E

Complex Variable Final Exam June 27, 2006

1. (21 %) True or false (If it is false, explain briefly why it isn't true)

   A. If $w = f(z)$ is analytic within a domain D of z-plane and maps D to the domain D' of w-plane, then $P(u, v)=P(w)$ is a harmonic function in D' will result in $p(x, y)=p(z)$ to be harmonic function in D.

   B. $f(z)=P(z)/Q(z)$, $P(x)$ is a complex polynomial of order n, and $Q(x)$ is a complex polynomial of power m, if $m > n$, $\alpha > 0$, and the contour C_R is $\{z \mid z = \mathrm{R}e^{i\theta}, 0 \le \theta \le 2\pi\}$, then $\oint_{C_R} f(z)e^{i\alpha z}dz$ approach 0 as R approaches infinity.

   C. $f(z)=\dfrac{1}{z^2(e^z-1)}$ possess a pole of order 2 at $z = 0$.

   D. For $f(z)=1/(z-3)$, the Laurent series valid for $|z|>3$ is $z^{-1}+3z^{-2}+9z^{-3}+\ldots$. Since there are an infinite number of negative powers of $z = z - 0$, $z = 0$ is an essential singularity.

   E. $f(z)=\dfrac{1}{\sin\left(\pi/z\right)}$ has an infinite number of singular points and all of the singular points are isolated singularities.

   F. The image of the circle $|z-1|=1$ under the complex mapping $T(z)=\dfrac{z-1}{z-2}$ is a circle since the linear fractional transformation has the so-called circle preserving property.

   G. The complex function $f(z)=z+1/z$ is analytic except at $z = 0$, therefore the mapping $w = f(z)$ from w-plane to z-plane is conformal except at $z = 0$.

2. (8 %) Find the Laurent series of $f(z)=\dfrac{1}{(z^2+1)(z-3i)}$, centered at $z = 0$, in the following domains:

   A. $|z|<1$ B. $1<|z|<3$ C. $|z|>3$

3. (24 %) Evaluate the following:

   A. $\displaystyle\int_{-\infty}^{\infty}\frac{x\cdot dx}{(x^2+1)(x^2+2x+2)}$

   B. $\displaystyle\int_0^{\pi}\sin^{2n}\theta\cdot d\theta$, n=1, 2, 3, …; hint: $(x+y)^n=\displaystyle\sum_{k=0}^{n}x^n y^{n-k}\binom{n}{n-k}$

   C. $\oint_C \csc z\cdot dz$, where C is the clockwise contour defined by $9x^2+25y^2=900$

   D. Solve the inverse Laplace transform of $F(s)=\dfrac{\coth(\pi s/2)}{s^2+1}$. Hint: $f(t)=\dfrac{1}{2\pi i}P.V.\displaystyle\int_{\gamma-i\infty}^{\gamma+i\infty}F(s)ds$

4. (15 %) In Fig. 1, find an appropriate conformal mapping and solve the Dirichlet problem.

$$\begin{cases} DE: \nabla^2 u = 0, R = \{z \mid 0 < Arg(z) < \pi/4\} \\ \quad BC: u(\Gamma_1) = 1, u(\Gamma_2) = 0 \\ \Gamma_1 = \{z \mid |z| < 1, Arg(z) = 0, Arg(z) = \pi/4\} \\ \Gamma_2 = \{z \mid |z| > 1, Arg(z) = 0, Arg(z) = \pi/4\} \end{cases}$$

Hint: first consider mapping Fig. 1 to the upper complex plane; then try to solve the new problem using a linear combination of $\frac{1}{\pi} Arg(w+a)$.

5. (16 %) (a) Find the linear fractional transformation that map the points x=0, x=1, x=∞ in the Z-plane to u=-1, u=0, and u=1 in the W-plane. This is in fact, the normalized Smith chart in transmission theory and microwave engineering. (b) Please find the mapping lines of x=0, x=1, y=1, and y=-1, in the right-half of the Z-plane to W-plane; mark each line accordingly.

6. (16 %) Schwarz-Christoffel Transformation

   Find the mapping function from the upper-half complex plane x-y (Fig. 2) to u-v (Fig. 3) plane.

   A. First, find $f'(z)$. Map the x-y plane using the Schwarz-Christoffel transformation to Fig. 4, first, then extent u_1 to infinity. Require that $f(-1) = u_1$, $f(0) = \pi i/2$, and $f(1) = u_1 + \pi i$.

   B. From A, if we require that $\text{Im}(f(t)) = 0$ for $t < -1$, $\text{Im}(f(t)) = \pi$, $t > 1$, and $\text{Im}(f(0)) = \pi i/2$, find $f(z)$.

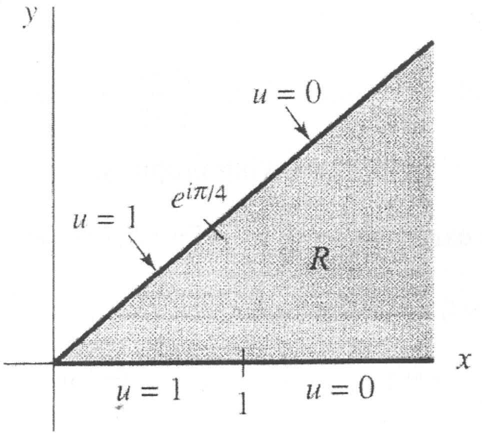

Fig. 1

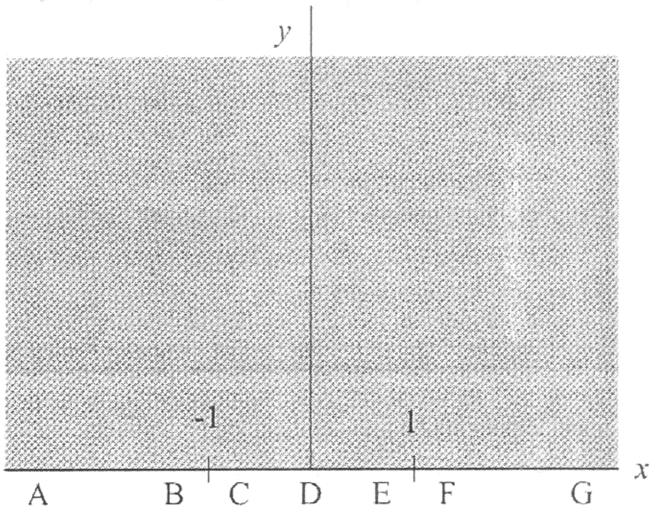

Fig. 2

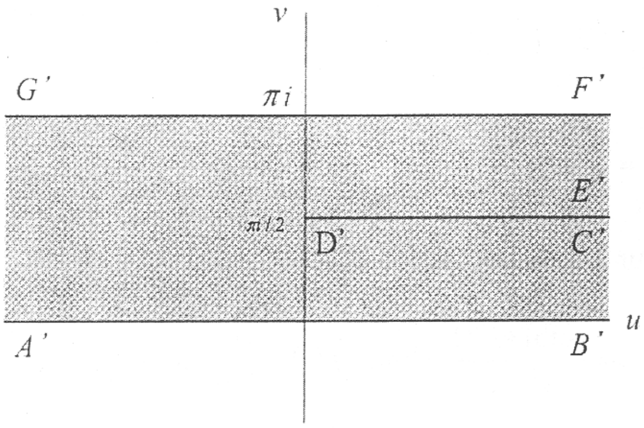

Fig. 3

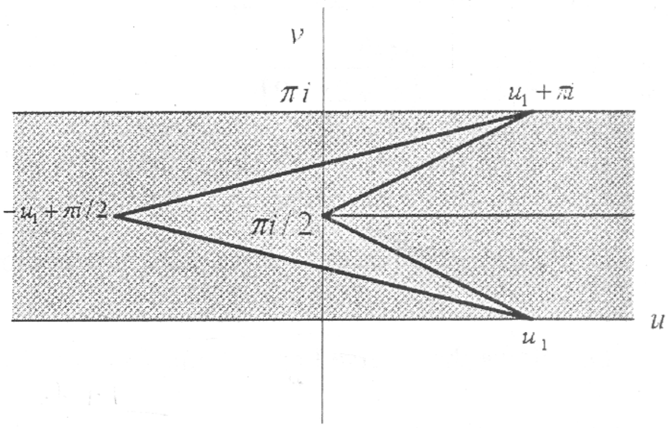

Fig. 4