Department of Electrical Engineering

National Taiwan University

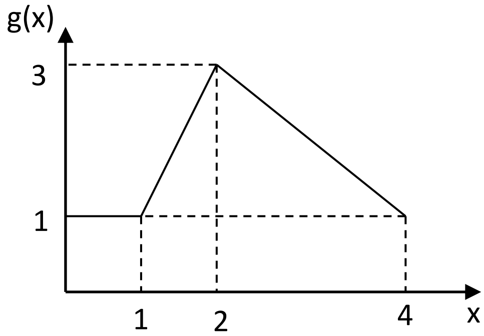
Probability and Statistics, Spring 2013

Midterm Examination

15:30-17:20, Thursday, April 18, 2013

（所有同學請先在答案卷上標註自己所屬的班級）

1. **(14%) The probability density of a continuous random variable X is 0 for any $X < 0$ and $X > 4$. The CDF of X follows the form of $F_X(x) = ax^2$.**
   i. Find the value of a (4%)
   ii. Let Y = g(X), where g(x) is illustrated in the figure below. Find P(Y = 1) (2%)
   iii. Find the PDF of Y for Y>1 (8%)



2. **(20%) The Sixers and the Celtics play the NBA final series this year. Let the Celtics is stronger than the Sixers and the probability that Celtics wins a game is 2/3. Assume the results of all games are independent.**
   i. Let the series is a best of 7 series and the series ends as soon as one of the teams has won 4 games. Find the PMF for the number W of games Celtics wins (6%) and the number N of games played in the series (6%).

   ii. Let the series ends as soon as one of the teams has 4 points. A team gets one point if it wins a game. Otherwise, it loses one point. For example, if the Sixers wins the first game, the Sixers is 1 and the Celtics is -1. Find the probability that the series ends in less than 7 games. (8%)

3. **(8%) Let the target of a dart game is a 4 inch by 4 inch square. The dart lands randomly and uniformly in the target. Find two different events (each with a non-zero probability), A and B, so that they are independent. (2%) Find three different events (each with a non-zero probability), A, B, and C so that they are mutually independent. (6%) Please define the events by drawing them in the target and prove your answer.**

4. **(8%) Let you have an exponential $(\lambda = 1)$ random number generator. Please find a way to generate a uniform random variable between [0, 1]. Please prove your answer.**

5. **(10%) Alex, Ben, Sam and Tim are four prisoners, one of whom is sentenced to die with equal probability. The four prisoners cannot communicate with each other. Alex asks a jailer to tell him who will be freed so that Alex could ask him to bring a letter to Alex's wife. The jailer tells Alex that Ben is going to be freed. So, Alex considers the probability of himself dying as 1/3 after learning this information. Is Alex correct? To answer the question, let us analyze P[A|JB] = P[A, JB]/P[JB], where A is the event that Alex is sentenced to death and JB is the event that jailer tells Alex that Ben is going to be freed. B, S, and T correspond to events that Ben, Sam and Tim are sentenced to death respectively.**
   i. Derive P[JB|A], P[JB], and P[A|JB]. (5%)
   ii. Discuss if Alex is correct based on what you get for (i) (5%)

6. **(10%) Consider throwing a fair die, where the sample space *S* = {⚀, ⚁, ..., ⚃, ⚄, ⚅}.**
   i. Please define any two different random variables, say, *X* and *Y*, based on *S* and write down the probability mass functions $P_X(x)$ and $P_Y(y)$ respectively. (6%)
   ii. Let event A={⚀, ⚁, ⚄}. Derive $P_{Y|A}(y)$. (4%)

7. **(10%) You go to a party with 500 guests. What is the probability that no more than 2 other guests have the same birthday as you? Please give your answer in a numerical value of three digits after the decimal point. For simplicity, exclude birthdays on February 29 and you may want to use Table 1 for your numerical calculation.**

8. **(8%) Let $Y \triangleq X^2 + 1$. Compute E[Y] and σ^2_Y if $f_X(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{else.} \end{cases}$**

9. **(12%) A transmitter transmits binary bits. When a transmitter transmits a "0" bit, the voltage of the signal is a Gaussian random variable $N(\mu,\sigma) = N(-3, 2)$, where μ is the mean and σ is the standard deviation. When it transmits a "1" bit, the voltage of the signal is a Gaussian random variable $N(3, 2)$. "0" and "1" are equally likely.**
   i. Let B_i be the event that the transmitter transmits bit "i," i = 0 or 1 and let V be the signal voltage level observed. $P[B_0|V \le 0.5]$ = ? $P[B_1|V \le 0.5]$ = ? (Table 2 gives the CDF values of the standard Gaussian r. v.) (4%)
   ii. You have a decision rule that if you observe $V \le -0.5$, you consider B_0 occurs, and if $V \ge 0.5$, you consider B_1 occurs; You have no decision otherwise. Let Y be a derived r.v. from r.v. V according to the decision

rule:

$$Y=g(V) = \begin{cases} 0, & \text{if } V \le -0.5; \\ 1, & \text{if } V \ge 0.5; \\ -1, & \text{otherwise.} \end{cases}$$

Calculate P[no decision] = P[Y=-1] (4%)

iii. Calculate the error probability that you decide “1” is transmitted but actually “0” is transmitted. (4%)

Appendix:

N	0	1	2	3	4	5	6	7	8	9
3.0	1.09861	1.10194	1.10526	1.10856	1.11186	1.11514	1.11841	1.12168	1.12493	1.12817
3.1	1.13140	1.13462	1.13783	1.14103	1.14422	1.14740	1.15057	1.15373	1.15688	1.16002
3.2	1.16315	1.16627	1.16938	1.17248	1.17557	1.17865	1.18173	1.18479	1.18784	1.19089
3.3	1.19392	1.19695	1.19996	1.20297	1.20597	1.20896	1.21194	1.21491	1.21788	1.22083
3.4	1.22378	1.22671	1.22964	1.23256	1.23547	1.23837	1.24127	1.24415	1.24703	1.24990
3.5	1.25276	1.25562	1.25846	1.26130	1.26413	1.26695	1.26976	1.27257	1.27536	1.27815
3.6	1.28093	1.28371	1.28647	1.28923	1.29198	1.29473	1.29746	1.30019	1.30291	1.30563
3.7	1.30833	1.31103	1.31372	1.31641	1.31909	1.32176	1.32442	1.32708	1.32972	1.33237
3.8	1.33500	1.33763	1.34025	1.34286	1.34547	1.34807	1.35067	1.35325	1.35584	1.35841
3.9	1.36098	1.36354	1.36609	1.36864	1.37118	1.37372	1.37624	1.37877	1.38128	1.38379

Table 1. Natural Logarithms Table (ln N) for Question 7

How to Use:

- You need to find the natural logarithm of 2.85
- Find the row with N = 2.8
- Find the column 5, the number inside the cell is 1.04732

Definition 1: Poisson (α) Random Variable

X is a Poisson (α) random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} \dfrac{\alpha^x e^{-\alpha}}{x!} & x = 0, 1, 2, \ldots, \\ 0 & otherwise \end{cases}$$

Where the parameter α is in range $\alpha > 0$.

Cumulative Distribution Function of the Standard Normal Distribution

Consider the standard normal random variable Z with mean 0 and variance 1. That is $Z \sim N(0,1)$

For values of z the table gives the cumulative probabilities $F(z) = P(Z < z)$

For example, $P(Z < 1.0) = 0.8413$

z	F(z)	z	F(z)	z	F(z)	z	F(z)	z	F(z)	z	F(z)
0.00	0.5000										
0.01	0.5040	0.31	0.6217	0.61	0.7291	0.91	0.8186	1.21	0.8869	1.51	0.9345
0.02	0.5080	0.32	0.6255	0.62	0.7324	0.92	0.8212	1.22	0.8888	1.52	0.9357
0.03	0.5120	0.33	0.6293	0.63	0.7357	0.93	0.8238	1.23	0.8907	1.53	0.9370
0.04	0.5160	0.34	0.6331	0.64	0.7389	0.94	0.8264	1.24	0.8925	1.54	0.9382
0.05	0.5199	0.35	0.6368	0.65	0.7422	0.95	0.8289	1.25	0.8944	1.55	0.9394
0.06	0.5239	0.36	0.6406	0.66	0.7454	0.96	0.8315	1.26	0.8962	1.56	0.9406
0.07	0.5279	0.37	0.6443	0.67	0.7486	0.97	0.8340	1.27	0.8980	1.57	0.9418
0.08	0.5319	0.38	0.6480	0.68	0.7517	0.98	0.8365	1.28	0.8997	1.58	0.9429
0.09	0.5359	0.39	0.6517	0.69	0.7549	0.99	0.8389	1.29	0.9015	1.59	0.9441
0.10	0.5398	0.40	0.6554	0.70	0.7580	1.00	0.8413	1.30	0.9032	1.60	0.9452
0.11	0.5438	0.41	0.6591	0.71	0.7611	1.01	0.8438	1.31	0.9049	1.61	0.9463
0.12	0.5478	0.42	0.6628	0.72	0.7642	1.02	0.8461	1.32	0.9066	1.62	0.9474
0.13	0.5517	0.43	0.6664	0.73	0.7673	1.03	0.8485	1.33	0.9082	1.63	0.9484
0.14	0.5557	0.44	0.6700	0.74	0.7704	1.04	0.8508	1.34	0.9099	1.64	0.9495
0.15	0.5596	0.45	0.6736	0.75	0.7734	1.05	0.8531	1.35	0.9115	1.65	0.9505
0.16	0.5636	0.46	0.6772	0.76	0.7764	1.06	0.8554	1.36	0.9131	1.66	0.9515
0.17	0.5675	0.47	0.6808	0.77	0.7794	1.07	0.8577	1.37	0.9147	1.67	0.9525
0.18	0.5714	0.48	0.6844	0.78	0.7823	1.08	0.8599	1.38	0.9162	1.68	0.9535
0.19	0.5753	0.49	0.6879	0.79	0.7852	1.09	0.8621	1.39	0.9177	1.69	0.9545
0.20	0.5793	0.50	0.6915	0.80	0.7881	1.10	0.8643	1.40	0.9192	1.70	0.9554
0.21	0.5832	0.51	0.6950	0.81	0.7910	1.11	0.8665	1.41	0.9207	1.71	0.9564
0.22	0.5871	0.52	0.6985	0.82	0.7939	1.12	0.8686	1.42	0.9222	1.72	0.9573
0.23	0.5910	0.53	0.7019	0.83	0.7967	1.13	0.8708	1.43	0.9236	1.73	0.9582
0.24	0.5948	0.54	0.7054	0.84	0.7995	1.14	0.8729	1.44	0.9251	1.74	0.9591
0.25	0.5987	0.55	0.7088	0.85	0.8023	1.15	0.8749	1.45	0.9265	1.75	0.9599
0.26	0.6026	0.56	0.7123	0.86	0.8051	1.16	0.8770	1.46	0.9279	1.76	0.9608
0.27	0.6064	0.57	0.7157	0.87	0.8078	1.17	0.8790	1.47	0.9292	1.77	0.9616
0.28	0.6103	0.58	0.7190	0.88	0.8106	1.18	0.8810	1.48	0.9306	1.78	0.9625
0.29	0.6141	0.59	0.7224	0.89	0.8133	1.19	0.8830	1.49	0.9319	1.79	0.9633
0.30	0.6179	0.60	0.7257	0.90	0.8159	1.20	0.8849	1.50	0.9332	1.80	0.9641