微分方程期末考

Dept. of Electrical Engineering

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1. (15%) A linear system, as shown by the block diagram in Fig 1.1, can be described by the following equation:

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = g(t), \quad t \ge 0$$

$y(0) = y'(0) = \ldots = y^{(n-1)}(0) = 0$

Fig 1.1

When input g(t) is an impulse, i.e. Dirac delta function δ (t),

then the output y(t) = $e^{-t} - e^{-2t}$

(a) Find the values of coefficients $a_n, a_{n-1}, \cdots, a_1, a_0$ (7%)

(b) If input g(t) becomes e^{-t}, then y(t) = ? (8%)

2. (5%) $F(s) = \dfrac{s+\pi}{s^2+\pi^2}e^{-s}$, Find the inverse Laplace transform $f(t) = L^{-1}\{F(s)\}$

3. (5%) Solve the following differential-integral equation to obtain f(t)

$$\frac{df}{dt} = 1 - \sin t - \int_0^t f(t-\tau)d\tau, \quad f(0) = 0$$

4. (10%) Let $\Phi(t) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be fundamental matrices associated with the system of linear equations $\dot{X}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} X(t)$. What are the conditions on a, b, c, d ?

5. (15%) Given $\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *with* $X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find X(t).

6. (7%) (a) Find the Fourier series of f(x) on the given interval

$$f(x)=\begin{cases}0 & -\pi<x<0\\ x^2 & 0\le x<\pi\end{cases}\qquad (5\%)$$

(b) Use the results of (a) to show

$$\frac{\pi^2}{12}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\qquad (2\%)$$

7. (12%) Use an appropriate infinite series method about x = 0 to find two linearly independent solutions of the given equation.

(a) $2xy'' - (3 + 2x)y' + y = 0$ (6%)

(b) $y'' + e^x y' - y = 0$ (6%)

8. (6%) Find a particular solution of the differential equation, $x'' + 10x = f(t)$

$$f(t)=\begin{cases}5 & 0<t<\pi\\ -5 & \pi<t<2\pi\end{cases},\quad f(t+2\pi)=f(t),\quad -\infty<t<\infty,$$

9. (25%) Consider the following Laplace equation,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\qquad 0<x<a,\ y>0$$

subject to $u(0, y) = 0$, $u(a, y) = \exp(-\beta y)$, and $u(x, 0) = V$

where a, β and V are all constants.

We want to find the solution with both the Fourier series and Fourier integral transform. Questions (a) and (b) are related to the former while Question (c) is to the latter.

(a) If we divide the solution into $u = u_1 + u_2$ where both u_1 and u_2 follow the Laplace equation, and u_1 is subject to $u_1(0, y) = 0$ and $u_1(a, y) = \exp(-\beta y)$ only, please find a particular solution u_1. Note that only a particular solution is enough, not the general solution. (5%)

(b) Find the boundary conditions followed by u_2 and solve the solution u_2 with the Fourier series. (10%)

(c) Solve the solution u with the Fourier integral transform. (10%)