2005 Fall 台大電機系 微分方程 期末考

1. (15%) Given the differential equation

$$2xy'' + y' - y = 0.$$

   (a) Show that $x = 0$ is a regular singular point of the differential equation. (3%)

   (b) What is the indicial equation of the differential equation and the related indicial roots? (3%)

   (c) Find a solution of the form $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$. (9%)

2. (20%) Given the nonhomogeneous differential equation

$$X'(t) = A\,X(t) + F(t),$$

where $X(t)$ and $F(t)$ are two $n \times 1$ column vectors. If A is an $n \times n$ constant matrix and has n distinct real eigenvalues: $\lambda_1, \lambda_2, \dots, \lambda_n$, with the corresponding eigenvectors: $K_1, K_2, \dots, K_n$.

(a) Find a fundamental set of solutions of the associated homogeneous equation. (2%)

(b) Find a fundamental matrix $\Phi(t)$ of the associated homogeneous equation. (2%)

(c) Show that the fundamental matrix $\Phi(t)$ is nonsingular. (2%)

(d) Show that the fundamental matrix $\Phi(t)$ satisfies $\Phi'(t) = A\,\Phi(t)$. (4%)

(e) Find a particular solution of the nonhomogeneous differential equation. (6%)

(f) Find the solution if the initial condition is $X(0) = X_0$. (4%)

3. (10%) Calculate

(a) $L\{te^{2t}\sin kt\}$ where k is a constant. (5%)

(b) $L^{-1}\{(5s^2-15s-11)/[(s+1)(s-2)^3]\}$. (5%)

4. (15%) Suppose y(t) is piecewise continuous and of the exponential order on $[0, \infty)$. It follows $t\dfrac{d^2y}{dt^2}+y=0$ subject to y(0)=A and y'(0)=B, where A and B are constants. Please use the Laplace transform to find the solution represented by A and B. Which of them is not arbitrary? Please find its value. (Notice that if no close form exist, please express your answer in terms of power series.)

5. (13%) Solve the Laplace's equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ for a square plate subject to the given boundary condition

$$u(0,y)=0,\quad u(2,y)=y(2-y)$$

$$u(x,0)=0,\quad u(x,2)=\begin{cases} x, & 0<x<1 \\ 2-x, & 1\le x<2 \end{cases}$$

(a) Find u(x,y). (10%)

(b) What is the maximum value of the temperature u for $0\le x\le 2,\ \ 0\le y\le 2$.

(3%)

6. (9%) Using the given set of functions to find a sub-set of orthogonal functions, and using this sub-set functions to expand the following meander function f(x).

$$\left\{1, \cos\frac{n\pi}{p}x, \sin\frac{m\pi}{p}x\right\}, n=1,2,3,\cdots, m=1,2,3,\cdots$$

(a) Find a sub-set of orthogonal functions on $[0,\frac{p}{2}]$. (3%)

(b) Prove the functions in (a) are orthogonal on $[0,\frac{p}{2}]$. (3%)

(c) Use the functions in (a) to expand meander function f(x),

$$f(x) = \begin{cases} -1, & -\dfrac{p}{2} < x < 0 \\ +1, & 0 \le x < \dfrac{p}{2} \end{cases} \quad (3\%)$$

7. (13%) Find the half-range cosine and sine expansions of

   $f(x) = x^2 + x, \quad 0 < x < 1$ (10%), and find the value of cosine and sine expansions of f(x) at x=99. (3%)

8. (10%) Find the solution of

   $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0<x<\pi,\ y>0$$

   u(0, y)=f(y),

   $$\left.\frac{\partial u}{\partial x}\right|_{x=\pi} = 0, \quad y>0,$$

   $$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad 0<x<\pi.$$