Electromagnetics II final exam

date: 2013/01/07

姓名：________________

學號：________________

注意事項：1. 題目卷(第一頁)，以及「Smith chart」，皆要填寫考生姓名與學號；
2. 考試完畢，請將題目卷、Smith chart、答案卷一併繳回，未繳回的部份，不予計分；
3. 題目一共六題，每題 20 分，採計個人最高分之五題。

1. (20%) For a transmission line of characteristic impedance 50 Ω, terminated by a load impedance (100 + j100) Ω, find the following quantities by using the provided Smith chart:
   (1) reflection coefficient at the load;
   (2) SWR on the line;
   (3) the distance of the first voltage minimum of the standing-wave pattern from the load;
   (4) the line impedance at d = 0.125 λ
   (5) the line admittance at d = 0.125 λ and
   (6) the location nearest to the load at which the real part of the line admittance is equal to the line characteristic admittance.
   (Make sure you obtain the answers by using the Smith chart)

(*Solution*)

(1) $\Gamma_R = 0.62\angle 29.7^o$

(2) SWR = 4.2656

(3) 0.29 λ

(4) (0.8 − j1.4) ×50=40 − j70

(5) (0.3 + j0.54) /50=0.006 + j0.0108

(6) 0.22 λ

2. (20%) The input impedance of a lossy line of length 100 m is measured at a frequency of 100 MHz for two cases: with the output short-circuited, it is (36 + j48) Ω, and with the output open circuited, it is (36 – j48) Ω. Find:

(1) (5%) the characteristic impedance of the line;

(2) (5%) the attenuation constant of the line; and

(3) (10%) the phase velocity in the line, assuming its approximate value to be 2 × 10^8 m/s.

(*Solution*)

(1) (5%) $Z_o = \sqrt{\overline{Z}_{in}^{s}\overline{Z}_{in}^{o}} = \sqrt{(36+j48)(36-j48)} = \sqrt{3600} = 60\ \Omega$

$where:$

$\overline{Z}_{in}^{s}$ = Input impedance of short-end transmission line.

$\overline{Z}_{in}^{o}$ = Input impedance of open-end transmission line.

(2) (5%) $\tanh\left(\overline{\gamma}\ell\right) = \sqrt{\dfrac{\overline{Z}_{in}^{s}}{\overline{Z}_{in}^{o}}} = \sqrt{\dfrac{36+j48}{36-j48}} = \sqrt{\dfrac{60\angle 53^o}{60\angle -53^o}} = 1\angle 53^o = 0.6+j0.8$

$$\overline{\gamma}\ell = \frac{1}{2}\ln\left(\frac{1.6+j0.8}{0.4-j0.8}\right) = \frac{1}{2}\ln\left(\frac{1.789\angle 26.565}{0.894\angle -63.435}\right) = \frac{1}{2}\ln\left(2\angle 90^o\right)$$

$$= \frac{1}{2}\ln\left(2e^{j\left(\frac{\pi}{2}+2n\pi\right)}\right) = \frac{1}{2}\ln(2) + \frac{1}{2}\ln\left(e^{j\left(\frac{\pi}{2}+2n\pi\right)}\right)$$

$$= \frac{1}{2}\ln(2) + j\left(\frac{\pi}{4}+n\pi\right) \quad n = 0,\ 1,\ 2,\ \ldots$$

$$\alpha\ell = \frac{1}{2}\ln(2) \rightarrow \alpha = 3.466\times 10^{-3}\ (\mathrm{Np/m})$$

(3) (10%) $\beta = \dfrac{1}{\ell}\left(\dfrac{\pi}{4}+n\pi\right) = \dfrac{\omega}{v_p} = \dfrac{2\pi\times 100\times 10^6}{2\times 10^8} = \dfrac{\pi\times 10^8}{10^8} = \pi$

$$\rightarrow \frac{\pi}{4}+n\pi = \ell\pi = 100\pi \rightarrow n = \frac{1}{\pi}\left(100\pi - \frac{\pi}{4}\right) = 99.75$$

$$n = 99 \Rightarrow v_p = \frac{\omega}{\beta} = \frac{2\pi\times 100\times 10^6}{\frac{1}{100}\left(\frac{\pi}{4}+99\pi\right)} = \frac{2\pi\times 100\times 10^8}{99.25\pi} = \frac{2\times 100\times 10^8}{99.25} = 2.015\times 10^8\ (m/s)$$

$$n = 100 \Rightarrow v_p = \frac{\omega}{\beta} = \frac{2\pi\times 100\times 10^6}{\frac{1}{100}\left(\frac{\pi}{4}+100\pi\right)} = \frac{2\pi\times 100\times 10^8}{100.25\pi} = \frac{2\times 100\times 10^8}{100.25} = 1.995\times 10^8\ (m/s)$$

The latter one is closer to the given value. Therefore, n is 100, and phase velocity is 1.995×10^8 (m/s).

3. (20 %) Consider a parallel-plate waveguide propagating waves in the $\hat{z}$ direction and filled with a dielectric medium (μ_0, ε_0) for $z < 0$ and (μ_0, $3\varepsilon_0$) for $z > 0$. The plate separation is $2\sqrt{3}\pi\,(cm)$, and the operating frequency is 30/2π GHz.

(1) (5 %) Consider the $z < 0$ region, i.e. the waveguide in the absence of the dielectric, which TE_m and TM_m modes can propagate in this waveguide?

(2) (5 %) Find the oblique incident angles θ_i corresponding to each propagating modes in (1).

(3) (5 %) What are the phase and group velocities in the $\hat{z}$ direction for the TM_2 mode at the $z < 0$ region and the operating frequency 30/2π GHz?

(4) (5 %) For waves propagating in the $-\hat{z}$ direction from $z > 0$ region, for which TE and TM modes be totally reflected at the dielectric boundary?

(*Solution*) 改分原則：可以參考解答中之分配比例給分，亦或是以一個地方扣一分(或扣等份的分數)之方式計分。

(1) (5 %) Consider the $z < 0$ region, i.e. the waveguide in the absence of the dielectric, which TE_m and TM_m modes can propagate in this waveguide?

$$f_c = \frac{m}{2a\sqrt{\mu\varepsilon}} = \frac{m}{2\times 2\sqrt{3}\pi\times 10^{-2}\times\sqrt{\mu_0\varepsilon_0}} = \left.\frac{mc}{4\pi\sqrt{3}\times 10^{-2}}\right|_{c=3\times 10^8} = \frac{\left(\sqrt{3}\right)m}{4\pi}\times 10^{10}\,Hz \rightarrow (+1\%)$$

If the operating frequency is 30/2π GHz, then

$$m=1,\ f_c = \frac{\left(\sqrt{3}\right)}{4\pi}\times 10^{10}\,Hz < f = \frac{3}{2\pi}\times 10^{10}\,Hz;\ m=2,\ f_c = \frac{\sqrt{3}}{2\pi}\times 10^{10}\,Hz < f = \frac{3}{2\pi}\times 10^{10}\,Hz$$

$$m=3,\ f_c = \frac{3\sqrt{3}}{4\pi}\times 10^{10}\,Hz < f = \frac{3}{2\pi}\times 10^{10}\,Hz;\ m=2,\ f_c = \frac{\sqrt{3}}{\pi}\times 10^{10}\,Hz > f = \frac{3}{2\pi}\times 10^{10}\,Hz$$

$\rightarrow$ Propagation mode: $m = 1,2,3 \rightarrow (+1\%)$

<u>Answer:</u> (TE_1, TE_2, TE_3), (TM_0, TM_1, TM_2, TM_3) can propagate. →(+3%)

(Note: There is no TE_0 mode existing in the parallel plate waveguide.)

(2) (5 %) Find the oblique incident angles θ_i corresponding to each propagating modes in (1).

<u>Answer:</u> $$\theta_i = \cos^{-1}\left(\frac{f_c}{f}\right) = \cos^{-1}\left(\frac{\frac{\left(\sqrt{3}\right)m}{4\pi}\times 10^{10}}{\frac{3}{2\pi}\times 10^{10}}\right) = \cos^{-1}\left(\frac{m}{2\sqrt{3}}\right), \rightarrow \frac{f_c}{f} = \frac{m}{2\sqrt{3}} \rightarrow (+1\%)$$

Propagation Mode	θ_i
TM_0	$90^{\circ} \rightarrow (+1\%)$
TE_1, TM_1	$\theta_i = \cos^{-1}\left(\frac{1}{2\sqrt{3}}\right) = 1.278 \rightarrow (+1\%)$
TE_2, TM_2	$\theta_i = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 0.9553 \rightarrow (+1\%)$
TE_3, TM_3	$\theta_i = \cos^{-1}\left(\frac{3}{2\sqrt{3}}\right) = 0.5236 \rightarrow (+1\%)$

(3) (5 %) What are the phase and group velocities in the $\hat{z}$ direction for the TM_2 mode at the $z < 0$ region and the operating frequency $30/2\pi$ GHz?

<u>*Answer*:</u>

$$v_{pz} = \left.\frac{v_p}{\sqrt{1-\left(f_c/f\right)^2}}\right|_{\begin{pmatrix} TM_2 \text{ mode} \\ v_p = c = 3\times10^8 \\ \frac{f_c}{f} = \frac{m}{2\sqrt{3}}, m=2 \end{pmatrix}} \rightarrow (+1\%) = \frac{3\times10^8}{\sqrt{1-\left(\frac{1}{\sqrt{3}}\right)^2}} = \frac{3\times10^8}{\sqrt{\frac{2}{3}}} = 3.67\times10^8\ m/s \rightarrow (+2\%)$$

$$v_g = v_p\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 3\times10^8 \times \sqrt{\frac{2}{3}} = 2.45\times10^8. \rightarrow (+2\%)$$

(4) (5 %) For waves propagating in the $-\hat{z}$ direction from $z > 0$ region, for which TE and TM modes be totally reflected at the dielectric boundary?

Consider the $z < 0$ region,

$z < 0$ —|—|—|—|→ f

$f(\mathrm{TM}_0)$ $f(TE_1)$ $f(TE_2)$ $f(TE_3)$

$f(\mathrm{TM}_1)$ $f(\mathrm{TM}_2)$ $f(\mathrm{TM}_3)$

$\rightarrow (+1\%)$

Consider the $z > 0$ region,

$z > 0$ —|—|—|—|—|—|—|→ f

$f(\mathrm{TM}_0)$ $f(TE_1)$ $f(TE_2)$ $f(TE_3)$ $f(TE_4)$ $f(TE_5)$ $f(TE_6)$

$f(\mathrm{TM}_1)$ $f(\mathrm{TM}_2)$ $f(\mathrm{TM}_3)$ $f(\mathrm{TM}_4)$ $f(\mathrm{TM}_5)$ $f(\mathrm{TM}_6)$

$\rightarrow (+1\%)$

<u>*Answer*:</u>

(TE_4, TM_4), (TE_5, TM_5), (TE_6, TM_6) are reflected. $\rightarrow (+3\%)$

4. (20 %) The electric field inside a certain dielectric-slab waveguide (where $\varepsilon = 9\varepsilon_0$, $\mu = \mu_0$ and thickness 3 m) is $\bar{E}(\bar{r}) = \hat{y}E_0(\sin x)e^{-jz}$ (V/m).

(1) (5 %) Is this a TE or TM mode, why?

(2) (5 %) What is the waveguide wavelength λ_z in meters?

(3) (5 %) What is ω in radians/s?

(4) (5 %) Find the effective thickness d_{eff} of this dielectric-slab waveguide for this mode.

(*Solution*)

(1) (5 %) Is this a TE or TM mode, why?

Answer:

This is a TE mode because there is no $\bar{E}(\bar{r})$ in the propagation direction. $\left(i.e.,\ \bar{E}(\bar{r}) = \hat{y}E_y \text{ only}\right)$

(2) (5 %) What is the waveguide wavelength λ_z in meters?

Answer:

$$\bar{E}(\bar{r}) = \hat{y}E_0(\sin x)e^{-jz}\ (V/m) \rightarrow \beta_z = 1m, \rightarrow \lambda_z = \frac{2\pi}{\beta_z} = 2\pi(m)$$

(3) (5 %) What is ω in radians/s?

Answer:

$$\bar{E}(\bar{r}) = \hat{y}E_0(\sin x)e^{-jz}\ (V/m) \rightarrow \beta_x = 1m,$$

$$\because \beta_x^2 + \beta_z^2 = \omega^2\mu_0\varepsilon_2\big|_{\varepsilon_2 = 9\varepsilon_0} = 9\omega^2\mu_0\varepsilon_0$$

$$\rightarrow \omega = \sqrt{\frac{\beta_x^2 + \beta_z^2}{9\mu_0\varepsilon_0}} = \frac{\sqrt{2}}{3}\frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{\sqrt{2}}{3}\times 3\times 10^8 = 1.414\times 10^8\,(rad/s)$$

(4) (5 %) Find the effective thickness d_{eff} of this dielectric-slab waveguide for this mode.

Answer:

$$\beta_{x1}^2 + \beta_z^2 = \omega^2\mu_0\varepsilon_1 \rightarrow -\alpha_{x2}^2 + \beta_z^2 = \omega^2\mu_0\varepsilon_2\big|_{\varepsilon_2 = \varepsilon_0} = \omega^2\mu_0\varepsilon_0$$

$$\rightarrow \alpha_{x2}^2 = \beta_z^2 - \omega^2\mu_0\varepsilon_0 = 1 - \left(1.414\times 10^8\right)^2 \times \frac{1}{9}\times 10^{-16} = 1 - (1.414)^2\frac{1}{9} = 1 - \frac{\left(\sqrt{2}\right)^2}{9} = \frac{7}{9}$$

$$\rightarrow \alpha_{x2} = \frac{\sqrt{7}}{3}$$

$$d_{eff} = d + \frac{2}{\alpha_{x2}} = 3 + \frac{2\times 3}{\sqrt{7}} = 5.27\,(m)$$

where:

$$\beta_{x1}^2 + \beta_z^2 = \omega^2\mu_0\varepsilon_2\big|_{\varepsilon_2 = 9\varepsilon_0} = 9\omega^2\mu_0\varepsilon_0 \rightarrow \omega = \sqrt{\frac{\beta_x^2 + \beta_z^2}{9\mu_0\varepsilon_0}} = 1.414\times 10^8\,(rad/s)$$

5. (20 %) Figure 3 shows a rectangular waveguide of height a and width b. Consider a wave guided in the z direction, with z-dependence of $e^{-j\beta_z z}$.

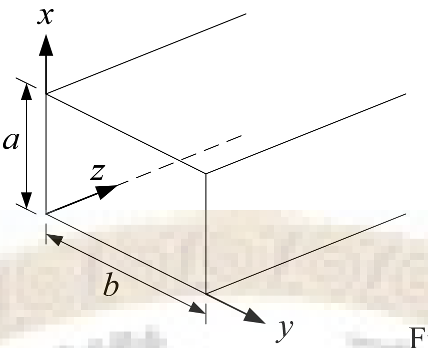

Fig. 5

(1) (5%) For TE modes with $E_z = 0$, $H_z \neq 0$, derive E_x, E_y, H_x and H_y in terms of H_z from the Faraday's law and the Ampere's law.

<u>*Answer*</u>:

We conclude that E_x, E_y, H_x and H_y in terms of $\bar{H}_z$ are shown as following, $\rightarrow (+1\%)$

$$\begin{cases} \bar{E}_x = \dfrac{j\omega\mu}{\beta_z^2 - \beta^2}\dfrac{\partial \bar{H}_z}{\partial y} \rightarrow (+1\%) \\ \bar{E}_y = -\dfrac{j\omega\mu}{\beta_z^2 - \beta^2}\dfrac{\partial \bar{H}_z}{\partial x} \rightarrow (+1\%) \end{cases} \quad \begin{cases} \bar{H}_x = j\dfrac{\beta_z}{\beta_z^2 - \beta^2}\dfrac{\partial \bar{H}_z}{\partial x} \rightarrow (+1\%) \\ \bar{H}_y = j\dfrac{\beta_z}{\beta_z^2 - \beta^2}\dfrac{\partial \bar{H}_z}{\partial y} \rightarrow (+1\%) \end{cases}$$

$(Derivation)$

$$\text{Faraday's law:} \nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \rightarrow \text{Phasor form:} \nabla \times \bar{E} = -j\omega\mu\bar{H} \rightarrow \begin{vmatrix} \hat{a}_x & -\hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \bar{E}_x & \bar{E}_y & \bar{E}_z \end{vmatrix} = -j\omega\mu\bar{H}$$

$$\hat{a}_x\left(\frac{\partial \bar{E}_z}{\partial y} - \frac{\partial \bar{E}_y}{\partial z}\right) + \hat{a}_y\left(\frac{\partial \bar{E}_x}{\partial z} - \frac{\partial \bar{E}_z}{\partial x}\right) + \hat{a}_z\left(\frac{\partial \bar{E}_y}{\partial x} - \frac{\partial \bar{E}_x}{\partial y}\right) = -j\omega\mu\left(\hat{a}_x\bar{H}_x + \hat{a}_y\bar{H}_y + \hat{a}_z\bar{H}_z\right)$$

$$\text{Ampere's law:} \nabla \times \bar{H} = \frac{\partial D}{\partial t} \rightarrow \text{Phasor form:} \nabla \times \bar{H} = j\omega\varepsilon\bar{E} \rightarrow \begin{vmatrix} \hat{a}_x & -\hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \bar{H}_x & \bar{H}_y & \bar{H}_z \end{vmatrix} = j\omega\varepsilon\bar{E}$$

$$\hat{a}_x\left(\frac{\partial \bar{H}_z}{\partial y} - \frac{\partial \bar{H}_y}{\partial z}\right) + \hat{a}_y\left(\frac{\partial \bar{H}_x}{\partial z} - \frac{\partial \bar{H}_z}{\partial x}\right) + \hat{a}_z\left(\frac{\partial \bar{H}_y}{\partial x} - \frac{\partial \bar{H}_x}{\partial y}\right) = j\omega\varepsilon\left(\hat{a}_x\bar{E}_x + \hat{a}_y\bar{E}_y + \hat{a}_z\bar{E}_z\right)$$

$\bar{E}_z = 0,\ \bar{E}_y = E_y e^{-j\beta_z z}$:

$$\begin{cases} -\dfrac{\partial \bar{E}_y}{\partial z} = -j\omega\mu\bar{H}_x \rightarrow j\beta_z\bar{E}_y = -j\omega\mu\bar{H}_x \quad (1.1) \\ \dfrac{\partial \bar{E}_x}{\partial z} = -j\omega\mu\bar{H}_y \rightarrow j\beta_z\bar{E}_x = j\omega\mu\bar{H}_y \quad (1.2) \\ \dfrac{\partial \bar{E}_y}{\partial x} - \dfrac{\partial \bar{E}_x}{\partial y} = -j\omega\mu\bar{H}_z \quad (1.3) \end{cases} \quad \begin{cases} \dfrac{\partial \bar{H}_z}{\partial y} - \dfrac{\partial \bar{H}_y}{\partial z} = j\omega\varepsilon\bar{E}_x \rightarrow \dfrac{\partial \bar{H}_z}{\partial y} + j\beta_z\bar{H}_y = j\omega\varepsilon\bar{E}_x \quad (1.4) \\ \dfrac{\partial \bar{H}_x}{\partial z} - \dfrac{\partial \bar{H}_z}{\partial x} = j\omega\varepsilon\bar{E}_y \rightarrow -j\beta_z\bar{H}_x - \dfrac{\partial \bar{H}_z}{\partial x} = j\omega\varepsilon\bar{E}_y \quad (1.5) \\ \dfrac{\partial \bar{H}_y}{\partial x} - \dfrac{\partial \bar{H}_x}{\partial y} = j\omega\varepsilon\bar{E}_z \rightarrow \dfrac{\partial \bar{H}_y}{\partial x} - \dfrac{\partial \bar{H}_x}{\partial y} = 0 \quad (1.6) \end{cases}$$

$$\begin{cases}(1.4)\rightarrow \bar{H}_y=\dfrac{1}{j\beta_z}\left(j\omega\varepsilon\bar{E}_x-\dfrac{\partial\bar{H}_z}{\partial y}\right)\\(1.2)\rightarrow j\beta_z\bar{E}_x=j\omega\mu\bar{H}_y=\dfrac{j\omega\mu}{j\beta_z}\left(j\omega\varepsilon\bar{E}_x-\dfrac{\partial\bar{H}_z}{\partial y}\right)\rightarrow -\beta_z^2\bar{E}_x=j\omega\mu\left(j\omega\varepsilon\bar{E}_x-\dfrac{\partial\bar{H}_z}{\partial y}\right)\\\rightarrow -\beta_z^2\bar{E}_x+\omega^2\mu\varepsilon\bar{E}_x=-j\omega\mu\left(\dfrac{\partial\bar{H}_z}{\partial y}\right)\rightarrow\left(\beta_z^2-\beta^2\right)\bar{E}_x=j\omega\mu\left(\dfrac{\partial\bar{H}_z}{\partial y}\right)\rightarrow\bar{E}_x=\dfrac{j\omega\mu}{\beta_z^2-\beta^2}\dfrac{\partial\bar{H}_z}{\partial y}\end{cases}$$

$$\begin{cases}-\dfrac{\partial\bar{E}_y}{\partial z}=-j\omega\mu\bar{H}_x\rightarrow j\beta_z\bar{E}_y=-j\omega\mu\bar{H}_x\ \ (1.1)\\\dfrac{\partial\bar{E}_x}{\partial z}=-j\omega\mu\bar{H}_y\rightarrow j\beta_z\bar{E}_x=j\omega\mu\bar{H}_y\ \ (1.2)\\\dfrac{\partial\bar{E}_y}{\partial x}-\dfrac{\partial\bar{E}_x}{\partial y}=-j\omega\mu\bar{H}_z\ \ (1.3)\end{cases}\quad\begin{cases}\dfrac{\partial\bar{H}_z}{\partial y}-\dfrac{\partial\bar{H}_y}{\partial z}=j\omega\varepsilon\bar{E}_x\rightarrow\dfrac{\partial\bar{H}_z}{\partial y}+j\beta_z\bar{H}_y=j\omega\varepsilon\bar{E}_x\ \ (1.4)\\\dfrac{\partial\bar{H}_x}{\partial z}-\dfrac{\partial\bar{H}_z}{\partial x}=j\omega\varepsilon\bar{E}_y\rightarrow -j\beta_z\bar{H}_x-\dfrac{\partial\bar{H}_z}{\partial x}=j\omega\varepsilon\bar{E}_y\ \ (1.5)\\\dfrac{\partial\bar{H}_y}{\partial x}-\dfrac{\partial\bar{H}_x}{\partial y}=j\omega\varepsilon\bar{E}_z\rightarrow\dfrac{\partial\bar{H}_y}{\partial x}-\dfrac{\partial\bar{H}_x}{\partial y}=0\ \ (1.6)\end{cases}$$

$$\begin{cases}(1.5)\rightarrow -j\beta_z\bar{H}_x-\dfrac{\partial\bar{H}_z}{\partial x}=j\omega\varepsilon\bar{E}_y\rightarrow\bar{H}_x=\dfrac{-1}{j\beta_z}\left(j\omega\varepsilon\bar{E}_y+\dfrac{\partial\bar{H}_z}{\partial x}\right)\\(1.1)\rightarrow j\beta_z\bar{E}_y=-j\omega\mu\bar{H}_x=\dfrac{j\omega\mu}{j\beta_z}\left(j\omega\varepsilon\bar{E}_y+\dfrac{\partial\bar{H}_z}{\partial x}\right)\rightarrow -\beta_z^2\bar{E}_y=j\omega\mu\left(j\omega\varepsilon\bar{E}_y+\dfrac{\partial\bar{H}_z}{\partial x}\right)\\\rightarrow -\beta_z^2\bar{E}_y+\omega^2\mu\varepsilon\bar{E}_y=j\omega\mu\left(\dfrac{\partial\bar{H}_z}{\partial x}\right)\rightarrow -\left(\beta_z^2-\beta^2\right)\bar{E}_y=j\omega\mu\left(\dfrac{\partial\bar{H}_z}{\partial x}\right)\rightarrow\bar{E}_y=-\dfrac{j\omega\mu}{\beta_z^2-\beta^2}\dfrac{\partial\bar{H}_z}{\partial x}\end{cases}$$

$$\begin{cases}(1.5)\rightarrow\bar{E}_y=\dfrac{-1}{j\omega\varepsilon}\left(j\beta_z\bar{H}_x+\dfrac{\partial\bar{H}_z}{\partial x}\right)\\(1.1)\rightarrow -j\omega\mu\bar{H}_x=j\beta_z\bar{E}=_y=\dfrac{-j\beta_z}{j\omega\varepsilon}\left(j\beta_z\bar{H}_x+\dfrac{\partial\bar{H}_z}{\partial x}\right)\rightarrow\omega^2\mu\varepsilon\bar{H}_x=-j\beta_z\left(j\beta_z\bar{H}_x+\dfrac{\partial\bar{H}_z}{\partial x}\right)\\\rightarrow -\left(\beta_z^2-\beta^2\right)\bar{H}_x=-j\beta_z\dfrac{\partial\bar{H}_z}{\partial x}\rightarrow\bar{H}_x=j\dfrac{\beta_z}{\beta_z^2-\beta^2}\dfrac{\partial\bar{H}_z}{\partial x}\end{cases}$$

$$\begin{cases}(1.4)\rightarrow\bar{E}_x=\dfrac{1}{j\omega\varepsilon}\left(\dfrac{\partial\bar{H}_z}{\partial y}+j\beta_z\bar{H}_y\right)\\(1.2)\rightarrow j\omega\mu\bar{H}_y=j\beta_z\bar{E}_x=\dfrac{j\beta_z}{j\omega\varepsilon}\left(\dfrac{\partial\bar{H}_z}{\partial y}+j\beta_z\bar{H}_y\right)\rightarrow -\omega^2\mu\varepsilon\bar{H}_y=j\beta_z\left(\dfrac{\partial\bar{H}_z}{\partial y}+j\beta_z\bar{H}_y\right)\\\rightarrow\left(-\omega^2\mu\varepsilon\bar{H}_y+\beta_z^2\right)\bar{H}_y=j\beta_z\dfrac{\partial\bar{H}_z}{\partial y}\rightarrow\bar{H}_y=j\dfrac{\beta_z}{\beta_z^2-\beta^2}\dfrac{\partial\bar{H}_z}{\partial y}\end{cases}$$

(2) (2%) If the waveguide is made of perfect electric conductor, list the boundary conditions on the four walls, $x = 0, x = a, y = 0, y = b$.

Answer:

$$\begin{cases} \bar{E}_x = 0,\ \forall y = 0, x \in (0,a) \\ \bar{E}_x = 0,\ \forall y = b, x \in (0,a) \end{cases} \begin{cases} \bar{E}_y = 0,\ \forall x = 0, y \in (0,b) \\ \bar{E}_y = 0,\ \forall x = a, y \in (0,b) \end{cases} \rightarrow (+1\%)$$

$$\text{From } (1): \begin{cases} \dfrac{\partial \bar{H}_z}{\partial y} = 0,\ \forall y = 0, x \in (0,a) \\ \dfrac{\partial \bar{H}_z}{\partial y} = 0,\ \forall y = b, x \in (0,a) \end{cases} \begin{cases} \dfrac{\partial \bar{H}_z}{\partial x} = 0,\ \forall x = 0, y \in (0,b) \\ \dfrac{\partial \bar{H}_z}{\partial x} = 0,\ \forall x = a, y \in (0,b) \end{cases} \rightarrow (+1\%)$$

(3) (3%) Assume $H_z(x,y,z) = X(x)Y(y)e^{-j\beta_z z}$, which satisfies the wave equation $\left(\nabla^2 + k^2\right)H_z(x,y,z) = 0$, where $k^2 = \omega^2\mu\varepsilon$. Derive the harmonic equations of $X(x)$ and $Y(y)$, with proper constants β_x and β_y, respectively.

Answer:

$$\left(\nabla^2 + k^2\right)H_z(x,y,z) = 0 \rightarrow \left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \omega^2\mu\varepsilon\right]H_z(x,y,z) = 0,\ \ where: H_z(x,y,z) = \bar{X}(x)Y(y)e^{-j\beta_z z}$$

$$\rightarrow \bar{X}''(x)\bar{Y}(y) + \bar{X}(x)\bar{Y}''(y) + \omega^2\mu\varepsilon\bar{X}(x)\bar{Y}(y) = 0 \rightarrow \frac{\bar{X}''(x)}{\bar{X}(x)} + \frac{\bar{Y}''(y)}{\bar{Y}(y)} = -\omega^2\mu\varepsilon$$

$$\rightarrow \frac{\bar{X}''(x)}{\bar{X}(x)} = -\beta_x^2 = \text{constant}, \rightarrow \frac{d^2\bar{X}}{dx^2} = -\beta_x^2\bar{X}(x) \rightarrow \bar{X}(x) = A_1e^{j\beta_x x} + A_2e^{-j\beta_x x}$$

$$\frac{\bar{Y}''(y)}{\bar{Y}(y)} = -\beta_y^2 = \text{constant}, \rightarrow \frac{d^2\bar{Y}}{dy^2} = -\beta_y^2\bar{Y}(y) \rightarrow \bar{Y}(y) = B_1e^{j\beta_y y} + B_2e^{-j\beta_y y}$$

(4) (3%) Use the relations derived in (1) to list the corresponding expressions of E_x, E_y, H_x and H_y in terms of the H_z expression obtained in (3).

Answer:

$$\bar{H}_z(x,y,z) = \bar{X}(x)\bar{Y}(y)e^{-j\beta_z z} = \left(A_1e^{j\beta_x x} + A_2e^{-j\beta_x x}\right)\left(B_1e^{j\beta_y y} + B_2e^{-j\beta_y y}\right)e^{-j\beta_z z}, \rightarrow (+1\%)$$

$$where: \bar{X}(x) = A_1e^{j\beta_x x} + A_2e^{-j\beta_x x}, \bar{Y}(y) = B_1e^{j\beta_y y} + B_2e^{-j\beta_y y} \rightarrow (+1\%)$$

$$\begin{cases} \bar{E}_x = \dfrac{j\omega\mu}{\beta_z^2 - \beta^2}\dfrac{\partial \bar{H}_z}{\partial y} = \dfrac{-\omega\mu\beta_y}{\beta_z^2 - \beta^2}\left(A_1e^{j\beta_x x} + A_2e^{-j\beta_x x}\right)\left(B_1e^{j\beta_y y} - B_2e^{-j\beta_y y}\right)e^{-j\beta_z z} \\ \bar{E}_y = -\dfrac{j\omega\mu}{\beta_z^2 - \beta^2}\dfrac{\partial \bar{H}_z}{\partial x} = \dfrac{\omega\mu\beta_x}{\beta_z^2 - \beta^2}\left(A_1e^{j\beta_x x} - A_2e^{-j\beta_x x}\right)\left(B_1e^{j\beta_y y} + B_2e^{-j\beta_y y}\right)e^{-j\beta_z z} \\ \bar{H}_x = j\dfrac{\beta_z}{\beta_z^2 - \beta^2}\dfrac{\partial \bar{H}_z}{\partial x} = -\dfrac{\beta_z\beta_x}{\beta_z^2 - \beta^2}\left(A_1e^{j\beta_x x} - A_2e^{-j\beta_x x}\right)\left(B_1e^{j\beta_y y} + B_2e^{-j\beta_y y}\right)e^{-j\beta_z z} \\ \bar{H}_y = j\dfrac{\beta_z}{\beta_z^2 - \beta^2}\dfrac{\partial \bar{H}_z}{\partial y} = -\dfrac{\beta_z\beta_y}{\beta_z^2 - \beta^2}\left(A_1e^{j\beta_x x} + A_2e^{-j\beta_x x}\right)\left(B_1e^{j\beta_y y} - B_2e^{-j\beta_y y}\right)e^{-j\beta_z z} \end{cases} \rightarrow (+1\%)$$

(5) (3%) Impose the boundary conditions listed in (2) upon the expressions of E_x, E_y in (4) to prove that $\beta_x = m\pi / a$ and $\beta_y = n\pi / b$. List all possible values of m and n, and explain why.

Answer:

Boudary condition:

$$\begin{cases} \bar{E}_x = 0,\ \forall y = 0, x \in (0,a) \\ \bar{E}_x = 0,\ \forall y = b, x \in (0,a) \end{cases} \quad \begin{cases} \bar{E}_y = 0,\ \forall x = 0, y \in (0,b) \\ \bar{E}_y = 0,\ \forall x = a, y \in (0,b) \end{cases}$$

$$\bar{E}_x = \frac{-\omega\mu\beta_y}{\beta_z^2 - \beta^2}\left(A_1 e^{j\beta_x x} + A_2 e^{-j\beta_x x}\right)\left(B_1 e^{j\beta_y y} - B_2 e^{-j\beta_y y}\right)e^{-j\beta_z z} \ldots (5.1)$$

$$\bar{E}_y = \frac{\omega\mu\beta_x}{\beta_z^2 - \beta^2}\left(A_1 e^{j\beta_x x} - A_2 e^{-j\beta_x x}\right)\left(B_1 e^{j\beta_y y} + B_2 e^{-j\beta_y y}\right)e^{-j\beta_z z} \ldots (5.2)$$

$$\because \bar{E}_x = 0,\ \forall y = 0, x \in (0,a) \to B_1 - B_2 = 0 \to B_1 = B_2$$

$$\because \bar{E}_y = 0,\ \forall x = 0, y \in (0,b) \to A_1 - A_2 = 0 \to A_1 = A_2, (c)(js)$$

$$(5.1) \to \bar{E}_x = \frac{-\omega\mu\beta_y}{\beta_z^2 - \beta^2} A\left(e^{j\beta_x x} + e^{-j\beta_x x}\right)\left(e^{j\beta_y y} - e^{-j\beta_y y}\right)e^{-j\beta_z z} = \frac{-j\omega\mu\beta_y}{\beta_z^2 - \beta^2} A \cos\beta_x x \cdot \sin\beta_y y \cdot e^{-j\beta_z z}, \ldots (5.3)$$

$$(5.2) \to \bar{E}_y = \frac{\omega\mu\beta_x}{\beta_z^2 - \beta^2} A\left(e^{j\beta_x x} - e^{-j\beta_x x}\right)\left(e^{j\beta_y y} + e^{-j\beta_y y}\right)e^{-j\beta_z z} = \frac{j\omega\mu\beta_x}{\beta_z^2 - \beta^2} A \cdot \sin\beta_x x \cdot \cos\beta_y y \cdot e^{-j\beta_z z}, \ldots (5.4)$$

$$\because \bar{E}_x = 0,\ \forall y = b, x \in (0,a) \to (5.3)$$

$$\to \sin\beta_y b = 0 \to \beta_y = \frac{n\pi}{b}, n = 0,1,2,\ldots \left(\because \text{If } \beta_y = \frac{n\pi}{b}, n = 0,1,2,\ldots, \text{ then } \sin\beta_y b = 0\right)$$

$$\because \bar{E}_y = 0,\ \forall x = a, y \in (0,b) \to (5.4)$$

$$\to \sin\beta_x a = 0 \to \beta_x = \frac{m\pi}{a}, m = 0,1,2,\ldots \left(\because \text{If } \beta_x = \frac{m\pi}{a}, m = 0,1,2,\ldots, \text{ then } \sin\beta_x a = 0\right)$$

(6) (4%) Given specific m and n, derive the expressions of cutoff frequency f_c, guided wavelength λ_g, apparent phase velocity v_{pz} and group velocity v_{gz}.

Answer:

$$\left(\nabla^2 + k^2\right)H_z(x,y,z) = 0 \to \left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \omega^2\mu\varepsilon\right]H_z(x,y,z) = 0,\quad where: H_z(x,y,z) = \bar{X}(x)Y(y)e^{-j\beta_z z}$$

$$\because \frac{\bar{X}''(x)}{\bar{X}(x)} = -\beta_x^2, \frac{\bar{Y}''(y)}{\bar{Y}(y)} = -\beta_y^2, \to \left(-\beta_x^2 - \beta_y^2 + \omega^2\mu\varepsilon\right)H_z(x,y,z) = 0 \to \omega^2\mu\varepsilon = \beta_x^2 + \beta_y^2$$

$$\to f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2}$$

$$\to \lambda_c = \frac{v_p}{f_c} = \frac{1}{\sqrt{\mu\varepsilon} \cdot f_c} = \left(\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2}\right)^{-1}$$

$$\to \lambda_g = \frac{2\pi}{\beta_z} = \frac{2\pi}{\beta\sin\theta} = \frac{\lambda}{\sqrt{1 - (\lambda/\lambda_c)^2}} = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}}$$

$$\rightarrow v_{pz}=\frac{\omega}{\beta_z}=\frac{\omega}{\beta\sin\theta}=\frac{v_p}{\sqrt{1-\left(\lambda/\lambda_c\right)^2}}=\frac{v_p}{\sqrt{1-\left(f_c/f\right)^2}}$$

$$\rightarrow v_{gz}=\frac{d\omega}{d\beta_z}=\frac{1}{\sqrt{\mu\varepsilon}}\sqrt{1-\left(\frac{f_c}{f}\right)^2}, where: \beta_z=\frac{2\pi}{\lambda_g}=\frac{2\pi}{\lambda}\sqrt{1-\left(\frac{f_c}{f}\right)^2}=\omega\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

Note: **Problem 6** is in the following page.

6. (20 %) Figure 4 shows a Hertzian dipole of current moment $\hat{z}I\ell$, located at the origin. The magnetic vector potential is derived as $\bar{A} = \mu \frac{I\ell}{4\pi r} e^{-jkr}\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right)$.

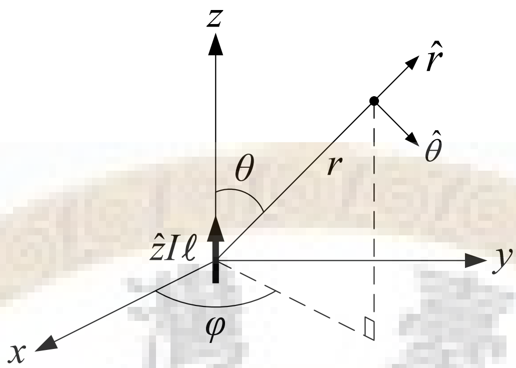

Fig. 4

(1) (4%) Derive the magnetic field by using the relation

$$\bar{H} = \frac{1}{\mu}\nabla\times\bar{A} = \frac{1}{\mu}\frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

<u>*Answer*:</u> $\bar{H} = \hat{\phi}\frac{I\ell}{4\pi}\left(\sin\theta e^{-jkr}\right)\left\{\frac{jk}{r} + \frac{1}{r^2}\right\}$

$(Derivation)$

$$\bar{A} = \mu\frac{I\ell}{4\pi r}e^{-jkr}\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right) = \hat{r}A_r + \hat{\theta}A_\theta, where: A_\phi = 0$$

$$\bar{H} = \frac{1}{\mu}\frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} = \frac{1}{\mu}\frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & 0 \end{vmatrix}$$

$$= \frac{1}{\mu}\frac{1}{r^2\sin\theta}\left[\hat{r}\left(-r\frac{\partial A_\theta}{\partial\phi}\right) + r\hat{\theta}\left(\frac{\partial A_r}{\partial\phi}\right) + r\sin\theta\hat{\phi}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\right] = \frac{1}{\mu}\frac{1}{r^2\sin\theta}\left[r\sin\theta\hat{\phi}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\right]$$

$$= \frac{1}{\mu r}\hat{\phi}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right) = \hat{\phi}\frac{1}{\mu}\frac{1}{r}\left\{jk\left(\frac{\mu I\ell}{4\pi}\sin\theta e^{-jkr}\right) + \frac{1}{r}\left(\frac{\mu I\ell}{4\pi}\sin\theta e^{-jkr}\right)\right\}$$

$$= \hat{\phi}\frac{1}{\mu}\left(\frac{\mu I\ell}{4\pi}\sin\theta e^{-jkr}\right)\left(\frac{1}{r}\right)\left\{jk + \frac{1}{r}\right\} = \hat{\phi}\frac{I\ell}{4\pi}\left(\sin\theta e^{-jkr}\right)\left\{\frac{jk}{r} + \frac{1}{r^2}\right\}$$

$where:$

$$A_r = \mu\frac{I\ell}{4\pi r}\cos\theta e^{-jkr}, A_\theta = -\mu\frac{I\ell}{4\pi r}\sin\theta e^{-jkr}$$

$$\frac{\partial(rA_\theta)}{\partial r} = -\frac{\partial}{\partial r}\left(\mu\frac{I\ell}{4\pi}\sin\theta e^{-jkr}\right) = -\mu\frac{I\ell}{4\pi}\sin\theta\frac{\partial}{\partial r}\left(e^{-jkr}\right) = jk\left(\frac{\mu I\ell}{4\pi}\sin\theta e^{-jkr}\right)$$

$$-\frac{\partial A_r}{\partial\theta} = -\mu\frac{I\ell}{4\pi r}e^{-jkr}\frac{\partial}{\partial\theta}(\cos\theta) = \frac{1}{r}\left(\frac{\mu I\ell}{4\pi}\sin\theta e^{-jkr}\right)$$

(2) (4%) Derive the electric field by applying the Ampere's law to the magnetic field derived in (1) as

$$\bar{E}=\frac{1}{j\omega\varepsilon}\nabla\times\bar{H}=\frac{1}{j\omega\varepsilon}\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ H_r & rH_\theta & r\sin\theta H_\phi\end{vmatrix}$$

Answer:

$$\bar{H}=\hat{\phi}\frac{I\ell}{4\pi}\left(\sin\theta e^{-jkr}\right)\left\{\frac{jk}{r}+\frac{1}{r^2}\right\}=\hat{\phi}H_\phi,\ where: H_r=H_\theta=0, H_\phi=\frac{I\ell}{4\pi}\left(\sin\theta e^{-jkr}\right)\left\{\frac{jk}{r}+\frac{1}{r^2}\right\}$$

$$\bar{E}=\frac{1}{j\omega\varepsilon}\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ 0 & 0 & r\sin\theta H_\phi\end{vmatrix}=\frac{1}{j\omega\varepsilon}\frac{1}{r^2\sin\theta}\left[\hat{r}\frac{\partial}{\partial\theta}\left(r\sin\theta H_\phi\right)-r\hat{\theta}\frac{\partial}{\partial r}\left(r\sin\theta H_\phi\right)\right]$$

$$=\frac{1}{j\omega\varepsilon}\frac{1}{r^2\sin\theta}\left[\hat{r}\left(\frac{I\ell}{4\pi}\right)\left(2\sin\theta\cos\theta e^{-jkr}\right)\left\{jk+\frac{1}{r}\right\}+r\hat{\theta}\left(\frac{I\ell}{4\pi}\right)\left(\sin^2\theta\right)\left\{-k^2e^{-jkr}+\frac{jk}{r}e^{-jkr}+\frac{1}{r^2}e^{-jkr}\right\}\right]$$

$$=\left(\frac{I\ell}{4\pi}\right)\frac{1}{j\omega\varepsilon}\frac{1}{\sin\theta}\frac{1}{r^2}\left[\hat{r}2\cos\theta\left(\sin\theta e^{-jkr}\right)\left\{jk+\frac{1}{r}\right\}+\hat{\theta}\sin\theta\left(\sin\theta e^{-jkr}\right)r\left(-k^2+\frac{jk}{r}+\frac{1}{r^2}\right)\right]$$

$$=\frac{I\ell}{4\pi}\frac{1}{j\omega\varepsilon}\frac{\sin\theta e^{-jkr}}{\sin\theta}\frac{1}{r^2}\left[\hat{r}2\cos\theta\left(jk+\frac{1}{r}\right)+\hat{\theta}\sin\theta\left(-k^2r+jk+\frac{1}{r}\right)\right]$$

$$=\frac{I\ell}{4\pi}\frac{e^{-jkr}}{j\omega\varepsilon}\left[\hat{r}2\cos\theta\left(\frac{jk}{r^2}+\frac{1}{r^3}\right)+\hat{\theta}\sin\theta\left(-\frac{k^2}{r}+\frac{jk}{r^2}+\frac{1}{r^3}\right)\right]$$

where:

$$r\sin\theta H_\phi=\left(r\sin\theta\right)\frac{I\ell}{4\pi}\left(\sin\theta e^{-jkr}\right)\left\{\frac{jk}{r}+\frac{1}{r^2}\right\}=\frac{I\ell}{4\pi}\left(\sin^2\theta e^{-jkr}\right)\left\{jk+\frac{1}{r}\right\}$$

$$\frac{\partial}{\partial\theta}\left(r\sin\theta H_\phi\right)=\frac{I\ell}{4\pi}\left(jk+\frac{1}{r}\right)e^{-jkr}\frac{\partial}{\partial\theta}\left(\sin^2\theta\right)=\frac{I\ell}{4\pi}\left(2\sin\theta\cos\theta e^{-jkr}\right)\left\{jk+\frac{1}{r}\right\}$$

$$\frac{\partial}{\partial r}\left(r\sin\theta H_\phi\right)=\frac{I\ell}{4\pi}\left(\sin^2\theta\right)\frac{\partial}{\partial r}\left\{jke^{-jkr}+\frac{e^{-jkr}}{r}\right\}=\frac{I\ell}{4\pi}\left(\sin^2\theta\right)\left\{k^2e^{-jkr}-\frac{jk}{r}e^{-jkr}-\frac{1}{r^2}e^{-jkr}\right\}$$

(3) (2%) In the far-field region, $kr >> 1$; by keeping only the terms with $(1/r)$ -dependence and neglecting all the higher-order terms, write down the approximate expression of the electric field derived in (2).

Answer:

$$\bar{E}=\frac{I\ell}{4\pi}\frac{e^{-jkr}}{j\omega\varepsilon}\left[\hat{r}2\cos\theta\left(\frac{jk}{r^2}+\frac{1}{r^3}\right)+\hat{\theta}\sin\theta\left(-\frac{k^2}{r}+\frac{jk}{r^2}+\frac{1}{r^3}\right)\right]_{r>>1}\approx-\hat{\theta}\frac{I\ell}{4\pi r}\left(\frac{k^2}{j\omega\varepsilon}\right)\left(\sin\theta e^{-jkr}\right)$$

(4) (3%) Calculate the time-averaged Poynting vector $\langle \bar{P} \rangle = \frac{1}{2}\mathrm{Re}\{\bar{E}\times\bar{H}^*\}$ in the far-field region.

Answer:

In the far-field region, $kr >> 1$

$$\bar{E} \approx -\hat{\theta}\frac{I\ell}{4\pi r}\left(\frac{k^2}{j\omega\varepsilon}\right)\left(\sin\theta e^{-jkr}\right), \bar{H} = \hat{\phi}\frac{I\ell}{4\pi}\left(\sin\theta e^{-jkr}\right)\left\{\frac{jk}{r}+\frac{1}{r^2}\right\} \approx \hat{\phi}\frac{I\ell}{4\pi r}(jk)\left(\sin\theta e^{-jkr}\right)$$

$$\langle P \rangle = \frac{1}{2}\mathrm{Re}\{\bar{E}\times\bar{H}^*\} = \hat{r}\frac{1}{2}\left(\frac{I\ell\sin\theta}{4\pi r}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right)$$

where:

$$\bar{E}\times\bar{H}^* = \left[-\hat{\theta}\frac{I\ell}{4\pi r}\left(\frac{k^2}{j\omega\varepsilon}\right)\left(\sin\theta e^{-jkr}\right)\right]\times\left[\hat{\phi}\frac{I\ell}{4\pi r}(jk)\left(\sin\theta e^{-jkr}\right)\right]^*$$

$$= -\hat{\theta}\times\hat{\phi}\left[\left(\frac{I\ell\sin\theta}{4\pi r}\right)\left(\frac{k^2}{j\omega\varepsilon}\right)\left(e^{-jkr}\right)\right]\left[\left(\frac{I\ell\sin\theta}{4\pi r}\right)(jk)^*\left(e^{-jkr}\right)^*\right]$$

$$= -\hat{r}\left(\frac{I\ell\sin\theta}{4\pi r}\right)^2\left(\frac{k^2}{j\omega\varepsilon}\right)(-jk) = \hat{r}\left(\frac{I\ell\sin\theta}{4\pi r}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right)$$

(5) (3%) Integrate $\langle \bar{P} \rangle$ derived in (4) over a spherical surface of radius *r*, centered at the origin, to derive the time-averaged total radiated power, $\langle P_{\mathrm{rad}} \rangle$. Note that an infinitesimal area on the spherical surface is $d\bar{a} = \hat{r}r^2\sin\theta d\theta d\phi$.

Answer:

$$\langle P_{\mathrm{rad}} \rangle = \int_S \langle P \rangle \cdot d\vec{a} = \int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\langle P \rangle r^2\sin\theta d\theta d\phi = \frac{1}{2}\left(\frac{I\ell}{4\pi}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right)\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\left(\frac{\sin\theta}{r}\right)^2 r^2\sin\theta d\theta d\phi$$

$$= \frac{1}{2}\left(\frac{I\ell}{4\pi}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right)\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\sin^3\theta d\theta d\phi = \frac{1}{2}\left(\frac{I\ell}{4\pi}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right)2\pi\int_{\theta=0}^{\theta=\pi}\frac{1}{4}(3\sin\theta-\sin 3\theta)d\theta d\phi$$

$$= \frac{\pi}{4}\left(\frac{I\ell}{4\pi}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right)\int_{\theta=0}^{\theta=\pi}(3\sin\theta-\sin 3\theta)d\theta = \frac{\pi}{4}\left(\frac{I\ell}{4\pi}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right)\left(-3\cos\theta\Big|_{\theta=0}^{\theta=\pi}+\frac{1}{3}\cos 3\theta\Big|_{\theta=0}^{\theta=\pi}\right)$$

$$= \frac{\pi}{4}\left(\frac{I\ell}{4\pi}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right)\left[-3(-1-1)+\frac{1}{3}(-1-1)\right] = \frac{\pi}{4}\left(\frac{I\ell}{4\pi}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right)\left(6-\frac{2}{3}\right) = -\frac{\pi}{4}\left(\frac{I\ell}{4\pi}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right)\left(\frac{16}{3}\right)$$

$$= \frac{4\pi}{3}\left(\frac{I\ell}{4\pi}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right) = \frac{(I\ell)^2}{12\pi}\left(\frac{k^3}{\omega\varepsilon}\right)$$

where:

$$\langle P \rangle = \frac{1}{2}\mathrm{Re}\{\bar{E}\times\bar{H}^*\} = \hat{r}\frac{1}{2}\left(\frac{I\ell\sin\theta}{4\pi r}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right) = \hat{r}\frac{1}{2}\left(\frac{I\ell}{4\pi}\right)^2\left(\frac{\sin\theta}{r}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right),$$

$$d\vec{a} = \hat{r}r^2\sin\theta d\theta d\phi$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \rightarrow \sin^3\theta = \frac{1}{4}(3\sin\theta-\sin 3\theta)$$

(6) (2%) Let $\langle P_{\text{rad}} \rangle = \frac{1}{2} R_{\text{rad}} I^2$, derive the expression of R_{rad}.

Answer:

$$\langle P_{\text{rad}} \rangle = \frac{(I\ell)^2}{12\pi}\left(\frac{k^3}{\omega\varepsilon}\right) = \frac{1}{2} I^2 \frac{(\ell)^2}{6\pi}\left(\frac{k^3}{\omega\varepsilon}\right) = \frac{1}{2} R_{\text{rad}} I^2 \rightarrow R_{\text{rad}} = \frac{(\ell)^2}{6\pi}\left(\frac{k^3}{\omega\varepsilon}\right)$$

(7) (2%) Define the directivity as $D = \dfrac{\left|\langle \bar{P} \rangle\right|_{\max}}{\langle P_{\text{rad}} \rangle / 4\pi}$. Find the directivity of a Hertzian dipole.

Answer:

$$D = \frac{\left|\langle \bar{P} \rangle\right|_{\max}}{\langle P_{\text{rad}} \rangle / 4\pi} = \frac{1}{2}\left(\frac{I\ell}{4\pi r}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right) \Bigg/ \left[\left(-\frac{(I\ell)^2}{12\pi}\left(\frac{k^3}{\omega\varepsilon}\right)\right)\left(\frac{1}{4\pi}\right)\right] = \frac{1}{2}\left(\frac{I\ell}{4\pi r}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right) \Bigg/ \left[-\frac{(I\ell)^2}{48\pi^2}\left(\frac{k^3}{\omega\varepsilon}\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{4\pi r}\right)^2 \Bigg/ \left[\frac{1}{48\pi^2}\right] = \frac{1}{2}\frac{1}{(4\pi r)^2} 48\pi^2 = \frac{3}{2}\frac{16\pi^2}{(4\pi r)^2} = \frac{3}{2r^2}$$

where:

$$\langle P \rangle = -\hat{r}\frac{1}{2}\left(\frac{I\ell}{4\pi}\right)^2\left(\frac{\sin\theta}{r}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right), \rightarrow \left|\langle P \rangle\right|_{\max} = \frac{1}{2}\left(\frac{I\ell}{4\pi r}\right)^2\left(\frac{k^3}{\omega\varepsilon}\right)$$

$$\langle P_{\text{rad}} \rangle = -\frac{(I\ell)^2}{12\pi}\left(\frac{k^3}{\omega\varepsilon}\right)$$

Note: There is some flaw in the definition of directivity D. It should be $D = \dfrac{\left|\langle \bar{P} \rangle\right|_{\max}}{\langle P_{\text{rad}} \rangle / (4\pi r^2)}$. Because of this indiscretion, the solution of $D = \dfrac{3}{2r^2}$ and $D = \dfrac{3}{2r^2}$ would both be considered as correct answers although $D = \dfrac{3}{2r^2}$ does not coincide with the physical meaning of directivity.

Useful formulas: One could refer to the useful formulas as shown in the followings.

$$\lambda_c = \frac{2a}{m},\quad f_c = \frac{m}{2a\sqrt{\mu\varepsilon}}$$

$$\lambda_g = \frac{2\pi}{\beta_z} = \frac{2\pi}{\beta\sin\theta} = \frac{\lambda}{\sqrt{1-(\lambda/\lambda_c)^2}} = \frac{\lambda}{\sqrt{1-(f_c/f)^2}},$$

$$v_{pz} = \frac{\omega}{\beta_z} = \frac{\omega}{\beta\sin\theta} = \frac{v_p}{\sqrt{1-(\lambda/\lambda_c)^2}} = \frac{v_p}{\sqrt{1-(f_c/f)^2}}$$

$$v_{pz} = \frac{\omega}{\beta_z};\frac{1}{v_g} = \frac{d\beta_z}{d\omega};v_g = \frac{\omega_B-\omega_A}{\beta_{zB}-\beta_{zA}}$$

$$\tan\left(\frac{\pi d\sqrt{\varepsilon_{r1}}}{\lambda_0}\cos\theta_i - \frac{m\pi}{2}\right) = \frac{\sqrt{\sin^2\theta_i-(\varepsilon_2/\varepsilon_1)}}{\cos\theta_i}, m=0,1,2,...$$

$$\begin{cases}\beta_{x1}^2+\beta_z^2=\omega^2\mu_0\varepsilon_1\\ -\alpha_{x2}^2+\beta_z^2=\omega^2\mu_0\varepsilon_2\end{cases},\quad \rightarrow \frac{\alpha_{x2}}{\beta_{x1}} = \sqrt{\frac{\omega^2\mu_0(\varepsilon_1-\varepsilon_2)}{\beta_{x1}^2}-1}$$

$$\tan\left(\beta_{x1}\frac{d}{2}\right) = \sqrt{\frac{\omega^2\mu_0(\varepsilon_1-\varepsilon_2)}{\beta_{x1}^2}-1},\quad \tan\left(\beta_{x1}\frac{d}{2}\cos\theta_i\right) = \sqrt{\frac{\omega^2\mu_0(\varepsilon_1-\varepsilon_2)}{\omega^2\mu_0\varepsilon_1\cos^2\theta_i}-1}$$

$$\tan\left(\frac{\pi d\sqrt{\varepsilon_{r1}}}{\lambda_0}\cos\theta_i\right) = \frac{\sqrt{\sin^2\theta_i-(\varepsilon_2/\varepsilon_1)}}{\cos\theta_i},\quad \tan\left[f(\theta_i)\right] = \begin{cases} g(\theta_i), & m=0,2,4,...\\ \dfrac{-1}{g(\theta_i)}, & m=1,3,5,...\end{cases}$$

$$f_c = \frac{mc}{2d\sqrt{\varepsilon_{r1}-\varepsilon_{r2}}}, m=0,1,2,...;\quad f_c = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$$

$$\lambda_g = \frac{\lambda}{\sqrt{1-(f_c/f)^2}},\quad v_{pz} = \frac{1}{\sqrt{\mu\varepsilon}\cdot\sqrt{1-(f_c/f)^2}}$$