Final Examination of Probability

(23%) 1. Let X be a **continuous** random variable with **probability density function (pdf)**

$$f(x)=\begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{elsewhere} \end{cases} \quad (1)$$

(a). Let $Y=\begin{cases} X & \text{if } X \leq 1 \\ 1/X & \text{if } X > 1 \end{cases}$, calculate the **cumulative** distribution function (**cdf**-also known as distribution function) and **pdf** of Y. (12%)

(b). Let X and Y be two ***independent*** random variables with the same pdf of Eq. (1), what is the joint probability function of X and Y ? (3 %) Let $Z = X/Y$, calculate the **cdf** and **pdf** of Z. (8%)

(15%)2. In this problem, two types of random variables are considered.

(a). Let the amount of water in a glass be a **normal** random variable. Assume that in 7% of the glasses containing water there are less than 15.5 ounces, and in 10% of them there are more than 16.3 ounces. Find the expectation and variance of the amount of water in a randomly selected glass? (7%)

(b). Let the number of trees that grow in a region of area A have a **Poission** distribution with expectation λA, where λ is a positive real number. Find the expected value of the distance form a certain tree to its nearest neighbor. (8%)

(12%) 3. In a sequence of ***independent*** **Bernoulli** trials, let X be the number of successes in the first m trials and Y the number of successes in the first n trials, $m < n$. Will the **conditional** probability function $p_{X|Y}(x \mid y)$ equal to $p_{Y|X}(y \mid x)$? You have to show your answer by evaluating the corresponding probability function.

(10%)4. A **random** point (X,Y) is selected form the rectangle $[0, \pi/2]\times[0,1]$. What is the probability that it lies below the curve $y=sinx$?

(20%)5.(a). Let X and Y be two random variables with **probability density function** $f(x,y)$. If $E(Y|X=x)$ is a linear function of x, that is, if $E(Y|X=x)=a+bx$ for some $a,b\in\mathbf{R}$, then $E(Y|X=x)=\mu_Y+\rho\dfrac{\sigma_Y}{\sigma_X}(x-\mu_X)$.(5%)

(b). Let the **joint** probability density function of two random variables X and Y be given by

$$f(x,y)=\begin{cases} 2 & \text{if } 0<y<x,\ 0<x<1 \\ 0 & \text{elsewhere} \end{cases}$$

Find $E(X|Y=y)$, $E(Y|X=x)$, $\rho(X,Y)$.(15%)

(20%)6. In this problem, we consider the application of **Moment-Generating functions** and **limit** theorems.

(a) .Let a random variable X have **Moment-Generating function** as follows

$$M_X(t) = \left(\frac{1}{3} + \frac{2}{3}e^t\right)^{10}$$

Find the variance of X and the Probability of $X \geq 8$.(10%)

(b) .We select 20 random numbers **independently** from the interval(0,1). Find the approximate probability that the sum of these numbers is at least eight.(10%)

Table 1 Area Under the Standard Normal Distribution to the Left of x

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy, \qquad \Phi(-x) = 1 - \Phi(x)$$

x	0	1	2	3	4	5	6	7	8	9
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.5772	.6808	.6844	.6879
.5	.6915	.6950	.5985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9889	.9889	.9889	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
3.6	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999