線性代數期末考

89/06/08

1.(10points)Find the determinant of the matrix $A \in M_{4\times 4}(R)$ define below. (*HINT*: Note that det(A)=0 when any 2 of the scalars x_i are equal.)

$$A=\begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{pmatrix}.$$

2.(10points) Let $B=I_n+\alpha uu^t$ where $\alpha \in R$, the $n\times 1$ vector $u \in R^n$ and $u^t u=1$ (u is a unit norm vector). Find det(B). (*HINT*: Find all eigenvectors of B.)

3.(5points) Let T: $M_{n\times n}(R) \to M_{n\times n}(R)$, $T(A)=A^t$. Assume that $n \geqq 2$. Find all eigenvalues of T.

4.(10points) Let $A,B \in M_{n\times n}(R)$. Show that every nonzero eigenvalue of the matrix AB is also an eigenvalve of the matrix BA.

5.(10points) Let U be a linear operator on a finite-dimensional inner product space V. If $\|U(x)\|=\|x\|$ for all x in V. Is it true that U is one-to-one? Justify your answer.

6)(5points)Consider the set of linear equations $Ax=0$, where A is an $n\times n$ matrix with component $A_{ij}\in Z_2$ for $i,j \in \{1,2,\ldots,n\}$ and $x=(x_1,x_2,\ldots,x_n)$, where Z_2 is the binary field. Suppose that the rank of A is k. Find the number of x in the solution set of $Ax=0$.

7.(10points) Let $W_1$and W_2 be subspaces of a finite-dimensional inner product space. Prove that $(W_1+W_2)^\perp=W_1^\perp \cap W_2^\perp$, where $W_1+W_2=\{y_1+y_2: y_1\in W_1, y_2\in W_2\}$.

8.(10points)Let $A\in M_{n\times n}(F)$. Prove that $\dim(\mathrm{span}(\{I_n,A,A^2,\ldots\}))\leqq n$.

9.(5points) Find the minimal solution to the following system of linear equations

$$\begin{aligned} 3x_1+x_2+2x_3+4x_4 &= -3 \\ -x_1-x_3+2x_4 &= -2 \end{aligned}$$

10.(10points) Let A be an $n\times n$ complex matrix with complex elements whose characteristic polynomial is given by

$$f(t)=a_n t^n+a_{n-1}t^{n-1}+\cdots+a_1 t+a_0$$

Prove that the characteristic poiynomial of A^* is given by

$$g(t)=\bar{a}_n t^n+\bar{a}_{n-1}t^{n-1}+\cdots+\bar{a}_1 t+\bar{a}_0$$

11.(10points)Let T:$P_2(R)\to P_2(R)$ be defined as

$$T(a_0+a_1x+a_2x^2)=(9a_0+2a_1)+(2a_0+6a_1)x+5a_2x^2$$

(a).(6points) Find the eigenvalues of T and the eigenspace corresponding to each eigenvalue.
(b).(4points) Is T diagonalizable? Explain your answer.

12.(10points) Let A be an $n\times n$ diagonalizable matrix. Prove that rank(A) = n - m, where m is the sum of the multiplicities of the nonzero eigenvalues of A.