Complex Analysis: Midterm Examination

10:20 AM - 12:00 PM, April 14, 2009.

[1] (10 %) Find all values of $\left(-8-i8\sqrt{3}\right)^{1/4}$ in the form of $a+ib$.

[2] (10 %) Prove $f(z)=e^{y}e^{ix}$ is nowhere analytic, where $z=x+iy$.

[3] (15 %) True or false (If it is false, explain briefly why it isn't true)

(a) (5 %) If $f(z)$ is analytic on a closed contour C, then $\oint_C f(z)dz=0$.

(b) (5 %) If $f(z)$ is differentiable at a point z_0 and at every point in some neighborhood of z_0, then $f(z)$ is an entire function.

(c) (5 %) $S=\{z\,|\mathrm{Re}(z)\neq 3\}$ is a domain (open connected set).

[4] (15 %) Verify that $u(x,y)=e^{x}(x\cos y-y\sin y)$ is harmonic. Find $v(x,y)$, the conjugate harmonic function of $u(x,y)$.

[5] (10 %) Evaluate $\int_C \frac{1}{z}dz$ in the form of $a+ib$, where C is the arc of the circle $z=4e^{it}$ with $-\pi/2\le t\le \pi/2$.

[6] (10 %) Evaluate $\oint_C \left(\frac{3}{z+2}-\frac{1}{z-2i}\right)dz$, where C is the circle $|z|=5$.

[7] (10 %) Expand $f(z)=\frac{1+z}{1-z}$ in the Taylor series centered at $z_0=i$, and give the radius of convergence of this series.

[8] (10 %) Let $f(z)=u(x,y)+iv(x,y)$ where the first partial derivatives of $u(x,y)$ and $v(x,y)$ are continuous. Prove that $f(z)$ is analytic at z <u>if and only if</u>

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ \text{and}\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

[9] (10 %) Assume $f(z)$ is analytic in a domain D, and C is a closed contour lying entirely in D. Use the fact that $f'(z_0)=\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$, with z_0 within C, to prove

$$f''(z_0)=\frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}dz.$$

Hint: $f''(z_0)=\lim_{\Delta z\to 0}\frac{f'(z_0+\Delta z)-f'(z_0)}{\Delta z}$.