Complex Analysis - Midterm Exam.

10:10AM to 12:30 PM, April 24, 2001

(1) (10 %) Assume $f(z) = u(x, y) + iv(x, y)$ is analytic in D, and $u(x, y) = x^3 - 3xy^2 - 5y$. Use the following Cauchy-Riemann equations to find $v(x, y)$.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(2) (15 %) Define $\mathrm{Ln}(z)$ as the principal value of $\ln(z)$.
(2a) (5 %) Find $\ln(i)$ and $\mathrm{Ln}(i)$.
(2b) (5 %) Find $\ln(\sqrt{3} + i)$ and $\mathrm{Ln}(\sqrt{3} + i)$.
(2c) (5 %) Find all possible values of i^{2i}.

(3) (20 %) Use the definitions that

$$\sin w = \frac{e^{iw} - e^{-iw}}{2i}, \qquad \cos w = \frac{e^{iw} + e^{-iw}}{2}$$

to derive w as a function of z in terms of $\underline{\underline{\ln(\cdot)}}$ function.
(3a) (10 %) $w = \sin^{-1} z$.
(3b) (10 %) $w = \tan^{-1} z$.

(4) (10 %) Prove that

$$\frac{d}{dz} \tanh^{-1} z = \frac{1}{1 - z^2}$$

(5) (20 %) Use Cauchy's integral formula to calculate
(5a) (10 %)

$$\oint_C \frac{5z + 7}{z^2 + 2z - 3}, \qquad \text{with } C : |z - 2| = 2$$

(5b) (10 %)

$$\oint_C \frac{z + 1}{z^4 + 4z^3}, \qquad \text{with } C : |z| = 1$$

(6) (10 %) Apply the root-test procedure to find the circle of convergence of the series

$$\sum_{k=1}^{\infty} \left(\frac{6k + 1}{2k + 5} \right)^k (z - 2i)^k$$

(7) (15 %) Expand $f(z) = (1 - z)^{-1}$ into a Taylor series

$$\sum_{k=0}^{\infty} a_k (z - 2i)^k$$

(7a) (10 %) Find the coefficient a_k.
(7b) (5 %) Find the circle of convergence of the series.