Signals and Systems Midterm

10:20a.m. ~ 12:20p.m., May 2, Fri., 2008

- Closed book, but open 1 sheet (both sides, 2 pages) of personal notes of A4 size
- Total score: 120
- Total 4 pages in one B4 sheet

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1. **[10]** Consider a system H to be tested as being **memoryless**, **causal**, **linear**, **time invariant**, and **invertible**. Three signals $x_1(t)$, $x_2(t)$, and $x_3(t)$ are sent to the system, and the corresponding output signals $y_1(t)$, $y_2(t)$, and $y_3(t)$ are obtained as shown in Figure 1.

   Based on the three input-output pairs, is it possible to determine each of the five properties for system H? If yes, what is it? If no, why? Justify your answer.

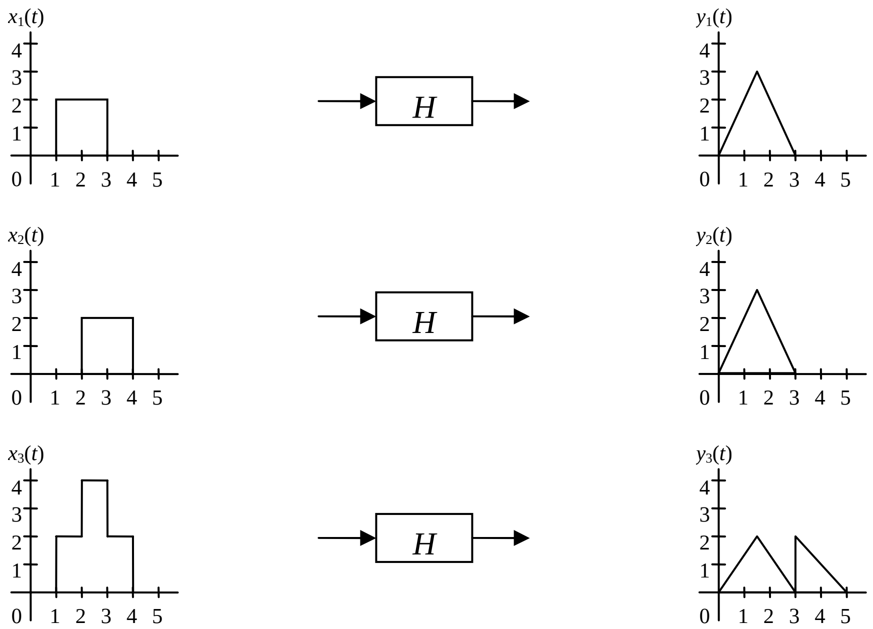

Figure 1

2. Consider a system as shown in Figure 2, where $h(t)$ is the impulse response of the LTI sub-system in the block, and 2D is the operation of time delay for 2 units.

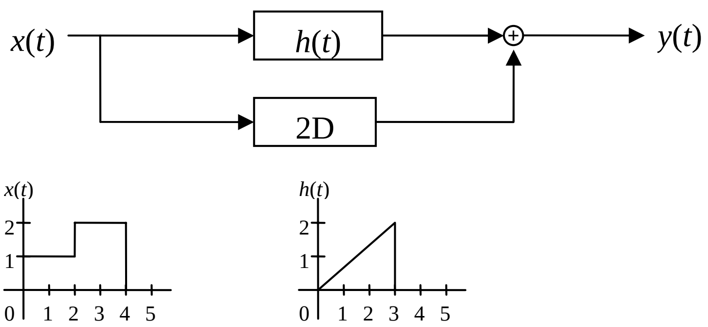

Figure 2

(a) **[4]** Plot the impulse response of the overall system.

(b) **[7]** Plot the output $y(t)$ of the system for input $x(t)$ shown in Figure 2.

(c) **[5]** Repeat $x(t)$ in time with a period of 6, and let $\tilde{x}(t)$ be the corresponding periodic version of $x(t)$. Plot the output $\tilde{y}(t)$ of the system for input $\tilde{x}(t)$.

3. **[6]** Let $x[n]$ be a periodic discrete-time sequence with period $N=8$ and Fourier series coefficients $a_k = -a_{k-4}$. Now generate a sequence

$$y[n] = \left(\frac{1+(-1)^n}{2}\right) x[n-1]$$

with period $N=8$ based on $x[n]$. Denoting the Fourier series coefficients of $y[n]$ as b_k, find a function $f[k]$ such that $b_k = f[k]\, a_k$.

4. Consult tables of Fourier transform pairs and answer the following questions:

(a) **[4]** $x(t) = te^{-3|t-1|}$, what is $X(j\omega)$?

(b) **[4]** $x[n] = \dfrac{\sin(\frac{\pi}{4}n)}{\pi n} * \dfrac{\sin[\frac{\pi}{4}(n-8)]}{\pi(n-8)}$, what is $X(e^{j\omega})$? (Note: * denotes convolution.)

(c) **[6]** $X(j\omega) = \dfrac{d}{d\omega}\left[\dfrac{4\sin(4\omega)\sin(2\omega)}{\omega}\right]$, what is $x(t)$?

(d) **[6]** What is $\displaystyle\int_{-\infty}^{\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt$?

5. **[10]** Let $r_{xy}[n] = \displaystyle\sum_{m=-\infty}^{\infty} x[m+n]y^*[m]$ be the cross-correlation of two discrete-time sequences $x[n]$ and $y[n]$, where $y^*[n]$ denotes the complex conjugate of $y[n]$. Let $X(e^{j\omega})$ and $Y(e^{j\omega})$ be the Fourier transform of $x[n]$ and $y[n]$, respectively. Find the Fourier transforms of $r_{xx}[n]$, $r_{xy}[n]$, $r_{yx}[n]$, and $r_{yy}[n]$.

Note: There are problems in the back.

6. The continuous-time Fourier transform pair is sometimes defined using the ordinary frequency f instead of the angular frequency ω (that is, $\omega = 2\pi f$) as follows:

$$\mathrm{F}\{x(t)\} = X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt, \quad \text{and}$$

$$\mathrm{F}^{-1}\{X(f)\} = x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df.$$

(a) **[4]** Derive the multiplication property for the new Fourier transform.

(b) **[4]** Derive the duality property for the new Fourier transform:

$$\text{if } x(t) \xleftrightarrow{\mathrm{F}} X(f), \text{ then } X(t) \xleftrightarrow{\mathrm{F}} \boxed{}\,?$$

(c) **[4]** Let $\mathrm{F}^2\{x(t)\} = \mathrm{F}\left\{\mathrm{F}\{x(t)\}\Big|_{f=t}\right\}$, and $\mathrm{F}^n\{x(t)\} = \mathrm{F}\left\{\mathrm{F}^{n-1}\{x(t)\}\Big|_{f=t}\right\}$ for $n \in N$ and $n > 2$.

Using the duality property of the Fourier transform, show that

$$\mathrm{F}^2\{x(t)\}\Big|_{f=t} = x(-t), \quad \mathrm{F}^3\{x(t)\}\Big|_{f=t} = \mathrm{F}^{-1}\{x(f)\}, \quad \text{and} \quad \mathrm{F}^4\{x(t)\}\Big|_{f=t} = x(t).$$

7. **[8]** Consider a discrete-time sequence $x[n]$ and its time-expanded version $x_k[n] = x\left[\lfloor n/k \rfloor\right]$, where $\lfloor z \rfloor$ is the greatest integer less than or equal to z. Let $X(e^{j\omega})$ be the Fourier transform of $x[n]$. Find the Fourier transform $X_k(e^{j\omega})$ for $x_k[n]$.

8. A causal and stable continuous-time LTI system H has the following frequency response

$$H(j\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega}.$$

(a) **[4]** Determine a differential equation relating the input $x(t)$ to the output $y(t)$ of the system.

(b) **[6]** What is the output $y(t)$ when the input is $x(t) = e^{-4t}u(t) - te^{-4t}u(t)$?

9. Consider the following transform for a continuous-time signal $x(t)$:

$$\mathrm{H}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)[\cos(\omega t) + \sin(\omega t)]dt.$$

(a) **[6]** Show that $X(j\omega) = X_e(\omega) - jX_o(\omega)$, where $X_e(\omega)$ and $X_o(\omega)$ are the even and odd parts of $X(\omega)$, and $X(j\omega)$ is the continuous-time Fourier transform of $x(t)$.

(b) **[6]** If $x(t)$ is a real-valued function, show that $X(\omega) = \Re e\{X(j\omega)\} - \Im m\{X(j\omega)\}$, where $\Re e\{X(j\omega)\}$ and $\Im m\{X(j\omega)\}$ is the real and imaginary part of $X(j\omega)$, respectively.

(c) **[4]** Evaluate $\mathrm{H}\{t^2 e^{-3t}u(t)\}$.

10. Let $x[n]$ be a discrete-time sequence of finite duration N_1 such that $x[n]=0$ outside the interval $0 \le n \le N_1 - 1$. The N-point discrete Fourier transform (DFT) of $x[n]$ is defined as

$$\tilde{X}[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk(2\pi/N)n}, \quad k \in Z,$$

where N is an integer larger than N_1.

(a) **[4]** Find the relation between $\tilde{X}[k]$ and $X(e^{j\omega})$, where $X(e^{j\omega})$ is the discrete-time Fourier transform (DTFT) of $x[n]$. Show that $\tilde{X}[k]$ can be considered as samples of $X(e^{j\omega})$ taken at discrete values of ω.

(b) **[4]** If $\tilde{X}[k]$ instead of $X(e^{j\omega})$ is used to recover $x[n]$ using the inverse discrete-time Fourier series (DTFS)

$$\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{X}[k]\, e^{jk(2\pi/N)n}, \quad n \in Z,$$

describe the difference between the resultant $\tilde{x}[n]$ and the original sequence $x[n]$.

(c) **[4]** Based on the results in (a) and (b), discuss the implication of recovering a finite-duration sequence $x[n]$ using the continuous-time function $X(e^{j\omega})$ and its discrete-time version $\tilde{X}[k]$.