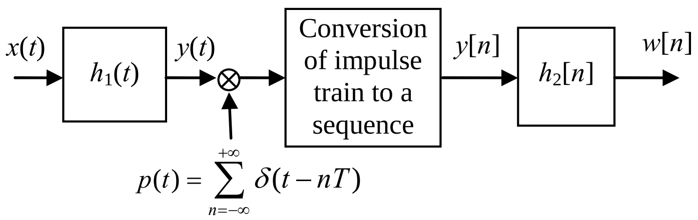
Signals and Systems, Final Exam Solutions

Spring 2005, Edited by bypeng

1. [21 points] Consider a continuous-time system with impulse response

$$h_1(t) = \delta(t) + e^{-3t}u(t) + 2e^{-t}u(t)$$

a) [3] Determine the transfer function of the *inverse* system of h_1.
b) [3] Is the inverse system causal and stable?
c) [3] Sketch the asymptotic approximation to the gain and phase of the Bode plot for the transfer function of the inverse system.
d) [3] Draw a direct-form representation of the inverse system using a minimum number of integrators and multipliers.
e) [3] Repeat Step d) and draw a parallel-form representation of the inverse system.
f) [3] Repeat Step d) and draw a cascade-form representation of the inverse system.
g) [3] Suppose the continuous-time system $h_1(t)$ is cascaded with a sampler, a converter that converts an impulse train to a sequence, and an discrete-time LTI system $h_2[n]$, as shown in the following block diagram. Determine the frequency response $H_2(e^{j\omega})$ such that $w[n] = \delta[n]$ when the input $x(t)$ is a unit impulse.



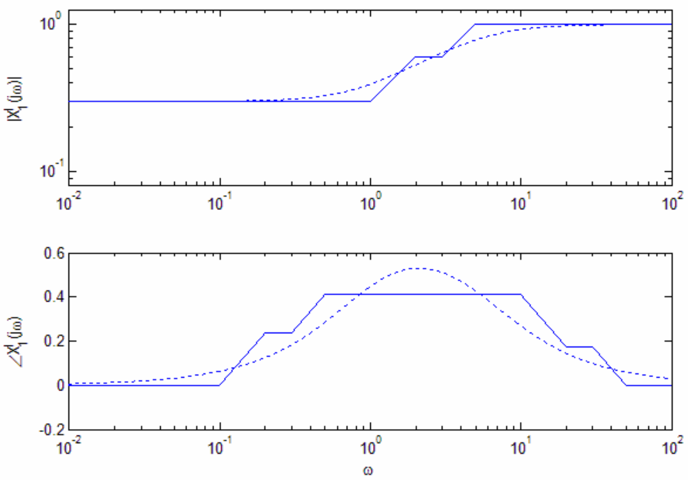
Solution:

a) We need the transfer function of the inverse system $H_1^I(s)$ satisfying $H_1^I(s)H_1(s) = 1$, and we have $H_1(s) = 1 + \dfrac{1}{3+s} + \dfrac{2}{1+s} = \dfrac{(3+s)(1+s)+(1+s)+2(3+s)}{(3+s)(1+s)} = \dfrac{10+7s+s^2}{(3+s)(1+s)} = \dfrac{(2+s)(5+s)}{(3+s)(1+s)}$

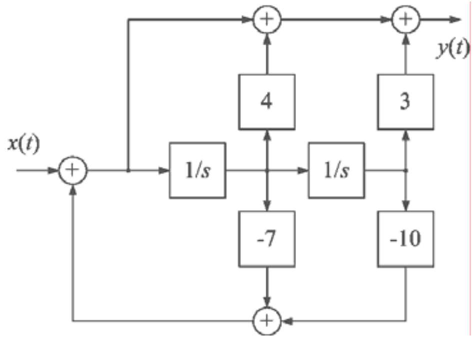
so $H_1(s) = \dfrac{(3+s)(1+s)}{(2+s)(5+s)}$; There are three possible ROCs $\mathrm{Re}[s] < -5$; $-5 < \mathrm{Re}[s] < -2$; $\mathrm{Re}[s] > -2$. But we need the ROC includes $\mathrm{Re}[s] > -1$, so the ROC is $\mathrm{Re}[s] > -2$.

b) Since $H_1^I(s)$ is rational and the ROC is right-hand-sided, the inverse system is causal.
Since $H_1^I(s)$ has all poles in the left half side of the s-plane, the inverse system is stable.

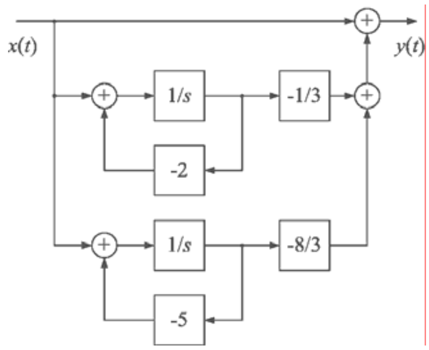
c) The sketch of the Bode plot is as the following figure. [Note that the dashed line is the actual response while the solid line is the approximation using the technique in Section 6.5/9.4 in the textbook. One should use the technique in Section 6.5/9.4 to get the full credit.]



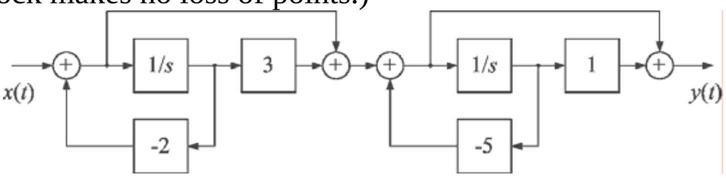
d) $H_1^I(s) = \dfrac{3+4s+s^2}{10+7s+s^2}$



e) $H_1^I(s) = 1 + \dfrac{-\frac{1}{3}}{2+s} + \dfrac{-\frac{8}{3}}{5+s}$



f) $H_1^I(s) = \left(\dfrac{s+3}{s+2}\right)\left(\dfrac{s+1}{s+5}\right)$. *One instance* of the cascade-form is as following figure. (The multiplier "1" *should* be omitted since we need a minimum number of integrators and multipliers; but the existence of this block makes no loss of points.)



g) When $x(t) = \delta(t)$, $y(t) = h_1(t)$, and then $y[n] = y(nT) = h_1(nT)$, and

$$Y(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Y\left(j\frac{\Omega - 2\pi k}{T}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_1\left(j\frac{\Omega - 2\pi k}{T}\right).$$ For $-\pi < \Omega < \pi$,

$$Y(e^{j\Omega}) = \frac{1}{T}\frac{\left(2+j\frac{\Omega}{T}\right)\left(5+j\frac{\Omega}{T}\right)}{\left(3+j\frac{\Omega}{T}\right)\left(1+j\frac{\Omega}{T}\right)},$$ and by $H_2(e^{j\Omega})Y(e^{j\Omega}) = W(e^{j\Omega}) = 1$, we have

$$H_2(e^{j\Omega}) = T\frac{\left(3+j\frac{\Omega}{T}\right)\left(1+j\frac{\Omega}{T}\right)}{\left(2+j\frac{\Omega}{T}\right)\left(5+j\frac{\Omega}{T}\right)}$$ for $-\pi < \Omega < \pi$ and with period 2π.

2. [15 points] Consider the sinusoidal modulating signal

$$x(t) = A_0 \cos(\omega_0 t)$$

with amplitude A_0 and frequency ω_0. The carrier wave is

$$c(t) = A_c \cos(\omega_c t)$$

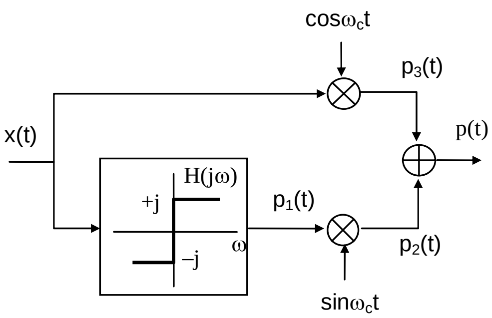
with amplitude A_c and frequency ω_c.

a) [3] Suppose the signal is transmitted through double-sideband/suppressed carrier (DSB/SC) modulation. What is the mathematical expression of the modulated signal $y(t)$?

b) [3] Show that $y(t)$ can be expressed as $y(t) = \frac{1}{2}A\cos\omega_1 t + \frac{1}{2}A\cos\omega_2 t$ and determine the value of A, ω_1, and ω_2.

c) [3] Depict the Fourier transform of $y(t)$.

d) [3] Suppose $x(t)$ is input to a~~n~~ single sideband (SSB) modulation system shown below. Sketch $P_1(j\omega)$, $P_2(j\omega)$, and $P_3(j\omega)$.



e) [3] Design a demodulator to recover $x(t)$ from $p(t)$.

Solution:

a) $y(t) = x(t)c(t) = A_0 A_c \cos(\omega_0 t)\cos(\omega_c t)$

b) By the product-to-sum identity, we have $y(t) = \frac{1}{2}A_0 A_c \cos[(\omega_c + \omega_0)t] + \frac{1}{2}A_0 A_c \cos[(\omega_c - \omega_0)t]$. We have $A = A_0 A_c$ and $\{\omega_1, \omega_2\} = \{\omega_c - \omega_0, \omega_c + \omega_0\}$.

c) $Y(j\omega) = \frac{1}{2}A_0 A_c \pi\delta(\omega - \omega_c - \omega_0) + \frac{1}{2}A_0 A_c \pi\delta(\omega + \omega_c + \omega_0)$
$+\frac{1}{2}A_0 A_c \pi\delta(\omega - \omega_c + \omega_0) + \frac{1}{2}A_0 A_c \pi\delta(\omega + \omega_c - \omega_0)$

d) $X(j\omega) = A_0\pi\delta(\omega - \omega_0) + A_0\pi\delta(\omega + \omega_0)$,

$P_1(j\omega) = A_0\pi j\delta(\omega - \omega_0) - A_0\pi j\delta(\omega + \omega_0)$;

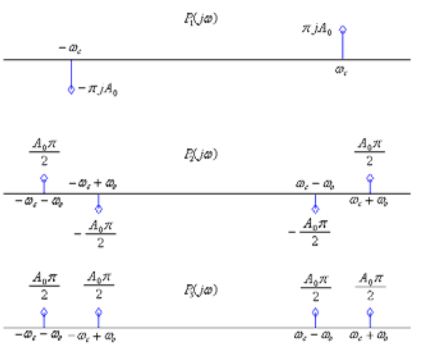
$p_1(t) = -A_0 \sin(\omega_0 t)$,

$p_2(t) = -A_0 \sin(\omega_0 t)\sin(\omega_c t)$

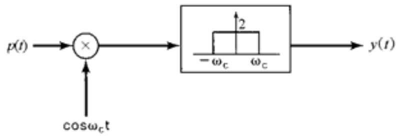
$= \frac{A_0}{2}(\cos[(\omega_c + \omega_0)t] - \cos[(\omega_c - \omega_0)t])$

$P_2(j\omega) = \frac{1}{2}A_0\pi\delta(\omega - \omega_c - \omega_0) + \frac{1}{2}A_0\pi\delta(\omega + \omega_c + \omega_0)$
$-\frac{1}{2}A_0\pi\delta(\omega - \omega_c + \omega_0) - \frac{1}{2}A_0\pi\delta(\omega + \omega_c - \omega_0)$;

$P_3(j\omega) = \frac{Y(j\omega)}{A_c}$



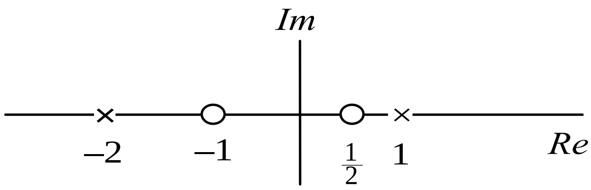
e) Now we have $P(j\omega) = A_0\pi\delta(\omega - \omega_c - \omega_0) + A_0\pi(\omega + \omega_c + \omega_0)$, $p(t) = A_0\cos[(\omega_c + \omega_0)t]$. Considering the system in the right figure, we have



$p(t)\cos\omega_c t = A_0\cos[(\omega_c + \omega_0)t]\cos(\omega_c t) = \frac{A_0}{2}\{\cos[(2\omega_c + \omega_0)t] + \cos(\omega_0 t)\}$, $y(t) = A_0\cos\omega_0 t$

The system behaves as a demodulator.

3. [12 points] Consider an LTI system for which the system function $H(s)$ has the following pole-zero pattern:



a) [3] Indicate all possible ROCs that can be associated with this pole-zero pattern.

b) [3] For each ROC identified in a) specify whether the associated system is stable and/or causal.

c) [3] Determine $H_1(s)$ such that $|H_1(j\omega)| = |H(j\omega)|$ and all poles and zeros of $H_1(s)$ are in the left-half of the s-plane.

d) [3] Determine $H_2(s)$ such that the phase of $H_2(j\omega)$ = the phase of $H(j\omega)$ and all poles and zeros of $H_2(s)$ are in the left-half of the s-plane.

Solution:

Note that since $H(s)$ has finite number of poles and zeros, it is rational.

a) & b)

Since the poles are $p_1 = 1$ and $p_2 = -2$, there are three different possible ROCs.

$\mathrm{Re}[s] > 1$: This ROC is right-hand-sided, so the system is causal; this ROC doesn't include $j\omega$-axis, so it is not stable.

$-2 < \mathrm{Re}[s] < 1$: This ROC is not right-hand-sided, so the system is not causal; this ROC includes $j\omega$-axis, so it is stable.

$\mathrm{Re}[s] < -2$: This ROC is neither right-hand-sided nor inclusive of $j\omega$-axis, so the system is neither causal nor stable.

c) We need the distance between $j\omega$ and $\frac{1}{2}$ the same as that between $j\omega$ and the to-be-found zero in the left-half of the s-plane, and the distance between $j\omega$ and 1 the same as that between

$j\omega$ and the to-be-found pole in the left-half of the s-plane. We conclude that the zero is moved to $-\frac{1}{2}$ and the pole is moved to -1. Note that the pole -1 cancels the effect of the zero -1.

Therefore, $H_1(s) = A_1 \dfrac{s+\frac{1}{2}}{s+2}$ for arbitrary constant A_1.

d) We need the angle between $j\omega - \frac{1}{2}$ and the real axis minus the angle between $j\omega - 1$ and the real axis remain the same. Since there are currently one zero and one pole (total 2, as a multiple of 2), a rather quick way is to "flip" the pole and the zero horizontally to the left-half of the s-plane and switch the pole to a zero, and the zero to a pole.

The conclusion is $H_2(s) = A_2 \dfrac{s+1}{s+2} \cdot \dfrac{s+1}{s+\frac{1}{2}} = A_2 \dfrac{(s+1)^2}{(s+2)(s+\frac{1}{2})}$ for arbitrary constant A_2.

4. [21 points] Suppose we are given the following information of an LTI system:

i) The output of the system in response to the input $x_1[n] = (1/2)^n u[n]$ is $y_1[n] = \left[a(1/6)^n + 8(1/3)^n \right] u[n]$.

ii) If the input is $x_2[n] = (-1)^n$, then the output is $y_2[n] = 18(-1)^n$.

a) [3] Determine the value of the constant a.
b) [3] Determine the system function $H(z)$ for this system.
c) [3] Determine the ROC of the system.
d) [3] Is the system stable and/or causal?
e) [3] Determine the response $y[n]$ if the input $x[n] = 1$ for all n.
f) [3] Draw a direct-form representation of the system.
g) [3] Draw a cascade-form representation of the system.

Solution:

a) By ii), since $x_2[n]$ is an eigenfunction, we have $H(-1) = 18$. Since $X_1(z) = \dfrac{1}{1-\frac{1}{2}z^{-1}}$ and

$Y_1(z) = \dfrac{a}{1-\frac{1}{6}z^{-1}} + \dfrac{8}{1-\frac{1}{3}z^{-1}} = \dfrac{(a+8) - \frac{1}{3}(a+4)z^{-1}}{(1-\frac{1}{6}z^{-1})(1-\frac{1}{3}z^{-1})}$, we have $H(z) = \dfrac{\left[(a+8) - \frac{1}{3}(a+4)z^{-1}\right](1-\frac{1}{2}z^{-1})}{(1-\frac{1}{6}z^{-1})(1-\frac{1}{3}z^{-1})}$,

and then $H(-1) = \dfrac{\left[(a+8) + \frac{1}{3}(a+4)\right](1+\frac{1}{2})}{(1+\frac{1}{6})(1+\frac{1}{3})} = \left(\dfrac{4}{3}a + \dfrac{28}{3}\right) \cdot \dfrac{3}{2} \cdot \dfrac{6}{7} \cdot \dfrac{3}{4} = 18$, $a = 7$.

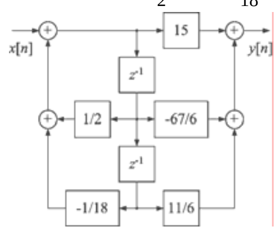
b) Therefore, $H(z) = \dfrac{15(1-\frac{11}{45}z^{-1})(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{6}z^{-1})(1-\frac{1}{3}z^{-1})}$

c) There are three different possible ROC of $H(z)$, say, $|z| > \frac{1}{3}$, $\frac{1}{6} < |z| < \frac{1}{3}$, and $|z| < \frac{1}{6}$. But the ROC of $Y(z)$, $|z| > \frac{1}{3}$, should include the intersection of that of $X(z)$, $|z| > \frac{1}{2}$, and that of $H(z)$. The only satisfying ROC is $|z| > \frac{1}{3}$.

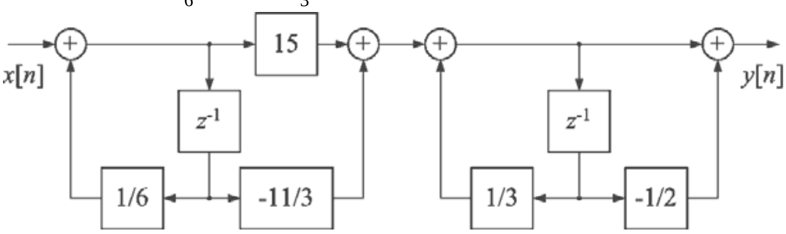
d) $H(z)$ is rational and the ROC is exterior of the circle $|z| = \frac{1}{3}$, so the system is causal; the ROC includes the circle $|z| = 1$, so the system is stable.

e) Observing that $x[n] = 1$ is another eigenfunction, we have $y[n] = 1 \cdot H(1) = \dfrac{15(1-\frac{11}{45})(1-\frac{1}{2})}{(1-\frac{1}{6})(1-\frac{1}{3})} = \dfrac{51}{5}$.

f) $H(z) = \dfrac{15 - \frac{67}{6}z^{-1} + \frac{11}{6}z^{-2}}{1 - \frac{1}{2}z^{-1} + \frac{1}{18}z^{-2}}$



g) $H(z) = \dfrac{15 - \frac{11}{3}z^{-1}}{1-\frac{1}{6}z^{-1}} \dfrac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{3}z^{-1}}$, *an instance*:



5. [12 points] Consider a system where the product $x(t)$ of two continuous-time signals $x_1(t)$ and $x_2(t)$ (that is, $x(t) = x_1(t)x_2(t)$) is sampled by a periodic impulse train $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$. Denote the sampled signal by $x_p(t)$, and suppose the two input signals are band limited:

$$X_1(j\omega) = 0, \quad |\omega| \geq \omega_1,$$
$$X_2(j\omega) = 0, \quad |\omega| \geq \omega_2.$$

a) [3] Derive a mathematical expression of this impulse-train sampling by showing how $X_p(j\omega)$ is related to $X_1(j\omega)$ and $X_2(j\omega)$

b) [3] Determine the maximum sampling interval T_M such that $x(t)$ can be reconstructed from $x_p(t)$ by using an ideal lowpass filter.

c) [3] Continue on Step b) and find the impulse response of the ideal low pass filter.

d) [3] Continue on Step c) and determine the reconstructed signal $x_r(t)$.

Solution:

a) $X(j\omega) = \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(j\omega_0) X_2(j(\omega - \omega_0)) d\omega_0$, $P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k \cdot \frac{2\pi}{T}\right)$,

so $X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(j\left(\omega - k \cdot \frac{2\pi}{T}\right)\right)$

$$= \frac{1}{2\pi T} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} X_1(j\omega_0) X_2(j(\omega - k \cdot \frac{2\pi}{T} - \omega_0)) d\omega_0$$

b) $X(j\omega) = \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$, so $X(j\omega)$ is band-limited and $X(j\omega) = 0$ for $|\omega| > \omega_1 + \omega_2$. The maximum sampling interval is given by $T_M = \frac{2\pi}{2(\omega_1 + \omega_2)} = \frac{\pi}{\omega_1 + \omega_2}$.

c) The frequency response of the ideal low-pass filter is given by $H(j\omega) = \begin{cases} T & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$, where T is the sampling interval and ω_c satisfies $\omega_1 + \omega_2 < \omega_c < \frac{2\pi}{T} - \omega_1 - \omega_2$. Then $h(t) = T \frac{\sin(\omega_c t)}{\pi t}$.

(As $T \to T_M$, we have $h(t) \to T_M \frac{\sin[(\omega_1 + \omega_2)t]}{\pi t} = \frac{\sin[(\omega_1 + \omega_2)t]}{(\omega_1 + \omega_2)t}$.)

d) Since both $x_1(t)$ and $x_2(t)$ are band-limited, $x(t)$ should also be band-limited, and then $x_r(t)$ can be completely reconstructed. Therefore, $x_r(t) = x(t) = x_1(t)x_2(t)$.

6. [15 points] Consider that we take the samples of a sinusoidal signal $x(t) = \cos(\omega_s t / 2 + \theta)$ by impulse-train sampling at a frequency ω_s equal to twice the frequency of $x(t)$. Let the sampled signal be denoted as $x_p(t)$.

a) [5] Find the resulting output $x_r(t)$ if $x_p(t)$ is applied as the input to an ideal lowpass filter with cutoff frequency = $\omega_s/2$.

b) [5] Find the necessary condition if we want to have perfect reconstruction of $x(t)$ from part a), i.e., $x_r(t) = x(t)$.

c) [5] Repeat part (a) if we set $x(t) = \sin(\omega_s t / 2)$.

Solution:

a) $x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) = \sum_{n=-\infty}^{\infty} \cos\left(\frac{\omega_s nT}{2} + \theta\right)\delta(t - nT)$

$$= \sum_{n=-\infty}^{\infty} \left(\cos\left(\tfrac{\omega_s nT}{2}\right)\cos\theta - \sin\left(\tfrac{\omega_s nT}{2}\right)\sin\theta\right)\delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} \left(\cos(n\pi)\cos\theta - \sin(n\pi)\sin\theta\right)\delta(t-nT) = \sum_{n=-\infty}^{\infty} \cos(n\pi)\cos\theta \cdot \delta(t-nT)$$

We conclude that $x_r(t) = \cos\theta \cdot \cos\frac{\omega_s t}{2}$.

b) We need $\cos\theta = 1$. So $\theta = 2k\pi$ for some arbitrary integer *k*.

c) $x(t) = \sin\frac{\omega_s t}{2} = \cos\left(\frac{\omega_s t}{2} - \frac{\pi}{2}\right)$, so it is the case when $\theta = \frac{\pi}{2}$, and then $x_r(t) = \cos\frac{\pi}{2}\cos\frac{\omega_s t}{2} = 0$.

7. [13 points] Consider that a ~~causal~~ LTI system has input and output related by the following differential equation:

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

Let the input $x(t) = 2u(t)$.

a) [5] Find the output *y*(*t*) of the LTI system with the condition of initial rest.

b) [8] Find the output *y*(*t*) of the LTI system with initial conditions given by $y(0^-) = 3$ and $y'(0^-) = -5$.

Solution:

For any (uni/bi)lateral Laplace transformable function $x(t)$, denote the bilateral Laplace transform as $X(s)$ and the unilateral Laplace transform as $\hat{X}(s)$.

a) Since the condition is initial rest, we use (bilateral) Laplace transform and set the ROC always the right-hand-sided part of the *s*-plane. We have $H(s) = \dfrac{1}{2+3s+s^2}$ and $X(s) = \dfrac{2}{s}$, then

$Y(s) = \dfrac{1}{2s+3s^2+s^3} = \dfrac{1}{s} + \dfrac{-2}{s+1} + \dfrac{1}{s+2}$, $y(t) = u(t) - 2e^{-t}u(t) + e^{-2t}u(t)$.

b) Since the condition is set at $t = 0^-$, we use unilateral Laplace transform to evaluate $y(t)$ as $t > 0^+$. We have $s^2\hat{Y}(s) - sy(0^-) - y'(0^-) + 3s\hat{Y}(s) - 3y(0^-) + 2\hat{Y}(s) = \frac{2}{s}$, and then

$(s^2+3s+2)\hat{Y}(s) = \frac{2}{s} + sy(0^-) + y'(0^-) + 3y(0^-) = \frac{2}{s} + 3s - 5 + 9 = \frac{2}{s} + 3s + 4$,

$\hat{Y}(s) = \dfrac{2+3s^2+4s}{s(s^2+3s+2)} = \dfrac{1}{s} + \dfrac{-1}{s+1} + \dfrac{3}{s+2}$, so $y(t) = u(t) - e^{-t}u(t) + 3e^{-2t}u(t)$ for $t > 0^+$;

for $t < 0^-$, we may let $y(t) = c_1 e^{-t} + c_2 e^{-2t}$, and then $y(0^-) = c_1 + c_2 = 3$, $y'(0^-) = -c_1 - 2c_2 = -5$, so $c_1 = 1$, $c_2 = 2$, and $y(t) = e^{-t} + 2e^{-2t}$.

The conclusion is $y(t) = e^{-t} + 2e^{-2t} + u(t) - 2e^{-t}u(t) + e^{-2t}u(t)$.

8. [11 points] Consider that a ~~causal~~ LTI system has input and output related by the following difference equation:

$$y[n] + 3y[n-1] = x[n]$$

Let the input $x[n] = 8u[n]$.

a) [5] Find the output $y[n]$ of the LTI system with the condition of initial rest.

b) [6] Find the output $y[n]$ of the LTI system with initial condition given by $y[-1] = 1$.

Solution:

For any (uni/bi)lateral z-transformable function $x[n]$, denote the bilateral z-transform as $X(z)$ and the unilateral z-transform as $\hat{X}(z)$.

a) Since the condition is initial rest, we use (bilateral) z-transform and set the ROC always the exterior part of the circle where the out-most pole lies in the *z*-plane. We have $H(z) = \dfrac{1}{1+3z^{-1}}$ and

$X(z) = \dfrac{8}{1-z^{-1}}$, then $Y(z) = \dfrac{8}{(1+3z^{-1})(1-z^{-1})} = \dfrac{6}{1+3z^{-1}} + \dfrac{2}{1-z^{-1}}$, $y[n] = 6\cdot(-3)^n u[n] + 2u[n]$.

b) Since the condition is set at $n = -1$, we use (unilateral) z-transform to evaluate $y[n]$ as $n \geq 0$.

We have $\hat{Y}(z)+3z^{-1}\hat{Y}(z)+3y[-1]=\dfrac{8}{1-z^{-1}}$, and then $(1+3z^{-1})\hat{Y}(z)=\dfrac{8}{1-z^{-1}}-3y[-1]=\dfrac{5+3z^{-1}}{1-z^{-1}}$, $\hat{Y}(z)=\dfrac{5+3z^{-1}}{(1-z^{-1})(1+3z^{-1})}=\dfrac{2}{1-z^{-1}}+\dfrac{3}{1+3z^{-1}}$, so $y[n]=2u[n]+3\cdot(-3)^{n}u[n]$ for $n\geq 0$;

for $n\leq -1$, we may find that $y[n]=-3y[n-1]$, so $y[n]=c(-3)^{n}$, and $y[-1]=-\dfrac{1}{3}c=1$, $c=-3$, so $y[n]=-3\cdot(-3)^{n}$. The conclusion is $y[n]=-3\cdot(-3)^{n}+2u[n]+6\cdot(-3)^{n}u[n]$.