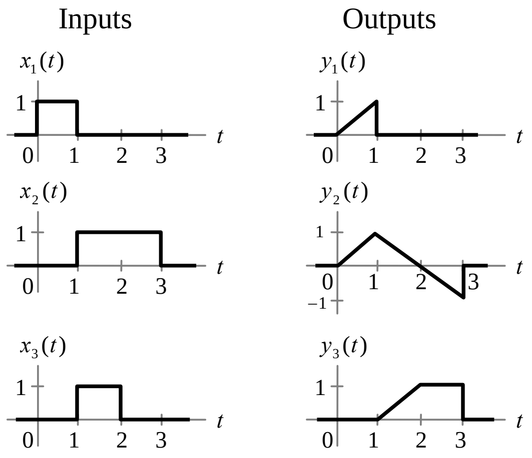
Signals and Systems, Midterm Exam Solutions

Spring 2005, Edited by bypeng

1. (12) Consider a continuous-time linear system with the input-output pairs depicted below.

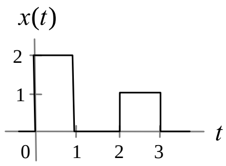


Answer the following question and justify your answer.

(a) (3) Is the system causal?

(b) (4) Is the system memoryless?

(c) (5) What would be the response of the system to the following signal $x(t)$? Give a sketch of the response.



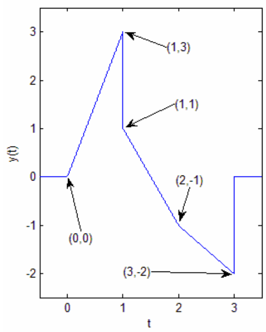
Solution:

The system is linear, therefore

(a) The system is NOT CAUSAL since $y_2(t)$ is non-zero in the interval $(0,1)$ but $x_2(t)$ is zero in the interval $(-\infty,1)$.

(b) A memoryless system implies a causal system, so the system is NOT MEMORYLESS by (a).

(c) Observing that $x(t)=2x_1(t)+x_2(t)-x_3(t)$, we have $y(t)=2y_1(t)+y_2(t)-y_3(t)$. The sketch is given as the following figure.



2. (12) Consider a discrete-time LTI system with unit sample response $h[n]=(n+1)\alpha^n u[n]$, where $|\alpha|<1$. Determine the step response of the system by

(a) (6) performing the convolution sum

(b) (6) using the discrete-time Fourier transform and its properties.

Solution:

(a) $h[n]*u[n]=u[n]*h[n]=\sum_{k=-\infty}^{\infty}u[k]h[n-k]=\sum_{k=0}^{\infty}(n-k+1)\alpha^{n-k}u[n-k]$. if $n<0$, then $h[n]*u[n]=0$;

if $n \geq 0$, then $h[n]*u[n] = \sum_{k=0}^{\infty}(n-k+1)\alpha^{n-k}u[n-k] = \sum_{k=0}^{n}(n+1)\alpha^{n-k} = (n+1)\alpha^n + n\alpha^{n-1} + \ldots + 2\alpha + 1$

$$= \frac{\alpha^n[1-(\frac{1}{\alpha})^{n+1}]}{1-\frac{1}{\alpha}} + \frac{\alpha^n[1-(\frac{1}{\alpha})^{n}]}{1-\frac{1}{\alpha}} + \ldots + \frac{\alpha^n[1-\frac{1}{\alpha}]}{1-\frac{1}{\alpha}} = \frac{(n+1)\alpha^n}{1-\frac{1}{\alpha}} - \frac{1}{1-\frac{1}{\alpha}}(\tfrac{1}{\alpha}+1+\alpha+\ldots+\alpha^{n-1})$$

$$= \frac{(n+1)\alpha^n}{1-\frac{1}{\alpha}} - \frac{\frac{1}{\alpha}(1-\alpha^{n+1})}{(1-\frac{1}{\alpha})(1-\alpha)} = \frac{(n+1)\alpha^{n+1}}{\alpha-1} - \frac{1-\alpha^{n+1}}{(\alpha-1)(1-\alpha)} = \frac{1-\alpha^{n+1}}{(1-\alpha)^2} - \frac{(n+1)\alpha^{n+1}}{1-\alpha}$$

Therefore, $h[n]*u[n] = \left(\frac{1-\alpha^{n+1}}{(1-\alpha)^2} - \frac{(n+1)\alpha^{n+1}}{1-\alpha}\right)u[n]$.

(b) $H(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^2}$, $U(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty}\pi\delta(\omega-2\pi k)$,

$$H(e^{j\omega})U(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^2}\frac{1}{1-e^{-j\omega}} + \frac{1}{(1-\alpha e^{-j\omega})^2}\sum_{k=-\infty}^{\infty}\pi\delta(\omega-2\pi k)$$

$$= \frac{\frac{1}{(1-\alpha)^2}}{1-e^{-j\omega}} + \frac{-\frac{\alpha}{(1-\alpha)^2}}{1-\alpha e^{-j\omega}} + \frac{-\frac{\alpha}{1-\alpha}}{(1-\alpha e^{-j\omega})^2} + \frac{1}{(1-\alpha\cdot \underbrace{e^{-j\omega}}_{1\text{ when }\omega=2\pi k})^2}\sum_{k=-\infty}^{\infty}\pi\delta(\omega-2\pi k)$$

$$= \frac{1}{(1-\alpha)^2}\left(\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty}\pi\delta(\omega-2\pi k)\right) + \frac{-\frac{\alpha}{(1-\alpha)^2}}{1-\alpha e^{-j\omega}} + \frac{-\frac{\alpha}{1-\alpha}}{(1-\alpha e^{-j\omega})^2}$$

$$F^{-1}\{H(e^{j\omega})U(e^{j\omega})\} = \frac{1}{(1-\alpha)^2}u[n] - \frac{\alpha}{(1-\alpha)^2}\alpha^n u[n] - \frac{\alpha}{1-\alpha}(n+1)\alpha^n u[n] = \left(\frac{1-\alpha^{n+1}}{(1-\alpha)^2} - \frac{(n+1)\alpha^{n+1}}{1-\alpha}\right)u[n].$$

3. (12) Consider two sequences

$$x[n] = \begin{cases} \alpha^n, & 0 \leq n \leq 6 \\ 0, & \text{otherwise} \end{cases}$$

and

$$y[n] = \begin{cases} 1, & 0 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

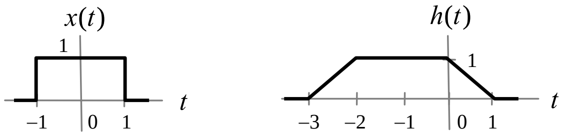
Calculate the convolution of the two signals.

Solution:

$$x[n]*y[n] = y[n]*x[n] = \sum_{k=-\infty}^{\infty} y[k]x[n-k] = \sum_{k=0}^{3} x[n-k] = x[n] + x[n-1] + x[n-2] + x[n-3]$$

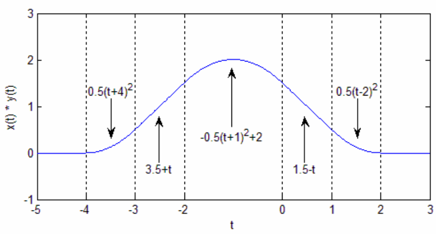
n	≤ -1	0	1	2	3 to 6	7	8	9	≥ 10
$x[n]*y[n]$	0	1	$1+\alpha$	$1+\alpha+\alpha^2$	$\alpha^{n-3}+\alpha^{n-2}+\alpha^{n-1}+\alpha^n$	$\alpha^4+\alpha^5+\alpha^6$	$\alpha^5+\alpha^6$	α^6	0

4. (12) Let the input $x(t)$ to an LTI system with impulse response $h(t)$ be given in the following figure. Find the output $y(t)$.



Solution:

$y(t) = x(t)*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-1}^{1} h(t-\tau)d\tau \underbrace{=}_{\tau_0 \triangleq t-\tau} \int_{t-1}^{t+1} h(\tau_0)d\tau_0$, so the output is as the following figure.



5. (12) Consider a first-order system described by $y[n]-\frac{1}{4}y[n-1]=x[n]$.

(a) (6) Find the output of the system if the input is $x[n]=(1/2)^n u[n]$ and the initial condition is $y[-1]=8$.

(b) (6) Determine the impulse response of the system.

Solution:

(a) $\forall n\geq 0$, $y[n]-\frac{1}{4}y[n-1]=\left(\frac{1}{2}\right)^n$, then $y[n]=c_1\left(\frac{1}{4}\right)^n+c_2\left(\frac{1}{2}\right)^n$, and by

$y[n]-\frac{1}{4}y[n-1]=c_2\left(\frac{1}{2}\right)^n-\frac{1}{4}c_2\left(\frac{1}{2}\right)^{n-1}=\frac{1}{2}c_2\left(\frac{1}{2}\right)^n=\left(\frac{1}{2}\right)^n$, we have $c_2=2$, and $y[0]=c_1+2$, but

$y[0]-\frac{1}{4}y[-1]=c_1+2-\frac{1}{4}\cdot 8=c_1=x[0]=1$, so $y[n]=\left(\frac{1}{4}\right)^n+2\left(\frac{1}{2}\right)^n$.

$\forall n<0$, $y[n]-\frac{1}{4}y[n-1]=0$, then $y[n]=c_3\left(\frac{1}{4}\right)^n$, and by $y[-1]=4c_3=8$, we have $y[n]=2\left(\frac{1}{4}\right)^n$.

The conclusion is $y[n]=2\left(\frac{1}{4}\right)^n+\left(-\left(\frac{1}{4}\right)^n+2\left(\frac{1}{2}\right)^n\right)u[n]$.

(b) The impulse response is defined as the output of an impulse as the input of a LTI system. Since we are going to find the impulse response, the LTI properties may be applied. Observing that

$x[n]-\frac{1}{2}x[n-1]=\left(\frac{1}{2}\right)^n u[n]-\frac{1}{2}\left(\frac{1}{2}\right)^{n-1}u[n-1]=\left(\frac{1}{2}\right)^n(u[n]-u[n-1])=\left(\frac{1}{2}\right)^n\delta[n]=\delta[n]$, we have

$$h[n]=y[n]-\frac{1}{2}y[n-1]=y[n]-\frac{1}{4}y[n-1]-\frac{1}{4}y[n-1]=x[n]-\frac{1}{4}y[n-1]$$

$$=\left(\frac{1}{2}\right)^n u[n]-\frac{1}{4}\left[2\left(\frac{1}{4}\right)^{n-1}+\left(-\left(\frac{1}{4}\right)^{n-1}+2\left(\frac{1}{2}\right)^{n-1}\right)u[n-1]\right]$$

$$=\left(\frac{1}{2}\right)^n u[n]-2\left(\frac{1}{4}\right)^n+\left(\frac{1}{4}\right)^n u[n-1]-\left(\frac{1}{2}\right)^n u[n-1]$$

$$=\left(\frac{1}{2}\right)^n \delta[n]-2\left(\frac{1}{4}\right)^n+\left(\frac{1}{4}\right)^n u[n-1]=\delta[n]-2\left(\frac{1}{4}\right)^n+\left(\frac{1}{4}\right)^n u[n-1]$$

$$=\left(\frac{1}{4}\right)^0 \delta[n]-2\left(\frac{1}{4}\right)^n+\left(\frac{1}{4}\right)^n u[n-1]=-2\left(\frac{1}{4}\right)^n+\left(\frac{1}{4}\right)^n u[n]$$

[Note: Here we may find that the system is not stable since the bounded input $x[n]=(1/2)^n u[n]$ results in the output $y[n]=2\left(\frac{1}{4}\right)^n+\left(-\left(\frac{1}{4}\right)^n+2\left(\frac{1}{2}\right)^n\right)u[n]$, which implies that $y[n]$ is not absolutely summable.

Since $y[n]$ is not (always) absolutely summable, the Fourier-transform approach does not work. Also notice that the system is not causal since the system is without initial rest, and then the impulse response is not a causal signal.]

6. (12) One way to identify that $1\xleftrightarrow{F}2\pi\delta(\omega)$ is an Fourier transform pair is to show that the inverse Fourier transform of $2\pi\delta(\omega)$ is indeed equal to the constant signal $x(t)=1$. Find an alternative way to directly prove this Fourier transform pair relationship.

Solution:

Consider $\int_{-T}^{T}1\cdot e^{-j\omega t}dt=\int_{-T}^{T}e^{-j\omega t}dt=\frac{e^{-j\omega T}-e^{j\omega T}}{-j\omega}=\frac{-j\sin\omega T+j\sin\omega T}{-j\omega}=\frac{2\sin\omega T}{\omega}$, and to evaluate $\int_{-\infty}^{\infty}1\cdot e^{-j\omega t}dt$, we find that $\lim_{T\to\infty}\frac{2\sin\omega T}{\omega}=2\pi\lim_{T\to\infty}\frac{\sin\omega T}{\pi\omega}=2\pi\lim_{T\to\infty}\frac{T}{\pi}\operatorname{sinc}\omega T$. Recall that $\delta(\omega)$ is (defined as) a function satisfying that for any integrable function $f(\omega)$, $\int_{-\infty}^{\infty}\delta(\omega)f(\omega)d\omega = f(0)$. It may be shown that $\lim_{T\to\infty}\frac{T}{\pi}\operatorname{sinc}\omega T$ satisfies this property, so we conclude that $\int_{-\infty}^{\infty}1\cdot e^{-j\omega t}dt = 2\pi\delta(\omega)$.

[Note: Any proof with the main idea attempting to evaluate $\int_{-\infty}^{\infty}\delta(x)e^{jkx}dx = e^{jk\cdot 0} = 1$ results in no credits.]

7. (12) Consider a discrete-time signal described by the periodic sequence $x[n] = \sum_{k=-\infty}^{\infty}\delta[n-kN]$.

(a) (6) Find the Fourier coefficients of the sequence.
(b) (6) Determine and plot the Fourier transform of the sequence.

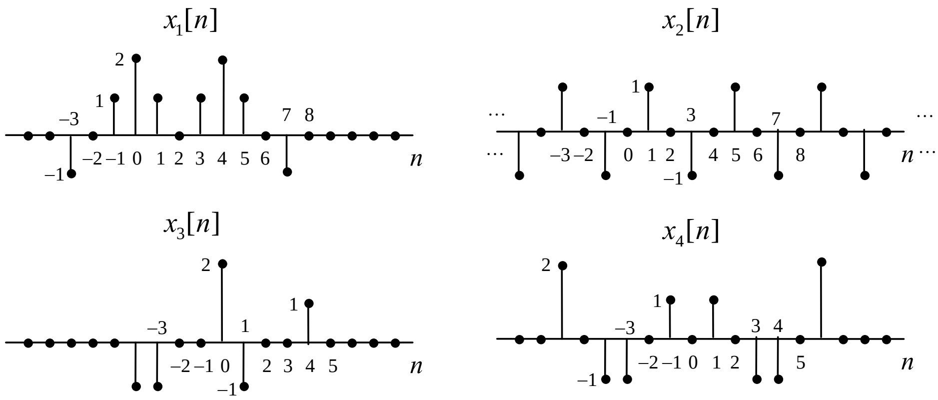
Solution:

(a) $x[n]$ is of fundamental period N and then frequency $\frac{2\pi}{N}$. Then

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-j\frac{2\pi}{N}nk} = \frac{1}{N}\sum_{n=0}^{N-1}\delta[n]e^{-j\frac{2\pi}{N}nk} = \frac{1}{N}e^{-j\frac{2\pi}{N}0\cdot k} = \frac{1}{N}$$

(b) $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty}2\pi\frac{1}{N}\delta(\omega-\frac{2\pi k}{N}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\omega-\frac{2\pi k}{N})$. See Figure 5.11(b) at pp. 372 in the textbook to find the plot.

8. (12) Consider the following signals:



For each signal, determine if its Fourier transform satisfies the following conditions. Justify your answers.

(a) $\mathcal{Re}\{X(e^{j\omega})\} = 0$.
(b) $\mathcal{Im}\{X(e^{j\omega})\} = 0$.
(c) There exists an integer α such that $e^{j\alpha\omega}X(e^{j\omega})$ is real.
(d) $\int_{-\pi}^{\pi}X(e^{j\omega})d\omega = 0$.
(e) $X(e^{j\omega})$ periodic.
(f) $X(e^{j0}) = 0$.

Solution:

(a) $\mathcal{Re}\{X(e^{j\omega})\} = 0 \Rightarrow X(e^{j\omega})$ pure imaginary $\Rightarrow x[n]$ real and odd. Only $x_2[n]$ satisfies this.

(b) $\mathcal{Im}\{X(e^{j\omega})\} = 0 \Rightarrow X(e^{j\omega})$ real $\Rightarrow x[n]$ real and even. Only $x_4[n]$ satisfies this.

(c) There exists an integer α such that $e^{j\alpha\omega}X(e^{j\omega})$ is real $\Rightarrow \alpha$ makes $x[n-\alpha]$ real and even. $x_1[n]$, $x_2[n]$, and $x_4[n]$ satisfy this.

(d) $\int_{-\pi}^{\pi}X(e^{j\omega})d\omega = \int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega 0}d\omega = 2\pi x[0] = 0$. $x_2[n]$ and $x_4[n]$ satisfy this.

(e) The periodicity of $X(e^{j\omega})$ is trivial. All of the four signals satisfy this.

(f) $X(e^{j0}) = \sum_{n=-\infty}^{\infty}x[n]e^{-j\omega 0} = \sum_{n=-\infty}^{\infty}x[n] = 0$. $x_2[n]$ and $x_3[n]$ satisfy this.

[As a summary, $x_1[n]$ satisfies (c)/(e), $x_2[n]$ satisfies (a)/(c)/(d)/(e)/(f), $x_3[n]$ satisfies (e)/(f), and $x_4[n]$ satisfies (b)/(c)/(d)/(e).]

9. (12) Consider a discrete-time signal $x[n]$ given by $x[n] = a^{|n|}$ with $|a| < 1$ and a continuous-time signal $y(t)$ given by $y(t) = 1/[5 - 4\cos(2\pi t)]$ with period $T = 1$.
   (a) (4) Find the discrete-time Fourier transform of $x[n]$.
   (b) (8) Use the concept of duality to determine the Fourier series coefficients of $y(t)$.

Solution:

(a) We have

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^{|n|}e^{-j\omega n} = 1+\sum_{n=1}^{\infty} a^n e^{-j\omega n} + \sum_{n=1}^{\infty} a^n e^{j\omega n} = 1+\frac{ae^{-j\omega}}{1-ae^{-j\omega}}+\frac{ae^{j\omega}}{1-ae^{j\omega}}$$

$$=1+\frac{ae^{-j\omega}-a^2+ae^{j\omega}-a^2}{(1-ae^{-j\omega})(1-ae^{j\omega})}=1+\frac{ae^{-j\omega}-a^2+ae^{j\omega}-a^2}{1-a(e^{j\omega}+e^{-j\omega})+a^2}=1+\frac{2a\cos\omega-2a^2}{1-2a\cos\omega+a^2}$$

$$=\frac{1-a^2}{1+a^2-2a\cos\omega}$$

(b) $y(t)=\dfrac{1}{5-4\cos 2\pi t}=\dfrac{\frac{1}{4}}{\frac{5}{4}-\cos 2\pi t}=\dfrac{1}{3}\cdot\dfrac{1-(\frac{1}{2})^2}{(1+(\frac{1}{2})^2)-2\cdot(\frac{1}{2})\cos 2\pi t}=\sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t}$,

$\sum_{k=-\infty}^{\infty}\left(3a_k e^{jk2\pi t}\right)=\dfrac{1-(\frac{1}{2})^2}{(1+(\frac{1}{2})^2)-2\cdot(\frac{1}{2})\cos 2\pi t}$. Letting $a=\frac{1}{2}$, $\omega=-2\pi t$, and then

$\sum_{k=-\infty}^{\infty}\left(3a_k e^{-jk\omega}\right)=\dfrac{1-a^2}{(1+a^2)-2\cdot a\cos(-\omega)}=\dfrac{1-a^2}{(1+a^2)-2a\cos\omega}$. Therefore $3a_k=a^{|k|}=(\frac{1}{2})^{|k|}$, and then $a_k=\frac{1}{3}\cdot(\frac{1}{2})^{|k|}$.

10. (12) Consider a discrete-time signal $x[n]$ and its discrete-time Fourier transform $X(e^{j\omega})$. We call $x[n]$ and $X(e^{j\omega})$ a discrete-time Fourier transform pair.
    (a) (6) Please prove the so-called Parseval's relation for this discrete-time Fourier transform pair.
    (b) (6) Assume that $x[n] = \sin(Wn)/(\pi n)$. Find the energy of $x[n]$.

Solution:

(a) The Parseval's relation for discrete-time Fourier transform pair is given by

$$\sum_{n=-\infty}^{\infty}|x[n]|^2=\frac{1}{2\pi}\int_{2\pi}\left|X(e^{j\omega})\right|^2 d\omega$$

The proof is written as the following derivation.

$$\frac{1}{2\pi}\int_{2\pi}\left|X(e^{j\omega})\right|^2 d\omega=\frac{1}{2\pi}\int_{2\pi}X(e^{j\omega})X^*(e^{j\omega})d\omega=\frac{1}{2\pi}\int_{2\pi}X(e^{j\omega})\sum_{n=-\infty}^{\infty}x^*[n]e^{j\omega n}d\omega$$

$$=\sum_{n=-\infty}^{\infty}x^*[n]\frac{1}{2\pi}\int_{2\pi}X(e^{j\omega})e^{j\omega n}d\omega=\sum_{n=-\infty}^{\infty}x^*[n]x[n]=\sum_{n=-\infty}^{\infty}|x[n]|^2$$

(b) If $0<|W|<\pi$, The Fourier transform of $x[n]=\dfrac{\sin Wn}{\pi n}$ is $X(e^{j\omega})=\begin{cases}1 & W>0,|\omega|<W\\ -1 & W<0,|\omega|<-W\\ 0 & \text{elsewhere}\end{cases}$ with period 2π. In this case, we denote $W_0=|W|$.

If $(2k-1)\pi<|W|<2k\pi$ for some positive integer k, The Fourier transform of $x[n]$ is

$X(e^{j\omega})=\begin{cases}-1 & W>0,|\omega|<2k\pi-W\\ 1 & W<0,|\omega|<-2k\pi+W\\ 0 & \text{elsewhere}\end{cases}$ with period 2π. In this case, we denote $W_0=|2k\pi-W|$.

If $2k\pi<|W|<(2k+1)\pi$ for some positive integer k, The Fourier transform of $x[n]$ is

$$X(e^{j\omega}) = \begin{cases} 1 & W > 0, |\omega| < W - 2k\pi \\ -1 & W < 0, |\omega| < 2k\pi - W \\ 0 & \text{elsewhere} \end{cases}$$ with period 2π. In this case, we denote $W_0 = |2k\pi - W|$.

In each case, by the Parseval's relation, we have $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-W_0}^{W_0} d\omega = \frac{W_0}{\pi}$. The conclusion is $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{W_0}{\pi}$, where $W_0 = |W - 2k\pi|$ and k is the integer such that $2k\pi$ is the closest to W.