2008-09 微分方程期末考

1. (5%) Find a power series solution of the differential equation $y''-4xy'-4y=e^x$ about $x=0$.

2. (5%) Is $x=0$ an ordinary or a singular point of the differential equation $xy''+(\sin x)y=0$? Defend your answer with sound mathematics.

3. (5%) If the Laplace transform of $f(t)$ is $F(s)$, and $k>0$, then find the Laplace transform of $e^{at}f(t-k)U(t-k)$.

4. (10%) Use the Laplace transform of the initial-value problem $ty''+y'+ty=0, y(0)=1, y'(0)=0$ to show that $L\{J_0(t)\}=\dfrac{1}{\sqrt{s^2+1}}$. (Hint: $F(s)=L\{f(t)\}, f(0)=\lim_{s\to\infty} sF(s)$)

5. (5%) Use the Laplace transform to solve the following BVP $y''+2y'+y=0, y'(0)=2, y(1)=2$.

6. (5%) Find the following inverse Laplace transform $L^{-1}\left\{\dfrac{8k^3 s}{(s^2+k^2)^3}\right\}$.

7. (20%) Let $A=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$,

(a) Solve $\dfrac{dX(t)}{dt}=AX(t)$ with initial condition $X(0)=\begin{bmatrix}1\\1\\1\end{bmatrix}$.

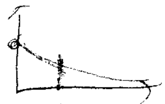
(b) Find $\Phi(t)$, such that the solution of (a) is $X(t)=\Phi(t)X(0)$ where $X(0)$ is the initial condition at $t=0$.

(c) Find $\Phi(t)$, such that the solution of (a) is $X(t)=\Phi(t)X(2)$ where $X(2)$ is the initial condition at $t=2$.

(d) Solve $\dfrac{dX(t)}{dt}=AX(t)+f(t)$ with initial condition $X(0)=\begin{bmatrix}0\\0\\0\end{bmatrix}$ and $f(t)=\begin{bmatrix}0\\0\\\sin(t)\end{bmatrix}$.

8. (15%)

(a) Show that the product of two odd functions is even.



(b) Show that $\int_{-T}^{T} f(t)\,dt = 0$ if $f(t)$ is an odd function.

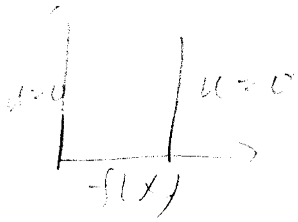
(c) Let $f(t) = e^{-t}$ for $0 \le t \le 1$. Please sketch the plots of the associated half range expansions of sine series, cosine series and Fourier series for $-2 \le t \le 2$, respectively.

9. (15%)

(a) (8) The wave equation $a^2 \dfrac{\partial^2 E}{\partial x^2} = \dfrac{\partial^2 E}{\partial t^2}$, $0 < x < L,\ t > 0$ is subject to the initial and boundary conditions:

$$E(0,t) = E(L,t) = 0, \quad t > 0$$

$$E(x,0) = f(x), \qquad \left(\frac{\partial E}{\partial t}\right)_{t=0} = 0, \qquad 0 < x < L$$



Please show that the solution of the wave equation can be written as

$$E(x,t) = \frac{1}{2}[f(x+at) + f(x-at)]$$

[Hint: use the identity $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$]

(b) (7) If the wave equation is defined for the infinite region $-\infty < x < \infty$ and $t > 0$ with the initial and boundary conditions removed, then please give a procedure that will lead to the general solution of the wave equation with the form

$$E(x) = F(x+at) + G(x-at)$$

where F and G are two arbitrary twice-differentiable functions.

10. (15%) Use the result that the Fourier transform of $e^{-x^2/4p^2}$ is $2\sqrt{\pi}pe^{-p^2\alpha^2}$ to solve the following equations

(a) (7) $\dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty,\ t > 0$

$u(x,0) = \delta(x), \quad -\infty < x < \infty.$

$\frac{du}{dt} = -x^2 D u$

$\int \frac{1}{u} du = \int -x^2 D\, dt$

$\ln u = -x^2 D t + c$

(b) (8) $\dfrac{\partial u}{\partial t} = -\dfrac{u}{\tau} - v\dfrac{\partial u}{\partial x} + D\dfrac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty,\ t > 0$

$u(x,0) = \delta(x), \quad -\infty < x < \infty.$

$\frac{\partial u}{\partial t} + \frac{u}{\tau}$

祝大家新年愉快，請留意助教公佈欄-微方成績及看考卷時間