Signals and Systems Final

10:20a.m. ~ 12:20p.m., June 19, Fri., 2009

- Closed book, but open 1 sheet (both sides, 2 pages) of personal notes of A4 size
- Total score: 110
- Total 4 pages in one B4 sheet

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1. **[16]** Consider that a continuous-time signal $x(t)$ with its Laplace transform given by

$$X(s) = \frac{(s+3)}{(s+1)(s+2)}$$

   (a) Find all possible inverse bilateral Laplace transforms and justify your answer.

   (b) Sketch the region of convergence (ROC) in each case.

   (c) Determine each time function as causal, anticausal, or two-sided and justify your answer.

   (d) Determine each time function as bound-input and bound-output (BIBO) stable or not BIBO stable and justify your answer.

2. **[16]** Consider that a linear time-invariant (LTI) system $h(t)$ has its transfer function $H(z)$ given by

$$H(z) = \frac{\sum_{i=0}^{M} b_i z^{-i}}{\sum_{i=0}^{N} a_i z^{-i}}$$

   Let $H_i(z)$ be the inverse system of $H(z)$.

   (a) Determine the constraint on the orders M and N if the LTI system $H(z)$ is causal and justify your answer.

   (b) Determine the constraint on the orders M and N if the inverse system is also causal and justify your answer.

   (c) Determine the constraint on the poles of $H(z)$ if $H(z)$ is stable and justify your answer.

   (d) Determine the constraint on the poles and zeros of $H(z)$ if $H_i(z)$ is also stable and justify your answer.

3. **[8]** Consider a causal LTI system specified by the following linear constant coefficient equation (LCCDE) with input $x(t)$ and output $y(t)$:

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

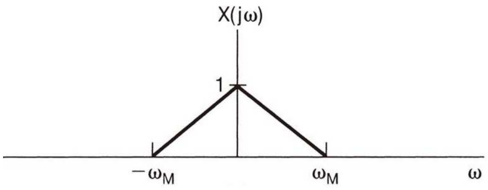
   (a) Sketch the Bode plot of the frequency response of the LTI system.

   (b) Determine the group delay of the LTI system.

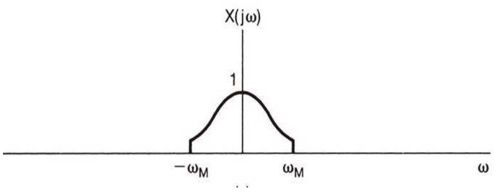
4. **[8]** Consider two message signals $x_1(t)$ and $x_2(t)$ with frequency spectra shown by Figures (a) and (b), respectively. We form the modulated signal $\Phi(t) = x_1(t)\cos\omega_c t + x_2(t)\sin\omega_c t$ for transmission, where $\omega_M \ll \omega_c$. In the receiver, we generate two signals $v_1(t)$ and $v_2(t)$ as

$$v_1(t) = \Phi(t)\cos\omega_c t \quad \text{and} \quad v_2(t) = \Phi(t)\sin\omega_c t,$$

respectively. Finally, $v_1(t)$ and $v_2(t)$ are filtered respectively by an ideal low-pass filter with cutoff frequency equal to $2\omega_M$ and unit amplitude to provide the output signals $y_1(t)$ and $y_2(t)$.



(a) $X_1(j\omega)$



(b) $X_2(j\omega)$

(a) Determine $v_1(t)$ and sketch its frequency spectrum.

(b) Determine $v_2(t)$ and sketch its frequency spectrum.

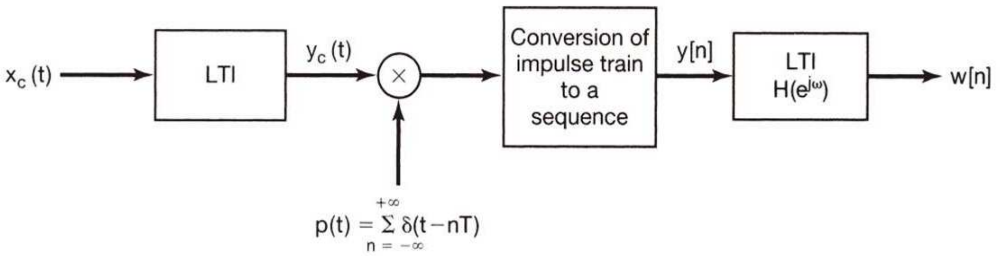
(c) Determine $y_1(t)$ and sketch its frequency spectrum.

(d) Determine $y_2(t)$ and sketch its frequency spectrum.

5. **[12]** Consider the system depicted below. Assume that the continuous-time LTI system is causal and specified by the LCCDE

$$\frac{dy_c(t)}{dt} + y_c(t) = x_c(t)$$

Now, we input a unit impulse to the system, i.e., $x_c(t) = \delta(t)$.



(a) Determine the output $y_c(t)$ of the continuous-time LTI system.

(b) Determine the frequency spectrum of $y[n]$.

(c) Determine the unit impulse response $h[n]$ of the discrete-time LTI system $H(e^{j\omega})$ such that $w[n] = \delta[n]$.

NOTE: There are problems in the back.

6. **[6]** For a real discrete-time sequence $x[n]$, prove that if $X(z)$, the z-transform of $x[n]$, has a pole (or zero) at $z = z_0$, then it must also have a pole (or zero) at $z = z_0^*$, where z_0^* denotes the complex conjugate of z_0.

7. **[6]** Denote the unilateral z-transform of a discrete-time sequence $x[n]$ by $\mathcal{X}(z)$. Prove that the unilateral z-transform of $x[n+1]$ is $z\mathcal{X}(z) - zx[0]$.

8. **[10]** Prove the initial-value theorem of Laplace transform. That is, if a continuous-time signal $x(t) = 0$ for $t < 0$ and contains no impulse or higher order singularities at the origin, then

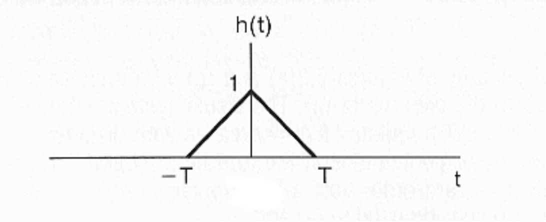
$$x(0^+) = \lim_{s\to\infty} sX(s).$$

(Hint: Make use of the Taylor series expansion of $x(t)$ at $t = 0^+$ and the Laplace transform pair $e^{-at}\left(\frac{t^n}{n!}\right)u(t) \stackrel{L}{\longleftrightarrow} \frac{1}{(s+a)^{n+1}}$, $\mathrm{Re}\{s\} > -a$.)

9. **[12]** Consider the case of frequency modulation where the modulated signal is expressed as $y(t) = \cos(\omega_c t + m\sin\omega_m t) = \cos(\omega_c t)\cos(m\sin\omega_m t) - \sin(\omega_c t)\sin(m\sin\omega_m t)$ with ω_c being the frequency of the carrier, ω_m the frequency of the modulating signal, and m the modulation index.
   (a) When m is sufficiently small (say, much smaller than $\pi/2$), the modulated signal can be approximated by $y(t) = \cos(\omega_c t) - m(\sin\omega_m t)(\sin\omega_c t)$. Sketch the spectrum of the approximated signal.
   (b) When m is large, the approximation shown above no longer applies. But it can be noted that the terms $\cos(m\sin\omega_m t)$ and $\sin(m\sin\omega_m t)$ represent periodic signals. What are the fundamental periods of these two periodic signals? Justify your answer.
   (c) Explain how the spectrum of $y(t)$ should look like for the case where m is large.

10. **[8]** Suppose $H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\sphericalangle H(e^{j\omega})}$ is the frequency response of a discrete-time system. The response of the system to the input $x[n] = \sin(\omega_0 n + \theta_0)$ would be of this form $y[n] = \left|H(e^{j\omega_0})\right| x[n - n_0]$ if $\sphericalangle H(e^{j\omega_0})$ is related to ω_0 in a certain way. Determine this relation.

11. **[8]** Show that, in terms of reconstruction of a signal from its samples, the linear interpolation method is equivalent to impulse-train sampling followed by convolution with a triangular impulse response $h(t)$ shown in the following figure:



where T is the sampling period.