Complex Variable Final Exam

2003/6/17

1. (25%) The statements below are all FALSE, please explain why they are wrong.

(a) If a complex function f has an isolated singularity at z_0 with a Laurent series $f(z)=\sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$ valid in some annular domain $R_1<|z-z_0|<R_2$, $0\le R_1<R_2$. The coefficient a_{-1} of $1/(z-z_0)$ is then defined as the residue of f at z_0.

(b) Suppose $f(z)=P(z)/Q(z)$, where P, Q are polynomials with degree of n & m and $m\ge n+1$. If C_R is a semicircular contour $z=Re^{i\theta}$, $0\le\theta\le\pi$, then $\int_{C_R} f(z)dz\to 0$ as $R\to\infty$.

(c) If $w=f(z)$ is analytic at z_0 in some domain D, then f is conformal at $z=z_0$ for a mapping from z-plane to w-plane.

(d) A linear fractional transformation $w=f(z)$ maps a circle in the z-plane to either a line or a circle in the w-plane. The image is a line if and only if the original circle passes through a zero of $f(z)$.

(e) The function $f(z)=\dfrac{1}{z(e^z-1)}$ has a simple pole at $z=0$.

2. (15%) $f(z)=\dfrac{1}{(z-i)(z-2i)}$, $z\ne i$, $z\ne 2i$. Find the Laurent series of $f(z)$ in the following domain (a) $|z|<1$ (b) $1<|z|<2$ (c) $2<|z|<\infty$

3. (10%) Evaluate $\oint_C \cot z dz$, where C is the contour defined by $|x|+|y|=5$, running clockwise.

4. (10%) Evaluate $\int_0^\infty \dfrac{dx}{x^3+1}$. Hint: use the contour in Fig. 1, and consider $\oint_C \dfrac{dz}{z^3+1}$, where $C=C_1+C_2+C_R$.

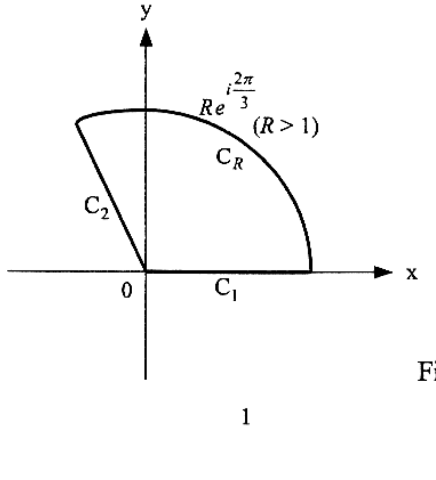

Fig. 1

5. (10%) Prove that $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots=\frac{\pi^4}{96}$

6. (15%) Solve the Dirichlet problem:

$$\begin{cases} DE:\ \nabla^2 u = 0 \quad in \quad R=\left\{z \,\middle|\, |z-2|>2, x>-\frac{1}{2}\right\} \\ BC:\ u(\Gamma_1)=1,\quad u(\Gamma_2)=0 \\ \Gamma_1=\{z \mid |z-2|=2\},\quad \Gamma_2=\left\{z \,\middle|\, x=-\frac{1}{2}\right\} \end{cases}$$

Explain why the solution level of curves must be circles.

Hint: Assume a linear fractional transformation that has w=1 as a pole and. try to map R in Fig. 2 into the annular region R' in Fig. 3.

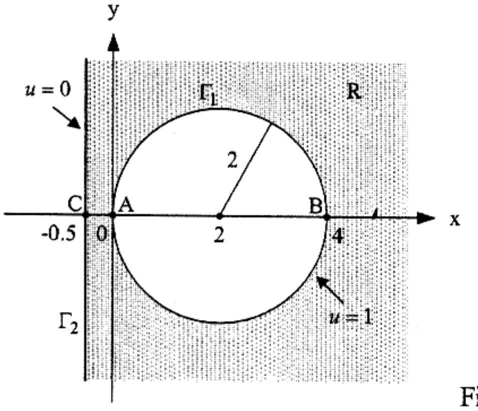

Fig. 2

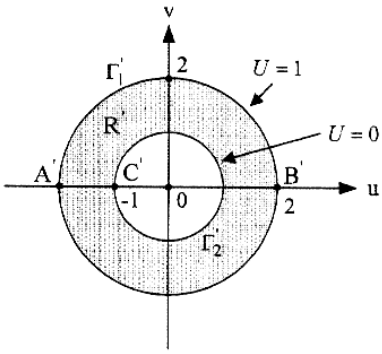

Fig. 3

7. (15%) Find the transformation (complex function $w=f(z)$) to map Fig. 4 to Fig. 5.

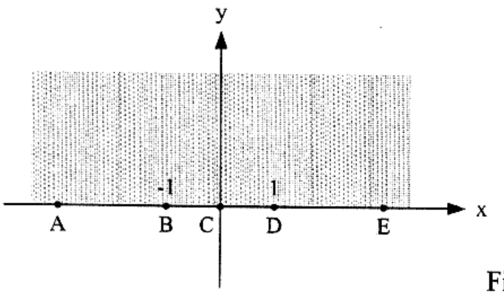

Fig. 4

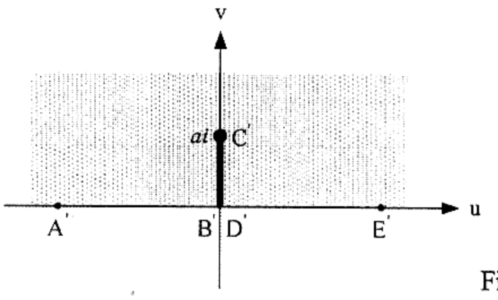

Fig. 5

Hint: Try to map the shaded area in Fig. 4. to that in Fig. 6 first, and let $f(-1)=-u_1$, $f(0)=ai$, then have $u_1 \to 0$.

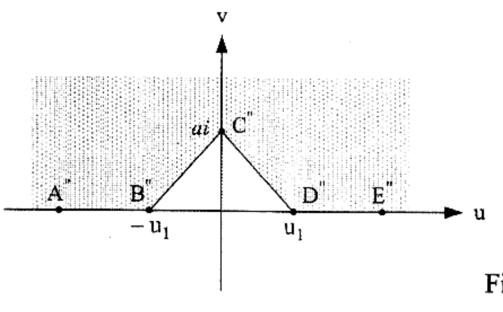

Fig. 6