Signals and Systems , Midterm
10:00-12:00 , Nov.27 , Sat , 1999

- Closed book, but open 1 sheet(both sides, 2 pages) of personal notes of A4 size
- Total score: 120 , time allocation:1 point/ minute

1. (14) For each system below , $x(t)$ or $x[n]$ is the input and $y(t)$ or $y[n]$ is the output. Determine if system(a) is : memoryless, time-invariant, linear, causal, or stable, and justify your answers. Determine if system(b) is invertible. If yes, find the inverse system. If no, find two input signals giving the same output.

   (a) (10)

$$y(t) = \int_{-\infty}^{2t} x(\tau)d\tau$$

   (b) (4)

$$y[n] = \sum_{k=-\infty}^{n} (\frac{1}{2})^{n-k}x[k]$$

2. (6) Consider an input signal $x[n] = a^n u[n]$, $0 < a < 1$, and a system with unit impulse response $h[n] = u[n]$. Find the output signal $y[n]$ by directly performing the convolution sum.

3. (8) Assume s_0 is a sulution of the equation $p(s) = \sum_{k=0}^{N} b_k s^k$. Show that $y(t) = Ae^{s_0 t}$ is a solution to the differential equation $\sum_{k=0}^{N} b_k \frac{d^k y(t)}{dt^k} = 0$, where A is an arbitrary complex constant.

4. (24) Verify the following expressions . If it is true, prove it. If not , prove it is not.

   (a) (8) x[n] is periodic, discrete signal. a_k are its Fourier Series coefficients, $x[n] \overset{FS}{\longleftrightarrow} a_k$. $x_{(m)}[n]$ is the time-expanded version,

$$x_{(m)}[n] = \begin{cases} x[n/m] & \text{,if n is an integer multiple of m.} \\ 0 & \text{,else.} \end{cases}$$

   The expression to be verified:

$$x_{(m)}[n] \overset{FS}{\longleftrightarrow} \frac{1}{m}a_k$$

   (b) (8) $x(t), y(t)$ are continuous-time, aperiodic signals. $X(j\omega), Y(j\omega)$ are their Fourier Transforms, $x(t) \overset{F}{\longleftrightarrow} X(j\omega), y(t) \overset{F}{\longleftrightarrow} Y(j\omega)$.
   The expression to be verified: generalized Parseval's relation:

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)Y^*(j\omega)d\omega.$$

(c) (8) $x[n], y[n]$ are discrete-time, aperiodic signals, $X(e^{j\omega}), Y(e^{j\omega})$ are their Fourier Transforms, $x[n] \xleftrightarrow{F} X(e^{j\omega})$, $y[n] \xleftrightarrow{F} Y(e^{j\omega})$
The expression to be verified : multiplication property:

$$x[n]y[n] \xleftrightarrow{F} \frac{1}{2\pi} \int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$$

5. (14)

   (a) (6) Assume a system has a unit impulse response $h(t) = \delta(t)$.What are the eigenfunctions of this system?

   (b) (8) If a system has a unit impulse response which is real and even , show that $x(t) = \cos\omega_0 t$ and $y(t) = \sin\omega_0 t$ are its eigenfunctions.

6. (10) A 2-dim signal $x(t_1, t_2)$ is periodic in both dimensions , with periods T_1 and T_2 in t_1 and t_2 dimensions respectively , i.e.,

$$x(t_1 + T_1, t_2 + T_2) = x(t_1, t_2), \quad all\ t_1, t_2$$

The 2-dim Fourier series representation of it is

$$x(t_1, t_2) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_{mn} e^{j(m\omega_1 t_1 + n\omega_2 t_2)} \quad , \omega_1 = \frac{2\pi}{T_1}, \omega_2 = \frac{2\pi}{T_2}.$$

Derive the expression to evaluate the coefficients a_{mn}.

7. (12) A system is specified by the following equation

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t),$$

where x(t) is the input and y(t) the output.

   (a) (6) Find the unit impulse response of this system.

   (b) (6) Find the output $y(t)$ if $x(t) = te^{-2t}u(t)$

8. (8) Show by induction that if $x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t), a > 0$, then $X(j\omega) = \frac{1}{(a+j\omega)^n}$

9. (8) Let $x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$. Find its Fourier transform $X(e^{j\omega})$.

10. (16) A system has a unit impulse response $h[n] = a^n u[n], |a| < 1$. Find the output signal $y[n]$ when the input signal is $x[n] = b^n u[n], |b| < 1$, from the frequency domain.

   (a) (4) Roughly sketch $|X(e^{j\omega})|$ as a function of ω for $b > 0$ and $b < 0$ respectively.

   (b) (6) Find $y[n]$ if $b \neq a$.

   (c) (6) Find $y[n]$ if $b = a$.