Signals and Systems, Midterm Exam Solutions

Spring 2004, Edited by bypeng

1. (5) A triangular pulse signal $x(2t+4)$ is depicted in Fig. P1. Sketch the signal $x(3t)+x(3t+2)$.

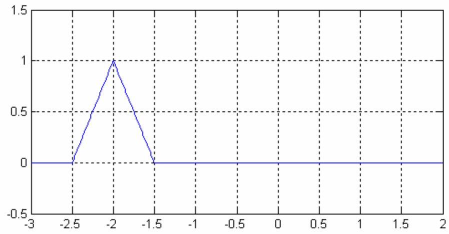
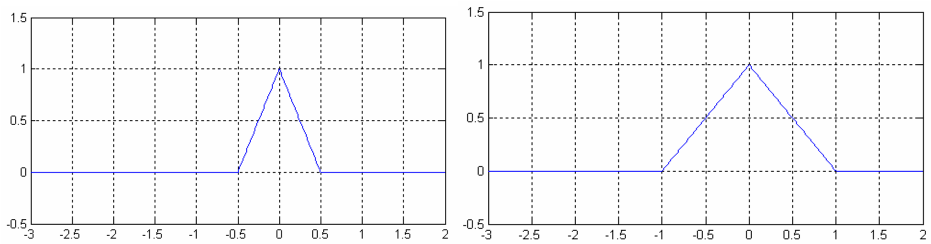


Fig. P1

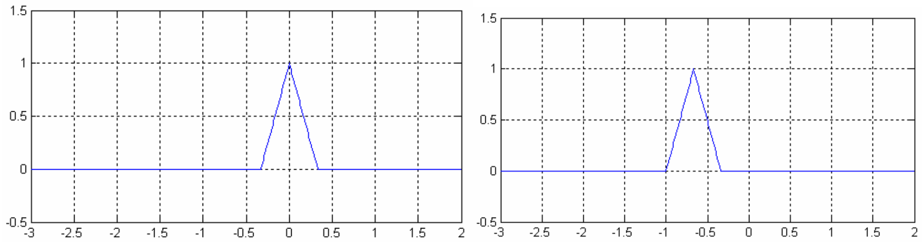
Solution:

$x(2t+4)=x(2(t+2))$, $\therefore$ $x(2t)$ and then $x(t)$ are depicted in Fig. P1-1(a) and (b), respectively:



(a) $x(2t)$ Fig. P1-1 (b) $x(t)$

Thus $x(3t)$ and $x(3t+2)$ can be sketched as Fig. P1-2(a) and P1-2(b), respectively:



(a) $x(3t)$ Fig. P1-2 (b) $x(3t+2)$

Therefore $x(3t)+x(3t+2)$ is given by Fig. P1-3.

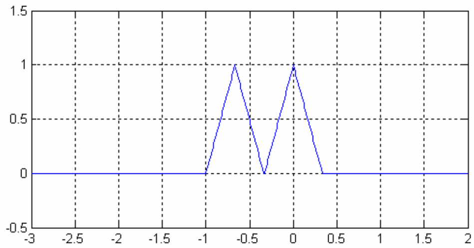


Fig. P1-3

2. (3) Determine and sketch the even and odd parts of the signal shown in Fig. P2. Label your sketch carefully.

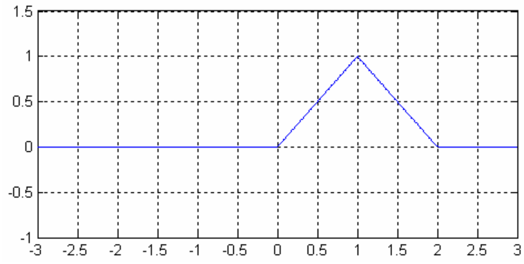
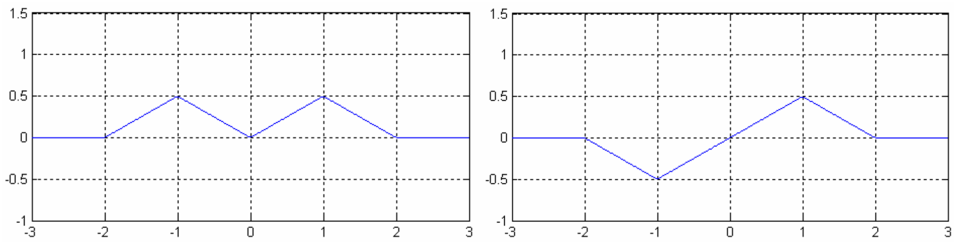


Fig. P2

Solution:

$Ev\{x(t)\} = \dfrac{x(t)+x(-t)}{2}$ and $Od\{x(t)\} = \dfrac{x(t)-x(-t)}{2}$. Therefore the sketches are given as Fig. P2-1.



(a) even part (b) odd part

Fig. P2-1

3. (10) Consider the following continuous-time system:

$$y(t) = \begin{cases} 0 & x(t) < 0 \\ x(t) + x(t-2) & x(t) \geq 0 \end{cases}$$

Determine whether the system is memoryless, time-invariant, linear, causal, stable? Justify your answers.

Solution:

Memorylessness: FALSE, since $y(t)$ depends not only on $x(t)$ but on $x(t-2)$.

Time invariance: TRUE. $y(t-t_0) = \begin{cases} 0 & x(t-t_0) < 0 \\ x(t-t_0) + x((t-t_0)-2) & x(t-t_0) \geq 0 \end{cases}$, trivially time-invariant.

Linearity: FALSE. Try $x_1(t) = t$ and $x_2(t) = t+2$, or any other possible input pairs.

(NOTE: you cannot explain it in such a brief way.)

Causalty: TRUE, since $y(t)$ depends only on $x(t)$ and $x(t-2)$, but not on $x(t+t_0)$ $\forall t_0 > 0$.

Stability: TRUE. $\forall x(t)$, if $\exists B \in \mathbf{R}$ such that $\forall t \in \mathbf{R}, |x(t)| < B$, then $\forall t \in \mathbf{R}, |y(t)| < 2B$.

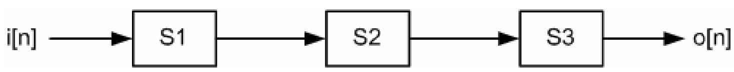
4. (5) Consider three systems with the following input-output relationships:

$$\text{S1}: y[n] = \begin{cases} x\left[\tfrac{1}{2}n\right] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

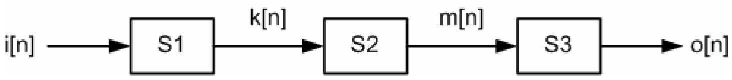
$$\text{S2}: y[n] = x[n] + \tfrac{1}{2}x[n-1] + \tfrac{1}{4}x[n-2]$$

$$\text{S3}: y[n] = x[2n]$$

Suppose that these systems are connected in series as follows. Find the input-output relationship (between $i[n]$ and $o[n]$) or the overall interconnected system.



Solution:



$$
\begin{aligned}
o[n] &= m[2n] \\
m[n] &= k[n] + \frac{1}{2}k[n-1] + \frac{1}{4}k[n-2] \Rightarrow \\
o[n] &= k[\underbrace{2n}_{\text{always even}}] + \frac{1}{2}k[\underbrace{2n-1}_{\text{always odd}}] + \frac{1}{4}k[\underbrace{2n-2}_{\text{always even}}] \\
&= i\left[\frac{2n}{2}\right] + \frac{1}{2}\cdot 0 + \frac{1}{4}i\left[\frac{2n-2}{2}\right] \\
&= i[n] + \frac{1}{4}i[n-1]
\end{aligned}
$$

Note that neither S1 nor S3 is a time-invariant system, so one cannot change the order S1-S2-S3.

5. (6) Consifer the following input-output relationship:

$$y[n] = \frac{1}{4}\sum_{k=0}^{3} x[n-k]$$

(a) (2 pt) Find the impulse response $h[n]$ of this system.

(b) (4 pt) Determine the output of the system when the input is the rectangular pulse defined as:

$$x[n] = u[n] - u[n-10]$$

Solution:

(a) $h[n] = \frac{1}{4}\sum_{k=0}^{3}\delta[n-k] = \frac{1}{4}(\delta[n]+\delta[n-1]+\delta[n-2]+\delta[n-3]) = \begin{cases} \frac{1}{4} & 0 \le n \le 3 \\ 0 & otherwise \end{cases}$

(b) $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=0}^{9} h[n-k] = \begin{cases} 0 & n \le -1 \\ \frac{n+1}{4} & -1 \le n \le 3 \\ 1 & 3 \le n \le 9 \quad \text{(Any boundary can be assign to both neighbor intervals.)} \\ \frac{13-n}{4} & 9 \le n \le 13 \\ 0 & n \ge 13 \end{cases}$

6. (8) Consider a causal discrete-time linear time-invariant system with input $x[n]$, output $y[n]$, and impulse response $h[n]$. Show that if the impulse response is absolutely summable, then the system is stable.

Solution:

$h[n]$ is absolutely summable if there is some $B_0 \in \mathbf{R}$ such that $\sum_{n=-\infty}^{\infty}|h[n]| = B_0$.

We know that $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$, and then $|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \le \sum_{k=-\infty}^{\infty}|h[k]x[n-k]| = \sum_{k=-\infty}^{\infty}|h[k]||x[n-k]|$.

If $x[n]$ is bounded, then we may find some $B \in \mathbf{R}$ such that $|x[n]| < B$ for any $n \in \mathbf{Z}$, and then

$|y[n]| \le \sum_{k=-\infty}^{\infty}|h[k]||x[n-k]| < \sum_{k=-\infty}^{\infty} B|h[k]| = B\sum_{k=-\infty}^{\infty}|h[k]| = BB_0$, $y[n]$ is bounded and therefore the system is stable.

7. (5) Consider a continuous-time linear time-invariant system with input $x(t)$, output $y(t)$, and impulse response $h(t)$. Show that $x(t) = e^{st}$ is an eigenfunction of the system and find the corresponding eigenvalue.

Solution:

$y(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau=\int_{-\infty}^{\infty}h(\tau)e^{s(t-\tau)}d\tau=\int_{-\infty}^{\infty}h(\tau)e^{st}e^{-s\tau}d\tau=e^{st}\int_{-\infty}^{\infty}h(\tau)e^{-s\tau}d\tau=x(t)\int_{-\infty}^{\infty}h(\tau)e^{-s\tau}d\tau$.

The corresponding eigenvalue is $\int_{-\infty}^{\infty}h(\tau)e^{-s\tau}d\tau$.

8. (4) Define a signal $\phi_k[n]=e^{jk\omega_0 n}$ where *k*, *n* are integer variables, ω_0 is the fundamental frequency. Assume that the fundamental period of $\phi_k[n]$ is *N*. Show that

$$\phi_{k+pN}[n]=\phi_k[n] \text{ and } \phi_k[n+qN]=\phi_k[n]$$

where *p*, *q* are any constant integers.

Solution:

Since ω_0 is the fundamental frequency and the fundamental period of $\phi_k[n]$ is *N*, we have $\omega_0 N=2\pi$.

Therefore,

$$\phi_{k+pN}[n]=e^{j(k+pN)\omega_0 n}=e^{jk\omega_0 n}e^{jpN\omega_0 n}=e^{jk\omega_0 n}e^{j2\pi pn}=e^{jk\omega_0 n}=\phi_k[n]$$
$$\phi_k[n+qN]=e^{jk\omega_0(n+qN)}=e^{jk\omega_0 n}e^{jk\omega_0 qN}=e^{jk\omega_0 n}e^{j2\pi kq}=e^{jk\omega_0 n}=\phi_k[n]$$

9. (10) Suppose that

$$x[n]\overset{FS}{\longleftrightarrow}a_k \text{ and } y[n]\overset{FS}{\longleftrightarrow}b_k$$

are both periodic with period *N*. Show that the coefficients c_k of the product $x[n]y[n]$ are given by

$$y[n]x[n]\overset{FS}{\longleftrightarrow}c_k=\sum_{r\in<N>}a_r b_{k-r}$$

Solution:

$a_k=\frac{1}{N}\sum_{n=<N>}x[n]e^{-j\frac{2\pi kn}{N}}$ and $b_k=\frac{1}{N}\sum_{n=<N>}y[n]e^{-j\frac{2\pi kn}{N}}$, and $x[n]=\sum_{k=<N>}a_k e^{j\frac{2\pi kn}{N}}$. Now we have

$$c_k=\frac{1}{N}\sum_{n=<N>}x[n]y[n]e^{-j\frac{2\pi kn}{N}}=\frac{1}{N}\sum_{n=<N>}\sum_{r=<N>}a_r e^{j\frac{2\pi rn}{N}}y[n]e^{-j\frac{2\pi kn}{N}}=\sum_{r=<N>}a_r\underline{\frac{1}{N}\sum_{n=<N>}y[n]e^{-j\frac{2\pi(k-r)n}{N}}}=\sum_{r=<N>}a_r b_{k-r}$$

10. (12) Suppose that the signal $x[n]$ has the following properties:

(a) $x[n]$ is a real and even signal.

(b) $x[n]$ has period $N=8$ and Fourier coefficients a_k .

(c) $a_9=5$.

(d) $\sum_{n=0}^{7}|x[n]|^2=400$.

Show that $x[n]=A\sin(Bn+C)$, and specify all possible numerical values for the constants *A*, *B*, and *C*.

Solution:

Since $x[n]$ is real and even signal, $\langle a_k\rangle$ is also real and even, and then $a_9=a_7=a_1=a_{-1}=5$. Now we have

$$\sum_{n=0}^{7}|x[n]|^2=400\Rightarrow\frac{1}{8}\sum_{n=0}^{7}|x[n]|^2=\sum_{k=0}^{7}|a_k|^2=|a_0|^2+\underbrace{|a_1|^2}_{25}+|a_2|^2+|a_3|^2+|a_4|^2+|a_5|^2+|a_6|^2+\underbrace{|a_7|^2}_{25}=50,$$

$a_0 = a_2 = a_3 = a_4 = a_5 = a_6 = 0$,

$$x[n] = 5e^{j\frac{2\pi\cdot 1n}{8}} + 5e^{j\frac{2\pi\cdot 7n}{8}} = 5e^{j\frac{2\pi n}{8}} + 5e^{-j\frac{2\pi n}{8}} = 10\cos\frac{n\pi}{4}$$

$$= 10\sin\left(n\left(\pm\frac{\pi}{4} + 2m_1\pi\right) + \left(2m_2\pi + \frac{\pi}{2}\right)\right) = -10\sin\left(n\left(\pm\frac{\pi}{4} + 2m_1\pi\right) + \left(2m_2\pi - \frac{\pi}{2}\right)\right)$$

$A = 10$, $B = \left(2m_1 \pm \frac{1}{4}\right)\pi$, $C = (2m_2 + \frac{1}{2})\pi$, where m_1 and m_2 are arbitrary integers; or

$A = -10$, $B = \left(2m_1 \pm \frac{1}{4}\right)\pi$, $C = (2m_2 - \frac{1}{2})\pi$.

11. (12) The input and the output of a stable and causal LTI system are related by the differential equation:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$$

(a) (4 pt) Find the impulse response of this system.

(b) (8 pt) What is the response of this system if $x(t) = e^{-2t}u(t)$?

Solution:

(a) We have

$$(j\omega)^2 Y(j\omega) + 5(j\omega)Y(j\omega) + 6Y(j\omega) = X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{(j\omega)^2 + 5(j\omega) + 6} = \frac{1}{(2+j\omega)(3+j\omega)} = \frac{1}{(2+j\omega)} - \frac{1}{(3+j\omega)}$$

Therefore

$$h(t) = e^{-2t}u(t) - e^{-3t}u(t)$$

(b)

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty}\left(e^{-2\tau}e^{-2(t-\tau)}u(\tau)u(t-\tau) - e^{-3\tau}e^{-2(t-\tau)}u(\tau)u(t-\tau)\right)d\tau$$

$$= u(t)\left(\int_0^t e^{-2t}d\tau - \int_0^t e^{-2t}e^{-\tau}d\tau\right) = \left(te^{-2t} + e^{-3t} - e^{-2t}\right)u(t)$$

12. (10) Consider the system shown in Fig. P13(b) and the spectrum $X(j\omega)$ of an input $x(t)$ is shown in Fig. P13(a).

Sketch

(a) (6 pt) the spectrum $F(j\omega)$ of the output $f(t)$,

(b) (2 pt) the spectrum of the equivalent filter $H_{eq}(j\omega)$ from $x(t)$ to $f(t)$, and

(c) (2 pt) the spectrum of the real part of the output, i.e., $\mathrm{Re}\{f(t)\}$.

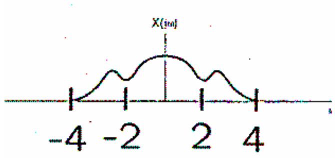


Fig. P13(a)

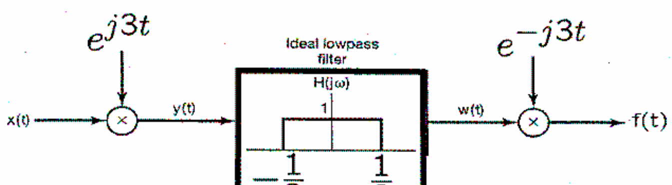
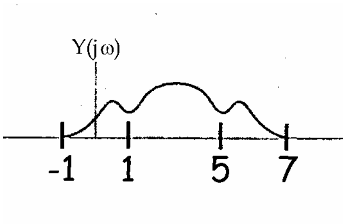


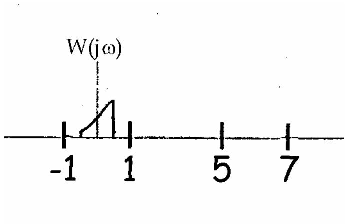
Fig. P13(b)

Solution:

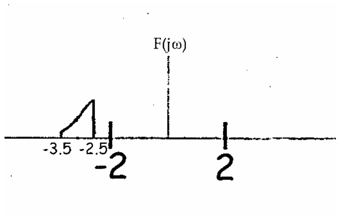
(a) We know that $x(t) \xleftrightarrow{CTFT} X(j\omega) \Rightarrow x(t)e^{j\omega_0 t} \xleftrightarrow{CTFT} X(j(\omega-\omega_0))$. Therefore:



$Y(j\omega)$

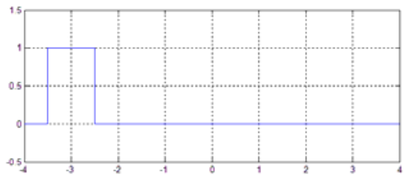


$W(j\omega)$

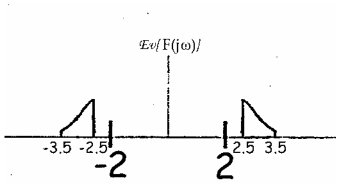


$F(j\omega)$

(b) By $H_{eq}(j\omega) = \dfrac{F(j\omega)}{X(j\omega)}$, we may sketch the spectrum as



(c) There is duality in Fourier transform therefore $\mathrm{Re}\{f(t)\} \xleftrightarrow{CTFT} \mathrm{Ev}\{F(j\omega)\}$. The spectrum is



13. (8) Given a discrete-time sequence $x[n]$ and assume that the Fourier transform of $x[n]$ is $X(e^{j\omega})$. If the signal $x_{(m)}[n]$ is obtained from $x[n]$ by inserting $m-1$ zeros between successive values of the original, derive the Fourier transform $X_{(m)}(e^{j\omega})$ of $x_{(m)}[n]$ in terms of $X(e^{j\omega})$.

Solution:

We know that $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}$, and then

$$X_{(m)}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(m)}[n]e^{j\omega n} = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{m-1} \underbrace{x_{(m)}[km+l]}_{0 \text{ if } l\neq 0} e^{j\omega(km+l)} = \sum_{k=-\infty}^{\infty} x_{(m)}[km]e^{j\omega km} = \sum_{k=-\infty}^{\infty} x[k]e^{jm\omega k} = X(e^{jm\omega})$$

14. (12) Consider a causal LTI system characterized by the difference equation:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

Find (a) the impulse response $h[n]$ of the system (6 pt)

and (b) the output $y[n]$ of the system if the input is $x[n] = \left(\frac{1}{3}\right)^n u[n]$. (6 pt)

Solution:

We have

$$Y(e^{j\omega})-\frac{3}{4}e^{-j\omega}Y(e^{j\omega})+\frac{1}{8}e^{-j2\omega}Y(e^{j\omega})=2X(e^{j\omega})$$

$$H(e^{j\omega})=\frac{Y(e^{j\omega})}{X(e^{j\omega})}=\frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}(e^{-j\omega})^2}=\frac{2}{(1-\frac{1}{4}e^{-j\omega})(1-\frac{1}{2}e^{-j\omega})}=\frac{4}{(1-\frac{1}{2}e^{-j\omega})}-\frac{2}{(1-\frac{1}{4}e^{-j\omega})}$$

Therefore

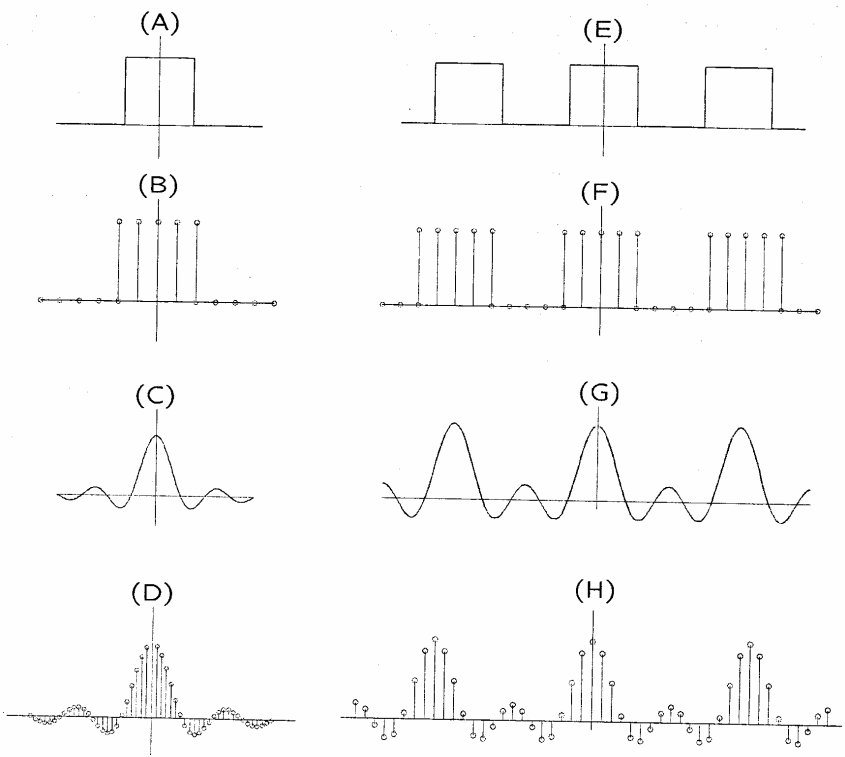
$$h[n]=\left(4\left(\frac{1}{2}\right)^n-2\left(\frac{1}{4}\right)^n\right)u[n]$$

$$\begin{aligned}y[n]&=\sum_{k=-\infty}^{\infty}h[k]x[n-k]=\sum_{k=-\infty}^{\infty}\left(4\left(\frac{1}{2}\right)^k-2\left(\frac{1}{4}\right)^k\right)\left(\frac{1}{3}\right)^{n-k}u[k]u[n-k]\\&=u[n]\left(4\sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(\frac{1}{3}\right)^{n-k}-2\sum_{k=0}^{n}\left(\frac{1}{4}\right)^k\left(\frac{1}{3}\right)^{n-k}\right)=u[n]\left(4\cdot 6\cdot\left(\left(\frac{1}{2}\right)^{n+1}-\left(\frac{1}{3}\right)^{n+1}\right)-2\cdot 12\left(\left(\frac{1}{3}\right)^{n+1}-\left(\frac{1}{4}\right)^{n+1}\right)\right)\\&=\left(12\left(\frac{1}{2}\right)^n-16\left(\frac{1}{3}\right)^n+6\left(\frac{1}{4}\right)^n\right)u[n]\end{aligned}$$

15. (10) Use the following plots (A), (B), …, (H) to pair the following relationships (a), (b), …, (e). Note that the actual numeral values on these plots are not important. Provide your answers based on their characteristics only. If you think the answer is none of them, fill out "None". Notations: 'FS': Fourier Series, 'FT': Fourier Transform, 'CT': Continuous-Time, and 'DT': Discrete-Time.

(a) $(A)\xleftrightarrow{CTFT}?$ (b) $(C)\xleftrightarrow{CTFT}?$ (c) $(D)\xleftrightarrow{DTFT}?$

(d) $(E)\xleftrightarrow{CTFS}?$ (e) $(H)\xleftrightarrow{DTFS}?$



Solution:

(a) (C) (b) (A) (c) (E) (d) (D) (e) (F)