Discrete Mathematics

Midterm Exam, Fall 2006

1. (20 pts) True (O) or False (X) ? Score=max$\{0, \lceil Right - \frac{1}{2}Wrong \rceil\}$.

   (a) $(p \vee q) \rightarrow (\neg p \rightarrow q)$ is a tautology

   (b) Propositional logic has a sound and complete deduction system.

   (c) Formula $(\forall x \exists y P(x,y)) \vee \exists y(\neg P(x,y))$ does not contain any free variable.

   (d) A propositional formula is said to be *satisfiable* if and only if it can always be satisfied regardless of the truth assignment.

   (e) $(\exists x \forall y P(x,y)) \rightarrow (\forall y \exists x P(x,y))$ is a valid statement.

   (f) $(\exists x P(x)) \vee (\exists y Q(y)) \rightarrow \exists x(P(x) \vee Q(x))$ is a valid formula.

   (g) If $A \cup B \subseteq A \cup C$ then $B \subseteq C$.

   (h) The logical connectives $\{\neg, \vee\}$ are logically complete.

   (i) The Godel Incompleteness Theorem says that the predicate logic does not have a sound and complete deduction system.

   (j) The power set of the empty set $\emptyset$, i.e., $2^{\emptyset}$, is the empty set.

   (k) There is a bijective function (i.e., one-to-one and onto function) from the set of integers Z to the set of natural numbers N.

   (l) If $f \circ g$ is injective (i.e., one-to-one), then g must be injective. (Here $f \circ g$ is the composition function, defined as $(f \circ g)(x) = f(g(x))$).

   (m) If $f \circ g$ is surjective (i.e., onto), then g must be surjective.

   (n) The set of rational numbers Q is countably infinite.

   (o) The union of two countably infinite sets is always countably infinite.

   (p) Mathematical induction is useful for showing that the set of real numbers is uncountably infinite.

   (q) Strong mathematical induction is more powerful than weak mathematical induction.

   (r) Consider formula $(p_1 \wedge p_2 \wedge ... \wedge p_k) \rightarrow q$. If we are able to show $\neg p_j$ for some j, $1 \leq j \leq k$, then the above formula is clearly true because the assumption $p_1 \wedge p_2 \wedge ... \wedge p_k$ is false. Such a proof method is called *proof by contradiction.*

   (s) For the set of natural numbers N, $N \in 2^N$. (Here 2^N denotes the power set of N.)

   (t) If set A is a proper subset of B (i.e., $A \subset B$ but $A \neq B$), then there cannot be a one-to-one correspondence (i.e., a bijection) between A and B.

2. (10 pts) Prove that $\neg p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \vee r)$ are logically equivalent using a truth table.

3. (10 pts) Using Tables 5, 6, and 7 to show that the following formula is a tautology.

$$((p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow r$$

4. (10 pts) Use the inference rules given in Table 1 to show that

$$\begin{array}{c} p \to q \\ \neg p \to r \\ r \to s \\ \hline \neg q \to s \end{array}$$

5. (15 pts) Let us define two new quantifiers: the *unique existentential quantification* and *almost universal quantification*. We shall use the notation $\exists! x P(x)$ to express the proposition: There exists a unique value for x such that $P(x)$ is true. We shall use the notation $\forall! x P(x)$ to express the proposition: $P(x)$ is true for all possible values of x except one. Answer the following questions:

   (a) (6 pts) Let the universe of discourse be the set of all positive and negative integers (0 is not in the universe). For each of the following propositions decide whether they are true or false. No explanations are needed. (i) $\exists! x(x^4 = 1)$ (ii) $\exists! x(x + 2 = 2x)$ (iii) $\exists! x(x^2 + 2 = 2x)$ (iv) $\forall! x(x < -1)$ (v) $\forall! x(x^2 > x)$ (vi) $\forall! x(x^2 \neq 4)$.

   (b) (4 pts) Are the two propositions $\forall! x P(x) \wedge \forall! x Q(x)$ and $\forall! x(P(x) \wedge Q(x))$ equivalent? Justify your answer.

   (c) (5 pts) We know that $\neg \exists x P(x) \Leftrightarrow \forall x \neg P(x)$. Does this also hold for $\neg \exists! x P(x) \Leftrightarrow \forall! x \neg P(x)$? Why? If not, can you find some equivalence involving $\forall!$, $\exists!$, $P(x)$ and $\neg P(x)$? Justify your answer.

6. (15 pts) Consider the following program fragment in which n is a positive integer and x is a real number. (':=' denotes the assignment operator.)

```
poly:= 1
i := 1
While i ≤ n
begin
        poly := poly * x + 1
        i := i + 1
end
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   (a) (5 pts) Find a loop invariant for the above program

   (b) (10 pts) Use the loop invariant found in (a) to prove that the program correctly computes $\sum_{j=0}^{n} x^j$ upon termination. That is, when the program terminates, poly $= 1 + x + ... + x^n$.

7. (20 pts) Define the following terms in a precise manner:

   (a) *Well-ordering property* of natural numbers

   (b) The *continuum hypothesis*

   (c) Two infinite sets having the same *Cardinality*

   (d) *Countable set*

   (e) *Partial correctness* in program verification