LINEAR ALGEBRA: Midterm Examination (May 3, 2001)

1. Let $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 4 & 1 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$. Let $T : \mathcal{R}^4 \longrightarrow \mathcal{R}^3$ be a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is a vector in $\mathcal{R}^4$. Please determine the following statement as being true or false. A proof or explanation is required for each answer.

   (a) (5%) The column space of A and the range of T have the same dimension.

   (b) (5%) If a set $\mathcal{B}$ is a basis of the null space of T, $\mathcal{B}$ contains at least one non-zero vector in $\mathcal{R}^4$.

   (c) (5%) The sum of the dimension of the range and the null space of T is 4.

   (d) (5%) The system of linear equation $A\mathbf{x} = \underset{\sim}{0}$, where $\mathbf{x} = [x_1\ x_2\ x_3\ x_4]^T$ has only the zero solution.

2. Let A be a 3×3 matrix and $A = [T]_{\mathcal{B}}$, where $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. Suppose $T(\mathbf{b}_1) = -2\mathbf{b}_1 + \mathbf{b}_3$, $T(\mathbf{b}_2) = 4\mathbf{b}_1 - 3\mathbf{b}_3$, $T(\mathbf{b}_3) = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$.

   (a) (6%) Please write the matrix A.

   (b) (6%) Please find $[T]_{\mathcal{B}'}$ that if $\mathcal{B}' = \{\mathbf{b}_3, \mathbf{b}_2, \mathbf{b}_1\}$.

3. (12%) Determine if the system of linear equations is consistent. If it is consistent, find its general solution.

$$\begin{aligned} x_1 + 0x_2 - x_3 - 2x_4 - 8x_5 &= -3 \\ -2x_1 + 0x_2 + x_3 + 2x_4 + 9x_5 &= 5 \\ 3x_1 + 0x_2 - 2x_3 - 3x_4 - 15x_5 &= -9 \end{aligned}$$

4. (10%) Suppose the reduced row echelon form R and three columns of A are given by:

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad \mathbf{a}_1 = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}, \ \mathbf{a}_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \ \mathbf{a}_4 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

   Determine the matrix A.

5. (10%) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent vectors in $\mathcal{R}^3$ Find the reduced row echelon form of $A = [\mathbf{a}_1\, \mathbf{a}_2\, \mathbf{a}_3\, \mathbf{a}_4]$, given that $\mathbf{a}_1 = \mathbf{u}$, $\mathbf{a}_2 = 2\mathbf{u}$, $\mathbf{a}_3 = 2\mathbf{u} + 3\mathbf{v}$ and $\mathbf{a}_4 = \mathbf{v}$.

6. (12%) Find the inverse of $\begin{bmatrix} \mathbf{0} & A \\ B & \mathbf{0} \end{bmatrix}$ if $A \in \mathcal{R}^{n\times n}$ and $B \in \mathcal{R}^{m\times m}$ are both invertible matrices, and $\mathbf{0}$ are zero matrices.

7. **(a)** (12%) Prove that $\det A = (d-a)(d-b)(d-c)(c-a)(c-b)(b-a)$, where

$$A = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{bmatrix}.$$

   (b) (12%) Let $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a nonempty subset of $\mathcal{R}^n$ and A be an $m \times n$ matrix with rank n. Prove that if $\mathcal{S}$ is a linearly independent set, then the set $\{A\mathbf{u}_1, A\mathbf{u}_2, \ldots, A\mathbf{u}_k\}$ is also linearly independent.