Electromagnetics (I) Midterm Examination

April 29, 2011

1. A current density due to flow of charges is given by $\mathbf{J} = -(x\mathbf{a}_x + 2y\mathbf{a}_y + z^2\mathbf{a}_z)$ A/m^2. (a) By using surface integrals directly, find the displacement current emanating from the closed surface of the cubical box bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$. (b) Redo (a) by using an appropriate volume integral. (16%)

2. In a Cartesian coordinate system, constant current I flows along a straight conducting wire from a point charge $Q_1(t)$ located at the origin to a point charge $Q_2(t)$ located at (0, 0, 4). (a) Find the line integral of magnetic field $\mathbf{H}$ along the square closed path having the vertices at (4, 4, 0), (-4, 4, 0), (-4, -4, 0), and (4, -4, 0) and traversed in that order. (Express the answer in terms of current I.) (b) If the current I is slowly time-varying, i.e., $I = I(t)$, $(0 < z < 4)$, does the answer of (a) need to be modified? Explain briefly. (c) If $Q_2(t)$ is slowly moved from (0, 0, 4) to the origin (keeping I constant), how does the answer of (a) gradually change? Explain briefly. (15%)

3. Find the divergences of the following vector fields and explain their *physical meaning* with the aid of the divergence meter: (a) the position vector field associated with points in three-dimensional space and (b) the linear velocity vector field associated with points inside Earth due to its spin motion. (14%)

4. Consider a vector field $\mathbf{A} = y \cos x\,\mathbf{a}_x + \sin x\,\mathbf{a}_y$. (a) Evaluate the line integral $\int \mathbf{A} \cdot d\mathbf{l}$ from point (x_1, y_1, z_1) to point (x_2, y_2, z_2) along the straight line connecting them. (b) Find $\nabla \times \mathbf{A}$. (c) Use the results of (a) and (b) to verify Stokes' theorem. (15%)

5. A *low-frequency* time-varying magnetic field $\mathbf{H}(r, \varphi, z, t)$ is distributed in free space (μ_0, ε_0) as
$$\mathbf{H}(r, \varphi, z, t) = \begin{cases} Ar^3 \cos(\omega t)\mathbf{a}_z, & r \le a \\ 0, & r > a, \end{cases}$$
where cylindrical coordinates are used. (a) Find the electric field $\mathbf{E}(r, \varphi, z, t)$ everywhere. (b) If the frequency ω is increased, do the above results of electric and magnetic fields have to be modified? Discuss it briefly. (15%)

6. Consider a time-varying magnetic field $\mathbf{H}(y, z, t) = H_0 \cos(\omega t - \alpha y - \beta z)\mathbf{a}_x$ in free space (μ_0, ε_0). (a) Find the corresponding electric field $\mathbf{E}(y, z, t)$. (b) Find the necessary condition relating ω, α, β, μ_0, and ε_0 for the fields to satisfy both of Maxwell's curl equations. (c) Find the ratio of $|\mathbf{E}(y, z, t)|$ to $|\mathbf{H}(y, z, t)|$. (15%)

7. To solve a wave equation $\dfrac{\partial^2 E(z,t)}{\partial z^2} - \dfrac{1}{c^2} \cdot \dfrac{\partial^2 E(z,t)}{\partial t^2} = 0$, one can change the independent variables from (z, t) to (ξ, η) by $\xi = z - ct$ and $\eta = z + ct$. (a) Show that, with these new independent variables, the original equation becomes $\dfrac{\partial^2 E}{\partial \xi \partial \eta} = 0$. (b) Show that the solution of the new equation can be written as $E = f(\xi) + g(\eta)$, where f and g are arbitrary differentiable functions. (10%)

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$ in spherical coordinates