Signals and Systems Midterm

9:10a.m. ~ 11:10a.m., May 4, Fri., 2007

- Closed book, but open 1 sheet (both sides, 2 pages) of personal notes of A4 size
- Total score: 120
- Total 3 pages in one B4 sheet

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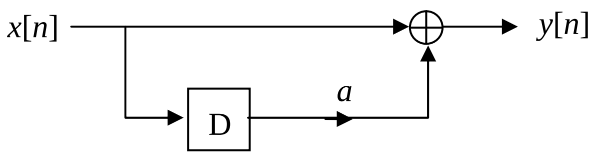
1. **[10]** Consider designing a discrete-time inverse system to eliminate the distortion associated with multipath propagation in a data communication problem, where the propagation channel consists of a direct path and a number of reflected paths. For simplicity, let us consider the discrete-time model of a two-path communication channel depicted as follows:



   a) **[2]** Write down the difference equation describing the two-path communication channel system.
   b) **[6]** Find the impulse response of a causal inverse system that will recover $x[n]$ from $y[n]$.
   c) **[2]** Check if the inverse system is stable and explain the physical meaning of the condition you have derived.

2. **[12]** Consider the signal

$$x[n] = \cos\left(\frac{7\pi}{16}n\right) + \cos\left(\frac{9\pi}{16}n\right).$$

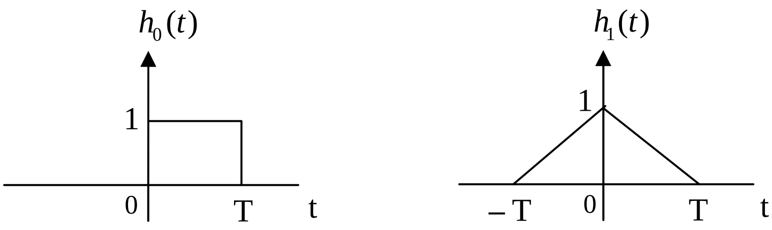
   a) **[4]** Compute the discrete-time Fourier transform (DTFT) of the signal.
   b) **[8]** Now compute the DTFT of only a portion of the signal by multiplying $x[n]$ with a windowing function $w[n]$,

$$w[n] = \begin{cases} 1, & |n| \le M \\ 0, & |n| > M \end{cases}$$

   Plot the DTFT of the truncated signal with $M = 8$ and $M = 40$ to evaluate the effect of truncating a signal on the DTFT.

3. **[12]** Given the Fourier transform pair $x[n] \xleftrightarrow{F} X(e^{j\omega})$, determine if each of the following statements is true or false. Justify your answer.
   a) **[3]** If $X(e^{j\omega}) = X(e^{j(\omega-1)})$, then $x[n] = 0$ for $|n| > 0$.
   b) **[3]** If $X(e^{j\omega}) = X(e^{j(\omega-\pi)})$, then $x[n] = 0$ for $|n| > 0$.
   c) **[3]** If $X(e^{j\omega}) = X(e^{j\omega/2})$, then $x[n] = 0$ for $|n| > 0$.
   d) **[3]** If $X(e^{j\omega}) = X(e^{j2\omega})$, then $x[n] = 0$ for $|n| > 0$.

4. **[16]** Let the impulse train $g(t)=\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$, where $x(t)$ is a continuous-time signal, be the input to two filters with impulse response $h_0(t)$ and $h_1(t)$ as depicted in the following figure:



Let $x_0(t)=g(t)*h_0(t)$ and $x_1(t)=g(t)*h_1(t)$.

a) **[6]** Suppose the Fourier transform (FT) of $x(t)$ is $X(j\omega)$. What is the FT of $g(t)$?

b) **[4]** In terms of filtering operation, what do these two filters do to the impulse train?

c) **[6]** Specify the frequency response of a filter that produces $x_1(t)$ as its output when $x_0(t)$ is the input.

5. **[10]** Let T be the period of a continuous-time periodic signal. Prove

a) **[5]** $T\sum_{k=-\infty}^{\infty}\delta(t-kT)=\sum_{k=-\infty}^{\infty} e^{j2\pi kt/T}$.

b) **[5]** $u[n]\overset{F}{\longleftrightarrow}\dfrac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{\infty}\pi\delta(\omega-2\pi k)$

6. **[14]** Answer the following questions.

a) **[6]** Consider a linear system with input $x(t)$ and output $y(t)$. Let $\Phi(t)$ be an eigenfunction of the system, i.e., if $x(t)=\Phi(t)$, then $y(t)=\lambda\Phi(t)$, where the complex number λ represents the eigenvalue associated with $\Phi(t)$. Assume that we input a signal $x(t)=\sum_{k=-\infty}^{\infty} a_k\Phi_k(t)$ to the system, where $\Phi_k(t)$ is an eigenfunction with a corresponding eigenvalue λ_k. Find the output $y(t)$ of the system in terms of $\{a_k\}$, $\{\Phi_k(t)\}$, and $\{\lambda_k\}$.

b) **[8]** Let the system be characterized by the differential equation

$$y(t)=t^2\frac{d^2x(t)}{dt^2}+t\frac{dx(t)}{dt}$$

Are $\Phi_k(t)=t^k$ the eigenfunctions of the system? You should justify your answer. If your answer is yes, then determine the corresponding eigenvalue λ_k.

[NOTE: There are problems in the back.]

7. **[10]** Consider an important concept of the correlation between two signals. Let $x(t)$ and $y(t)$ be two signals; then the correlation function of $x(t)$ and $y(t)$ is defined as follows:

$$R_{xy}(t) = \int_{-\infty}^{\infty} x(\alpha+t)y(\alpha)d\alpha$$

a) **[5]** Prove that $R_{xy(t)} = x(t) * y(-t)$, where $*$ denotes the convolution integral.

b) **[5]** Find $R_{xy}(t)$ for $x(t) = 2e^{-t}u(t) - 3e^{-2t}u(t)$ and $y(t) = \cos(2\pi t) \cdot [u(t+2) - u(t-2)]$, where $u(t)$ denotes the unit step function.

8. **[14]** Consider a system with the relationship of its input and output given by

$$y(t) = \int_{0}^{t} e^{-\tau} x(t-\tau)d\tau$$

a) **[5]** Find the system impulse response $h(t)$ of the system.

b) **[4]** Is the system causal ? You must justify your answer.

c) **[5]** Determine the output $y(t)$ of the system if the input $x(t) = u(t+1)$.

9. **[12]** Consider a real continuous-time (CT) signal $x(t)$. Assume that $x(t)$ is periodic with period being 6. Some Fourier coefficients of $x(t)$ are given as follows: $a_k = 0$ for $k = 0$ and $k > 2$, and a_1 is positive real number. Moreover, $x(t) = -x(t-3)$ and the total average power of $x(t)$ equals $1/2$. Find the CT signal $x(t)$.

10. **[10]** Consider a system with the relationship of its input and output given by

$$y(t) = \int_{t}^{t+1} x(\tau-\alpha)d\tau$$

where α is a constant.

a) **[3]** Is the system invertible? You must justify your answer.

b) **[3]** Is the system stable? You must justify your answer.

c) **[4]** Find the values of α so that the system is causal.