$\sigma_1=\sigma_2$ $\quad 2\sigma_1^2(1-\rho^2) = -\frac{3}{4}$ $\quad 2\sigma_1\sigma_2\sqrt{1-\rho^2} = \frac{\sqrt{3}}{2}$

$2\sigma_2^2(1-\rho^2) = -\frac{3}{4}$ $\quad \sigma_1\sigma_2 \; 1-\rho^2 = \frac{3}{4}$

$\sigma_1^2\sigma_2^2(1-\rho^2)^2 = \frac{9}{16}$

$F_Z(0) =$

Probability and Statistics Final Exam

2009

1. (16%) X1 and X2 are independent random variables, each following a uniform distribution U(-1, 1). Let X_{min}=min(X1, X2) and X_{max}=max(X1, X2).
   (a). Are X_{min} and X_{max} independent? (Please show your answer based on the definition of independence, not your intuition) (2%)
   (b). Please find the joint CDF of X_{min} and X_{max} : $F_{Xmin,\, Xmax}$ (x, y) (8%)
   (c). Please find E(X_{max} - X_{min}) (6%)

2. (12%) K students participate in a party game where each student picks a number from 1 to N uniformly and independently. The rule is that whoever picks the same number as others is the winner. For example, if David and Mary both pick 7, they are both winners. Let X be the number of winners and Y is the number of those numbers being picked by the winners.
   (a). Are X and Y independent? (Please show your answer based on the definition of independence, not your intuition) (2%)
   (b). Please find E(X) and E(Y) (10%)

$-2vy$ $\quad \frac{\rho}{\sigma_1\sigma_2} =$

3. (13%) X and Y have a joint PDF as follows:

$$f_{X,Y}(x,y) = \frac{e^{-2[x^2-xy+y^2]/3}}{\sqrt{3}\pi}$$

$-\dfrac{(x^2 - xy + y^2)}{\frac{3}{2}}$

   (a). Are X and Y independent? (2%)
   (b). Let V=aX+bY and W=cX+dY. Please find a, b, c, and d such that V and W are independent random variables. (6%)
   (c). Z = X-Y. Derive f_Z. (5%)
   (Hint: For a real-valued symmetric matrix A, we can find that A=UDU' where D is a diagonal matrix with its element being the eigenvalues of A, and U is a unitary matrix with columns that are n orthonormal eigenvectors of A.)

4. (21%) David lives near the area where 3 MTA lines merge as shown in the figure below. Based on his experience, he finds out
   i. The trains arrive in the blue-line, red-line and green-line stations following Poisson distribution, each with a rate of 1/4, 1/2, and 1/4 per minute, respectively. $\lambda e^{-\lambda T}$ $\quad \lambda = \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$
   ii. When a train arrives at any one of the above three stations, the probability that the train has an empty seat is 0.5.

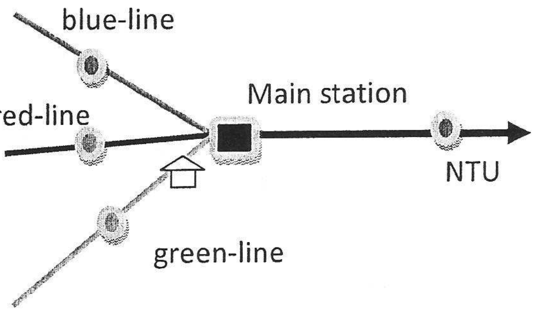
exactly one

T=?

iii. Whenever a train arrives in any of the three stations, there will ~~be~~ K people ~~(excluding David)~~ waiting for the train ~~bus~~. K is uniformly distributed between 0 and 4. However, more people are waiting in the main station. Therefore, K (again, excluding David) is uniformly distributed between 0 and 16 for the main station.

Uniform

iv. Everyone waiting for the train has an equal probability to get the empty seat.



David can take the train at any train station to go to NTU. However, ~~he does not~~ like to wait nor stand in a train. Let's find a station for him so that E(waiting time)*Prob(standing in a train) is minimized.

P[Standing]

(a). When David waits in the main station, is the probability that there is an empty seat still 0.5? If not, what is the probability that there is a seat left? (2%)

(b). When waiting in the main station, what is the probability that David will get an empty seat? (2%) (Note that we assume that none gets off the train in the main station)

(c). Show that the total number of train arrivals in one period of time is still Poisson and the total rate is 1 per minute. (7%) (Hint: Use moment generating function)

先做

(d). When waiting in the main station, what is the distribution of inter-arrival time (i.e., the time between two consecutive train arrivals)? (6%) (Hint: the time interval between two consecutive events that follow a Poisson distribution follows an exponential distribution)

(e). Given your above answers, which station should David choose to achieve his objective. (4%)

$p^k (1-p)^{N-k}$.

$P_k = \binom{M}{k}$,

5. (14%) A semiconductor wafer has M VLSI chips on it and these chips have the same circuitry. Each VLSI chip consists of N interconnected transistors. A transistor may fail (not function properly) with a probability p because of its fabrication process, which we assume to be independent among individual transistors. A chip is considered a failure if there are n or more transistor failures. Let K be the number of failed transistors on a VLSI chip, which is therefore a random variable (R.V.).

(a). What is a random variable? (4%)

(b). What is the sample space (also called outcome set) over which R.V. K is defined? (2%)

(c). Let X_i = 1 if a chip *i* fails and X_i = 0 if a chip *i* is good. Derive the probability that a chip is good, i.e., $p_g \equiv \Pr\{X_i = 0\} = ?$ (3%)

(d). Whether one chip is good or fails is independent of other chips. Let the yield of a wafer be defined as the percentage of good chips in the wafer, i.e., $Y = (1 - \frac{1}{M}\sum_{i=1}^{M} X_i) \times 100\%$. Then derive $\mu_Y \equiv E[Y]=?$ $\sigma_Y^2 = Var[Y]=?$

$Y \le y$

(Hint: Utilize p_g obtained from 5.iii)) (5%)

6. (24%) You are observing a radar signal sequence

$Y_k = \theta + \omega_k$, k=1, 2, 3, ...

where θ is an unknown constant, and ω_k is $N(\mu, \sigma^2)$ and independent and identical over time index k.

(a). Now you have one observation $Y_1 = x$, where x is an observed value. And you set an estimate of θ as $\Theta = x - \omega_1$. Derive $f_{\Theta|Y_1}(s \mid x)$, $E[\Theta \mid x]$ and $Var[\Theta \mid x]$. (8%)

(b). Given N observations of Y_k, how do you estimate the value of θ ? (3%) Is your estimate biased or unbiased, why? (5%)

(c). When N=100, propose an approximation method of your confidence (in terms of probability) that your estimate in (6.i) is within 0.2σ from the true parameter θ. Please explain quantitatively why. (8%)

$Y_n - E$

$$\left[\frac{\frac{1}{N}e^s}{1-(1-\frac{1}{N})e^s}\right]^{n} \quad (1-p+pe^s)^{n-1} \cdot pe$$

$n \cdot \frac{1}{k} \cdot (1-\frac{1}{k})^{n-1}$

Random Variable	PMF or PDF	MGF $\phi_X(s)$
Bernoulli (p)	$P_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & \text{otherwise} \end{cases}$	$1-p+pe^s$
Binomial (n, p)	$P_X(x) = \binom{n}{x} p^x(1-p)^{n-x}$	$(1-p+pe^s)^n$
Geometric (p)	$P_X(x) = \begin{cases} p(1-p)^{x-1} & x=1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$	$\frac{pe^s}{1-(1-p)e^s}$
Pascal (k, p)	$P_X(x) = \binom{x-1}{k-1} p^k(1-p)^{x-k}$	$(\frac{pe^s}{1-(1-p)e^s})^k$
Poisson (α)	$P_X(x) = \begin{cases} \alpha^x e^{-\alpha}/x! & x=0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$	$e^{\alpha(e^s-1)}$
Disc. Uniform (k, l)	$P_X(x) = \begin{cases} \frac{1}{l-k+1} & x=k,k+1,\ldots,l \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{sk}-e^{s(l+1)}}{(l-k+1)\,1-e^s}$
Constant (a)	$f_X(x) = \delta(x-a)$	e^{sa}
Uniform (a, b)	$f_X(x) = \begin{cases} \frac{1}{b-a} & a<x<b \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{bs}-e^{as}}{s(b-a)}$
Exponential (λ)	$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$	$\frac{\lambda}{\lambda-s}$
Erlang (n, λ)	$f_X(x) = \begin{cases} \frac{\lambda^n x^{n-1}e^{-\lambda x}}{(n-1)!} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$	$(\frac{\lambda}{\lambda-s})^n$
Gaussian (μ, σ)	$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$	$e^{s\mu+s^2\sigma^2/2}$

Table 6.1 Moment generating function for families of random variables.

Definition 4.17 ***Bivariate Gaussian Random Variables***

*Random variables X and Y have a **bivariate Gaussian PDF** with parameters μ_1, σ_1, μ_2, σ_2, and ρ if*

$$f_{X,Y}(x,y) = \frac{\exp\left[-\frac{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)}\right]}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}},$$

$$\frac{\frac{2\rho}{\sigma_1\sigma_2}}{2(1-\rho^2)} = \frac{2}{3}$$

where μ_1 and μ_2 can be any real numbers, $\sigma_1 > 0$, $\sigma_2 > 0$, and $-1 < \rho < 1$.

Theorem 4.28 *If X and Y are the bivariate Gaussian random variables in Definition 4.17, X is the Gaussian (μ_1, σ_1) random variable and Y is the Gaussian (μ_2, σ_2) random variable:*

$$f_X(x) = \frac{1}{\sigma_1\sqrt{2\pi}}e^{-(x-\mu_1)^2/2\sigma_1^2}. \qquad f_Y(y) = \frac{1}{\sigma_2\sqrt{2\pi}}e^{-(y-\mu_2)^2/2\sigma_2^2}.$$

Theorem 4.29 *If X and Y are the bivariate Gaussian random variables in Definition 4.17, the conditional PDF of Y given X is*

$$f_{Y|X}(y|x) = \frac{1}{\tilde{\sigma}_2\sqrt{2\pi}}e^{-(y-\tilde{\mu}_2(x))^2/2\tilde{\sigma}_2^2},$$

where, given $X = x$, the conditional expected value and variance of Y are

$$\tilde{\mu}_2(x) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1), \qquad \tilde{\sigma}_2^2 = \sigma_2^2(1-\rho^2).$$

$F(x,y) = \int \int$