Probability Midterm Exam

2:20-4:00 pm Thursday, 28 April 2005

Total: 3 pages, 100 points.

(1)You may use a calculator, but may not consult the textbook or neighbor.
(2)Please turn off your cell phones before the exam.
(3)Please show your work for partial credit, and underline your answers. Points are awarded for solutions, not answers, so correct answers without justification will not receive full credit.

PROBLEM 1: ***Traffic Lights I.*** **(10%)**

In this problem, we consider that traffic engineers have coordinated the timing of two traffic lights to encourage a run of green lights. Assume that the timing was designed so that with probability 0.8 a driver will find the second light to have the same color as the first. Let the first light be equally likely to be red or green.

(a) Find the probability $P(G_2)$ that the second light is green. (3%)

(b) Find the probability $P(W)$ that a driver waits for at least one light. (3%)

(c) Find the probability $P(G_1 \mid R_2)$ that the first light is green given a red second light. (4%)

PROBLEM 2: ***Traffic Lights II.*** **(15%)**

In this problem, we consider that three traffic lights are encountered when driving down a road. A light was red if the driver was required to come to a complete stop at that light; otherwise we call the light green. For the sake of simplicity, these definitions were carefully chosen to exclude the case of the yellow light. Assume that each outcome (a sequence of three lights, each either red or green) is equally likely.

(a) Find the sample space S of this experiment. (3%)

(b) Find the probability $P(R_2)$ that the second light was red. (3%)

(c) Find the probability $P(G_2)$ that the second light was green. (3%)

(d) Find the probability $P(R_1)$ that the first light was red. (3%)

(e) Are the events R_1 and R_2 independent? Are the events G_2 and R_2 independent?
Note: You should justify your answer for (e). (3%)

$(x+y)^n = \sum_{k=0}^{n} x^k y^{n-k}$

$\binom{n}{k}$

左 右

PROBLEM 3: ***The Banach Match Problem.*** **(15%)**

A pipe-smoking mathematician carries, at all times, 2 matchboxes, 1 in his left-hand pocket and 1 in his right-hand pocket. Each time he needs a match he is equally likely to take it from either pocket. Consider the moment when the mathematician first discovers that one of his matchboxes is empty. If it is assumed that both matchboxes initially contained N matches, what is the probability that there are exactly K matches in the other box, k=0, 1, …, N?

$2N-K$

$\sum ($

GO ON TO THE NEXT PAGE.

$\boxed{N}\ \boxed{N}$

$\sum_{k=0}$

$\boxed{N}\ \boxed{N}$

$\boxed{K}\ \boxed{0}$

$\binom{2N}{2N-K}\left(\frac{1}{2}\right)^N\left(\frac{1}{2}\right)^K$

PROBLEM 4: ***Hunting Game.*** **(10%)**

Ten hunters are waiting for ducks to fly by. When a flock of ducks flies overhead, the hunters fire at the same time, but each chooses his target at random, independently of the others. If each hunter independently hits his target with probability p, compute the expected number of ducks that escape unhurt when a flock of size 10 flies overhead.

PROBLEM 5: ***Internet Flow Control.*** **(10%)**

There are N Internet users in the NTU campus. During a minute, the probability that a user is on-line for the Internet service is p and is off-line with probability $1-p$. When a user is on-line, the user requires a bandwidth of b bits per second. .

(a) Let **B** be the total bandwidth needed by on-line users in a minute. Derive its probability distribution function. (4%)

(b) Let **T** be the number of minutes that a user continuously stays on-line. Derive Var(**T**). (6%)

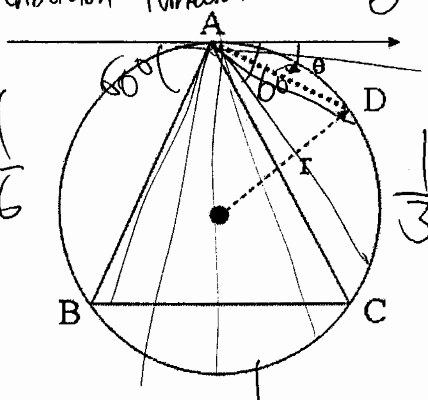
PROBLEM 6: ***Circle and Cord.*** **(15%)**

Consider a circle of radius **r** in the figure below, where ΔABC is an equilateral triangle inscribed in the circle and $\widehat{AB}$, $\widehat{BC}$ and $\widehat{AC}$ are three segments of the circle. Now select a point D from the circle to form a cord $\overline{AD}$ with $P(D \in \widehat{AB}) = \frac{1}{6}$, $P(D \in \widehat{BC}) = \frac{1}{2}$, $P(D \in \widehat{AC}) = \frac{1}{3}$, and the selection of D within each segment is random.

(a) Let θ be the angle between $\overline{AD}$ and the horizontal line, $0 \le \theta \le \pi$. What is the sample space of this experiment? (3%)

(b) Write down the event corresponding to $\overline{AD} > \overline{AC}$. (3%)

(c) Derive the CDF $F_\theta(t)$ and the PDF $f_\theta(t)$. (9%)



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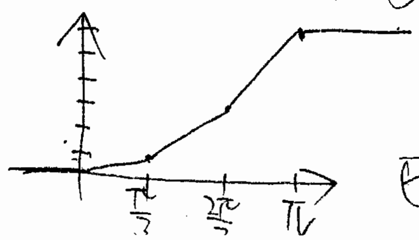

$1-e^{-\lambda t}$ $\quad \frac{1}{\lambda}$ $\quad \frac{1}{\lambda^2}$

PROBLEM 7: ***Queuing at a restaurant.*** **(10%)**

At a restaurant, the customers form a line waiting to be seated. Suppose that from the time a person becomes the first guy in the line, it takes X minutes for him/her to be seated. The PDF of X is

$\lambda e^{-\lambda t}$ $\quad e$

$$\cancel{f(x)}\ f(t) = \begin{cases} c \cdot \exp(-\lambda(t - t_0)), & t \geq t_0 \\ 0, & \text{otherwise} \end{cases}$$

$\lambda e^{-\lambda(t-t_0)}$ $\quad t_0 + \frac{1}{\lambda}$ $\quad \frac{1}{\lambda^2}$

for some $t_0 > 0$.

(a) Find c. (2%)

(b) Find the mean and variance of X. (2%)

(c) Tom arrived at this restaurant and waited in a line. The last person in front of Tom was seated t_1 minutes ago. How many more minutes on average does he have to wait?

Note: Answer the question separately for $t_1 < t_0$ and $t_1 > t_0$. (6%)

$-t_1 > -t_0$

PROBLEM 8: ***Yield analysis.*** **(5%)**

A factory produces batteries. The lifetime of these batteries are normal distributed with mean 100 hours and standard deviation 5 hours. Is the company right in saying that 97% of the batteries will last longer than 93 hours?

Note: Use the approximation

$$\Phi(x) \approx 0.5 + \frac{x(4.4-x)}{10} \quad \text{for } 0 \leq x \leq 2.$$

1.4 $\quad \frac{1.4 \times 3}{10}$ $\quad 0.47$ $\quad 0.92$

t^2, $-2t$, 2, 0 $\quad \lambda e^{-\lambda(t-t_0)}$, $-e^{-\lambda(t-t_0)}$, $\frac{1}{\lambda}e^{-\lambda(t-t_0)}$, $-\frac{1}{\lambda^2}e^{-\lambda(t-t_0)}$

PROBLEM 9: ***Joint & marginal PDF.*** **(10%)**

Let the joint probability density function of random variables X and Y be given by

$$f(x,y) = \begin{cases} 8xy & \text{if } 0 \leq y \leq x \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Calculate the marginal probability distribution function of Y. (5%)

(b) Find the conditional expectation of X given $Y = 1/4$. (5%)

$\frac{93-100}{5}$ $\quad -1.4$

97% $\quad 93$

STOP

PLEASE RECHECK YOUR ANSWERS! THANK YOU!

Probability Midterm Exam Solutions

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Total: 3 pages, 100 points.

PROBLEM 1: ***Traffic Lights I.* (10%)**

In this problem, we consider that traffic engineers have coordinated the timing of two traffic lights to encourage a run of green lights. Assume that the timing was designed so that with probability 0.8 a driver will find the second light to have the same color as the first. Let the first light be equally likely to be red or green.

(a) Find the probability $P(G_2)$ that the second light is green. (3%)

(b) Find the probability $P(W)$ that a driver waits for at least one light. (3%)

(c) Find the probability $P(G_1 \mid R_2)$ that the first light is green given a red second light. (4%)

Ans:

Given: $P(R_1) = P(G_1) = 0.5$, $P(G_2|G_1) = P(R_2|R_1) = 0.8$ =>

$P(R_1 \cap R_2) = 0.5 * 0.8 = 0.4$ (a) $P(G_2) = P(R_1 \cap G_2) + P(G_1 \cap G_2) = 0.1 + 0.4 = 0.5$

$P(R_1 \cap G_2) = 0.5 * 0.2 = 0.1$ => (b) $P(W) = 1 - P(G_1 \cap G_2) = 1 - 0.4 = 0.6$

$P(G_1 \cap R_2) = 0.5 * 0.2 = 0.1$ (c) $P(G_1 \mid R_2) = P(G_1 \cap R_2) / P(R_2) = 0.1 / 0.5 = 0.2$

$P(G_1 \cap G_2) = 0.5 * 0.8 = 0.4$

PROBLEM 2: ***Traffic Lights II.* (15%)**

In this problem, we consider that three traffic lights are encountered when driving down a road. A light was red if the driver was required to come to a complete stop at that light; otherwise we call the light green. For the sake of simplicity, these definitions were carefully chosen to exclude the case of the yellow light. Assume that each outcome (a sequence of three lights, each either red or green) is equally likely.

(a) Find the sample space S of this experiment. (3%)

(b) Find the probability $P(R_2)$ that the second light was red. (3%)

(c) Find the probability $P(G_2)$ that the second light was green. (3%)

(d) Find the probability $P(R_1)$ that the first light was red. (3%)

(e) Are the events R_1 and R_2 independent? Are the events G_2 and R_2 independent?
Note: You should justify your answer for (e). (3%)

Ans:

(a) sample space S = {RRR, RRG, RGR, RGG, GRR, GRG, GGR, GGG}

(b) $P(R_2) = 4/8 = 1/2$ (RRR, RRG, GRR, GRG)

(c) $P(G_2) = 4/8 = 1/2$ (RGR, RGG, GGR, GGG)

(d) $P(R_1) = 4/8 = 1/2$ (RRR, RRG, RGR, RGG)

(e) $P(R_1R_2) = 2/8 = 1/4$ (RRR, RRG)

=> $P(R_1R_2) = P(R_1) * P(R_2)$ => R_1 and R_2 are independent.

$P(G_2R_2) = 0$

=> $P(G_2R_2) \neq P(G_2) * P(R_2)$ => G_2 and R_2 are not independent.

PROBLEM 3: ***The Banach Match Problem.* (15%)**

A pipe-smoking mathematician carries, at all times, 2 matchboxes, 1 in his left-hand pocket and 1 in his right-hand pocket. Each time he needs a match he is equally likely to take it from either pocket. Consider the moment when the mathematician first discovers that one of his matchboxes is empty. If it is assumed that both matchboxes initially contained N matches, what is the probability that there are exactly K matches in the other box, k=0, 1, …, N ?

Ans: (textbook pp.220)

Every time that the left pocket is selected we say that a success has occurred. When the mathematician discovers that the left box is empty, the right one contains k matches if and only if the (N+1)st success occurs on the (N−k) + (N+1) = (2N−k+1)st trial. The probability of this event is

$$\binom{(2N-k+1)-1}{(N+1)-1}\left(\frac{1}{2}\right)^{N+1}\left(\frac{1}{2}\right)^{(2N-k+1)-(N+1)} = \binom{2N-k}{N}\left(\frac{1}{2}\right)^{2N-k+1}.$$

By symmetry, when the mathematician discovers that the right box is empty, with probability $\binom{2N-k}{N}\left(\frac{1}{2}\right)^{2N-k+1}$, the left box contains k matches. Therefore, the desired probability is

$$2\binom{2N-k}{N}\left(\frac{1}{2}\right)^{2N-k+1} = \binom{2N-k}{N}\left(\frac{1}{2}\right)^{2N-k}.$$

PROBLEM 4: ***Hunting Game.*** **(10%)**

Ten hunters are waiting for ducks to fly by. When a flock of ducks flies overhead, the hunters fire at the same time, but each chooses his target at random, independently of the others. If each hunter independently hits his target with probability p, compute the expected number of ducks that escape unhurt when a flock of size 10 flies overhead.

Ans: (Sheldon M. Ross, "A first course in probability", Ch.7, prob.55)

$$X_i = \begin{cases} 1\ , & \text{if the } i_{th} \text{ duck escapes unhurt} \\ 0\ , & \text{otherwise} \end{cases}$$

=> $E[X_1+X_2+\ldots+X_{10}] = E[X_1] + E[X_2] + \ldots + E[X_{10}]$, because X_i are independent

Also, each hunter will hit i_{th} duck with probability $p/10$, because P(hunter choose i_{th} duck as his target) * P(hunter hit his target | hunter choose i_{th} duck) = (1/10) * p.

=> $P(X_i = 1) = (1-\frac{p}{10})^{10}$ => $E[X] = E[X_1+X_2+\ldots+X_{10}] = 10 * (1-\frac{p}{10})^{10}$.

Alternative Solution: $E[X] = 10 * P(\text{one duck unhurt}) = 10 * \sum_{k=0}^{10} \binom{10}{k} \underbrace{(\frac{1-p}{10})^k}_{\text{be aimed but not be hit}} \underbrace{(\frac{9}{10})^{10-k}}_{\text{not be aimed}}$

PROBLEM 5: ***Internet Flow Control.*** **(10%)**

There are N Internet users in the NTU campus. During a minute, the probability that a user is on-line for the Internet service is p and is off-line with probability $1-p$. When a user is on-line, the user requires a bandwidth of b bits per second. .

(a) Let **B** be the total bandwidth needed by on-line users in a minute. Derive its probability distribution function. (4%)

(b) Let **T** be the number of minutes that a user continuously stays on-line. Derive Var(**T**). (6%)

Ans:

(a) $P(k \text{ users online}) = \binom{N}{k} p^k (1-p)^{N-k}$

=> $P(\mathbf{B} = t) = \binom{N}{\frac{t}{60b}} p^k (1-p)^{N-\frac{t}{60b}}$, because **B** is BW/min and b is BW/sec

=> $F_B(X \le 60kb) = \sum_{i=0}^{k} P_i = \sum_{i=0}^{k} \binom{N}{i} P^i (1-p)^{N-i}$, k=0,1,...,N

or $F_B(X \le t) = \sum_{i=0}^{\frac{t}{60b}} P_i = \sum_{i=0}^{\frac{t}{60b}} \binom{N}{i} P^i (1-p)^{N-i}$, t=0,60b,120b,...,60Nb

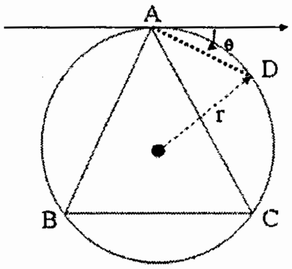
(b) $E[T] = \sum_{k=1}^{\infty} kp^k(1-p) = (1-p)p(\frac{d}{dp}(\sum_{k=1}^{\infty} p^k)) = \frac{p}{1-p}$

$E[T^2] = \sum_{k=1}^{\infty} k^2 p^k(1-p) = \frac{p}{1-p} + \frac{2p^2}{(1-p)^2}$

$\Rightarrow Var(T) = E[T^2] - (E[T])^2 = \frac{p}{(1-p)^2}$

PROBLEM 6: ***Circle and Cord.*** **(15%)**

Consider a circle of radius **r** in the figure below, where ΔABC is an equilateral triangle inscribed in the circle and $\widehat{AB}$, $\widehat{BC}$ and $\widehat{AC}$ are three segments of the circle. Now select a point D from the circle to form a cord $\overline{AD}$ with $P(D \in \widehat{AB}) = \frac{1}{6}$, $P(D \in \widehat{BC}) = \frac{1}{2}$, $P(D \in \widehat{AC}) = \frac{1}{3}$, and the selection of D within each segment is random.



(a) Let θ be the angle between $\overline{AD}$ and the horizontal line, $0 \le \theta \le \pi$. What is the sample space of this experiment? (3%)

(b) Write down the event corresponding to $\overline{AD} > \overline{AC}$. (3%)

(c) Derive the CDF $F_\theta(t)$ and the PDF $f_\theta(t)$. (9%)

Ans:

(a) sample space $S = \{\theta \mid 0 \le \theta \le \pi\}$

(b) event $\overline{AD} > \overline{AC} = \{\theta \mid \frac{\pi}{3} < \theta < \frac{2\pi}{3}\}$

(c) $0 \le \theta = t \le \frac{\pi}{3} : D \in \widehat{AC}$, $\Rightarrow P(\theta = t,\ 0 \le t \le \frac{\pi}{3}) = F_\theta(\frac{\pi}{3}) = \frac{1}{3}$

also D is selected randomly => t is a uniform distributed random variable $\Rightarrow \frac{F_\theta(t)}{t} = \frac{\frac{1}{3}}{\frac{\pi}{3}}$

$\Rightarrow F_\theta(t) = \frac{t}{\pi}$ if $0 \le t \le \frac{\pi}{3}$

$\frac{\pi}{3} \le \theta = t \le \frac{2\pi}{3} : D \in \widehat{BC}$, $\Rightarrow P(\theta = t,\ \frac{\pi}{3} \le t \le \frac{2\pi}{3}) = F_\theta(\frac{2\pi}{3}) - F_\theta(\frac{\pi}{3}) = \frac{1}{2}$,

also D is selected randomly => t is uniform distributed $\Rightarrow \frac{F_\theta(t) - \frac{1}{3}}{t - \frac{\pi}{3}} = \frac{F_\theta(\frac{2\pi}{3}) - \frac{1}{3}}{\frac{2\pi}{3} - \frac{\pi}{3}} = \frac{3}{2\pi}$

$$=> F_\theta(t)=\frac{3}{2\pi}(t-\frac{\pi}{3})+\frac{1}{3}=\frac{3}{2\pi}t-\frac{1}{6} \text{ if } \frac{\pi}{3}\le t\le\frac{2\pi}{3}$$

$\frac{2\pi}{3}<\theta=t<\pi : D\in \widehat{AB}$, => $P(\theta=t,\ \frac{2\pi}{3}\le t\le\pi) = F_\theta(\pi)-F_\theta(\frac{2\pi}{3})=\frac{1}{6}$,

also D is selected randomly => t is uniform distributed => $\dfrac{F_\theta(t)-\frac{5}{6}}{t-\frac{2\pi}{3}}=\dfrac{F_\theta(\pi)-\frac{5}{6}}{\pi-\frac{2\pi}{3}}=\dfrac{1}{2\pi}$

$$=> F_\theta(t)=\frac{1}{2\pi}(t-\frac{2\pi}{3})+\frac{5}{6}=\frac{1}{2\pi}t+\frac{1}{2} \text{ if } \frac{2\pi}{3}\le t\le\pi$$

So that $F_\theta(t)=\begin{cases}\dfrac{t}{\pi} & \text{if } 0\le t\le\dfrac{\pi}{3}\\ \dfrac{3}{2\pi}t-\dfrac{1}{6} & \text{if } \dfrac{\pi}{3}\le t\le\dfrac{2\pi}{3}\\ \dfrac{t}{2\pi}+\dfrac{1}{2} & \text{if } \dfrac{2\pi}{3}\le t\le\pi\end{cases}$, $f_\theta(t)=\dfrac{d}{dt}F_\theta(t)=\begin{cases}\dfrac{1}{\pi} & \text{if } 0\le t\le\dfrac{\pi}{3}\\ \dfrac{3}{2\pi} & \text{if } \dfrac{\pi}{3}\le t\le\dfrac{2\pi}{3}\\ \dfrac{1}{2\pi} & \text{if } \dfrac{2\pi}{3}\le t\le\pi\end{cases}$

<u>PROBLEM 7:</u> ***Queuing at a restaurant.*** **(10%)**

At a restaurant, the customers form a line waiting to be seated. Suppose that from the time a person becomes the first guy in the line, it takes X minutes for him/her to be seated. The PDF of X is

$$f(x)=\begin{cases}c\cdot\exp(-\lambda(t-t_0)), & t\ge t_0\\ 0, & \text{otherwise}\end{cases}$$

for some $t_0>0$.

(a) Find c. (2%)

(b) Find the mean and variance of X. (2%)

(c) Tom arrived at this restaurant and waited in a line. The last person in front of Tom was seated t_1 minutes ago. How many more minutes on average does he have to wait?
<u>Note:</u> Answer the question separately for $t_1<t_0$ and $t_1>t_0$. (6%)

<u>Ans:</u>

(a) $\int_{t_0}^{\infty} ce^{-\lambda(t-t_0)}dt=1$ => $c=\lambda$

(b) E[X] = $\int_{t_0}^{\infty} t\lambda e^{-\lambda(t-t_0)}dt=\lambda e^{\lambda t_0}\int_{t_0}^{\infty} te^{-\lambda t}dt=t_0+\frac{1}{\lambda}$

$$\text{Var}[X] = [t_0^2 + \frac{2}{\lambda}(t_0 + \frac{1}{\lambda})] - (t_0 + \frac{1}{\lambda})^2 = \frac{1}{\lambda^2}$$

(c) We want to solve $E[t - t_1 \mid t \geq t_1]$

so we need to solve $F_{X|X \geq t_1}(t) = P(X < t \mid X \geq t_1) = \dfrac{P(t_1 \leq X < t)}{P(X \geq t_1)}$ first.

$$t_1 < t_0: \quad F_{X|X\geq t_1}(t) = \begin{cases} P(X<t) = \int_{t_0}^{t} \lambda e^{-\lambda(t-t_0)} dt = 1 - e^{-\lambda(t-t_0)} & \text{if } t \geq t_0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow E[t - t_1 \mid t \geq t_1] = \int_{t_0}^{\infty} (t - t_1) f_{X|X\geq t_1}(t) dt = \lambda e^{\lambda t_0} \int_{t_0}^{\infty} (t - t_1) e^{-\lambda t} dt = (t_0 - t_1) + \frac{1}{\lambda}$$

$$t_1 > t_0: \quad F_{X|X\geq t_1}(t) = \begin{cases} \dfrac{F_X(t) - F_X(t_1)}{1 - F_X(t_1)} = 1 - e^{-\lambda(t-t_1)} & \text{if } t \geq t_0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow E[t - t_1 \mid t \geq t_1] = \int_{t_1}^{\infty} (t - t_1) f_{X|X\geq t_1}(t) dt = \lambda e^{\lambda t_0} \int_{t_1}^{\infty} (t - t_1) e^{-\lambda(t-t_1)} dt = \int_{0}^{\infty} x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

PROBLEM 8: ***Yield analysis.*** **(5%)**

A factory produces batteries. The lifetime of these batteries are normal distributed with mean 100 hours and standard deviation 5 hours. Is the company right in saying that 97% of the batteries will last longer than 93 hours?

Note: Use the approximation

$$\Phi(x) \approx 0.5 + \frac{x(4.4-x)}{10} \qquad \text{for } 0 \leq x \leq 2.$$

Ans:

Let t be the random variable of battery lifetime, and $X \sim N(0,1)$

$$P(t > 93) = P(\frac{t-100}{5} > \frac{93-100}{5}) = P(X > -1.4) = 1 - \Phi(-1.4) = \Phi(1.4) = 0.92$$

=> only 92% of the batteries will last longer than 93 hours => wrong

PROBLEM 9: ***Joint & marginal PDF.*** **(10%)**

Let the joint probability density function of random variables X and Y be given by

$$f(x, y) = \begin{cases} 8xy & \text{if } 0 \leq y \leq x \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Calculate the marginal probability distribution function of Y. (5%)

(b) Find the conditional expectation of X given $Y = 1/4$. (5%)

Ans:

(a) $f_Y(y) = \int_y^1 f(x,y)dx = \int_y^1 8xydy = 4y(1-y^2)$

(b) $E[X \mid Y=\frac{1}{4}] = \int_{\frac{1}{4}}^{1} xf_{X|Y}(x \mid y)dx = \int_{\frac{1}{4}}^{1} x\frac{f(x,\frac{1}{4})}{f_Y(\frac{1}{4})}dx = \frac{7}{10}$