09 Fall 微分方程期末考

1. (10%) Find the power series solution of the following equation about $x = 0$.

$$\frac{1}{2}(1-x^2)y'' - xy' + 3y = 0$$

2. (8%) Find the Laplace transform of $e^{-t}t\cosh(3t) + \delta(t-2)$

3. (7%) Find the inverse Laplace transform of $\dfrac{1}{(s-1)(s+1)s}$

4. (10%) Solve $x(t)$ by the Laplace transform

$$\frac{d^3}{dt^3}x(t) + 2\frac{d^2}{dt^2}x(t) + 3\frac{d}{dt}x(t) + 2x(t) = 0, \qquad x''(0) = 1, \quad x'(0) = -1, \quad x(0) = 1.$$

5. (a) (3%) Please write down the 1-D wave equation.
   (b) (7%) Please find the general solution of the 1-D wave equation.

6. (a) (3%) Please write down the 2-D Laplace's equation with the two independent variables represented by x and y, respectively, and the dependent variable represented by u(x,y).
   (b) (10%) Please solve the above 2-D Laplace's equation with the boundary conditions:

   $u(0,y)=0$, $u(a,y)=G(y)$ for $0<y<b$
   $u(x,0)=0$, $u(x,b)=g(x)$ for $0<x<a$

7. (7%) Use the result that the Fourier transform of $e^{-\frac{x^2}{4p^2}}$ is $2\sqrt{\pi}pe^{-p^2\alpha^2}$ to solve the following equation:

$$\frac{\partial u}{\partial t} = -\frac{u}{\tau} + D\frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty \qquad t > 0$$

$$u(x,0) = \delta(x) \qquad -\infty < x < \infty$$

8. (5%) If y_1 and y_2 are linearly independent solution of the associated homogeneous DE for $y''+P(x)y'+Q(x)y = f(x)$, show in the case of a non-homogeneous linear second-order DE that $Xp = \Phi(t)\int \Phi^{-1}(t)F(t)dt$ reduced to the form of

$$y_p = y_1\int \frac{-y_2 f(x)}{W}dt + y_2\int \frac{y_1 f(x)}{W}dt$$

9. (10%) Solve the given linear system

(a) $$X' = \begin{pmatrix} 1 & 2 \\ -0.5 & 1 \end{pmatrix} X + \begin{pmatrix} 0 \\ e^t \tan t \end{pmatrix}$$

(b) $$X' = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix} X$$

10. (3%) Please find and prove the smallest interval near the origin for the set of functions is said to be orthogonal on that smallest interval:

(a) $\{\cos nx\}, n = 1,3,5,\cdots$

(b) $\{1, \cos nx\}, n = 1,2,3,\cdots$

(c) $\{1, \cos nx, \sin mx\}, n = 1,2,3,\cdots, m = 1,2,3,\cdots$

11. (a) (4%) Find the Fourier series of $f(x)$ on the given interval

$$f(x) = \begin{cases} 0, -\pi < x < 0 \\ x^2, 0 \le x < \pi \end{cases}$$

(b) (3%) Use the result of (a) to show

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

12. (a) (5%) Find the c_n of the Complex Fourier Series of function $f(x)$ defined on an interval (−p, p)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}$$

(b) (5%) Use (a) to expand the following $f(x)$ and prove c_n has the form of Sinc function, $\frac{\sin(x)}{x}$

$$f(x) = \begin{cases} 0, & -\frac{1}{2} < x < -\frac{1}{4} \\ 1, & -\frac{1}{4} < x < \frac{1}{4} \\ 0, & \frac{1}{4} < x < \frac{1}{2} \end{cases}$$