Department of Electrical Engineering

National Taiwan University

Probability and Statistics, Spring 2012

Midterm Examination

15:30-17:20, Thursday, April 26, 2012

1. Dr. Jones got so addicted to the Mr. REE's (mysterious) drink "frog-egg-hit-the-milk" during the visit to Prof. Hey last year (see P&S Midterm 2011). As a result, he decided to visit the top university in an Asia-Pacific island again this summer!

   - **Heartbroken boy**
     During the visit, Dr. Jones was arranged to live in a dormitory room. Dr. Jones notices that every once a while, there would be some poor heartbroken boy shouting wholeheartedly outside the dorm in the late night. Based on Dr. Jones' experiences, it seems that the number of shouts in any time interval is a Poisson random variable, with an average of 6 shouts per hour.

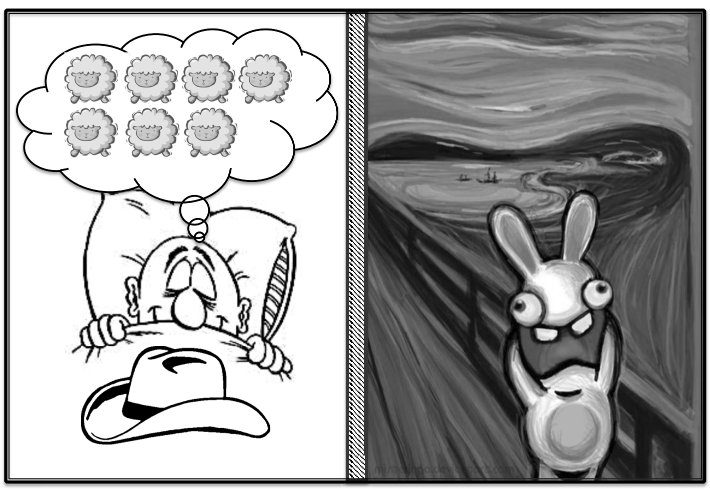

Figure 1: Dr. Jones and the heartbroken boy diagram.

   (a) (5%) Usually when Dr. Jones hears the 5th shout, he would go out say something to cheer the heartbroken boy up. What is the probability of Dr. Jones going out to cheer up the boy within x hours after the boy's arrival to the dorm?

   (b) (5%) From the probability derived in subproblem(a), try to derive the PDF of Dr. Jones' waiting time before cheering up the boy (i.e. the waiting time between the boy's arrival and the 5th shout). Show the derivation in detail to get the full credits. (Hint: The resulted PDF is one of the distributions introduced in Chapter 3.)

(c) (10%) When Dr. Jones goes to bed, he always needs to count 100 sheep before he falls asleep (the counting time is very short and thus negligible). For every time he gets awaken, he would need to double the sheep count before he can fall asleep again, i.e. counting 200 sheep when he gets awaken for the 1st time, 400 sheep for being awaken the 2nd time, etc.
On a given night at 23:00, Dr. Jones was about to go to bed. Yet, he happened to see the heartbroken boy just came to the dorm. Dr. Jones was really tired and decided that he will not leave the bed that day (i.e. he would keep sleeping and would not go out to cheer up the boy). Also note that every time the heartbroken boy shouts, Dr. Jones would be waken by the shout. Dr. Jones got out of bed at 7:00 in the next morning. Can you compute the expected number of sheep that Dr. Jones counted in that night?

- **Everybody loves Yao Ming!!**
Upon his arrival, Dr. Jones immediately found that the students in the university seems to have big love for the former NBA player, Mr. Yao Ming. Mr. Yao's pictures seems to appear everywhere on students' Facebook pages.

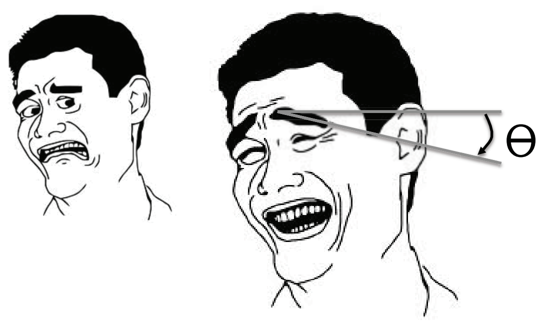

Figure 2: Mr. Yao Ming.

Dr. Jones noticed that the most fascinating feature of Mr. Yao is his eyebrows. According to Dr. Jones' observation, the angle θ of the eyebrow (as shown in Fig. 2), largely depends on the mood of Mr. Yao. When Mr. Yao is calm, $\theta = 0$ rad. And when Mr. Yao is in a good mood, θ is uniformly distributed in $[-\frac{\pi}{2}, -\frac{\pi}{4}]$ rad. But when he has a bad mood, θ is uniformly distributed in $[-\frac{\pi}{3}, 0)$ rad. On any given day, Mr. Yao's mood is equally likely to be calm, good, or bad.

(d) (5%) Can you find out the **PDF** of θ?

(e) (5%) Please find the mean and the variance of θ.

(f) (5%) Dr. Jones was watching a reality TV show "One day with Yao!!". He observed that Mr. Yao's eyebrow angle θ that day falls within the range of $[-\frac{\pi}{3}, 0]$. Condition on this, can you compute the conditional probabilities of Mr. Yao's moods (calm, good, bad) respectively?

- **Continuity of bicycles**

  Dr. Jones always goes to office early in the morning. He always parks his bicycle in a bicycle parking lot shown in Fig. 3 which as 4 legal parking spaces. However, Dr. Jones noticed that the students in this university seems to have very strange confidence, or should say, very strong belief, that "one can always squeeze in another bike between any two bikes". The bicycle that squeezes in between any two neighboring legally parked bicycles, is called "squeezer". So from the students' point of view, the capacity of the parking lot in Fig. 3 is actually 7 bicycles (4 legal spaces + 3 squeezers). When a student parks a bicycle, he/she would first randomly choose a legal space to park if available. If no legal space left, one would randomly pick a pair of neighboring legally parked bicycles (without squeezer in between), and then squeezes one's bicycle in between.

  Dr. Jones observes that there are 6 students who usually park in that area. On any given day, it is equally likely for a student to come, or not to come to the school. The decisions of different students are made independently. Today, Dr. Jones was the earliest to arrive and he parked as shown in Fig. 3.

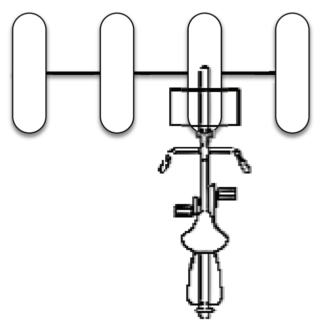

Figure 3: The bicycle parking lot that Dr. Jones parks.

(g) (5%) What is the PMF of the total number of students that come to the school that day?

(h) (5%) At the time when Dr. Jones leaves the office, what is the probability of Dr. Jones seeing a squeezer parked at one side (only one side) of his bike?

(j) (5%) When there are no squeezers at either side of the bike, Dr. Jones can move out his bike immediately. If there is one squeezer at one side, he would need 10 minutes to remove his bike. Yet if there are squeezers at both sides, 30 minutes would be needed.

Dr. Jones has an coffee break appointment with Prof. Hey at 16:30 and he leaves the office at 16:00. The biking time from the office to the coffee shop is 15 minutes and Prof. Hey always arrives 5 minutes before the appointment. Can you tell who will be expected to wait for the other person? And what is the expected amount time that he waits?

Once again, Dr. Jones enjoyed his life in the island very much, and he left happily. Without a doubt, Dr. Jones shall be back, again, in 2013...

2. (4%) What is a random variable? How does learning random variables help us find the probability of events in the random experiments that we are interested in? Please illustrate your arguments with examples.

3. (4%) What is the standard normal random variable? How can it be used to find the probability that the value of an arbitrary Gaussian random variable X is within a specific interval (a, b)? Please detail the procedure involved in finding the numerical value of $P[a < X < b]$.

4. (5%) Let A and B be events with $P[A] > 0$ and $P[B] > 0$. We say that an event B *implies* an event A if $P[A|B] > P[A]$, and an event B does not imply event A if $P[A|B] < P[A]$. Assuming that $P[B^c] > 0$, show that B implies A if and only if B^c does not imply A.

5. (17%) You and your friend plan to go mountain hiking on a Sunday morning. The plan is to arrive at the bus stop at 6:00am and take the bus to the trail-head (where the trail begins). The traveling speed of the bus is 60 km/hr and the traveling time is 1 hour. It is known that buses arrive at the bus stop with the inter-arrival time being the exponential (λ) random variable and the expected value being 10 minutes. In addition, it is known that the sum of n exponential (λ) random variables is an Erlang (n, λ) random variable.

   (a) (4%) You show up late at the bus stop at 6:10am and find that your friend has left with the previous bus. The cleaning woman near the bus stop tells you that the bus departed 5 minutes ago. What is the probability that the next bus will arrive within the next 5 minutes?

   (b) (4%) After waiting for 5 minutes at the bus stop, you find that there is still no sign of the bus. You make a decision that if the bus does not arrive in the next 5 minutes, you will take one of the standby taxis near the bus stop to the trail-head instead. It is known that the traveling speeds of different taxis follow the continuous uniform distribution between 60 km/hr and 90 km/hr, although each taxi travels at a constant speed. What is the probability that you will arrive at the trail-head before your friend does?

   (c) (4%) You finally arrive at the trail-head at 7:00am and find that your friend's bus has not arrived yet. What is the probability that you have past at most two buses (including the bus that your friend took) on your way to the trail-head?

   (d) (5%) Continued from (c), derive the probability distribution of the number of buses that you have past on your way to the trail-head.

6. (20%) It is said that a larger portion of the wealth of any society is owned by a smaller percentage of the people in that society. In fact, it has been shown that

wealth X of an individual in a society roughly follows the power law distribution as follows:

$$f_X(x) = \begin{cases} Cx^{-\alpha} & x \geq x_{\min}, \\ 0 & \text{otherwise}, \end{cases}$$

where C is a constant and $\alpha > 1$.

(a) (5%) Find the normalizing constant C and the CDF $F_X(x)$ for the probability that an individual has personal wealth less than or equal to x.

(b) (5%) Find the m^{th} moment $E[X^m]$ of X and the condition for which the moment exists.

(c) (5%) It has been observed that W of the wealth of a society is in the hands of the richest R of the population, where $0 < W, R < 1$. If $W = 80\%$ and $R = 20\%$, this is the famous "80-20 rule" for wealth distribution. Find α in terms of W and R.

(d) (5%) Explain how you generate random samples that follow the power law distribution to proceed with computer simulation for validating the properties of the power law distribution.