微分方程期中考

1. Find the general solution of

$c_1x^2+c_2x^{-2}$

$4c_1x^4-4c_2x$

$2c_1x-2c_2x^{-3}$

$2c_1x^4-2c_2$

$$(x-2)^3\frac{d^3y}{dx^3}+5(x-2)^2\frac{d^2y}{dx^2}+7(x-2)\frac{dy}{dx}+8y=0 \quad (5\%)$$

$2c_1+6c_2x^{-4}$

2. A differential equation has the form: $x^4y''+x^3y'-4x^2y=1$

$2c_1x^4+6c_2x$

   (a) Find the solution of the associated homogeneous equation. (5%)

$$\begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix}$$

   (b) Find the general equation of the above non-homogeneous equation. (5%)

   solution

3. Please show that the third-order differential equation with the form $F(\frac{dy}{dx},\frac{d^2y}{dx^2},\frac{d^3y}{dx^3})$ can be reduced to a first-order differential equation. (5%)

4. The electron and the nucleus of an atom in the materials under an electromagnetic (EM) wave will be oscillating. Assume that the bonding force between the electron and the nuclei can be modeled as a spring. The nucleus has its mass much larger than the mass of the electron, so the interaction of the atom under the EM wave can be simplified as the oscillation of the electron. The electron with a mass m due

$M_{核} >> M_{e^-}$

$M_{核} >> m$

to the spring has the restoring force proportional to the distance from its original position without the EM wave with the proportional constant k. In addition, the electron in the materials experiences a damping factor. This damping factor leads to a force proportional to the velocity of the electron with the proportional constant β, but opposite to the moving direction. The electron under the EM wave also experiences a Coulomb force equal to $-eE(t)$, where e is the charge of the electron and $E(t)$ is the electric field of the EM wave. $E(t) = A\cos\omega t$. For the electron without the EM wave, its position is at the origin. Under the EM wave, the electron is at the position $x(t)$. Please solve $x(t)$ for the condition with a small damping factor, so the oscillation is underdamped. (5%)

-kx

$-\beta v$
$-\beta \frac{dx}{dt}$

$f(t) = -eE(t)$
$= -e A\cos\omega t$

β小

β小

$e^{-5000t} = -\frac{1}{3}$

5. A model for the population $P(t)$ of a small town is given by the initial-value problem

$$\frac{dP}{dt} = \frac{5}{4}P(4000 - P) \qquad P(0) = 1000$$

(a) Solve $P(t)$。(7%)

$4000(1+3e^{-5000t})^{-1}$

$P(t) = \frac{4000}{1+3e^{-5000t}}$

× (b) Plot $P(t)$ versus t, for $t \geq 0$。(3%)

$P(t) = \frac{4000}{1+4000Ce^{-5000t}}$

(c) What is the limiting value of the population? (2%)

(d) If P(0) = 6000, what is the limiting value of the population? (3%)

6. Given $\frac{dy}{dx} = 5y - 5$

(a) Draw several plausible solution curves for $-\infty < x < \infty$. (7%)

(b) On the plot in (a), identify the solution curve y(x), $-\infty < x < \infty$, with the condition y(0) = 2. (3%)

7. Please find the critical points (2%) and phase portrait (2%) of the given differential equation:

$$\frac{dy}{dx} = \frac{ye^y - 9y}{e^y}$$

8. Please solve the given initial-value problem (5%) Find the largest interval over which the solution is defined. (2%)

$$y\frac{dy}{dx} - x = 2y^2, y(1) = 5$$

9. Please find an appropriate integrating factor to make the given non-exact differential equation exact. (3%) Solve this initial value differential equation. (4%)

$$\left(x^2+y^2-5\right)dx=\left(y+xy\right)dy,\ y(0)=1$$

10. Please show that the substitution $u=y'$ leads to Bernoulli equation。(3%) Solve this equation (4%)：

$$xy''=y'+x\left(y'\right)^2$$

11. Consider the following differential equation：

$k^{-2} \leftarrow -k^{-1}$

$$\left(x^2-2x\right)y''-2(x-1)y'+2y=f(x)$$

$(x^2-2x)=(x-1)^2-1$

$k^{-3} \leftarrow (-\frac{1}{2})k^{-2}$

(a) Find the general solution for f(x)=0，where a solution form $y=x^m$ is known。(9%)

(b) Find the solution intervals containing x = 1 for f(x) = 0、f(x) = x^2-2x、and f(x) = csc(2x)。You have to describe the reason。(7%)

△ (c) Find the general solution for f(x) = x^2-2x，subject to the solution interval in (b)。(9%)

$y_2(x)=x^m\left(\int\frac{dx}{x^{2m-2}}-2\int\frac{dx}{x^{2m-1}}\right)$

$y_2(x)=x^m\int\frac{dx}{x^m(x^2-2x+2)}$

$\frac{Ax^m+B}{x^{m+1}}+\frac{C}{x-2}$

$x=k+2$

$x-2=k$

$dk=dx$

$\frac{dx}{x^{m+1}(x-2)}$

$x^{m+2}=$

$\frac{dx}{x^{m+2}-2x^{m+1}}$

$\frac{A}{x^{m+1}}+\frac{B}{x-2}$

$Ax-2A+Bx^{m+1}$

$x^{m+1}(x-2)$