Signals and Systems, Final Exam

Solutions (Draft)

Spring 2008, Edited by bypeng

1. **[4]** What is the group delay $\tau(\omega)$ for a system with frequency response $H(j\omega)$? Explain what that means.

Solution:

The group delay $\tau(\omega)$ is given by

$$\tau(\omega) = -\frac{d}{d\omega}\{\sphericalangle H(j\omega)\}$$

This implies that the effective common time delay experienced by the small band or group of frequencies center at $\omega = \omega_0$ is the negative of the slope of the phase at that frequency.

2. **[4]** A linear, time-invariant continuous-time system $H(j\omega)$ is distortionless within a signal band, $|\omega| < \omega_c$, if for any input signal $x(t)$ with $X(j\omega) = 0$, $|\omega| \geq \omega_c$, the output is of the form $y(t) = kx(t - t_0)$ for some fixed k and t_0. What is the condition for the frequency response of this system, $H(j\omega)$, to be distortionless within a signal band, $|\omega| < \omega_c$?

Solution:

$y(t) = kx(t - t_0)$, so $Y(j\omega) = ke^{-j\omega t_0}X(j\omega) = H(j\omega)X(j\omega)$. The conclusion is $H(j\omega) = ke^{-j\omega t_0}$ for $|\omega| < \omega_c$.

3. **[8]** $x_c(t)$ is a continuous-time signal, $x_d[n] = x_c(nT)$, and

$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT).$$

The discrete-time Fourier transform of $x_d[n]$ is $X_d(e^{j\Omega})$, while the continuous-time Fourier transform of $x_p(t)$ is $X_p(j\omega)$. Find the relationship between $X_d(e^{j\Omega})$ and $X_p(j\omega)$.

Solution:

$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT) \quad \Rightarrow \quad X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\omega nT};$$

$$x_d[n] = x_c(nT) \quad \Rightarrow \quad X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\frac{\Omega}{T}nT} = X_p\left(j\frac{\Omega}{T}\right)$$

4. **[12]** A signal $x_c(t)$ is echoed, so the actually received signal is

$$y_c(t) = x_c(t) + ax_c(t - T_0)$$

An echo canceller is shown in Figure 4, with a goal to have $z_c(t) = x_c(t)$.

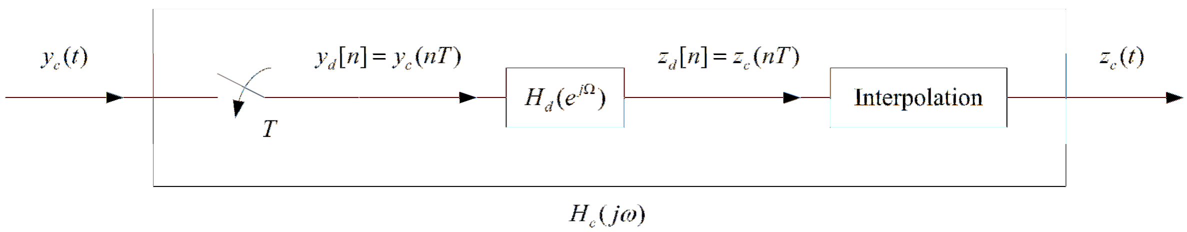


Figure 4

(a) **[8]** Assume the sampling theorem is satisfied, the interpolation is perfect so $z_c(nT) = z_d[n]$, and $T_0 = n_0 T$, n_0 is an integer. Find the difference equation relating $y_d[n]$ and $z_d[n]$ and the frequency response $H_d(e^{j\Omega})$.

(b) **[4]** What is the continuous-time frequency response $H_c(j\omega)$ between $y_c(t)$ and $z_c(t)$?

Solution:

(a) $y_d[n] = y_c(nT) = x_c(nT) + ax_c(nT - T_0) = x_c(nT) + ax_c((n-n_0)T) = x_d[n] + ax_d[n-n_0]$. Since we want $z_c(t) = x_c(t)$, we need $z_d[n] = x_d[n]$, so $z_d[n] + az_d[n-n_0] = y_d[n]$, and

$$Y_d(e^{j\Omega}) = (1 + ae^{-j\Omega n_0})Z_d(e^{j\Omega}) \;\Rightarrow\; H_d(e^{j\Omega}) = \frac{Z_d(e^{j\Omega})}{Y_d(e^{j\Omega})} = \frac{1}{1 + ae^{-j\Omega n_0}}$$

(b) Since we want $z_c(t) = x_c(t)$, $y_c(t) = z_c(t) + az_c(t - T_0)$, $Y_c(j\omega) = (1 + ae^{-j\omega T_0})Z_c(j\omega)$, then

$$H_c(j\omega) = \frac{Z_c(j\omega)}{Y_c(j\omega)} = \frac{1}{1 + ae^{-j\omega T_0}}$$

5. **[14]** Let $x[n]$ be a discrete-time signal with

$$\left|X(e^{j\omega})\right| = 0, \;\; \tfrac{2}{9}\pi \le |\omega| \le \pi$$

So the sampling frequency seems to be too high. The spectrum is shown in Figure 5.

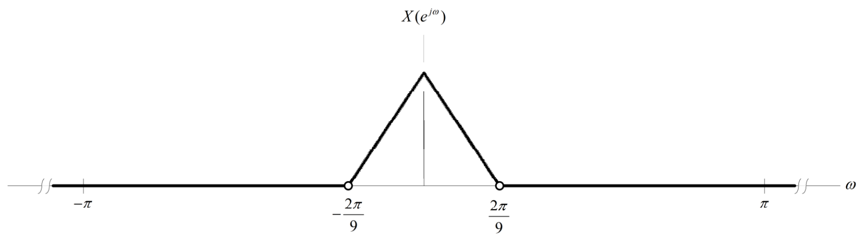


Figure 5

(a) **[4]** $x[n]$ can be decimated by a factor of integer N, i.e.,

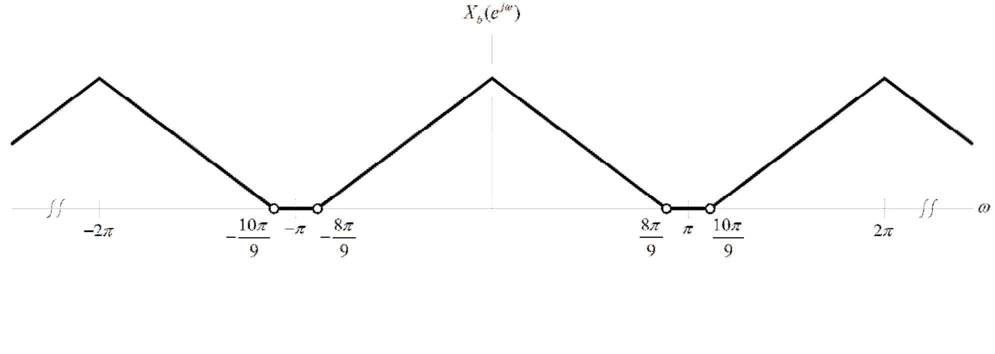
$$x_b[n] = x[nN].$$

Find the maximum integer N such that no aliasing will be introduced. Sketch $X_b(e^{j\omega})$ obtained in this case.

(b) **[10]** Is there any way to further reduce the sampling frequency without introducing aliasing effect? If yes, write it down, sketch the spectrum after each process, and explain what that means. If no, explain why.

Solution:

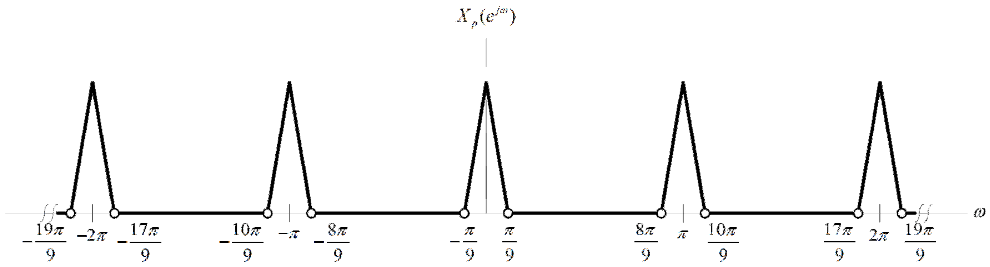
(a) We must find the largest integer N such that $\frac{2\pi}{N} \ge 2 \cdot \frac{2\pi}{9}$, so $N \le \frac{9}{2}$. The conclusion is $N = 4$. The sketch is as follows.



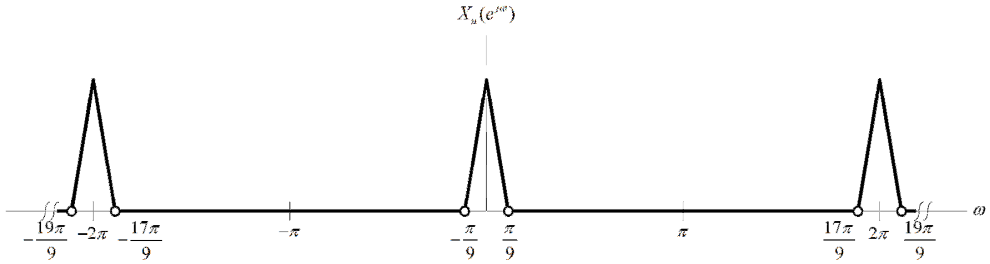
(b) We may upsample $x[n]$ by a factor of 2 and then downsample by a factor of 9. The first step is to construct the sequence $x_p[n]$, which is given by

$$x_p[n] = \begin{cases} x\left[\frac{n}{2}\right] & 2 \mid n \\ 0 & else \end{cases}$$

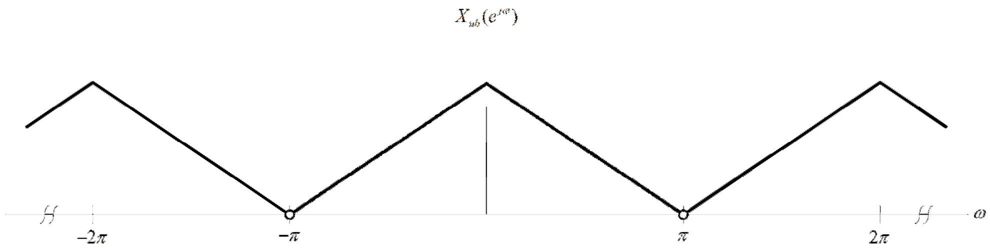
and the spectrum of $x_p[n]$ is as follows:



The second step is to apply a discrete-time lowpass filter with the bandwidth $\omega_{lp} = \frac{\pi}{2}$. The output is the upsampled sequence $x_u[n]$, whose spectrum is as follows:



The third step is to downsample $x_u[n]$ by a factor of 9. The spectrum of the output $x_{ub}[n]$ is as follows:



The interpolated and decimated sequence represents the maximum possible downsampling of $x[n]$.

6. **[8]** Explain why phase modulation with a signal $x(t)$ corresponds to frequency modulation with a signal $\frac{d}{dt}x(t)$.

Solution:

In phase modulation, the modulated signal $y_p(t)$ is given by

$$y_p(t) = A\cos[\omega_c t + \theta_c(t)]$$

where the phase function $\theta_c(t)$ is of the form

$$\theta_c(t) = \theta_0 + k_p x(t)$$

In frequency modulation, the modulated signal $y_f(t)$ is given by

$$y_f(t) = A\cos\theta_f(t)$$

where $\theta_f(t)$ satisfies

$$\frac{d\theta_f(t)}{dt} = \omega_c + k_f x(t)$$

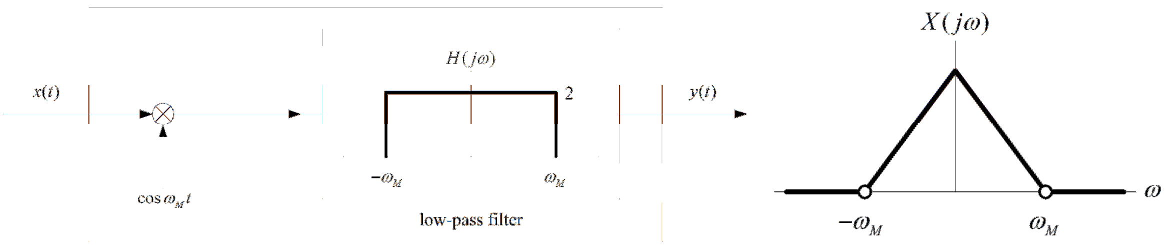
Compare both of the modulations, we may find that $y_p(t) = A\cos\theta_p(t)$, where

$$\theta_p(t) = \omega_c t + \theta_c(t) = \omega_c t + \theta_0 + k_p x(t), \quad \frac{d\theta_p(t)}{dt} = \omega_c + k_p \frac{dx(t)}{dt}$$

This means that phase modulation with a signal $x(t)$ exactly corresponds to frequency modulation with a signal $\frac{d}{dt}x(t)$.

7. **[8]** The system shown in Figure 7(a) is called a "frequency inverter" in early days, where ω_M is the frequency upper bound for the signal $x(t)$, i.e., $x(t)$ has a continuous-time Fourier transform $X(j\omega)$ shown in Figure 7(b), where $|X(j\omega)| = 0$ for $|\omega| \geq \omega_M$. $H(j\omega)$ is

$$H(j\omega) = \begin{cases} 2 & |\omega| \leq \omega_M \\ 0 & else \end{cases}.$$



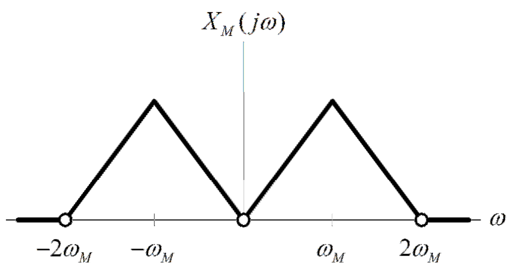
(a) (b)

Figure 7

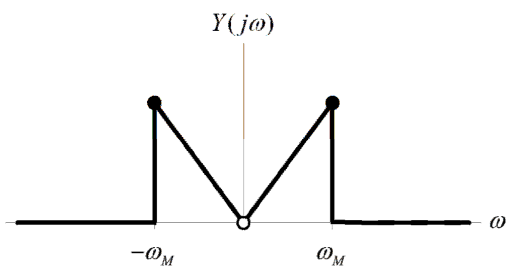
Explain why this system is called a "frequency inverter", and discuss whether the system can be its own inverse system.

Solution:

The spectrum of $x_M(t) = x(t)\cos\omega_M t$ is as follows:



The spectrum of $y(t)$ is as follows:



We can see

$$Y(j\omega) = \begin{cases} X(j(\omega_M - \omega)) & 0 \leq \omega \leq \omega_M \\ X(j(-\omega_M - \omega)) & -\omega_M \leq \omega \leq 0 \end{cases}$$

For $|\omega| \leq \omega_M$, and $Y(j\omega_M) = Y(-j\omega_M) = X(j0)$. So the system "inverts" the frequencies, matches the name "frequency inverter," and is the inverse system of itself, given $|X(j\omega)| = 0$ for $|\omega| \geq \omega_M$.

8. **[6]** Let $X(s)=\dfrac{1}{(s+1)(s+2)}$ be the Laplace transform of $x(t)$. Find $x(t)$ if the region of convergence is

(a) **[2]** $\Re e\{s\}>-1$

(b) **[2]** $\Re e\{s\}<-2$

(c) **[2]** $-2<\Re e\{s\}<-1$

Solution:

$$X(s)=\frac{1}{(s+1)(s+2)}=\frac{1}{s+1}-\frac{1}{s+2}$$

(a) $x(t)=e^{-t}u(t)-e^{-2t}u(t)$

(b) $x(t)=-e^{-t}u(-t)+e^{-2t}u(-t)$

(c) $x(t)=-e^{-t}u(-t)-e^{-2t}u(t)$

9. **[8]** Find the inverse Laplace transform of $\dfrac{1}{(s+a)^n}$ with region of convergence $\Re e\{s\}>-a$.

Solution:

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$$

10. **[6]** If $x(t)$ is right-sided, and if the line $\Re e\{s\}=\sigma_0$ is in the ROC of $X(s)$, are all values of s for which $\Re e\{s\}>\sigma_0$ also in the ROC of $X(s)$? If yes, show it. If no, explain why.

Solution:

$x(t)$ is right-sided, so we may let $x(t)=0$ for $t<T_0$. Since $\Re e\{s\}=\sigma_0$ is in the ROC of $X(s)$, we know that

$$\int_{T_0}^{\infty}\left|x(t)\right|e^{-\sigma_0 t}dt<\infty$$

That is, $\left|x(t)\right|e^{-\sigma_0 t}$ is absolutely integrable. For the case $\Re e\{s\}=\sigma_1>\sigma_0$, we may find that $\left|x(t)\right|e^{-\sigma_1 t}$ must be also absolutely integrable, therefore we conclude that all values of s for which $\Re e\{s\}>\sigma_0$ also in the ROC of $X(s)$.

11. **[9]** Use pole-zero plots to determine each of the following system functions to be lowpass, highpass, or bandpass:

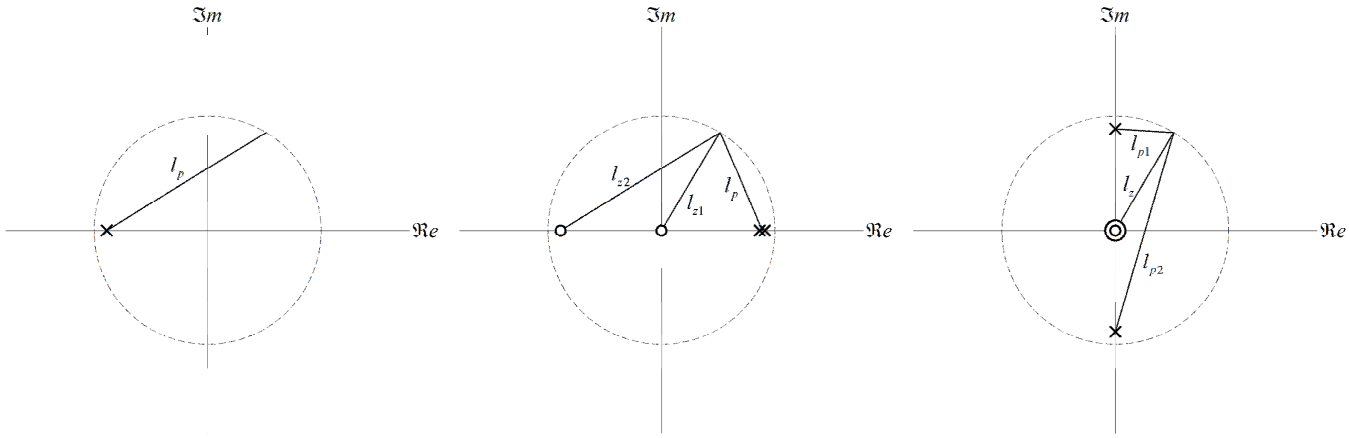
(a) **[3]** $H(z)=\dfrac{z^{-1}}{1+\frac{8}{9}z^{-1}},\ \ |z|>\dfrac{8}{9}$

(b) **[3]** $H(z)=\dfrac{1+\frac{8}{9}z^{-1}}{1-\frac{16}{9}z^{-1}+\frac{64}{81}z^{-2}},\ \ |z|>\dfrac{8}{9}$

(c) **[3]** $H(z)=\dfrac{1}{1+\frac{64}{81}z^{-2}},\ \ |z|>\dfrac{8}{9}$

Solution:

The three pole-zero plots are given in the following graph.



(a) (b) (c)

(a) $H(z)=\dfrac{z^{-1}}{1+\frac{8}{9}z^{-1}}=\dfrac{1}{z+\frac{8}{9}}$, the only pole is $p_1=-\frac{8}{9}$, and there are no zeros, so the pole-zero plot is shown in figure (a). We may find that the magnitude is $1/l_p$, where l_p becomes smaller when ω becomes closer to $\pm\pi$. The system is highpass.

(b) $H(z)=\dfrac{1+\frac{8}{9}z^{-1}}{1-\frac{16}{9}z^{-1}+\frac{64}{81}z^{-2}}=\dfrac{z^2+\frac{8}{9}z}{z^2-\frac{16}{9}z+\frac{64}{81}}=\dfrac{z(z+\frac{8}{9})}{(z-\frac{8}{9})^2}$, the poles are $p_1=p_2=\frac{8}{9}$, and the zeros are $z_1=0$, $z_2=-\frac{8}{9}$, so the pole-zero plot is shown in figure (b). We may find that the magnitude is $l_{z1}l_{z2}/l_p^{\,2}$, where $l_{z1}=1$, l_{z2} becomes smaller and l_p becomes bigger when ω becomes closer to $\pm\pi$. The system is lowpass.

(c) $H(z)=\dfrac{1}{1+\frac{64}{81}z^{-2}}=\dfrac{z^2}{z^2+\frac{64}{81}}=\dfrac{z^2}{(z+\frac{8}{9}j)(z-\frac{8}{9}j)}$, the poles are $p_1=\frac{8}{9}j$, $p_2=-\frac{8}{9}j$, and the zeros are $z_1=z_2=0$, so the pole-zero plot is shown in figure (c). We may find that the magnitude is $l_z^{\,2}/l_{p1}l_{p2}$, where $l_z=1$, $l_{p1}l_{p2}$ becomes smaller when ω becomes closer to $\pm\frac{\pi}{2}$. The system is bandpass.

12. **[18]** $X(z)$ is the z-transform of $x[n]$ with region of convergence $\Re$. Prove the following properties of z-transform, and discuss the possible changes of region of convergence. For each case, write down the corresponding properties of Laplace transform and discrete-time Fourier transform, if any.

(a) **[9]** $x[n-n_0]\xleftrightarrow{Z}z^{-n_0}X(z)$

(b) **[9]** $x^*[n]\xleftrightarrow{Z}X^*(z^*)$

Solution:

(a) $\mathcal{Z}\{x[n-n_0]\}=\sum_{n=-\infty}^{\infty}x[n-n_0]z^{-n}\overset{m\triangleq n-n_0}{=}\sum_{m=-\infty}^{\infty}x[m]z^{-m-n_0}=z^{-n_0}X(z)$

For $X(z)$ part, we need the region of convergence is in $\Re$. For z^{-n_0} part, we need $z\neq 0$ if $n_0>0$. Therefore the region of convergence is $\Re\backslash\{0\}$ if $n_0>0$ and is $\Re$ otherwise.

The corresponding Laplace transform property: $x(t-t_0)\xleftrightarrow{\mathcal{L}}e^{-st_0}X(s)$

The corresponding discrete-time Fourier transform property: $x[n-n_0]\xleftrightarrow{\mathcal{F}}e^{-j\omega n_0}X(j\omega)$

(b) $\mathcal{Z}\{x^*[n]\} = \sum_{n=-\infty}^{\infty} x^*[n]z^{-n} = \sum_{n=-\infty}^{\infty} x^*[n]((z^{-n})^*)^* = \sum_{n=-\infty}^{\infty} \left[x[n](z^{-n})^*\right]^* = \left(\sum_{n=-\infty}^{\infty} x[n](z^*)^{-n}\right)^* = X^*(z^*)$

To make the evaluation work, we need $\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty$, which is the same as the original z-transform. The conclusion is that the ROC is still $\Re$.

The corresponding Laplace transform property: $x^*(t) \overset{\mathcal{L}}{\longleftrightarrow} X^*(s^*)$

The corresponding discrete-time Fourier transform property: $x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-j\omega})$

13. **[6]** Consider a system with $H(z) = \dfrac{1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$. Draw the block diagram of the system with minimum number of delays in

(a) **[2]** direct form.

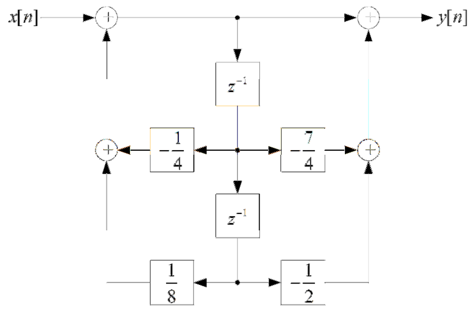
(b) **[2]** cascade form.

(c) **[2]** parallel form.

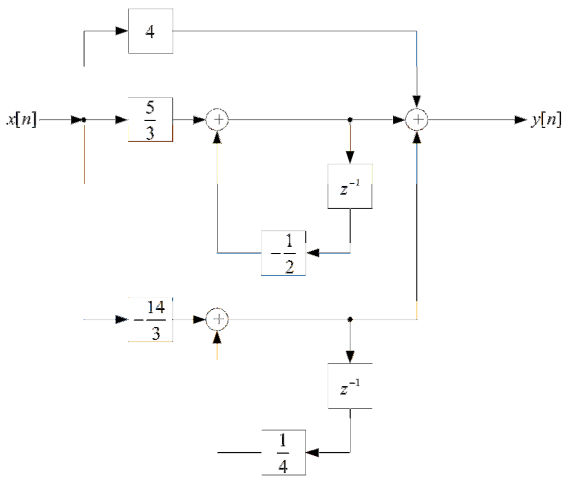
Solution:

$$H(z) = \frac{1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}} = \frac{(1-2z^{-1})(1+\frac{1}{4}z^{-1})}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{4(1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2})-3-\frac{11}{4}z^{-1}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}} = 4+\frac{\frac{5}{3}}{1+\frac{1}{2}z^{-1}}+\frac{-\frac{14}{3}}{1-\frac{1}{4}z^{-1}}$$

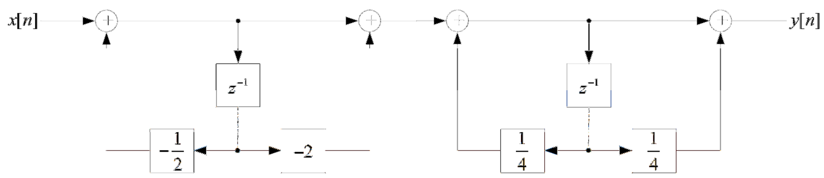
An instance is as the following graph.



(a)



(c)



(b)

14. **[9]** A causal, linear and time-invariant system is described by the following difference equation, where $x[n]$ is the input and $y[n]$ is the output,

$$y[n]-\frac{1}{2}y[n-1] = x[n]+\frac{1}{3}x[n-1].$$

Find the system function $H(z)$, its region of convergence, and discuss if it is stable.

Solution:

We have

$$Y(z)-\frac{1}{2}z^{-1}Y(z)=X(z)+\frac{1}{3}z^{-1}X(z) \quad \Rightarrow \quad H(z)=\frac{Y(z)}{X(z)}=\frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

The pole is $p=\frac{1}{2}$, so the possible ROC is $|z|<\frac{1}{2}$ and $|z|>\frac{1}{2}$. Since the system is causal, $|z|>\frac{1}{2}$ is adopted. Now we may find that the unit circle $|z|=1$ is inside the ROC, so the system is stable.