複變
100-2 期中.

COMPLEX ANALYSIS

Midterm (2012/4/17, 10:20 AM-12:10 PM)

1. Find the derivative of $\tanh^{-1}(z)$ at $z = \sqrt{2}$. **(10%)**

2. Use the formula $\sin z = \sin x \cosh y + i \cos x \sinh y$ to find all values of z satisfying the equation $\sin z = \cosh \pi$. **(10%)**

3. Show that $u(x, y) = \log_e (x^2+y^2)$ is harmonic in the domain D: $x = \mathrm{Re}(z) > 0$, $y = \mathrm{Im}(z) > 0$ and find a corresponding function $f(z) = u + iv$ that is analytic in D (v is the conjugate harmonic function of u). Please express f in terms of z. **(10%)**

   [Hint: you might need the Cauchy-Riemann equations in polar coordinates $\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \\ \dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta} \end{cases}$]

4. Find all values of $\ln\left(i^{3i}\right)$. **(10%)**

5. Evaluate the given integrals along the indicated contour C. **(10% each)**

   (1) $\oint_C \dfrac{2z^2 - z - 2}{z - z_0} dz$, where $|z_0| \neq 3$ and C: $|z| = 3$ in the positive direction

   (2) $\oint_C z^m (\bar{z})^n dz$, where m and n are integers and C: $|z| = 1$ in the positive direction.

   (3) $\oint_C \dfrac{\sinh z}{(z - \pi i)^4} dz$, where C: $\mathrm{Im}\left(\dfrac{1}{z}\right) = \dfrac{1}{4}$ in the positive direction.

   (4) $\oint_C \dfrac{\tan(z/2)}{(z - x_0)^2} dz$, where $-2 < x_0 < 2$ and C is the negatively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$.

   (5) $\oint_C z^{-2n} e^{-z} dz$, where n is a positive integer and C: $z = e^{-it}$ for $0 \leq t \leq 2\pi$.

6. Show that $\int_0^{\pi} e^{a\cos\theta} \cos(a \sin\theta) d\theta = \pi$, where a is real. **(10%)**

   [Hint: evaluate the integral $\oint_C \dfrac{e^{az}}{z} dz$, with C: $z = e^{i\theta}$ $(-\pi \leq \theta \leq \pi)$]