Signals and Systems, Final Exam Solutions (Draft)

Spring 2004, Edited by bypeng

1. (8) The output $y(t)$ of a causal LTI system is related to its input $x(t)$ by $\frac{dy(t)}{dt}+3y(t)=x(t)$.

   (a)(4) Determine the frequency response $H(j\omega)$ of the system.

   (b)(4) Find the group delay of this system.

Solution:

(a) From $\frac{dy(t)}{dt}+3y(t)=x(t)$, we have $j\omega Y(j\omega)+3Y(j\omega)=X(j\omega)$, and then $H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{1}{3+j\omega}$.

(b) The definition of the group delay is given by $\tau(\omega)\triangleq-\frac{d}{d\omega}\{\sphericalangle H(j\omega)\}$, and $\sphericalangle H(j\omega)=\sphericalangle 1-\sphericalangle\frac{\omega}{3}=-\tan^{-1}\frac{\omega}{3}$.

Therefore, the group delay of this system is $\tau(\omega)=-\frac{d}{d\omega}\left(-\tan^{-1}\frac{\omega}{3}\right)=\frac{d}{d\omega}\tan^{-1}\frac{\omega}{3}=\frac{1}{3}\cdot\frac{1}{1+(\frac{\omega}{3})^2}=\frac{3}{9+\omega^2}$.

Grading:

(a) Finding $H(j\omega)=\frac{1}{3+j\omega}$ gets **4 pts**. Otherwise, writing $H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}$ gets **2 pts**.

(b) Finding $\tau(\omega)=\frac{3}{9+\omega^2}$ gets **4 pts**. Otherwise, writing $\tau(\omega)=-\frac{d}{d\omega}\{\sphericalangle H(j\omega)\}$ and $\sphericalangle H(j\omega)=-\tan^{-1}\frac{\omega}{3}$ gets **3 pts**, and any one of them costs **2 pts**.

2. (10) For the discrete-time causal LTI system described by $y[n]-2r\cos(\theta)y[n-1]+r^2y[n-2]=x[n]$.

   (a)(6) Find the impulse response $h[n]$ of this system.

   (b)(4) What is/are the conditions for the system to be stable?

Solution:

(a) $y[n]-2r\cos(\theta)y[n-1]+r^2y[n-2]=x[n] \Rightarrow Y(e^{j\omega})-2r\cos(\theta)e^{-j\omega}Y(e^{j\omega})+r^2e^{-2j\omega}Y(e^{j\omega})=X(e^{j\omega}) \Rightarrow$

$H(e^{j\omega})=\frac{Y(e^{j\omega})}{X(e^{j\omega})}=\frac{1}{1-2r\cos(\theta)e^{-j\omega}+r^2e^{-2j\omega}}=\frac{1}{(1-re^{j\theta}e^{-j\omega})(1-re^{-j\theta}e^{-j\omega})}$. If $\theta\neq k\pi$ for any $k\in\mathbf{Z}$,

$$H(e^{j\omega})=\frac{\frac{-e^{2j\theta}}{1-e^{2j\theta}}}{(1-re^{j\theta}e^{-j\omega})}+\frac{\frac{1}{1-e^{2j\theta}}}{(1-re^{-j\theta}e^{-j\omega})} \Rightarrow h[n]=\left(\frac{-e^{2j\theta}}{1-e^{2j\theta}}(e^{j\theta})^n+\frac{1}{1-e^{2j\theta}}(e^{-j\theta})^n\right)r^nu[n]$$

$$=\frac{-e^{j\theta}}{1-e^{2j\theta}}\left(e^{j\theta}e^{jn\theta}-e^{-j\theta}e^{-jn\theta}\right)r^nu[n]=\frac{1}{e^{j\theta}-e^{-j\theta}}\left(e^{j(n+1)\theta}-e^{-j(n+1)\theta}\right)r^nu[n]=\frac{\sin[(n+1)\theta]}{\sin\theta}r^nu[n].$$

If $\theta=k\pi$ for some $k\in\mathbf{Z}$, $H(e^{j\omega})=\frac{1}{(1\pm re^{-j\omega})^2} \Rightarrow h[n]=(n+1)(\mp r)^nu[n]$.

(b) For both cases, we need $|r|<1$ to make $h[n]$ absolutely summable, and then to make the system stable.

Grading:

(a) Finding $h[n]$ in both cases gets **full 6 pts**. Otherwise: finding $H(e^{j\omega})=\frac{1}{(1-re^{j\theta}e^{-j\omega})(1-re^{-j\theta}e^{-j\omega})}$ gets **2 pts**.

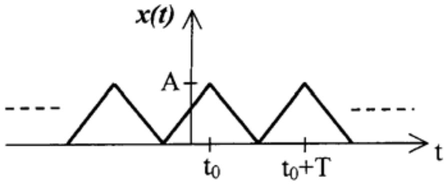
Finding $H(e^{j\omega})=\frac{\frac{-e^{2j\theta}}{1-e^{2j\theta}}}{(1-re^{j\theta}e^{-j\omega})}+\frac{\frac{1}{1-e^{2j\theta}}}{(1-re^{-j\theta}e^{-j\omega})}$ in the first case gets **1 another pt**, and finding $h[n]=\frac{\sin[(n+1)\theta]}{\sin\theta}r^nu[n]$ in the first case gets **1 more pt**. Finding $H(e^{j\omega})=\frac{1}{(1\pm re^{-j\omega})^2}$ in the second case gets **1 another pt** and finding $h[n]=(n+1)(\mp r)^nu[n]$ in the second case gets **1 more pt**.

(b) Check the answer case by case.

3. (14) A periodic triangle wave, given below, is to be sampled periodicly using impulse-train sampling.

(a)(8) Draw the system diagram and show the required operations to sample and reconstruct it.

(b)(6) Discuss performances of the system you designed in (a).



Solution:

There is no precise solution to this problem, but there is one key view point. One should mention that the triangle wave is NOT band-limited, so one cannot perfectly reconstruct (or can perfectly reconstruct with first-order hold interpolation and with probability 0) the triangular wave after the impulse-train sampling before knowing that the sampled signal is triangular.

Grading: Check case by case. Mentioning the sampling theorem and the band-limitlessness of the triangular wave should get more pts.

4. (12) For two band limited signals $x_1(t)$ and $x_2(t)$ with

$$X_1(j\omega)=0, |\omega|\geq\omega_1 \text{ and } X_2(j\omega)=0, |\omega|\geq\omega_2$$

(a)(6) Determine the minimum sampling rate required for $u(t)=x_1(t)x_2(t)$ such that $u(t)$ is recoverable by the use of an ideal lowpass filter.

(b)(6) Same as the above problem (a), but for $v(t)=x_1(t)-x_2(t)$.

Solution:

(a) We know that $u(t)=x_1(t)x_2(t)\xleftrightarrow{CTFT}U(j\omega)=X_1(j\omega)*X_2(j\omega)$. Therefore, if $X_1(j\omega)=0, |\omega|\geq\omega_1$ and $X_2(j\omega)=0, |\omega|\geq\omega_2$, with the property of the convolution, we have $U(j\omega)=0, |\omega|\geq\omega_1+\omega_2$. And then the minimum sampling rate required is $2(\omega_1+\omega_2)$.

(b) We know that $v(t)=x_1(t)-x_2(t)\xleftrightarrow{CTFT}V(j\omega)=X_1(j\omega)-X_2(j\omega)$. Therefore, if $X_1(j\omega)=0, |\omega|\geq\omega_1$ and $X_2(j\omega)=0, |\omega|\geq\omega_2$, we have $V(j\omega)=0, |\omega|\geq\max\{\omega_1,\omega_2\}$. And then the minimum sampling rate required is $2(\max\{\omega_1,\omega_2\})$.

Grading:

(a) Finding the minimum sampling rate out gets **full 6 pts**. Otherwise, mentioning that $u(t)=x_1(t)x_2(t)\xleftrightarrow{CTFT}U(j\omega)=X_1(j\omega)*X_2(j\omega)$ gets **3 pts**. (b) Same as in (a).

5. (10)

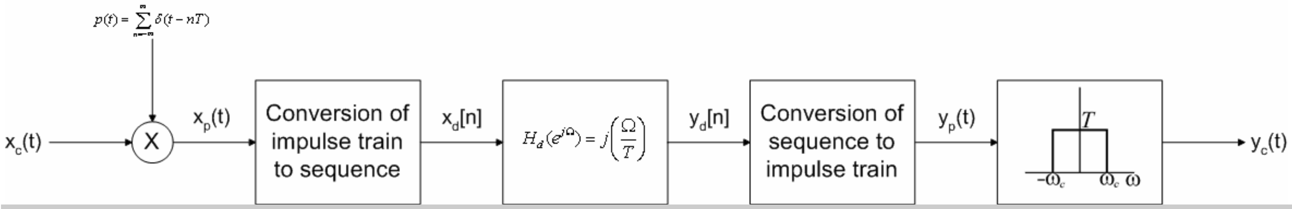
(a)(4) What is the corresponding discrete-time transfer function of the continuous-time band-limited differentiating filter $H_c(j\omega)=\begin{cases} j\omega & |\omega|<\omega_c \\ 0 & |\omega|>\omega_c \end{cases}$?

(b)(6) Draw the system diagram for the required operations to implement the differentiating filter $H_c(j\omega)$ digitally, with input $x_c(t)$ and output $y_c(t)$.

Solution:

(a) We have $H_c(j\omega)=\begin{cases} H_d(e^{j\omega T}) & |\omega|<\omega_s/2 \\ 0 & |\omega|>\omega_s/2 \end{cases}$, therefore $H_d(e^{j\Omega})=j\left(\frac{\Omega}{T}\right), |\Omega|<\pi$, where $T=\frac{2\pi}{\omega_s}=\frac{\pi}{\omega_c}$.

(b) With $T=\frac{\pi}{\omega_c}$:



Grading:
(a) Check case by case. (b) Whole diagram costs **6 pts**. CD- $H_d(e^{j\Omega})$ -DC part costs **4 pts**. The sampling part costs **2 pts**, and the filter part cost **2 pts**. Equivalent model will be checked case by case.

6. (10) There is intersymbol interference (ISI) problem for transmitting PAM signals (with symbol spacing T_1) over band-limited channels, shape of the transmitted pulses can be designed to allow PAM signals be free from the ISI problem. Prove that a pulse shape $p(t)$ with

$$P(j\omega)=\begin{cases}\frac{1}{2}(1+\cos\frac{\omega T_1}{2}) & 0\le|\omega|\le\frac{2\pi}{T_1}\\ 0 & elsewhere\end{cases}$$ can achieve ISI free transmission.

Solution:
To show the ISI freedom of the transmission, we need to show that $p(0)=p_0$ for some $p_0\neq 0$ and $p(kT_1)=0$ for any $k\in\mathbf{Z}-\{0\}$. By the inverse Fourier transform, $p(t)=\frac{1}{2\pi}\int_{-\frac{2\pi}{T_1}}^{\frac{2\pi}{T_1}}\left(\frac{1}{2}+\cos\frac{\omega T_1}{2}\right)e^{j\omega t}d\omega$. Then

$$p(0)=\frac{1}{2\pi}\int_{-\frac{2\pi}{T_1}}^{\frac{2\pi}{T_1}}\left(\frac{1}{2}+\cos\frac{\omega T_1}{2}\right)d\omega=\frac{1}{2\pi}\left(\frac{1}{2}\omega+\frac{2}{T_1}\sin\frac{\omega T_1}{2}\right)\Bigg|_{\omega=-\frac{2\pi}{T_1}}^{\frac{2\pi}{T_1}}=\frac{1}{T_1}$$

$$p(kT_1)=\frac{1}{2\pi}\int_{-\frac{2\pi}{T_1}}^{\frac{2\pi}{T_1}}\left(\frac{1}{2}+\cos\frac{\omega T_1}{2}\right)e^{j\omega kT_1}d\omega=\frac{1}{2\pi}\left(\int_{-\frac{2\pi}{T_1}}^{\frac{2\pi}{T_1}}\frac{1}{2}e^{j\omega kT_1}d\omega+\int_{-\frac{2\pi}{T_1}}^{\frac{2\pi}{T_1}}(e^{j\frac{\omega T_1}{2}}+e^{-j\frac{\omega T_1}{2}})e^{j\omega kT_1}d\omega\right)=0$$

Grading: Check case by case.

7. (8) The signal $x(t)=\frac{\sin\pi t}{\pi t}$ is amplitude-modulated to be $w(t)=(x(t)+A)\cos(10\pi t)$. Find the minimum value of the constant A such that $x(t)$ can be recovered from $w(t)$ using asynchronous demodulation.

Solution:
$x(t)$ can be recovered from $w(t)$ using asynchronous demodulation if and only if $x(t)+A\ge 0$ for any $t\in\mathbf{R}$.
We have $\frac{d}{dt}x(t)=\frac{\pi^2 t\cos\pi t-\pi\sin\pi t}{\pi^2t^2}=\frac{\pi t\cos\pi t-\sin\pi t}{\pi t^2}\equiv 0$, $t=\frac{1}{\pi}\tan\pi t$. We may find the minimum value of $x(t)$ occurs at $t\simeq\pm 1.43$, at which $x(\pm1.43)\simeq -0.217\simeq 0.22$. Therefore, $A=0.22$.
Alternative approach to find t satisfying $t=\frac{1}{\pi}\tan\pi t$: Since $\tan\pi t$ is bluff around $t=\frac{1}{2}+k$ for any $k\in\mathbf{Z}$, an approximation of the minimum value of $x(t)$ occurs at $t\simeq\pm1.5$, at which $x(\pm1.5)\simeq-\frac{2}{3\pi}\simeq 0.212$. Therefore $A=0.22$.

Grading: Finding the minimum A gets full **8 pts**. Otherwise, mentioning $x(t)+A\ge 0$ gets **5 pts**.

8. (8) Let $X(s)$ be the Laplace transform of $x(t)$ and its region of convergence (ROC) be **R**. Find the Laplace transform's and the ROC's for the following signals: $x(-t)$ and $x^*(t)$.

Solution:

$$X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}dt\ \Rightarrow\ \int_{-\infty}^{\infty}x(-t)e^{-st}dt\underbrace{=}_{t=-\tau;dt=-d\tau}\int_{\infty}^{-\infty}-x(\tau)e^{-s(-\tau)}d\tau=\int_{-\infty}^{\infty}x(\tau)e^{-(-s)\tau}d\tau=X(-s).$$

Since $X(s)$ has ROC **R**, the ROC of $X(-s)$ is $-\mathbf{R}$, the complement of **R**.

$$\int_{-\infty}^{\infty} x^*(t)e^{-st}dt = \int_{-\infty}^{\infty} (x(t)(e^{-st})^*)^* dt = \left(\int_{-\infty}^{\infty} x(t)(e^{-st})^* dt\right)^* = \left(\int_{-\infty}^{\infty} x(t)e^{-s^*t}dt\right)^* = X^*(s^*) .$$

And the ROC of $X^*(s^*)$ is also **R**.

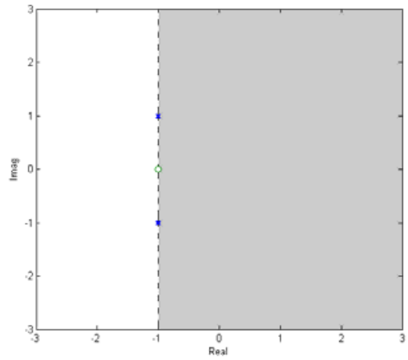
Grading: Derivation is necessary and costs **1 pt** each function. Finding the Laplace transform of each function gets **2 pts** and its ROC costs **1 pt**.

9. (10) For the causal LTI system $H(s) = \dfrac{s+1}{s^2+2s+2}$,

(a)(4) Draw the pole-zero plot and identify the ROC of it.

(b)(6) Determine the response $y(t)$ when the input is $x(t) = e^{-|t|}$, $-\infty < t < \infty$.

Solution:



(a) $H(s) = \dfrac{s+1}{s^2+2s+2} = \dfrac{s+1}{(s+1+j)(s+1-j)} = \dfrac{s+1}{(s+1)^2+1}$

Poles: $-1+j$, $-1-j$; Zeros: -1, ∞.

Since the system is causal, we need $h(t)$ to be right-sided, and then the ROC is $\mathrm{Re}\{s\} > -1$.

(b) $x(t) = e^{-|t|} = e^{-t}u(t) + e^{t}u(-t)$, $X(s) = \dfrac{1}{s+1} - \dfrac{1}{s+1} = \dfrac{-2}{s^2-1}$, with ROC $-1 < \mathrm{Re}\{s\} < 1 \;\Rightarrow$

$$Y(s) = \frac{-2}{(s^2+2s+2)(s-1)} = -\frac{2}{5}\frac{1}{s-1} + \frac{2}{5}\frac{s+3}{s^2+2s+2} = -\frac{2}{5}\left(\frac{1}{s-1} - \frac{s+1}{(s+1)^2+1} - \frac{2}{s^2+2s+2}\right)$$ with ROC $-1 < \mathrm{Re}\{s\} < 1 \Rightarrow$

$$y(t) = \frac{2}{5}e^{t}u(-t) + \frac{2}{5}e^{-t}\cos t\, u(t) + \frac{4}{5}e^{-t}\sin t\, u(t) .$$

Grading:

(a) The poles, the zeros, and the ROC are necessary to be drawn in the pole-zero plot. One loss of the necessary part will **decrease** the pts one get by **1 pt**.

(b) Finding $y(t) = \frac{2}{5}e^{t}u(-t) + \frac{2}{5}e^{-t}\cos t\, u(t) + \frac{4}{5}e^{-t}\sin t\, u(t)$ gets **6 pts**. Otherwise, Finding $Y(s)$ gets **4 pts**.

10. (10) Find the possible inverse Laplace transform's of $H(s) = \dfrac{b}{(s-a)^n}$ with ROC's specified.

Solution:

(There are two approaches to find the inverse Laplace transform of $H(s)$. The first approach is to use induction from $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\}$ and some properties of Laplace transform. The second approach is to use the residue approach one just learned in the course of complex analysis. BTW, the answer is $b\dfrac{t^{n-1}}{(n-1)!}e^{at}u(t)$ if the ROC is $\mathrm{Re}\{s\} > a$, and $-b\dfrac{t^{n-1}}{(n-1)!}e^{at}u(-t)$ if the ROC is $\mathrm{Re}\{s\} < a$.)

Grading:

Using either approach to find each possible inverse Laplace transform gets **5 pts**. Note that if one directly apply the table from his/her personal note, then at least the first approach is needed. Otherwise only **3 pts** for each possible inverse Laplace transform are credited.

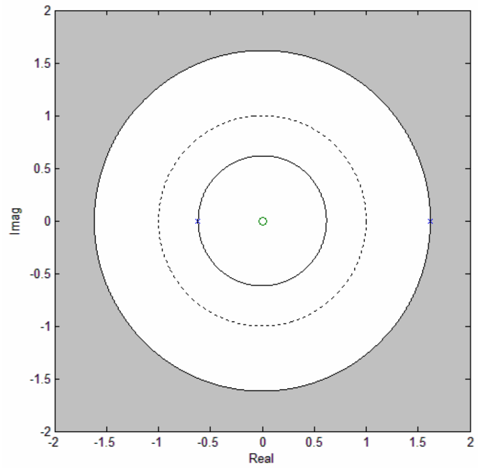
11. (12) For the causal LTI system described by $y[n] = y[n-1] + y[n-2] + x[n-1]$.

(a)(4) Find the system function $H(z)$ for the system.

(b)(4) Plot the poles and zeros of $H(z)$ and indicate the ROC.

(c)(6) Find a stable unit sample response that satisfies the difference equation.

Solution:



(a) We have $y[n] - y[n-1] - y[n-2] = x[n-1] \;\Rightarrow$

$$Y(z) - z^{-1}Y(z) - z^{-2}Y(z) = z^{-1}X(z) \;\Rightarrow\; H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}}.$$

(b) $H(z) = \dfrac{z^{-1}}{1 - z^{-1} - z^{-2}} = \dfrac{z^{-1}}{(1 - \frac{1+\sqrt{5}}{2} z^{-1})(1 - \frac{1-\sqrt{5}}{2} z^{-1})}$.

Poles: $\dfrac{1+\sqrt{5}}{2}$, $\dfrac{1-\sqrt{5}}{2}$.

Zeros: 0, ∞.

According to the time-domain relationship, the system is causal, and then the impulse response is right-sided. Therefore, the ROC is $|z| > \dfrac{1+\sqrt{5}}{2}$.

(c) To find a stable unit sample response, we need the ROC being $\dfrac{\sqrt{5}-1}{2} < |z| < \dfrac{1+\sqrt{5}}{2}$, which results in absolute summability of $h[n]$. And then $H(z) = \dfrac{-\frac{1}{\sqrt{5}}}{1 - \frac{1+\sqrt{5}}{2} z^{-1}} + \dfrac{\frac{1}{\sqrt{5}}}{1 - \frac{1-\sqrt{5}}{2} z^{-1}} \Rightarrow h[n] = -\dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n u[-n-1] + \dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)^n u[n]$.

Grading: (Note there is a grade overflow in this problem. The total credit to this problem will be the summation of the three sub-problems until the summation is more than 12 pts, which results in 12 pts.)

(a) Full or zero. (b) Same as in 9(a). (c) Finding $h[n]$ gets **6 pts**. Otherwise, conclusion with ROC modified to $\dfrac{\sqrt{5}-1}{2} < |z| < \dfrac{1+\sqrt{5}}{2}$ gets **3 pts**.

12. (10) A discrete-time LTI system, with input $x[n]$ and output $y[n]$, is known to have the following properties:

If $x[n] = (-2)^n$ for n, then $y[n] = 0$ for all n, and

If $x[n] = (1/2)^n u[n]$ for all n, then $y[n] = \delta[n] + a(1/4)^n u[n]$ for all n,

Determine the value of the constant a.

Solution:

By $x[n] = (-2)^n \;\Rightarrow\; y[n] = 0$, we know that $H(z)$ satisfies $H(-2) = 0$.

And by $x[n] = (1/2)^n u[n]$, $X(z) = \dfrac{1}{1 - \frac{1}{2} z^{-1}}$ with ROC $|z| > \frac{1}{2}$, we have $Y(z) = 1 + \dfrac{a}{1 - \frac{1}{4} z^{-1}} = \dfrac{1 + a - \frac{1}{4} z^{-1}}{1 - \frac{1}{4} z^{-1}}$ with ROC $|z| > \frac{1}{4}$, $\Rightarrow$ $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{(1 + a - \frac{1}{4} z^{-1})(1 - \frac{1}{2} z^{-1})}{1 - \frac{1}{4} z^{-1}}$ with ROC $|z| > \frac{1}{4}$. Now

$$H(-2) = \frac{(1 + a + \frac{1}{8})(1 + \frac{1}{4})}{1 + \frac{1}{8}} = 0 \;\Rightarrow\; a = -\frac{9}{8}.$$

Grading:

Finding $a = -\dfrac{9}{8}$ gets **10 pts**. Otherwise, finding $H(z) = \dfrac{(1 + a - \frac{1}{4} z^{-1})(1 - \frac{1}{2} z^{-1})}{1 - \frac{1}{4} z^{-1}}$ with ROC $|z| > \frac{1}{4}$ gets **4 pts**, and finding $H(-2) = 0$ gets **4 pts**.