$w = \frac{1}{x^2}$ $\quad -\frac{2}{x^3}\cdot x + \frac{2}{x^2} = 0$ $\quad \frac{e^{y^2}x}{2yx\cdot e^{y^2}}$, $e^{y^2}(x^2y+4y)$

$\cancel{2ye^{y^2}(x^2y+4y)}$

$2xy\cdot e^{y^2}$

微分方程期中考 {2005 Fall}

1. Solve the given initial-value problem by using an appropriate substitution

(a) $y^{1/2}\dfrac{dy}{dx} + y^{3/2} = 1, y(0) = 4$ (7%)

$w = B$ $\quad kx+2$ $\quad \frac{64}{9}$ $\quad 576$

(b) $xv\dfrac{dv}{dx} + v^2 = 32x, v(3) = 0$ (7%)

$\frac{x}{2}\left(-\frac{2A}{x^3} + \frac{64}{3}\right) = -\frac{A}{x^2} + \frac{32}{3}x + \frac{A}{x^2} + \frac{64}{3}x = 32x$

$+\frac{A}{x^2} + \frac{64}{3}x$

2. Solve the given initial-value problem

$$xdx + (x^2y + 4y)dy = 0, y(4) = 0 \qquad (7\%)$$

3. The number 0 is a critical point of the autonomous differential equation $\dfrac{dx}{dt} = x^n$, where n is a positive integer. For what values of n is 0 asymptotically stable? Semi-stable? Unstable? (7%)

$n=3.$ $\quad \frac{1}{x^2} = -2t+c$ $\quad x^2 = \frac{1}{c-2t}$

? ? ?

$\int x^{-n}\,dx = \frac{x^{-n+1}}{-n+1}$ $\quad \frac{1}{x} = -t+c$ $\quad x = \frac{1}{-t+c}$

4. Find a one-parameter family of solutions for the differential equation $x^2\dfrac{dy}{dx} + xy + 4 = x^2y^2$, where $y_1 = \dfrac{2}{x}$ is a known solution of the equation。 (7%)

$u - 1 = kx^4u + kx^4$

$-8+28-32+12$. $\quad u(1-kx^4) = 1+kx^4$

$$\begin{array}{r|rrrr} & 1 & 5 & 6 & \\ (-2) & 1 & 9 & 16 & 12 \\ & & 2 & & \\ \hline & & 5 & 16 & \\ & & 5 & 10 & \\ \hline & & & 6 & 12 \end{array}$$

5. Consider the third-order differential equation:

$$\frac{d^3y(x)}{dx^3} + 7\frac{d^2y(x)}{dx^2} + 16\frac{dy(x)}{dx} + 12y(x) = 0$$

$(-2x+1)e^{-2x}$

(a) Find a fundamental set of solutions for the differential equation. You need to show that the set of solutions you find is a fundamental set of solutions. (8%)

$= -2e^{-2x} - 2(-2x+1)e^{-2x}$

(b) Find the general solution $y(x)$ of the differential equation. (2%)

$= (4x-4)e^{-2x}$

$(xe^{-2x})' = -2xe^{-2x} + e^{-2x}$

$(m^2+4m+4)(m+3)$ $\quad m^3+7m^2+16m+12$

$(1+2)(1+2)(1+3)$

$m^3+4m^2+4m+3m^2+12m+12$

$-D^2-5D-6$ $\quad$ $28e^{2x}-1$

D^2+5D-6 $\quad$ $(xe^{2x})' = 2xe^{2x}+e^{2x}$

$-2D^2-10D$ $\quad$ $= [(2x+1)e^{2x}]' = 2e^{2x} + \frac{3}{2}e^{2x} + 2(2x+1)e^{2x} = (4x+4)e^{2x}$

6. Find the general solution of

$$\frac{d^2y(x)}{dx^2} - 2\frac{dy(x)}{dx} = 3e^{2x} + 4x^2 \text{, with the initial condition:}$$

$\frac{3}{2}(4e^{2x}x + 4e^{2x}) - \frac{3}{2}(4xe^{2x}+2e^{2x})$ $\quad$ $-4-4F=0$ $\quad$ $y''-2y'=4x^2$

$$y(0) = -1, \frac{dy(0)}{dx} = 1$$ (10%)

$\cancel{4e^{2x}} - 2 - \cancel{4x} - \cancel{4e^{2x}} + 2 + \cancel{4x} + 4x^2$ $\quad$ $2Ce^{2x}$

7. Consider the system of differential equations:

$$2\frac{dx(t)}{dt} + \frac{dy(t)}{dt} - y(t) = \sin t$$

$$\frac{dx(t)}{dt} + 6x(t) + \frac{dy(t)}{dt} + 5y(t) = \cos t$$

$(5c_1+5c_2)\cos t + (5c_2-5c_1)\sin t$

$-c_1\cos t - c_2\sin t$

$5(-c_1\sin t + c_2\cos t)$

$6(c_1\cos t + c_2\sin t)$

Find the general solution, $x(t)$ & $y(t)$ of the system of differential equations and find the steady-state solution of the system. (15%)

$3E+2A=0$

$4F+3B=0$

$c_1+c_2 = -\frac{1}{5}$

$c_1-c_2 = \frac{8}{5}$

$c_1 = \frac{7}{10}$

$c_2 = -\frac{9}{10}$

8. It is required to find the solution based on the mathematical reasoning and physical law. No credit will be given if the answer is just based on the physical law. We are going to consider the trajectory of a body moving under a central force in the two-dimensional case. The position of the body is represented by (x(t), y(t)) in the rectangular coordinate where x(t) = r(t)cosθ(t) and y(t) = r(t)sinθ(t) in the representation of the polar coordinate. The central force in the position (x, y) is given by $f\left(\sqrt{x^2+y^2}\right)$ or f(r). The direction of the force can be written as (cosθ, sinθ) and f(r) can be positive or negative to represent a repulsive or attractive forces. A special case is f(r) = $-\lambda/r^2$ for the electrical or gravitational forces where λ is a certain positive constant. Please follow the mathematical procedure to find whether there is a solution corresponding to the circular trajectory.

$V(t)\hat{r}$

(a) Assume the body mass is m and the force is the general case, $f\left(\sqrt{x^2+y^2}\right)$. Please find the simultaneous differential equations followed by x(t) and y(t) according to the Newton law. (7%)

$m\sqrt{x''(t)^2+y''(t)^2} = f(\sqrt{x^2+y^2})$

$\frac{4}{5}\sin t + \frac{8}{5}\cos t - \frac{7}{10}\sin t - \frac{9}{10}\cos t$ $\quad$ $-\frac{7}{10}\cos t + \frac{9}{10}\sin t$

$\sin t\left(\frac{4}{5}-\frac{7}{10}+\frac{9}{10}\right) + \cos t\left(\frac{8}{5}-\frac{9}{10}-\frac{7}{10}\right) = 0.$

(b) Please find the corresponding simultaneous differential equations followed by r(t) and θ(t). (5%)

(c) If there exists a circular trajectory, what is the differential equation which can be assumed for r(t)? Please find the solution r(t) subject to the initial condition r(0) = R. $f(x) = m\cdot x^4$ (5%)

(d) Bring the solution r(t) into the simultaneous equations of r(t) and θ(t) to find the solution θ(t) subject to the initial condition $\theta(0) = \theta_0$. At the same time, please find the condition f(r) has to follow and then there exists a circular-trajectory solution. $f(r(t)) = m\cdot r''(t)$. (10%)

(e) Because of the assumption made in Question (c), the solutions have to be brought back into the original simultaneous equations in Question (b) to see whether the assumption is correct or not. This checking procedure maybe gives an additional condition for f(r). Does f(r) has to be in the form, $-\lambda/r^2$ or in any form, and then there is a solution for circular trajectories? How about $f(r) = \lambda\sin(kr)$, does it have a solution for the circular trajectory? Give your answer and the associated reason. (8%)

$\theta''\hat{\theta} + 2\theta'(-\theta'\hat{r}) + \boxed{(-\theta'\hat{r})'}\theta$

$-\theta''\hat{r} - \theta'^2\hat{\theta}$

$r''(t) = 3r(t)$

$r(t) = e^{\sqrt{3}t}$.

$= \hat{r}(-2\theta' - \theta'') + \hat{\theta}(\theta'' - \theta'^2)$

$\cos\theta \to -\sin\theta\cdot\theta' \to (-\cos\theta)\cdot\theta' + \theta''(-\sin\theta)$

$\cancel{r(t)\cos\theta}$

$(\theta\hat{\theta})'$
$= \theta'\hat{\theta} + \theta\cdot(-\theta')\hat{r}$
$= \theta'\hat{\theta} - \theta\cdot\theta'\hat{r}$

$x' = r'\cos\theta - \sin\theta\cdot\theta'\cdot r$

$x'' = r''\cos\theta - 2\sin\theta\cdot\theta'\cdot r' + r(-\cos\theta\cdot\theta' - \sin\theta\cdot\theta'')$

$(\theta\hat{\theta})''$
$= \theta''\hat{\theta} + \theta'(-\theta')\hat{r}$
$- [\theta'\theta' + \theta\cdot\theta'']\hat{r}$
$- \theta\cdot\theta'[\theta'\hat{\theta}]$
$= \hat{r}[\theta\cdot\theta'' - 2\theta'^2] + \hat{\theta}[\theta'' - \theta\cdot\theta'^2]$

$\int \frac{f(r)}{m}\,dr = u\cdot du = \frac{u^2}{2}$

$\frac{2}{m}\int f(r)\,dr = u'$

$r(t) = r$

$r'(t) = u$

$r''(t) = \frac{du}{dt} = \frac{du}{dr}\frac{dr}{dt} = u\cdot\frac{du}{dr}$

$f(r) = mu\cdot\frac{du}{dr}$