**LINEAR ALGEBRA: Final Examination** (June 14, 2001)

1. Determine whether or not the set $V$ is a subspace of the vector space $P$. *Need to justify your answer.*

   (a) (5%) $V$ is the set consisting of the zero polynomial and all polynomials of the form $c_0 + c_1 x + \ldots + c_m x^m$ such that $c_0 + c_1 = 0$.

   (b) (5%) $V$ is the set consisting of the zero polynomial and all polynomials of the form $c_0 + c_1 x + \ldots + c_m x^m$ such that $c_0 = 0$ or $c_1 = 0$.

2. (10%) Let $V$ be a finite dimensional vector space and $W$ be a subspace of $V$. Prove that if $\dim W = \dim V$, then $W = V$.

3. (15%) Given a set of data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, let $y = c_0 + c_1 x$ be the least-square line that best fits the data. Let $\mathbf{x} = [x_1\ x_2\ \ldots\ x_n]^T$, $\mathbf{y} = [y_1\ y_2\ \ldots\ y_n]^T$, $\bar{x} = (x_1 + x_2 + \ldots + x_n)/n$, and $\bar{y} = (y_1 + y_2 + \ldots + y_n)/n$. Show that if $\bar{x} = 0$ then $c_0 = \bar{y}$ and $c_1 = (\mathbf{x}^T\mathbf{y})/(\mathbf{x}^T\mathbf{x})$.

4. (a) (5%) Show that $(I - 2\mathbf{u}\mathbf{u}^T)$ is an orthogonal matrix where $I$ is the identity matrix and $\mathbf{u}$ is a unit vector.

   (b) (5%) Assume $\mathbf{x}$ and $\mathbf{y}$ are two vectors of the same norm. Show that $\|\mathbf{x}-\mathbf{y}\|^2 = 2(\mathbf{x}-\mathbf{y})^T\mathbf{x}$.

   (c) (10%) Assume $Q = I - 2\mathbf{u}\mathbf{u}^T$ where $\mathbf{u} = (\mathbf{x}-\mathbf{y})/\|\mathbf{x}-\mathbf{y}\|$ and $\|\mathbf{x}\| = \|\mathbf{y}\|$. Show that $Q\mathbf{x} = \mathbf{y}$ and $Q\mathbf{y} = \mathbf{x}$.

5. Let $T$ be the linear operator defined as:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 12x_1 + 10x_3 \\ -5x_1 + cx_2 - 5x_3 \\ -5x_1 - 3x_3 \end{bmatrix}.$$

   (a) (7%) Find the characteristic polynomial of $T$ in terms of $c$.

   (b) (8%) Determine all values of $c$ for which $T$ is not diagonalizable.

6. (10%) Determine whether the following transformation $T$ is an isomorphism:

$$T : \mathcal{P}_2 \longrightarrow \mathcal{R}^3, \text{ defined by } T(f(x)) = \begin{bmatrix} f(0) \\ f'(0) \\ f''(0) \end{bmatrix},$$

   where $\mathcal{P}_2$ is the set of all polynomials with degree $\leq 2$.

7. Let $A$ be any $n \times n$ matrix. Define the following two matrices:

$$A_0 = (A + A^T)/2, \qquad A_1 = (A - A^T)/2.$$

   (a) (5%) Prove that $A_0$ is diagonalizable.

   (b) (5%) Prove that $\lambda = 0$ is an eigenvalue of $A_1$.

   (c) (5%) Prove that if $\lambda$ is an eigenvalue of $A_1$, then $\lambda = 0$.

   (d) (5%) Prove that if $A$ is not symmetric, then $A_1$ is not diagonalizable.