G

Probability Midterm Exam

2:20-4:20 pm Thursday, 20 April 2006

Total: 3 pages, 100 points.

(1) A calculator is NOT allowed. You may not consult the textbook or neighbor.
(2)Please turn off your cell phones before the exam.
(3)Please show your work for partial credit, and underline your answers. Points are awarded for solutions, not answers, so correct answers without justification will not receive full credit.

PROBLEM 1: (8%)

Define a function of two events A and B by

$$g(A;B) \triangleq \frac{P(A \cup B)}{P(B)}$$

For all events A, B, with $P(A)$ denoting the probability of an event A. Now, for any given event B, can $g(A;B)$ be a probability measure?

(Hint: A probability measure $h(A)$ has to satisfy the Three Axioms of Probability. Namely, (1) $h(A) \geq 0$ for any event A, (2) $h(S) = 1$ for the sample space S, and (3) If $A \cap C$ is an empty set for any events A and C, then $P(A \cup C) = P(A) + P(C)$.)

PROBLEM 2: (6%)

Consider rolling three fair dice (of the same size) at a time in a casino. The game is to bet on the numbers that the three dice show up. Let n_1, n_2, and n_3 be the three numbers that show up in a single roll.

(a) Find the probability of the event that $n_1 = n_2 = n_3$. (2%)
(b) Find the probability of the event that $n_1 < n_2 < n_3$. (2%)
(c) Find the probability of the event that $n_1 + n_2 + n_3 = 12$. (2%)

PROBLEM 3: (6%)

For events A, B, C, solve the following questions:

(a) If $P(A) = 0.2$, $P(A \cup B) = 0.5$, and $P(A|B) = 0.5$, find $P(B)$. (2%)
(b) If A and B are mutually exclusive and independent and if $B \subset A$, find $P(B)$. (2%)
(c) Show that $2P(A \cap B) \leq P(A) + P(B)$. (2%)

PROBLEM 4: (5%)

Prove that the inequality

$$P\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} P(A_k)$$

holds for arbitrary events $A_1, A_2, \ldots, A_n$, with n a positive integer.

GO ON TO THE NEXT PAGE.

PROBLEM 5: (7%)

A communication channel transmits the digits 0 and 1. However, due to static, the digit transmitted is incorrectly received with probability *0.2*. Suppose that we want to transmit an important message consisting of one binary digit. To reduce the chance of error, we transmit 00000 instead of 0 and 11111 instead of 1. If the receiver of the message uses "majority" decoding, what is the probability that the message will be incorrectly decoded? What independence assumptions are you making? (By majority decoding we mean that the message is decoded as "0" if there are at least three zeros in the message received and as "1" otherwise.)

PROBLEM 6: (9%)

According to a representative for an automobile manufacturer, the company uses 3000 lock-and-key combinations on its vehicles. Suppose that you find a key for one of these cars.

(a) Give the expected number of vehicles that you would have to check to find one that your key fit. (3%)

(b) Give the probability that you would have to check at least 3000 vehicles to find one that your key fit. (3%)

(c) Give the probability that at most 2000 vehicles would have to be checked to find one that your key fit. (3%)

(Note. Please give simplest form of your answers as you can.)

PROBLEM 7: (9%)

The number of times that an individual contracts a cold in a given year is a Poisson random variable with parameter $\lambda = 3$, Suppose a new wonder drug (based on large quantities of vitamin C) has just been marketed that reduces that Poisson parameter to $\lambda = 2$ for 75 percent of the population. For the other 25 percent of the population, the drug has no appreciable effect on colds. If an individual tries the drug for a year and has 0 colds in that time, how likely is it that the drug is beneficial for him or her? ($e \approx 2.718$)

PROBLEM 8: (10%)

The time it takes for a student to finish an aptitude test (in hours), say X, has a density function of the form

$$f(x) = \begin{cases} c(x-1)(2-x) & \text{if } 1 < x < 2 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Probability $\{1 < X < 1.5\}$ =? Why does "$1 < X < 1.5$" have a probability? (6%)

(b) Var[X] =? (4%)

GO ON TO THE NEXT PAGE.

PROBLEM 9: (15%)

Random Number Generation

(a) Suppose X is a random variable with a cumulative distribution function F. Suppose that F is continuous. Define $Y = F(X)$. Prove that Y has the uniform distribution on (0, 1). That is, show that Y satisfies $P(Y \le y) = y$ for all $y \in (0, 1)$. (8%)

(b) You have a computer code that generates random numbers uniformly distributed in [0, 1]. Let the corresponding random variable be **X**. Now you want to generate a random variable **Y** of Gamma distribution with parameters $(2, \lambda)$, where $\lambda > 0$. Describe how you may apply (a) to generate **Y** by using the given uniform random number generator code and explain why. (7%)

PROBLEM 10: (15%)

Some city draws electricity completely from a given power plant. The city is connected to this power plant with two power lines. The times before failure for these two lines are independent. They are exponentially distributed with parameters $\lambda 1$ and $\lambda 2$ (hour^{-1}), respectively. Now, both lines are fine at time 0.

(a) One of these two lines fails first at time t. Find the distribution of t. (5%)

(b) Find the probability that the first failed line is line 1. (5%)

(c) Find the distribution of t given that it is known that line 1 fails first. (5%)

PROBLEM 11: (10%)

A point *(X, Y)* is selected at random from a triangle whose three vertices are (0, 0), (1, 1), and (-1, 1) on the $x - y$ plane.

(a) Find the joint probability density function of X and Y. (5%)

(b) Find the probability density function of X + Y. (5%)

STOP

PLEASE RECHECK YOUR ANSWERS! THANK YOU!