Linear Algebra Final Exam

You may NOT use any automatic computing device such as calculators or computers.

1. [15 % = 4 + 3 + 2 + 3+3]

Consider the vector space C[-1, 1], which is the set of all continuous functions defined in [-1, 1], with vector sum $f+g$ defined by $(f+g)(t) = f(t) + g(t)$ and scalar multiple cf defined by $(cf)(t) = c(f(t))$, where c is a scalar. Let $B_1 = \{1\}$ and $W_1 = \text{Span } B_1$, let $B_2 = \{1, x\}$, $W_2 = \text{Span } B_2$, and let $B_3 = \{1, x, |x|\}$ and $W_3 = \text{Span } B_3$.

(a) Show that W_1 and W_2 are subspaces of W_3.

(b) Please find the dimensions of W_1, W_2 and W_3.

(c) What is the zero vector in W_3?

(d) Show that B_2 is a basis for W_2.

(e) Is B_3 a basis for W_3? Please explain.

2. [18 % = 4 + 4 + 4 + 6]

Let A be a diagonalizable $n\times n$ matrix. Assume that A has one set of orthonormal eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$, $\mathbf{v}_i \in \mathcal{R}^n$, corresponding to the following set of eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_n\}$, $\lambda_i \in \mathcal{R}$. Suppose that $a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0$ is the characteristic polynomial of A, that is, $a_n \lambda_i^n + a_{n-1} \lambda_i^{n-1} + \ldots + a_1 \lambda_i + a_0 = 0$. Please answer the following questions.

(a) Please derive one set of eigenvectors and the corresponding eigenvalues of $cA + bI_n$ in terms of $\mathbf{v}_i$, λ_i, c, b, etc., where c, b are nonzero scalars and $I_n \in \mathcal{R}^{n\times n}$ is the identity matrix;

(b) Show that A^k is diagonalizable, where k is any positive integer;

(c) Please derive one set of eigenvectors and the corresponding eigenvalues of A^k in terms of $\mathbf{v}_i$, λ_i, k, etc.; and

(d) Show that $a_n A^n + a_{n-1} A^{n-1} + \ldots + a_1 A + a_0 I_n = O$, where O is the zero matrix.

3. [12 % = 4 + 4 + 4]

Let T be a linear operator on $\mathcal{R}^n$ and B = $\{b_1, b_2, ..., b_n\}$ and C = $\{c_1, c_2, ..., c_n\}$ be two distinct sets of orthonormal bases for $\mathcal{R}^n$. Furthermore, let the B-matrix and C-matrix of T be $A_B = [T]_B$ and $A_C = [T]_C$, respectively. Please answer the following questions.

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(a) Please prove that A_B and A_C are similar, i.e., there exists an invertible matrix P such that $A_B = P^{-1}A_C P$.

If you cannot prove (a), you can still use the result of (a) to answer the following questions.

(b) Please show that A_B and A_C have the same eigenvalues.

(c) Please derive the relationship between the eigenvectors of A_B and the eigenvectors of A_C.

4. [7 % = 2 + 5]

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ be linearly independent vectors in $\mathcal{R}^n$, where $1 \le k \le n$. Furthermore, let

$$A = I_n - 2C(C^TC)^{-1}C^T,$$

where C is an $n \times k$ matrix given by $C = [\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_k]$.

(a) Show that A is an orthogonal matrix.

(b) Find the eigenvalues of A and describe the corresponding eigenspaces.

5. [5 %]

Let S be a nonempty subset of $\mathcal{R}^n$. Prove that $(S^\perp)^\perp = \text{Span } S$.

6. [10 %]

Let A be an $n \times n$ symmetric matrix with rank r, and suppose that the sum of the multiplicities of the positive eigenvalues of A is α. Prove that there exists an $n \times r$ matrix G such that $A = GJG^T$, where J is an $r \times r$ diagonal matrix whose diagonal entries are given by

$$j_{ii} = \begin{cases} 1, & i = 1, 2, \cdots, \alpha \\ -1, & i = \alpha + 1, \alpha + 2, \cdots, r \end{cases}.$$

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7. [10 %]

Let V_1, V_2, W_1, and W_2 be finite dimensional vector paces such that V_1 is isomorphic to V_2 and W_1 is isomorphic to W_2. Let $\mathcal{L}(V_1,W_1)$ and $\mathcal{L}(V_2,W_2)$ respectively be the vector spaces of all linear transformations from V_1 to W_1 and from V_2 to W_2 under the operations of addition of linear transformations and product of a linear transformation by a scalar. Prove that $\mathcal{L}(V_1,W_1)$ is isomorphic to $\mathcal{L}(V_2,W_2)$.

8. [23 % = 5 + 5 + 5 + 4 + 4]

Let

$$W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

and

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(a) Find an orthonormal basis for W.

(b) Find the orthogonal projection of **v** onto W.

(c) Find the orthogonal projection of **v** onto $W^{\perp}$.

(d) Use the result of (a) and (c) to form a basis $B = \{$ **b**1, **b**2, **b**3 $\}$ for R^3. What is $[\mathrm{v}]_B$, the B-coordinate vector of **v** ?

(e) Let T be a linear transformation on $\mathcal{R}^3$, with $[T(\mathbf{b}1)]_B = [0, 0, 0]^T$, $[T(\mathbf{b}2)]_B = [0,0,0]^T$,$[T(\mathrm{b}3)]_B = [0, 0, -1]^T$Please find $[T(\mathbf{v})]_B$.