$ is an orthogonal matrix for some orthonormal basis $\mathcal{B}$ of $\mathcal{V}$. Prove that T is an orthogonal operator if and only if for any orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ is also an orthonormal basis for $\mathcal{V}$. (15%)