Complex Analysis: Final Exam

June 22, 2010

1. (20%) True or false (If it is false, explain briefly why it isn't true)

   (a) Suppose $f(z)=P(z)/Q(z)$, where the degree of $P(z)$ is m, the degree of $Q(z)$ is n, and $m \geq n+2$. If C_R is a semicircular contour $z=Re^{i\theta}$, $0 \leq \theta \leq \pi$, then $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$

   (b) Since $f(z)=\dfrac{1}{z(1-z)}$ can be expressed as $\dfrac{1}{z^2}+\dfrac{1}{z^3}+\dfrac{1}{z^4}+\ldots\ldots$ for $|z|>1$, $z=0$ is an essential singularity for $f(z)$.

   (c) In Fig. 1, $w=f(z)$ maps the curves C_1 and C_2 which intersect at z_0 in z-plane to the curves C_1' and C_2' which intersect at w_0 in w-plane, respectively. The angle between the tangent vectors of C_1 and C_2 at z_0 is θ, and the angle between the tangent vectors of C_1' and C_2' at w_0 is ϕ. If $\theta=\phi$, then $w=f(z)$ is conformal.

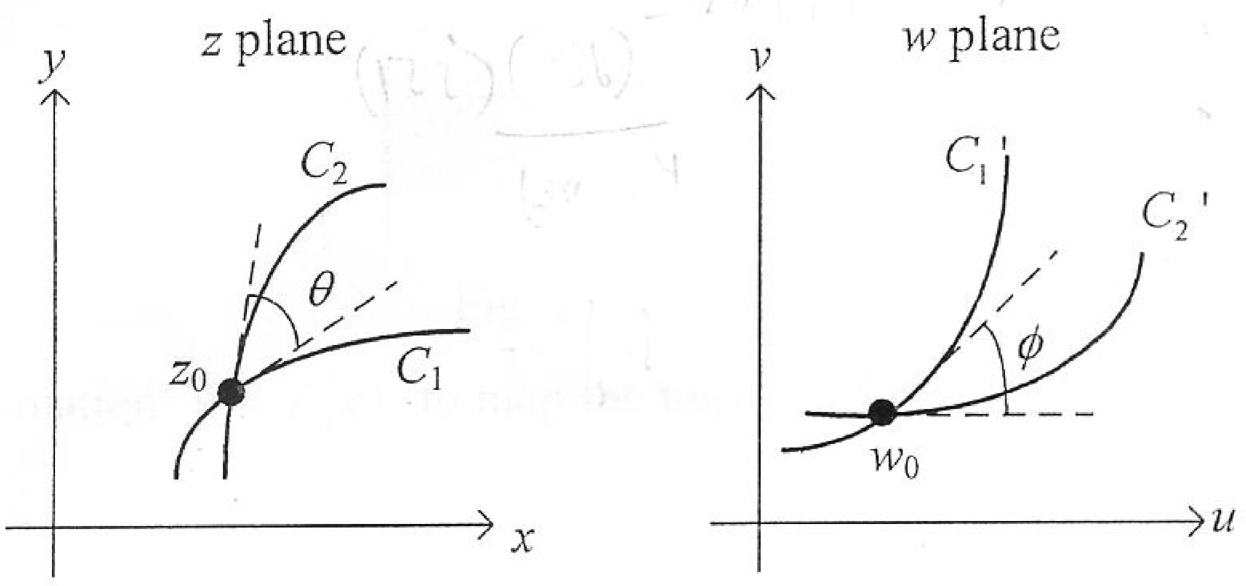


Fig. 1

   (d) A linear fractional transformation $w=f(z)$ maps a circle in the z-plane to either a line or a circle in the w-plane. The image is a line if and only if the original circle passes through a zero of $f(z)$.

2. (10%) Find the Laurent series which is expanded in powers of $(z-1)$ for $ze^{1/(z-1)}$ and give the largest annular domain in which the series is valid.

3. (20 %) Evaluate the Cauchy principal value of following integrals

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{2x^2+3}{(x^2+9)^2}dx$

   (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2-16}dx$

4. (20%) Evaluate the following integrals

(a) $\oint_C \dfrac{1}{z\sin^2 z}dz$, where $C = \{z \mid |z| = 1\}$

(b) $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\sin\theta} d\theta$

5. (15%) Find an appropriate conformal mapping and solve the Dirichlet problem as shown in Fig. 2.

D.E. $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) p(x,y) = 0,\ R = \left\{ z \,\middle|\, \left|z - \dfrac{1}{2}\right| > \dfrac{1}{2} \text{ and } \mathrm{Re}[z] > 0 \right\}$

B.C. $\begin{cases} p(x,y) = 1 & \text{for } \left\{ z \,\middle|\, \left|z - \dfrac{1}{2}\right| = \dfrac{1}{2} \right\} \\ p(x,y) = 0 & \text{for } \{ z \mid \mathrm{Re}[z] = 0 \} \end{cases}$

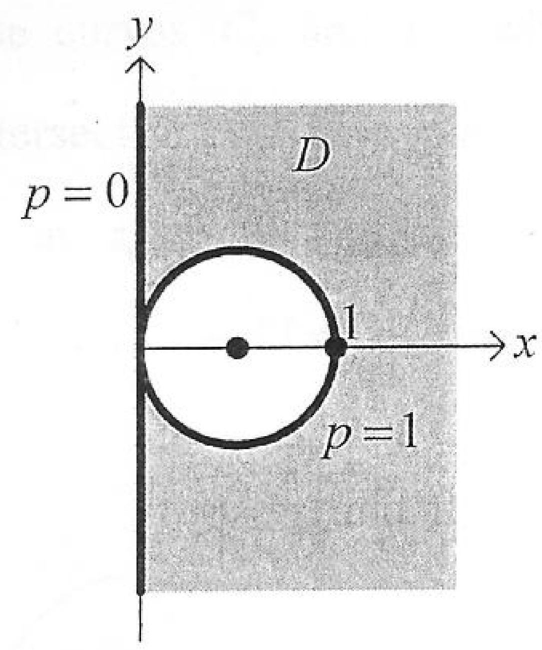
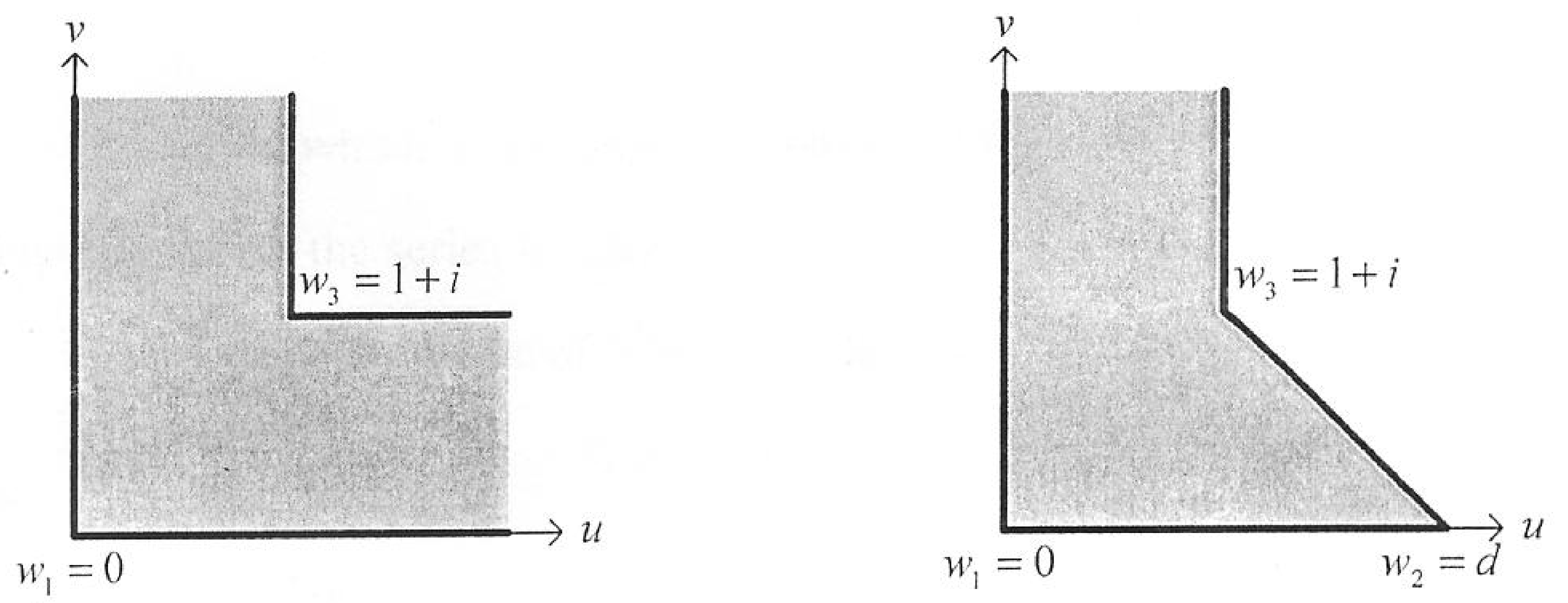


Fig. 2

6. (15%) Find the transformation $w = f(z)$ to map the upper half plane to the domain shown in Fig. 3(a).

Hint: (1) Map the upper half plane to the domain as shown in Fig. 3(b) and let $d \to \infty$

(2) $\displaystyle\int \frac{1}{(1-z^2)^{\frac{1}{2}}} dz = \sin^{-1} z,\ \int \frac{1}{z(z^2-1)^{\frac{1}{2}}} dz = -\sin^{-1}\frac{1}{z}$



(a) (b)

Fig. 3