Signals and Systems, Midterm Exam

Solutions (Draft)

Spring 2009, Edited by bypeng

1. **[12]** Suppose x and y denote input and output, respectively, of each of the three systems:

System A: $y(t) = x(t+2)\sin(\omega t+2), \quad \omega \neq 0$

System B: $y[n] = \left(-\frac{1}{2}\right)^n (x[n]+1)$

System C: $y[n] = \sum_{k=1}^{n} x^2[k+1] - x[k]$

Answer the following questions for each system and justify your answer.

(a) Is the system linear?
(b) Is the system time invariant?
(c) Is the system causal?
(d) Is the system stable?

Solution:

System A:

(a) Linear. For any $y_1(t) = x_1(t+2)\sin(\omega t+2)$ and $y_2(t) = x_2(t+2)\sin(\omega t+2)$, if $x(t) = x_1(t) + x_2(t)$,

$$y(t) = x(t+2)\sin(\omega t+2) = \left[x_1(t+2) + x_2(t+2)\right]\sin(\omega t+2)$$
$$= x_1(t+2)\sin(\omega t+2) + x_2(t+2)\sin(\omega t+2) = y_1(t) + y_2(t)$$

(b) NOT time invariant. For any $y_1(t) = x_1(t+2)\sin(\omega t+2)$, if $x(t) = x_1(t-T)$,

$$y(t) = x(t+2)\sin(\omega t+2) = x_1(t+2-T)\sin(\omega t+2)$$

But

$$y_1(t-T) = x_1(t-T+2)\sin(\omega(t-T)+2) = x_1(t+2-T)\sin(\omega t+2-\omega T) \neq y(t)$$

(c) NOT causal. Consider $x_1(t) = u(t-1.5)$ and $x_2(t) = u(t-2.5)$, then $y_1(t) = u(t+0.5)\sin(\omega t+2)$ and $y_2(t) = u(t-0.5)\sin(\omega t+2)$ .We know that $x_1(t) = x_2(t)$ for $t \le 0$, but $y_1(0) = u(0.5)\sin 2 = \sin 2$ and $y_2(0) = u(-0.5)\sin 2 = 0$.

(d) Stable. For any $y_1(t) = x_1(t+2)\sin(\omega t+2)$, if $|x_1(t)| < B$, then $|y_1(t)| < B \cdot 1 = B$.

System B:

(a) NOT linear. For any $y_1[n] = \left(-\frac{1}{2}\right)^n (x_1[n]+1)$ and $y_2[n] = \left(-\frac{1}{2}\right)^n (x_2[n]+1)$, if $x[n] = x_1[n] + x_2[n]$,

$$y[n] = \left(-\tfrac{1}{2}\right)^n (x[n]+1) = \left(-\tfrac{1}{2}\right)^n (x_1[n]+x_2[n]+1)$$
$$= \left(-\tfrac{1}{2}\right)^n (x_1[n]+1+x_2[n]+1-1) = y_1[n] + y_2[n] - \left(-\tfrac{1}{2}\right)^n \neq y_1[n] + y_2[n]$$

(b) NOT time invariant. For any $y_1[n] = \left(-\frac{1}{2}\right)^n (x_1[n]+1)$, if $x[n] = x_1[n-N]$,

$$y[n] = \left(-\tfrac{1}{2}\right)^n (x[n]+1) = \left(-\tfrac{1}{2}\right)^n (x_1[n-N]+1)$$

But

$$y_1[n-N] = \left(-\tfrac{1}{2}\right)^{n-N} (x_1[n-N]+1) = \left(-\tfrac{1}{2}\right)^{-N} \left(-\tfrac{1}{2}\right)^n (x_1[n-N]+1) = \left(-\tfrac{1}{2}\right)^{-N} y[n] \neq y[n]$$

(c) Causal. For any $y_1[n] = \left(-\frac{1}{2}\right)^n (x_1[n]+1)$ and $y_2[n] = \left(-\frac{1}{2}\right)^n (x_2[n]+1)$, if $x_1[n] = x_2[n]$ for $n \le N$, then for $n = N$, $y_1[n] = \left(-\frac{1}{2}\right)^n (x_1[n]+1) = \left(-\frac{1}{2}\right)^n (x_2[n]+1) = y_2[n]$.

(d) NOT stable. For any $y_1[n] = \left(-\frac{1}{2}\right)^n (x_1[n]+1)$, if $|x_1[n]| < B$ and for any $B_x < B$ $\exists n\, |x_1[n]| > B_x$, then for any $B_0 > 0$, we may find that $|y_1[n]| \ge 2^{-1} \cdot 2^{-n}(B+1) > B_0$ if $n < -1 - \log_2\left(\frac{B_0}{B+1}\right)$.

System C:

(a) NOT linear. For $y_1[n] = \sum_{k=1}^{n} {x_1}^2[k+1] - x_1[k]$ and $y_2[n] = \sum_{k=1}^{n} {x_2}^2[k+1] - x_2[k]$, if $x[n] = x_1[n] + x_2[n]$,

$$y[n]=\sum_{k=1}^{n}(x_1[k+1]+x_2[k+1])^2-(x_1[k]+x_2[k])$$
$$=\sum_{k=1}^{n}x_1^{\ 2}[k+1]+x_2^{\ 2}[k+1]+2x_1[k+1]x_2[k+1]-(x_1[k]+x_2[k])$$
$$=y_1[n]+y_2[n]+\sum_{k=1}^{n}2x_1[k+1]x_2[k+1]\neq y_1[n]+y_2[n]$$

(b) NOT time invariant. For $y_1[n]=\sum_{k=1}^{n}x_1^{\ 2}[k+1]-x_1[k]$, if $x[n]=x_1[n-N]$ for any $N>0$,

$$y[n]=\sum_{k=1}^{n}x_1^{\ 2}[k+1-N]-x_1[k-N]\ \overset{\text{substitute } k' \text{ for } k-N}{=}\ \sum_{k'=1-N}^{n-N}x_1^{\ 2}[k'+1]-x_1[k']$$

But

$$y_1[n-N]=\sum_{k=1}^{n-N}x_1^{\ 2}[k+1]-x_1[k]=y_1[n-N]-\sum_{k=1-N}^{0}x_1^{\ 2}[k+1]-x_1[k]\neq y_1[n-N]$$

(c) NOT causal. Consider $x_1[n]=\delta[n-2]$ and $x_2[n]=\delta u[n-3]$, then

$$y_1[n]=\sum_{k=1}^{n}x_1^{\ 2}[k+1]-x_1[k]=\sum_{k=1}^{n}\delta^2[k-1]-\delta[k-2]$$
$$y_2[n]=\sum_{k=1}^{n}x_2^{\ 2}[k+1]-x_2[k]=\sum_{k=1}^{n}\delta^2[k-2]-\delta[k-3]$$

We know that $x_1[n]=x_2[n]$ for $n\leq 1$, but for $n=1$,

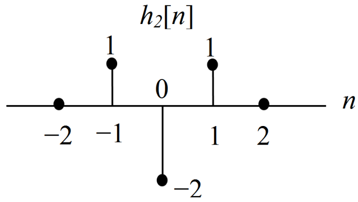
$$y_1[1]=\sum_{k=1}^{1}\delta^2[k-1]-\delta[k-2]=\delta^2[0]-\delta[-1]=1$$
$$y_2[1]=\sum_{k=1}^{1}\delta^2[k-2]-\delta[k-3]=\delta^2[-1]-\delta[-2]=0$$

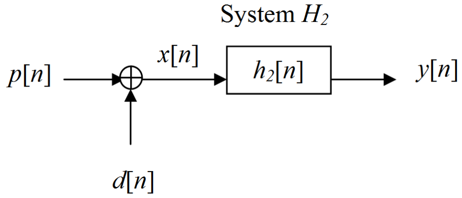
(d) NOT stable. Consider $x_1[n]=2$, then

$$y_1[n]=\sum_{k=1}^{n}x_1^{\ 2}[k+1]-x_1[k]=\sum_{k=1}^{n}4-2=2n$$

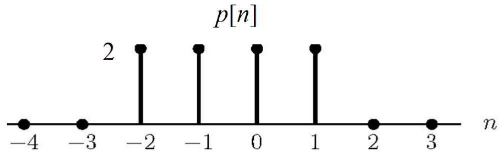
which is not bounded.

2. We want to develop an edge detector that is robust against additive noise. Consider a discrete-time (DT) linear time-invariant (LTI) system H_2 with $h_2[n]=h[n]*h[n+1]$ as its impulse response shown below, where $h[n]=\delta[n]-\delta[n-1]$.



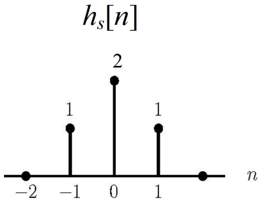


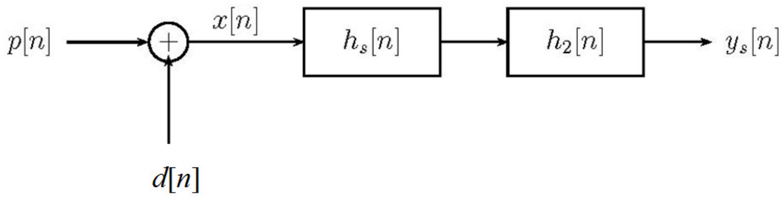
(a) **[4]** Assume there is no noise, i.e., $d[n]=0$ and $x[n]=p[n]$. Sketch the output $y[n]$ of the system assuming the input $p[n]$ to the system is the following signal:



(b) **[4]** Assume the noise is $d[n]=-\delta[n+1]$ and the input $p[n]$ remains the same. Sketch the output $y[n]$ of the system.

(c) **[4]** In order to use system H_2 as a part of an edge detector, we would like to add an LTI system H_s whose unit impulse response $h_s[n]$ is shown below. System H_s smoothes out effect of noise on $x[n]$. The overall system can be represented as below:

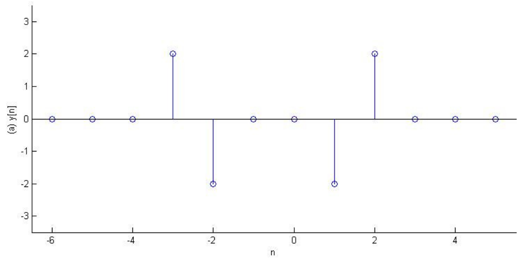




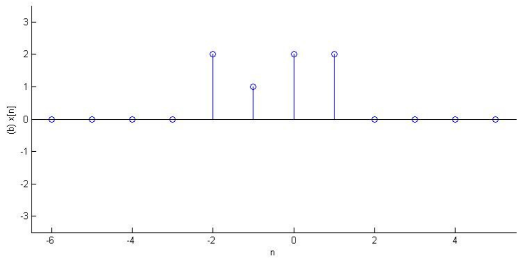
Sketch the output $y_s[n]$ of the system with $d[n]$ and $p[n]$ specified in Part (b).

Solution:

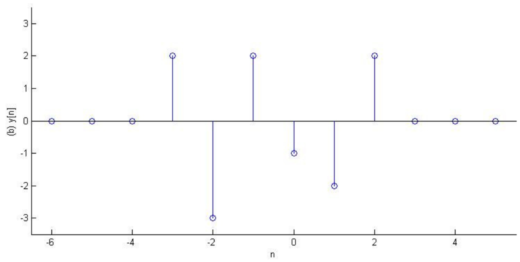
(a)



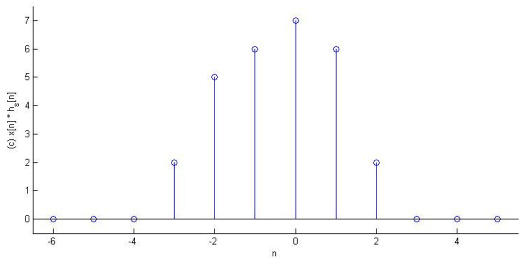
(b) $p[n]$ is given by



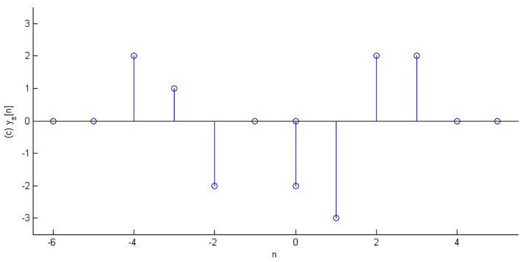
then $y[n]$ is given by



(c) $x[n]*h_s[n]$ is given by



then $y[n]$ is given by



3. **[15]** You are given the following 5 facts about a discrete time sequence $x[n]$:

i. $x[n]$ is real and odd.

ii. $x[n]$ is periodic with period $N = 6$.

iii. $\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = 10$.

iv. $\sum_{n=<N>} (-1)^{n/3} x[n] = 6j$.

v. $x[1] > 0$.

Find an expression of $x[n]$ in the form of sines and cosines.

Solution:

Given facts i and ii, we know that

$$x[n] = a_1 \sin\frac{\pi n}{3} + a_2 \sin\frac{2\pi n}{3}$$

for real numbers a_1, a_2. Then

$$x[0] = x[3] = -x[-3] = 0$$

$$x[1] = -x[-1] = a_1 \sin\tfrac{\pi}{3} + a_2 \sin\tfrac{2\pi}{3} = \tfrac{\sqrt{3}}{2}a_1 + \tfrac{\sqrt{3}}{2}a_2$$

$$x[2] = -x[-2] = a_1 \sin\tfrac{2\pi}{3} + a_2 \sin\tfrac{4\pi}{3} = \tfrac{\sqrt{3}}{2}a_1 - \tfrac{\sqrt{3}}{2}a_2$$

By fact iii, we have

$$\tfrac{1}{6}\cdot\left[2\left(\tfrac{\sqrt{3}}{2}a_1 + \tfrac{\sqrt{3}}{2}a_2\right)^2 + 2\left(\tfrac{\sqrt{3}}{2}a_1 - \tfrac{\sqrt{3}}{2}a_2\right)^2\right] = \tfrac{1}{3}\cdot\left(\tfrac{3}{4}a_1^{\,2} + \tfrac{3}{4}a_2^{\,2} + \tfrac{3}{2}a_1a_2 + \tfrac{3}{4}a_1^{\,2} + \tfrac{3}{4}a_2^{\,2} - \tfrac{3}{2}a_1a_2\right)$$

$$= \tfrac{1}{2}a_1^{\,2} + \tfrac{1}{2}a_2^{\,2} = 10$$

By fact iv, we have

$$(-1)^{1/3}\left(\tfrac{\sqrt{3}}{2}a_1 + \tfrac{\sqrt{3}}{2}a_2\right) + (-1)^{2/3}\left(\tfrac{\sqrt{3}}{2}a_1 - \tfrac{\sqrt{3}}{2}a_2\right) + (-1)^{-1/3}\left(-\tfrac{\sqrt{3}}{2}a_1 - \tfrac{\sqrt{3}}{2}a_2\right) + (-1)^{-2/3}\left(-\tfrac{\sqrt{3}}{2}a_1 + \tfrac{\sqrt{3}}{2}a_2\right)$$

$$= (-1)^{1/3}\left(\tfrac{\sqrt{3}}{2}a_1 + \tfrac{\sqrt{3}}{2}a_2\right) + (-1)^{2/3}\left(\tfrac{\sqrt{3}}{2}a_1 - \tfrac{\sqrt{3}}{2}a_2\right) + (-1)^{2/3}\left(\tfrac{\sqrt{3}}{2}a_1 + \tfrac{\sqrt{3}}{2}a_2\right) + (-1)^{1/3}\left(\tfrac{\sqrt{3}}{2}a_1 - \tfrac{\sqrt{3}}{2}a_2\right)$$

$$= (-1)^{1/3}\left(\tfrac{\sqrt{3}}{2}a_1 + \tfrac{\sqrt{3}}{2}a_2 + \tfrac{\sqrt{3}}{2}a_1 - \tfrac{\sqrt{3}}{2}a_2\right) + (-1)^{2/3}\left(\tfrac{\sqrt{3}}{2}a_1 - \tfrac{\sqrt{3}}{2}a_2 + \tfrac{\sqrt{3}}{2}a_1 + \tfrac{\sqrt{3}}{2}a_2\right)$$

$$= (-1)^{1/3}\sqrt{3}a_1 + (-1)^{2/3}\sqrt{3}a_1 = \sqrt{3}a_1\cdot\left[(-1)^{1/3} + (-1)^{2/3}\right] = \sqrt{3}a_1\cdot\sqrt{3}j = 3ja_1 = 6j$$

and then $a_1 = 2$, making $a_2 = 4$ since $x[1] > 0$. Therefore $x[n] = 2\sin\frac{\pi n}{3} + 4\sin\frac{2\pi n}{3}$.

(Note: We usually define $(-1)^{1/3} = \frac{1+\sqrt{3}j}{2}$, but since $(-1)^{1/3} = \frac{1\pm\sqrt{3}j}{2}$ is not well defined, $-3ja_1 = 6j$ is still a reasonable derivation and then $x[n] = -2\sin\frac{\pi n}{3} + 4\sin\frac{2\pi n}{3}$ is also an OK answer.)

4. **[6]** Suppose $x(t) = \left(\frac{\sin(4\pi t)}{\pi t}\right)\left(\frac{\sin(2\pi t)}{\pi t}(-1)^t\right)$. Determine the frequency range of $x(t)$.

Solution:

$$\mathcal{F}\left(\tfrac{\sin(4\pi t)}{\pi t}\right) = \begin{cases} 1 & |\omega| < 4\pi \\ 0 & |\omega| > 4\pi \end{cases}$$

$$\mathcal{F}\left(\tfrac{\sin(2\pi t)}{\pi t}(-1)^t\right) = \mathcal{F}\left(\tfrac{\sin(2\pi t)}{\pi t}e^{j\pi t}\right) = \begin{cases} 1 & |\omega - \pi| < 2\pi \\ 0 & |\omega - \pi| > 2\pi \end{cases} = \begin{cases} 1 & -\pi < \omega < 3\pi \\ 0 & \omega < -\pi \text{ or } \omega > 3\pi \end{cases}$$

The frequency range of $\frac{\sin(4\pi t)}{\pi t}$ is $(-4\pi, 4\pi)$ and the frequency range of $\frac{\sin(2\pi t)}{\pi t}(-1)^t$ is $(-\pi, 3\pi)$, then the frequency range of $x(t)$ is $(-5\pi, 7\pi)$.

(Note: Similar to problem 3, since $(-1)^t$ is not a well defined function, $(-7\pi, 5\pi)$ is an OK answer.)

5. Let $x(t)$ be a periodic signal with fundamental period T and Fourier series coefficients a_k. Derive the Fourier series coefficients of each of the following signals in terms of a_k:

   (a) **[3]** The odd part of $x\left(t - \frac{T}{2}\right)$

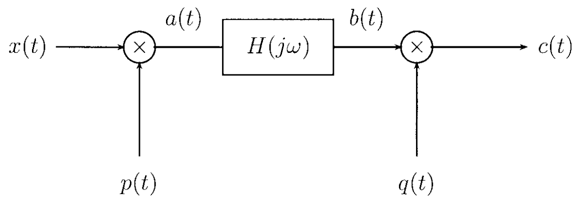
   (b) **[3]** $x\left(\frac{T}{4} - t\right)$

Solution:

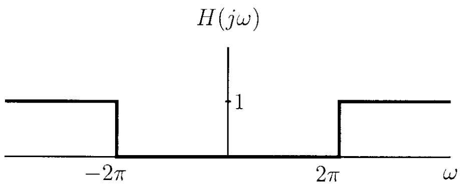
(a) $x\left(t - \frac{T}{2}\right) \xleftrightarrow{\mathcal{FS}} a_k e^{-jk\frac{2\pi}{T}\frac{T}{2}} = a_k e^{-jk\pi} = (-1)^k a_k$, and then $\mathcal{O}d\left\{x\left(t - \frac{T}{2}\right)\right\} \xleftrightarrow{FS} j\,\mathrm{Im}\{(-1)^k a_k\}$.

(b) $x\left(\frac{T}{4} - t\right) = x\left(-\left(t - \frac{T}{4}\right)\right)$, $x(-t) \xleftrightarrow{\mathcal{FS}} a_{-k}$, and then $x\left(\frac{T}{4} - t\right) = x\left(-\left(t - \frac{T}{4}\right)\right) \xleftrightarrow{FS} a_{-k}e^{-jk\frac{2\pi}{T}\frac{T}{4}} = a_{-k}e^{-jk\frac{\pi}{2}}$.

6. **[10]** Suppose $x(t)=\frac{\sin(4\pi t)}{\pi t}$, $p(t)=\cos(2\pi t)$, and $q(t)=\frac{\sin(2\pi t)}{\pi t}$. Consider the system depicted below:



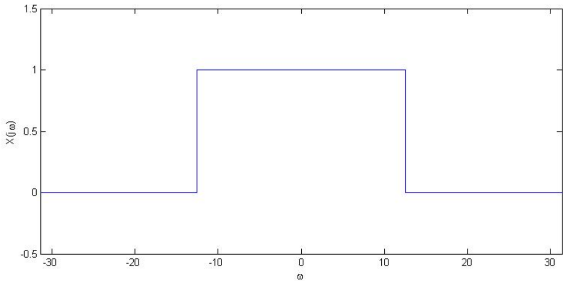
where the frequency response of $H(j\omega)$ is given by



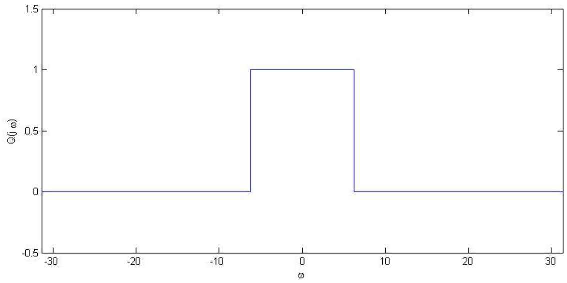
Determine the output $c(t)$ and its Fourier transform $C(j\omega)$. Sketch and clearly label $C(j\omega)$.

Solution:

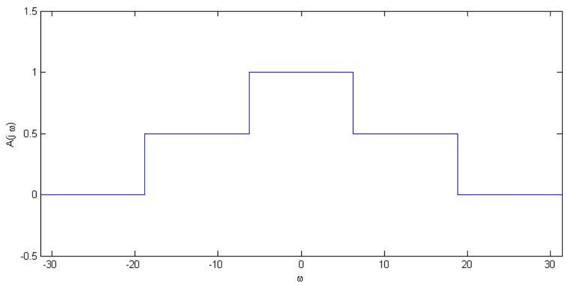
$X(j\omega)=\begin{cases}1 & |\omega|<4\pi \\ 0 & |\omega|>4\pi\end{cases}$, whose sketch is given by



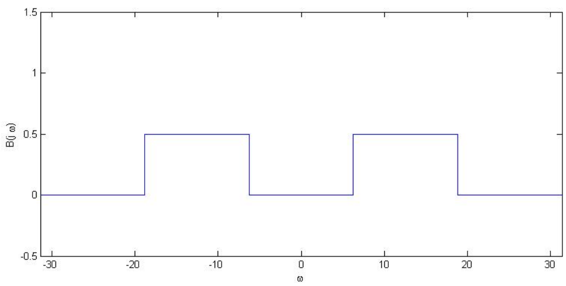
$Q(j\omega)=\begin{cases}1 & |\omega|<2\pi \\ 0 & |\omega|>2\pi\end{cases}$, whose sketch is given by



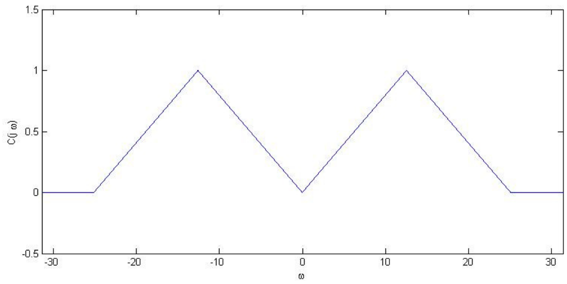
$P(j\omega)=\pi\left[\delta(\omega-2\pi)+\delta(\omega+2\pi)\right]$, so $A(j\omega)$ is



Then $B(j\omega)$ is given by



So $C(j\omega)$ is given by



Observing that

$$B(j\omega)=\tfrac{1}{2\pi}R(j\omega)*\pi\left[\delta(\omega-4\pi)+\delta(\omega+4\pi)\right]$$

where

$$R(j\omega)=\begin{cases}1 & |\omega|<2\pi\\ 0 & |\omega|>2\pi\end{cases}$$

we know that

$$b(t)=\frac{\sin 2\pi t}{\pi t}\cos 4\pi t$$

and then

$$c(t)=\left(\frac{\sin 2\pi t}{\pi t}\right)^2\cos 4\pi t$$

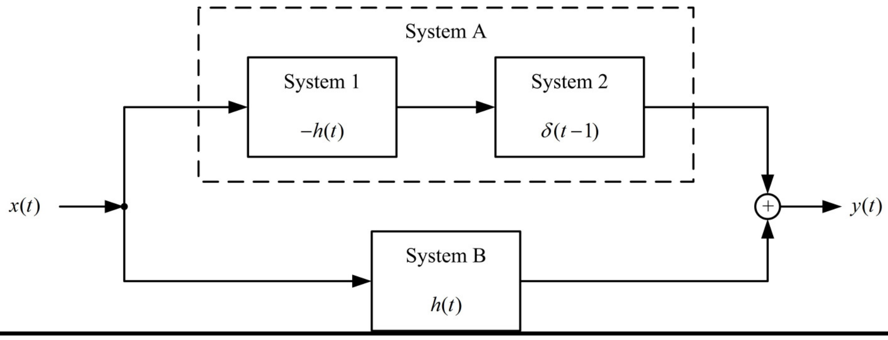
7. Assume that a linear time-invariant (LTI) system has an input $x(t)$ and output $y(t)$ relationship given by $y(t)=\int_0^\infty e^{-\alpha}x(t-\alpha)d\alpha$.

   (a) **[4]** Find the system impulse response $h(t)$.

   (b) **[4]** Is the system causal? Why?

   (c) **[4]** Determine $y(t)$ if the input $x(t)$ is set to $u(t+1)$.

   (d) **[4]** Following (a), consider an interconnection of LTI systems given as following graph: System A is a series interconnection of System 1 with impulse response $-h(t)$ and System 2 with impulse response $\delta(t-1)$. Then the overall system is a parallel interconnection of System A and System B with impulse response $h(t)$. Find the impulse response of the overall system. (Note: Remember to substitute your result in (a) for $h(t)$.)



Solution:

(a) $h(t)=\int_0^\infty e^{-\alpha}\delta(t-\alpha)d\alpha$

For $t<0$, we can see $\delta(t-\alpha)=0$, so $h(t)=0$.

For $t>0$, we can see $h(t)=\int_0^\infty e^{-\alpha}\delta(t-\alpha)d\alpha=\int_{-\infty}^\infty e^{-\alpha}\delta(t-\alpha)d\alpha=e^{-t}$

The conclusion is $h(t)=e^{-t}u(t)$.

(b) Yes, because $h(t)$ is only valued when $t>0$.

(c) $y(t)=h(t)*u(t+1)=u(t+1)\int_0^{t+1}e^{-\tau}d\tau=\left(1-e^{-(t+1)}\right)u(t+1)$.

(d) $h_{\text{total}}(t)=h_A(t)+h_B(t)=h_1(t)*h_2(t)+h_B(t)=-h(t)*\delta(t-1)+h(t)=-e^{-(t-1)}u(t-1)+e^{-t}u(t)$

8. Consider a first-order LTI system with the linear constant coefficient differential equation (LCCDE) given by $\frac{dy(t)}{dt}+2y(t)=x(t)$ for $t\ge 0$ and $y(0)=4$.

   (a) **[4]** Find the natural response of the LTI system.

   (b) **[4]** Find the forced response of the LTI system when the input $x(t)=2$.

   (c) **[4]** Find the forced response of the LTI system when the input $x(t)=e^{3t}u(t)$.

Solution:

(a) $\frac{dy(t)}{dt} + 2y(t) = 0$, $y_h(t) = Ae^{-2t}$ for some constant A.

(b) $\frac{dy(t)}{dt} + 2y(t) = 2$, it can be observed that $y_p(t) = 1$.

(c) $\frac{dy(t)}{dt} + 2y(t) = e^{3t}$, letting $y_p(t) = Ye^{3t}$, we have $3Ye^{3t} + 2Ye^{3t} = e^{3t}$, $Y = \frac{1}{5}$, so $y_p(t) = \frac{1}{5}e^{3t}$.

9. Suppose that an LTI system with unit impulse response $h[n]$ has an input signal $x[n]$.
   (a) **[5]** Find the output $y[n]$ of the system at $n = 0$ in terms of $X(e^{j\omega})$ and $H(e^{j\omega})$ by using the convolution property of Fourier transform, where $X(e^{j\omega})$ and $H(e^{j\omega})$ are the Fourier transforms of $x[n]$ and $h[n]$, respectively.
   (b) **[5]** Find the output $y[n]$ of the system at $n = 0$ in terms of $x[n]$ and $h[n]$ by using the convolution sum.
   (c) **[5]** Based on the above results, derive the Parseval's relation between the DT signal $x[n]$ and its Fourier transform $X(e^{j\omega})$.

Solution:

(a) $y[n] = \frac{1}{2\pi}\int_{2\pi} Y(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})H(e^{j\omega})e^{j\omega n}d\omega$, $y[0] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})H(e^{j\omega})d\omega$.

(b) $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$, $y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k]$.

(c) By (a) and (b), $\sum_{k=-\infty}^{\infty} x[k]h[-k] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})H(e^{j\omega})d\omega$. Now letting $h[n] = x^*[-n]$, then

$$\sum_{k=-\infty}^{\infty} x[k]x^*[k] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})X^*(e^{j\omega})d\omega$$

or

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

10. Here, we consider a scheme to generate signals for communication applications. Assume that two continuous-time (CT) signals $f_1(t)$ and $f_2(t)$ are given as follows: $f_1(t) = 2\cos(200\pi t)$ and $f_2(t) = 5\cos(1000\pi t)$, respectively. Let the signal $f_3(t)$ be the product of the two CT signals $f_1(t)$ and $f_2(t)$.
   (a) **[3]** Find the frequency spectrum of $f_3(t)$.
   (b) **[3]** Describe the properties of the Fourier transform you use to derive the result in part (a).

Solution:

$f_3(t) = f_1(t)f_2(t) = 2\cos(200\pi t)\cdot 5\cos(1000\pi t) = 5[\cos(1200\pi t) + \cos(800\pi t)]$,

so applying the linearity property on $\cos\omega_0 t \overset{\mathcal{F}}{\longleftrightarrow} \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$,

$F_3(j\omega) = 5\pi[\delta(\omega-1200\pi) + \delta(\omega+1200\pi) + \delta(\omega-800\pi) + \delta(\omega+800\pi)]$.

11. Consider the following DT signals $x_1[n]$ and $x_2[n]$:

$$x_1[n] = \frac{\sin\frac{n\pi}{2}}{n\pi}$$

$$x_2[n] = \begin{cases} \dfrac{\sin\frac{5n\pi}{9}}{9\sin\frac{n\pi}{9}} & \text{if } n \text{ is not a multiple of 9} \\ \dfrac{5}{9} & \text{if } n \text{ is a multiple of 9} \end{cases} \quad \text{with period } N = 9$$

(a) **[5]** Find the discrete-time Fourier transform of $x_1[n]$ by using the duality between the discrete-time Fourier transform and continuous-time Fourier series expansion.

(b) **[5]** Find the Fourier series expansion of $x_2[n]$ by using the duality for the discrete-time Fourier series expansion.

Solution:

(a) We know that for periodic signal

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < \frac{T}{2} \end{cases}$$

with period T, the corresponding Fourier series coefficients are $\dfrac{\sin k\omega_0 T_1}{k\pi}$. To use the duality property, we let $T = 2\pi$ and then $\omega_0 = 1$, resulting in the Fourier series coefficients $\dfrac{\sin kT_1}{k\pi}$.

Comparing $\dfrac{\sin kT_1}{k\pi}$ with $x_1[n]$, we may find that $T_1 = \dfrac{\pi}{2}$, and then

$$X_1(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\omega| \le \pi \end{cases}$$

with frequency period 2π.

(b) We know that

$$g[n] = \begin{cases} 1 & |n| \le 2 \\ 0 & 2 < |n| \le 4 \end{cases} \quad \stackrel{\mathcal{FS}}{\longleftrightarrow} \quad b_k = \begin{cases} \frac{\sin\frac{5k\pi}{9}}{9\sin\frac{k\pi}{9}} & k \text{ is not a multiple of 9} \\ \frac{5}{9} & k \text{ is a multiple of 9} \end{cases}$$

with period $N = 9$. Therefore

$$x_2[n] = \begin{cases} \frac{\sin\frac{5n\pi}{9}}{9\sin\frac{n\pi}{9}} & \text{if } n \text{ is not a multiple of 9} \\ \frac{5}{9} & \text{if } n \text{ is a multiple of 9} \end{cases} \quad \stackrel{\mathcal{FS}}{\longleftrightarrow} \quad \tfrac{1}{9}\, g[-k] = \begin{cases} \frac{1}{9} & |k| \le 2 \\ 0 & 2 < |k| \le 4 \end{cases}$$