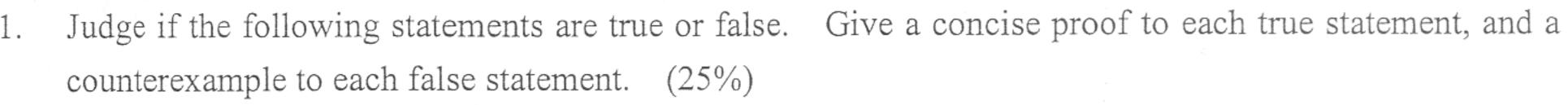
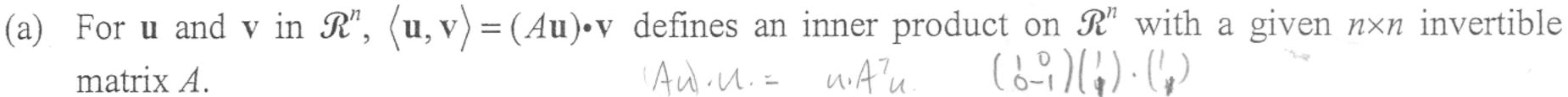
## 為域即

## Linear Algebra Final Examination Dept. of Elec. Eng., National Taiwan University June 21, 2006

## USE OF ALL AUTOMATIC COMPUTING MACHINES IS PROHIBITED





(b) For any two subspaces V and W of  $\mathcal{R}^n$ ,  $V \subset W$  implies  $W^{\perp} \subset V^{\perp}$ .

(c) For any two subspaces v and w of  $\mathcal{R}^n$ , Pv + Pw is an orthogonal projection matrix, where Pv and Pw are the orthogonal projection matrices for v and w, respectively.

(d) If A is a symmetric matrix, then its largest singular value equals its largest eigenvalue.

(e) For any  $n \times n$  invertible matrix A,  $A^{\dagger} = A^{-1}$ .

2. For  $V = \text{Span}\{e^t, te^t, t^2e^t\}$  and the differential operator D, find all eigenvalues of D and an eigenvector for each eigenvalue. (15%)

For any  $n \times n$  matrix A, the minimal polynomial of A is defined as the polynomial  $p(t) = t^m + p_{m-1}t^{m-1} + \dots + p_1t + p_0$  with the lowest degree m such that  $p(A) = A^m + p_{m-1}A^{m-1} + \dots + p_1A + p_0I_n = O$ . Find the minimal polynomials of the following matrices and justify your answers.

(a) The elementary matrix E obtained from  $I_n$  by interchanging its ith and jth rows  $(i \neq j)$ . (10%)

(b) The  $n \times n$  diagonalizable matrix M with k distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$ , where  $k \le n$  and  $\lambda_i$  has the algebraic multiplicity  $n_i$  for i = 1, 2, ..., k. (10%)

4. Find all eigenvalues and their corresponding eigenspaces of the matrix E in Problem 3(a). (10%)

5. Given the SVD of  $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{30}} & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} \\ \frac{5}{\sqrt{30}} & 0 & -\frac{1}{\sqrt{6}} \end{bmatrix}^T$ , plot the image of the

unit sphere  $\{(x_1 \ x_2 \ x_3) \ | \ x_1^2 + x_2^2 + x_3^2 = 1, \ x_i \in \mathcal{R} \text{ for } i = 1, 2, 3.\}$  in  $\mathcal{R}^3$  under the mapping of  $T_A$ . (15%)

6. A linear operator T on a finite-dimensional inner product space V is called an orthogonal operator if  $[T]_{\mathcal{B}}$  is an orthogonal matrix for some orthonormal basis  $\mathcal{B}$  of V. Prove that T is an orthogonal operator if and only if for any orthonormal basis  $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$  the set  $\{T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_n)\}$  is also an orthonormal basis for V. (15%)