Department of Electrical Engineering, National Taiwan University

Probability and Statistics, 2011, Spring

Midterm Examination

2011/4/28 Thursday, 15:30-17:20

1. Due to the recent political chaos in Egypt, Dr. Jones had to pause his mummy research (see P&S Midterm 2009, Final 2010) and leave Egypt. He was soon invited by his friend Prof. Hey to visit a top university in an Asia-Pacific island for a year.

   - **People can't read**

     Upon Dr. Jones' arrival, Prof. Hey brought Dr. Jones to a store near the front entrance of the university. There were so many people waiting in long line to buy some mysterious drink which seems to have something to do with frogs and milk according to its name. The mischievous Prof. Hey tried to tease Dr. Jones by treating him the drink. As the most renowned adventurer in the world, though feeling so sick about it, Dr. Jones still forced himself to drink it. Surprisingly, the taste was so good and Dr. Jones got addicted to it ever since then. Everyday he would bike across the intersection of the Eisenhower Rd. and the Newborn South Rd. to buy the drink.

     At the intersection, there are two huge signs written "Pedestrians should go across the road through the underground passage". Strangely, there are many student-like pedestrians walking directly across the road. Being really curious about the Chinese reading ability of the students from the university, Dr. Jones always like to ask them "Can't you read?". Here are the typical responses he would get from the pedestrians:

     - "Sorry...I am from other city."
     - "Mind your own business!!"
     - "Get a life, you Halloween freak!"
     - "...(total silence)"

     Based on Dr. Jones' his experience, it is equally likely for a pedestrian to respond him with either one of the four responses.

     (a) (5%) On a given day, Dr. Jones saw there were 20 pedestrians about to walk directly across the intersection. He once again sincerely asked everyone the magical question. What is the probability of 3 people saying they were from other cities, and 4 people telling him to mind his own business, and 5 people calling him a Halloween freak, and 8 people remaining in total silence?

     (b) (5%) In the subproblem (a), what is the expected number of people that would call Dr. Jones a Halloween freak on that day?

(c) (6%) One day before Dr. Jones left the office to buy the drink, Prof. Hey had a bet with him on the expected number of pedestrians that would call Dr. Jones Halloween freak at the intersection that day. According to Dr. Jones' experience, the number of pedestrians at the intersection N is usually distributed as Poisson distribution with variance equal to 24. What is the the expected number of pedestrians that would call Dr. Jones Halloween freak at the intersection that day.?

- **Frenzy of umbrella fashion**

  Dr. Jones noticed that the students in the university seems to have great interest in the umbrella fashion because it seems that everyone is constantly buying new umbrella all the time. This is particularly true for hardworking students who often study hard in the university library. Dr. Jones figured that buying new umbrellas is probably a special way of relieving pressure for the people in this university. He always thought so until one day he visited the university library to borrow a book and left his umbrella outside the library entrance...

(d) (6%) Dr. Jones always regards any person who ever purchases 10 or more umbrellas within a month to be an umbrella fashion specialist. Dr. Jones arrived the university on April 1 and bought an umbrella right away. On each day of April, the chance of raining is 0.5, the chance of Dr. Jones visiting the library is 0.8, and the chance of any umbrella at the library entrance getting stolen is 0.25. Dr. Jones only brings umbrella to school on rainy days, and he would always buy a new umbrella right away whenever his umbrella gets stolen. What is the probability of Dr. Jones finally becomes an umbrella specialist right on the date of April 20? (Note: Note before, not after, but right on April 20.)

(e) (6%) Dr. Jones visited the library with Prof. Hey one day during the midterm break. There were so many students studying in the library. A student fell asleep and snored terribly loud. A girl was obviously being very annoyed by it. Being so annoyed too, Prof. Hey and Dr. Jones were not able to work at all. The very bored couple started to bet again on how much longer would the girl remain on her seat during the next snore. According to their observation, the duration of each snore is exponentially distributed with mean equal to 10 seconds. It is known that most girls can only tolerate loud noise for at most 15 seconds before they give up and leave. Can you tell Prof. Hey the expected amount of time that the girl would remain on her seat during the next snore? (Note: If the snore lasts shorter than 15 seconds, then the time remaining on the seat = the duration of the snore.)

- **Parked, moved, and towed**

  Dr. Jones found that the university had a very serious issue in bicycle parking. There are simply too many bikes in the campus. If a bike is not parked properly within a bike parking slot, it will soon be towed by the university staff. Dr. Jones always follows the rule by parking his bike at the parking slot. Yet, his bike got towed one day and he was really pissed because it took him six hours to find his bike at the tow yard. He couldn't understand why his bike got towed even though he parked properly within a parking slot.

One day, he arrived the campus late and had problem finding parking slot. There was another guy having trouble finding parking slot too. Dr. Jones noticed that the guy approached a bike which was properly parked near the edge of the parking area, and the guy quickly moved the bike outside the parking slot and then parked his own bike in the emptied slot instead. Dr. Jones soon realized why his bike got towed the other day.

(f) (5%) Each day Dr. Jones arrives the bike parking area next to his office around 10:00AM. To be more specific, Dr. Jones' arrival time is Gaussian distributed with expected value 10:00AM and a standard deviation of 10 min. Since the students of the university usually wake up late, so most of them arrive around 10:20. The parking area is usually empty before 10:15, half full during 10:15 ∼10:20, and full after 10:20. What is the probability of Dr. Jones arriving the campus to see the parking area is full?

(g) (6%) When Dr. Jones arrives before 10:15, he can always park at the central area. On the other hand, he can only park at the edge of the parking area if he arrives during 10:15 ∼10:20. Any bike parked near the edge will have the high risk of being removed by some shameless people who arrive late, with a probability of 0.3. Since the shameless people are generally lazy to move bikes that are far away. So any bike parked at the central area will have a lower probability of 0.1 of being removed. All bikes that are removed from the parking area will definitely be towed. On any given day, what is the provability of Dr. Jones' getting pissed again to find his bike getting towed?

- **Circus in the rain**

  There is a Chinese old saying "An orange becomes a sunkist when it gets across the border". After visiting for months, Dr. Jones unconsciously built up some inappropriate habit. For instance, he started to ride bike with one hand holding the umbrella in rainy days.

(h) (6%) On any day without rain, Dr. Jones would not accidentally hit anyone during the ride and the amount of time needed to ride to the office is exactly 10 minutes. On any rainy day, if Dr. Jones does not hit anyone during the ride, the time it takes to arrive the office is uniformly distributed within the range [10, 20] (minutes); yet if Dr. Jones hits someone during the ride, the time it takes to arrive is exponentially distributed with mean equal to 30 minutes. The probability of raining on any day is 0.5. The probability of Dr. Jones hitting people during a ride on any rainy day is 0.6 (ya...he is not very good at it yet). Please find the CDF and the PDF of the time needed for Dr. Jones to arrive the office.

(j) (5%) Find the mean of the time needed for Dr. Jones to arrive the office.

Though being annoyed by some of the minor things he saw during the visit, Dr. Jones still likes the university a lot, especially after he attended one of the most popular annual campus event called 20EE. Upon his departure at the airport, his good friend Prof. Hey brought him another food called "large intestine wrapping the small intestine". Well, enough is enough, Dr. Jones did not dare to taste it this time...

2. (8%) Union bound

   (a) (4%) Let $A_1, A_2, \ldots, A_n$ be a set of n events. Starting from the axioms of probability, show the following bound for the probability of the union of $\{A_k\}$:

$$P\left[\bigcup_{k=1}^{n} A_k\right] \leq \sum_{k=1}^{n} P\left[A_k\right].$$

   When will the equality hold?

   (b) (4%) Given a set of numbers $p_1, p_2, \ldots, p_n \in [0, 1]$, use the union bound in (a) to show that

$$(1-p_1)(1-p_2)\cdots(1-p_n) \geq 1 - p_1 - p_2 \cdots - p_n.$$

3. (8%) In mathematics, a graph consists of a set of nodes together with a set of links connecting pairs of nodes (each pair of nodes in the graph has at most one link of no directionality). A random graph is a graph that is generated from a given set of nodes by randomly connecting nodes with links. Now given a set of $n > 1$ nodes, generate random graphs by connecting each pair of nodes with probability p independently of others in the graph.

   (a) (4%) Let D be the degree of a node in the graph, where the degree of a node is the number of nodes connected to it. Find the probability distribution of D.

   (b) (4%) Adjust p such that the expected number of links in the graph is n. Find the expected number of isolated nodes that are not connected to any other node in the graph. What does the ratio of isolated nodes (normalized to the total number of nodes) converge to as n becomes sufficiently large?

4. (16%) A continuous random variable X has PDF as shown in the following:

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad \forall x \in \mathcal{R}.$$

   Let random variable $Y = aX + b$, where $a > 0$ and b are constants.

   (a) (4%) Find the PDF of random variable Y.

   (b) (4%) The median of a continuous random variable is the value at which the CDF is equal to 0.5, whereas the mode is the value at which the PDF achieves its maximum value. Find the median and mode of Y.

(c) (4%) Let U be a continuous uniform $(0, 1)$ random variable. Explain how samples of Y can be generated from samples of U.

(d) (4%) Let Z be the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is Y. Find the probability distribution of Z.

5. (8%) Let T be the lifetime of a system. Define the failure rate function $r(t)$ of the system as follows:

$$r(t) = \lim_{\Delta \to 0} \frac{P\left[t < T \leq t + \Delta | T > t\right]}{\Delta}.$$

That is, given that the system has functioned up to time t, the probability that it will fail in the next Δ seconds is $r(t)\Delta$ for an infinitesimal value of Δ. If a system has a constant failure rate function such that $r(t) = \lambda > 0$, find the probability distribution of the system lifetime T.

6. (10%) Consider a biased coin that shows head with probability p when tossed. Now conduct an experiment by tossing the coin repeatedly until the head comes up twice in a row (i.e. two successive heads) or the tail comes up twice in a row. Let X be the number of tosses until the experiment is over.

(a) (4%) Let H_k (or T_k) be the event that the head (or tail) comes up at the k^{th} toss, where $k \geq 1$. Show that $E[X|H_1] = 1 + p + (1-p)E[X|T_1]$.

(b) (6%) Find the expected number of tosses $E[X]$. What happens if $p = 0.5$?

(*Hint: Use the concept of the conditional expected value for solving this problem.*)