Complex Variable Final Exam

1. (a) Write down the definition of the "residue" of a complex function with an isolated singularity at the point z_0. (5%)
   (b) Write down the "Cauchy's residue theorem". (5%)

2. The linear fraction transformation $w = \dfrac{az+b}{cz+d}$ can map a circle/straight line in z-plane onto a circle/straight line in w-plane. Describe the conditions, under which we can map
   (a) a circle to a straight line, and (5%)
   (b) a straight line to a circle. (5%)

3. Suppose $f(z) = \dfrac{3z-6}{z(z-3)}$, (10%)
   (a) expand $f(z)$ in a Laurent series valid for $|z| > 3$,
   (b) use Laurent series to find the residue of $f(z)$ at $z = 0$.

4. Evaluate $\oint_C \dfrac{\cos z}{(z-1)^2(z^2+9)}dz$, $C: |z-1| = 1$. (10%)

5. Evaluate $\int_0^\infty \dfrac{x^{-p}}{1+x}dx$, $0 < p < 1$. (10%)

6. Use Cauchy's residue theorem to sum the series $\sum_{n=-\infty}^{\infty} \dfrac{(-1)^n}{n^2+2}$. (15%)
   (You need to explain how to obtain your answer. Simply plugging the formula can only get minor credits.)

7. Find a transformation that will map the domain $0 \le y \le 2$ in z-plane onto $0 \le \arg w \le \dfrac{\pi}{3}$ in the w-plane. (10%)

8. Use the linear fraction transformation to solve the Dirichlet problem in Fig 8(a). Explain why, with one exception, all level curves must be circles. Which level curve is a line? Hint: Fig 8(b) can be the possible mapping image in w-plane. (15%)

9. Use the Schwarz-Christoffel formula to construct a conformal mapping from the upper half-plane $y \ge 0$ to the region in Fig 9. Require that $f(-1) = -ai$ and $f(1) = ai$. Hint :
   $\int \sqrt{u^2 - a^2}\,du = \dfrac{u}{2}\sqrt{u^2-a^2} - \dfrac{a^2}{2}\ln|u + \sqrt{u^2-a^2}| + C$ (10%)

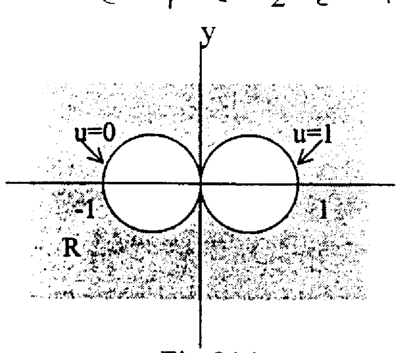

Fig 8(a)

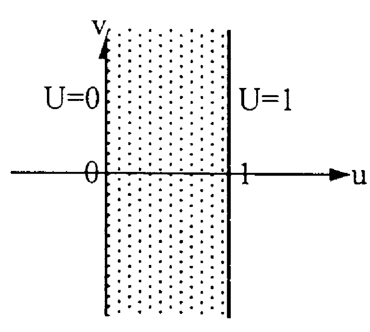

Fig 8(b)

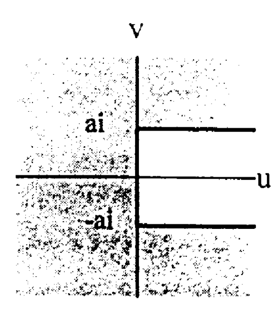

Fig 9