Linear Algebra Midterm Exam

You may NOT use any automatic computing device such as calculators or computers.

1. Please determine if the following statements are true or false. Explanations are required. (30% totally, 3% each)
   (a) If the rank of an $m \times n$ matrix A is m, then the rows of A are linearly independent.
   (b) If the nullity of an $n \times n$ matrix A is zero, then the columns of A span $\mathbf{R}^n$.
   (c) If a set of vector S is linearly independent, then $\mathbf{0}$ is not contained in S.
   (d) Let R be the reduced row echelon form of a matrix A. Then the rank of A and the rank of R are always equal.
   (e) Let T: V→W be a linear transformation, where V and W are subspaces of R^n. If T is one-to-one, then dim(V) $\geq$ dim(W).
   (f) Let A be an $n \times n$ matrix. A**x**=**0** has infinite number of solutions if and only if A is not invertible.
   (g) If A and B are $n \times m$ matrices such that $A\mathbf{x}=\mathbf{0}$ if and only if $B\mathbf{x}=\mathbf{0}$, then A and B have the same reduced row echelon form.
   (h) A *linear* transformation T: $\mathrm{R}^n \rightarrow \mathrm{R}^m$ is one-to-one if and only if it is onto.
   (i) The determinant of an upper triangular $n \times n$ matrix or a lower triangular $n \times n$ matrix equals to the sum of its diagonal entries.
   (j) The columns of any matrix form a basis for its column space.

2. (a) Let **u** be a solution to the A**x**= **b** and **v** be a solution to A**x**= **0**, show that **u - v** is a solution to **Ax=b**. (5%)
   (b) Let **u** be a solution to the A**x**= **b** and **v** be a solution to A**x**= **0**, show that **u** + **v** = **u** if all columns of A are linearly independent. (6%)

3. Please solve the general solution of the following system of linear equations. (12%)

$$\begin{aligned} x_1 + 2x_2 \quad + \quad x_4 + x_5 &= 3 \\ x_3 + \ 3x_4 \qquad &= 7 \end{aligned}$$

4. Let T: $\mathbf{R}^n \rightarrow \mathbf{R}^m$ be a linear transformation that is onto. Furthermore, let V be a finite subset of $\mathbf{R}^n$ and T(V) be the image of V. Prove that if T(V) is linearly independent, then V is linearly independent. (8%)

5. Let $A=\begin{bmatrix} ab & a^2 \\ -b^2 & -ab \end{bmatrix}$ where $ab \neq 0$. Find the column space and null space of A and show that they are equal. (8%)

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6. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$. Use Gaussian elimination to find A^{-1}(if it exists). Please show your work. (7%)

7. Determine the values of a, b, and c for which the following matrix is not invertible. (6%)

$$M = \begin{bmatrix} 4 & a & 3 & 9 & 5 \\ 3 & -6 & 2 & b & 0 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 0 & 4 \\ 0 & 0 & 2 & 1 & c \end{bmatrix}$$

8. For the matrix

$$A = \begin{bmatrix} 3 & 1 & -2 & 1 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ -5 & -2 & 5 & -5 & -3 \\ -2 & -1 & 3 & 2 & -10 \end{bmatrix}$$

its reduced row echelon form is

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -5 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Please find (a) a basis for the column space of A (6%) and (b) a basis for the null space of A, (6%) and determine (c) the dimension of the column and null spaces of A, respectively. (6%) For every answer, you need to justify (or, provide a brief proof) why the sets of vectors you choose are bases for them and how to determine their dimensions.