Probability and Statistics, Final Exam, 2007

1. (12%; 3% each) Let X be a continuous random variable with the probability density function $f_X(x)$ which is an even function, i.e. $f_X(x) = f_X(-x)$. Also, define a new random variable $Y = X^2$. Determine whether each of the following statements is true or false. Prove it if the statement is true. Explain it if the statement is false.

   true (a) X has zero mean, i.e., $E[X] = 0$.

   true (b) X and Y are orthogonal, i.e., $E[XY] = 0$

   true (c) X and Y are uncorrelated, i.e., $E[XY] = E[X]E[Y]$.

   (d) If $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp[-x^2/2]$, then X and Y are independent.

   $f_Y(y) = f[x^2 \le y] = f[-\sqrt{y} \le x \le \sqrt{y}]$ $\quad f_{X,Y}$

2. (8%) Let X_n's, $n = 1, ..., N$, be independent and identically distributed (iid) normal random variables with zero mean, i.e., $E[X_n] = 0$, and unit variance, i.e., $Var(X_n) = 1$. Also, let Y_n's, $n = 1, ..., N$, be iid binary-valued random variables with the common probability density function $f_Y(y) = \frac{1}{2}$ if $y = +1$ or $y = -1$, and $f_Y(y) = 0$ otherwise. In addition, X_n's, Y_n's, $n = 1, ..., N$, are independent. Now, define a new random variable $Z = \sum_{n=1}^{N} X_n Y_n$. Derive the probability density function of Z.

3. (10%) Let X_n's, $n = 1, ..., N$, be independent and identically distributed exponential random variables with common mean $1/\lambda$. Find the moment generating function of $X = \min(X_1, X_2, ..., X_n)$.

4. (10%) A fair coin is tossed until N tails occur successively, with N a positive integer. Find the expected number of tosses required.

5. (10%) Let X be a random variable and k be a constant. Prove that

$$P(X > t) \le \frac{E(\exp[kX])}{\exp[kt]}; k > 0.$$

6. (10%; 5% each)

   (a) Prove that the expected value of the sample variance S^2 is equal to the population variance σ^2 (5%)

   (b) Show that $\frac{(n-1)S^2}{\sigma^2}$ having a chi-square distribution with n-1 degree of freedom. (5%)

7. (10%) Let $X_1, X_2, X_3, ...X_n$, be a random sample from the geometric distribution with *p.m.f.* $f(x; p) = (1-p)^{x-1} p$, $x = 1,2,3....$, Find the Maximum Likelihood Estimation of p.

8. (10%) Suppose that the proportion q of defective items in a large population of notebook computers is unknown, and that it is desired to test the following hypotheses:

H_0: $q = 0.3$,

H_1: $q \neq 0.3$

Suppose also that a random sample of 20 items is drawn from the population. Let X denote the number of defective items in the sample, and consider a test procedure such that the critical region contains all the outcomes for which either $X \geq 7$ or $X \leq 1$.

0.3

(a) Determine the significance level (or size) of the test. (4%)

(b) Determine the power of the test at $q = 0.2$ and 0.4 respectively. (4%) 0 1 6 7

(c) Determine the confidence interval of the test. (2%) 0.05 0.35

9. (9%; 3% each) The capacities (in ampere-hours) of 10 batteries were recorded as follows:
140, 136, 150, 144, 148, 152, 138, 141, 143, 151 E(X) Var(X)

(a) Estimate the population variance σ^2 (3%)

(b) Compute a 99 percent two-sided confidence interval for σ^2. (3%)

(c) Compute a value v that enables us to state, with 90 percent confidence, that σ^2 is less than v. (3%)

10 (4%; 2% each) A population distribution is known to have standard deviation 20. Determine the p-value of a test of the hypothesis that the population mean is equal to 50, if the average of a sample of 64 observations is :

(a) 52.5 (2%)

(b) 55.0 (2%)

11 (7%)In a certain chemical process, it is very important that particular solution that is to be used as a reactant have a pH of exactly 8.20. A method for determining pH that is available for solutions of this type is known to give measurements that are normally distributed with a mean equal to the actual pH and with a standard deviation of 0.02. Suppose 10 independent measurements yielded the following pH values:

8.18	8.17
8.16	8.15
8.17	8.21
8.22	8.16
8.19	8.18

(a) What conclusion can be drawn at the $\alpha = 0.10$ level of significance? (3%)

(b) What about at the $\alpha = 0.05$ level of significance? (4%)

Note: Please use the table in the other page.