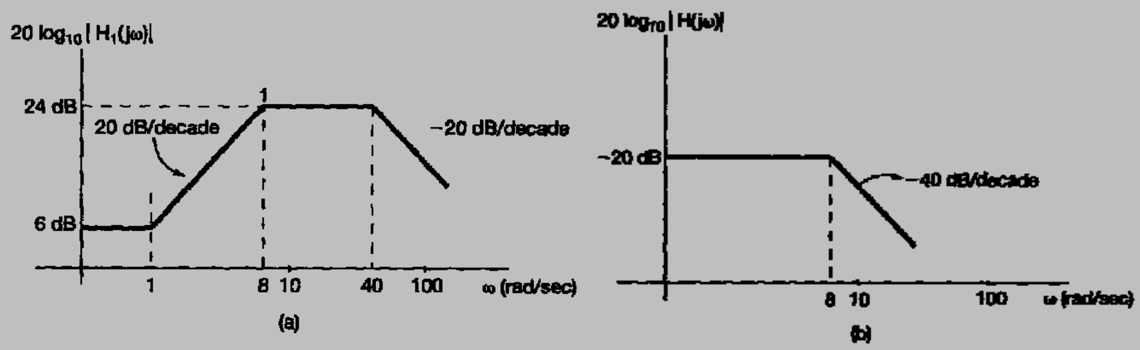
Signals and Systems, Final Exam

Solutions (Draft)

Spring 2007, Edited by bypeng

1. **[12]** A continuous-time LTI system with frequency response $H(j\omega)$ is constructed from two continuous-time LTI systems with frequency responses $H_1(j\omega)$ and $H_2(j\omega)$, respectively. The straight-line approximations of the Bode magnitude plots of $H_1(j\omega)$ and $H(j\omega)$ are shown in the following figure. All of the poles and zeros of $H_1(s)$ and $H(s)$ are on the real axis.



a) **[4]** Specify $H_2(j\omega)$ if $H_1(j\omega)$ and $H_2(j\omega)$ are connected in cascade form.
b) **[4]** Specify $H_2(j\omega)$ if $H_1(j\omega)$ and $H_2(j\omega)$ are connected in parallel form.
c) **[4]** Specify $H_2(j\omega)$ if $H_1(j\omega)$ and $H_2(j\omega)$ are connected in negative feedback form with $H_2(j\omega)$ in the feedback loop.

Solution:

We know that

1) $H_1(j\omega) = \dfrac{A(j\omega+1)}{(j\omega+8)(j\omega+40)}$, and by $H_1(0)=2$ (since 6dB), $\dfrac{A\cdot 1}{8\cdot 40}=2$, $A=640$.

2) $H(j\omega) = \dfrac{B}{(j\omega+8)^2}$, any by $H(0)=0.1$ (since -20dB), $\dfrac{B}{8\cdot 8}=0.1$, $B=6.4$.

Therefore:

a) Cascade form: $H(j\omega)=H_1(j\omega)H_2(j\omega)$,

$$H_2(j\omega)=\frac{H(j\omega)}{H_1(j\omega)}=\frac{6.4}{(j\omega+8)^2}\cdot\frac{(j\omega+8)(j\omega+40)}{640(j\omega+1)}=\frac{0.01(j\omega+40)}{(j\omega+8)(j\omega+1)}$$

b) Parallel form: $H(j\omega)=H_1(j\omega)+H_2(j\omega)$,

$$H_2(j\omega)=H(j\omega)-H_1(j\omega)=\frac{6.4}{(j\omega+8)^2}-\frac{640(j\omega+1)}{(j\omega+8)(j\omega+40)}$$

$$=\frac{6.4(j\omega+40)-640(j\omega+1)(j\omega+8)}{(j\omega+8)^2(j\omega+40)}=\frac{-640(j\omega)^2-5753.6(j\omega)-4864}{(j\omega+8)^2(j\omega+40)}$$

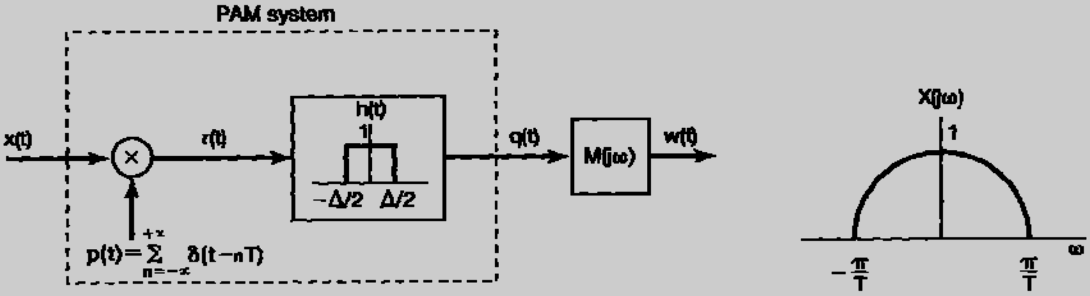
$$=-\frac{32}{5}\frac{100(j\omega)^2+899(j\omega)+760}{(j\omega+8)^2(j\omega+40)}\approx-\frac{640(j\omega+0.94465)(j\omega+8.04535)}{(j\omega+8)^2(j\omega+40)}$$

c) Feedback form: $H(j\omega)=\dfrac{H_1(j\omega)}{1+H_1(j\omega)H_2(j\omega)}$,

$$H_2(j\omega)=\frac{1}{H(j\omega)}-\frac{1}{H_1(j\omega)}=\frac{(j\omega+8)^2}{6.4}-\frac{(j\omega+8)(j\omega+40)}{640(j\omega+1)}$$

$$=\frac{(j\omega+8)\left[100(j\omega)^2+899(j\omega)+760\right]}{640(j\omega+1)}\approx\frac{(j\omega+8)(j\omega+0.94465)(j\omega+8.04535)}{6.4(j\omega+1)}$$

2. **[12]** Figure (a) below shows a model of amplitude modulation system using a pulse-train carrier.



(a) (b)

a) **[4]** Let $x(t)$ be a band-limited signal with $X(j\omega)=0$ for $|\omega|\ge \pi/T$, as shown above. Determine and sketch $R(j\omega)$ and $Q(j\omega)$.

b) **[4]** Find the maximum value of Δ such that $w(t)=x(t)$ with an appropriate filter $M(j\omega)$.

c) **[4]** Determine and sketch the compensating filter $M(j\omega)$ such that $w(t)=x(t)$.

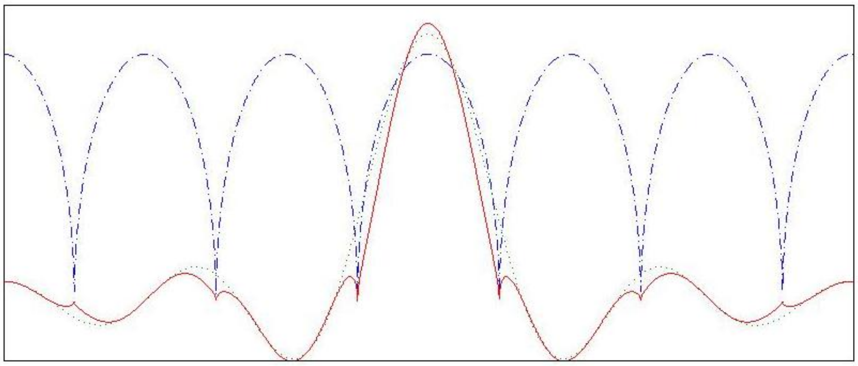
Solution:

a) We have

$$R(j\omega)=\frac{1}{2\pi}\left[X(j\omega)*P(j\omega)\right]=X(j\omega)*\left[\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{T}\right)\right]=\frac{1}{T}\sum_{k=-\infty}^{\infty}X\left(j\left(\omega-\frac{2\pi k}{T}\right)\right)$$

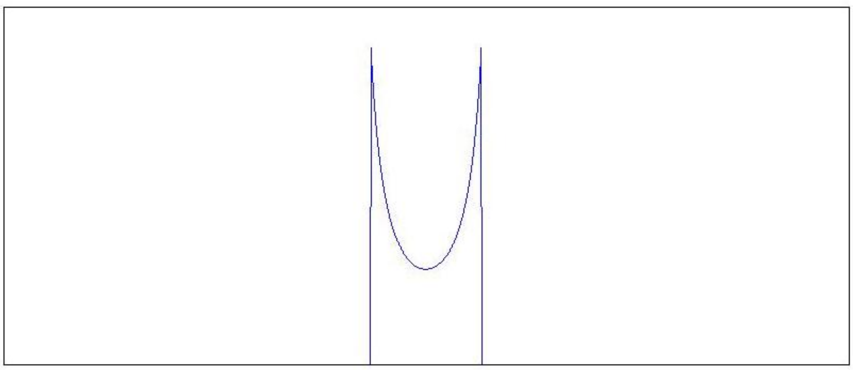
$$Q(j\omega)=R(j\omega)H(j\omega)=\frac{2R(j\omega)\sin\frac{\omega\Delta}{2}}{\omega}$$

The sketch is as the following, where the dash-dot line is $R(j\omega)$ and the solid line is $Q(j\omega)$.



b) We need $\frac{2\pi}{\Delta}>\frac{\pi}{T}$ to keep $Q(j\omega)$ in the interval $\left\{\omega:-\frac{\pi}{T}<\omega<\frac{\pi}{T}\right\}$ not distorted. $\Delta<2T$.

c) $M(j\omega)=\begin{cases}\dfrac{T}{H(j\omega)} & -\dfrac{\pi}{T}<\omega<\dfrac{\pi}{T}\\ 0 & \text{elsewhere}\end{cases}=\begin{cases}\dfrac{\omega T}{2\sin\frac{\omega\Delta}{2}} & -\dfrac{\pi}{T}<\omega<\dfrac{\pi}{T}\\ 0 & \text{elsewhere}\end{cases}$, the sketch is as the following.



3. **[8]** Determine $X(s)$ and its region of convergence based on the following 5 descriptions about a signal $x(t)$, which is real, and its Laplace transform $X(s)$:

i) $X(0) = 4$

ii) $X(s)$ has exactly two poles

iii) $X(s)$ has no zeros in the finite s-plane

iv) $X(s)$ has a pole at $s = j - 2$

v) $e^{3t}x(t)$ is absolutely integrable

Solution:

By ii) and iii), we know that

$$X(s) = \frac{A}{(s+p_1)(s+p_2)}$$

Since $x(t)$ is real, by iv) we know that $p_2 = -j - 2$ and then

$$X(s) = \frac{A}{(s+j+2)(s-j+2)} = \frac{A}{s^2+4s+5}$$

By $X(0) = 4$, the gain can be solved to be $A = 20$, and then

$$X(s) = \frac{20}{s^2+4s+5}$$

According to the positions of the poles, the possible ROC must be $\mathrm{Re}[s] > -2$ or $\mathrm{Re}[s] < -2$. Since $e^{3t}x(t)$ is absolutely integrable, the ROC of $X(s-3)$, whose ROC must be $\mathrm{Re}[s] > 1$ or $\mathrm{Re}[s] < 1$, must contain the imagery axis. We conclude that the ROC of $X(s-3)$ is $\mathrm{Re}[s] < 1$, and then the ROC of $X(s)$ is

$$\mathrm{Re}[s] < -2$$

4. **[12]** Consider an LTI system with $H(z) = \dfrac{64}{49z^{-2}+64}$, $|z| > \frac{7}{8}$.

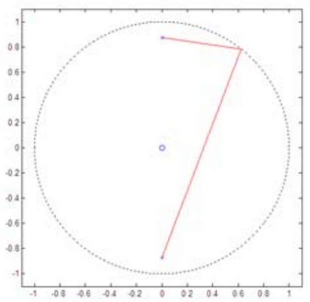
a) **[4]** Use geometric evaluation of the magnitude of the Fourier transform from the pole-zero plot to determine if the system is approximately lowpass, bandpass, or highpass.

b) **[4]** Determine if the system is causal.

c) **[4]** Determine $h[n]$.

Solution:

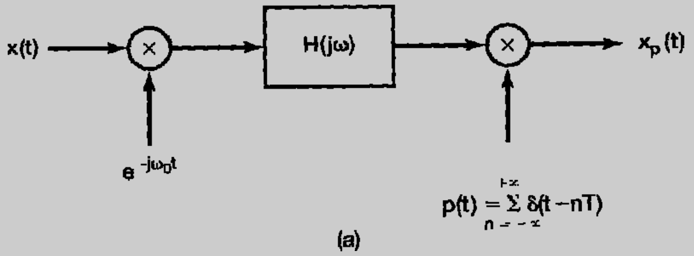
a) $H(z) = \dfrac{64}{49z^{-2}+64} = \dfrac{1}{\left[1+\left(\frac{7}{8}j\right)z^{-1}\right]\left[1-\left(\frac{7}{8}j\right)z^{-1}\right]}$, $p_1 = \dfrac{7}{8}j$, $p_2 = \dfrac{7}{8}j$. Referring to the right figure, we may see that the product of the two red line becomes smaller as the frequency becomes closer to $\pi/2$. So the downstairs part is smaller as the frequency becomes closer to $\pi/2$. We conclude that this system is bandpass.



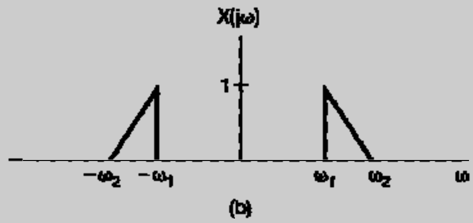
b) Since the ROC is exterior of the circle $|z| < \frac{7}{8}$, the system is causal.

c) $h[n] = \left(\dfrac{7}{8}\right)^n \cos\left(\dfrac{\pi n}{2}\right)u[n]$.

5. **[16]** The sampling theorem states that a signal must be sampled at a rate larger than twice its highest frequency. For a bandpass signal with its energy concentrates in a narrow band, however, it is possible to have a sampling rate that is lower than twice the highest frequency of the signal to be sampled. To examine the possibility, let us apply the system shown in Figure (a) below to sample and reconstruct a bandpass signal $x(t)$, which is real, with $X(j\omega)$ nonzero only for $\omega_1 < |\omega| < \omega_2$, as shown in Figure (b) below. The system consists of multiplying the signal $x(t)$ by a complex-exponential $e^{-j\omega_0 t}$, sending the product to an ideal lowpass filter $H(j\omega)$, and then sampling the filtered signal.



(a)



(b)

Suppose $\omega_1 > \omega_2 - \omega_1$ and the cutoff frequency of $H(j\omega)$ is $(\omega_2 - \omega_1)/2$.

a) **[4]** Determine the frequency ω_0 for the complex exponential.

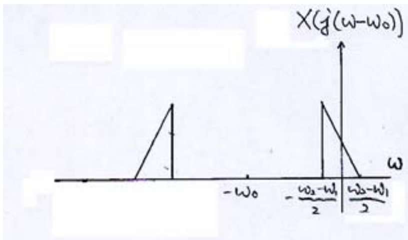
b) **[4]** Determine the maximum sampling period T such that $x(t)$ is recoverable from $x_p(t)$.

c) **[4]** Sketch $X_p(j\omega)$.

d) **[4]** Determine a system to recover $x(t)$ from $x_p(t)$.

Solution:

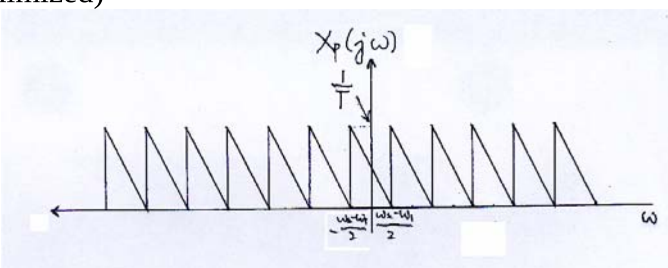
a) We want to shift the spectrum of $x(t)$ such that $H(j\omega)$ can "cover." The spectrum of the output of the multiplier is as the right figure. So



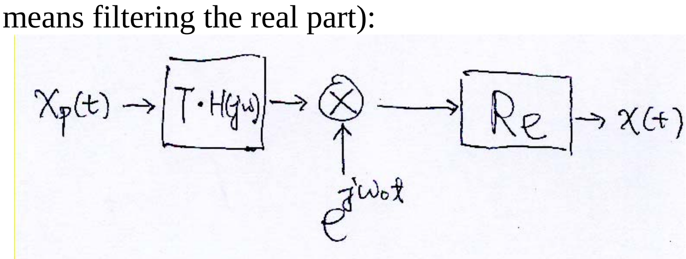
$$\omega_0 = \frac{1}{2}\left(\frac{\omega_2 - \omega_1}{2} + 2\omega_1 + \frac{\omega_2 - \omega_1}{2}\right) = \frac{\omega_1 + \omega_2}{2}$$

b) We need $\omega_s > 2\dfrac{\omega_2 - \omega_1}{2} = \omega_2 - \omega_1$, so $T = \dfrac{2\pi}{\omega_s} = \dfrac{2\pi}{\omega_2 - \omega_1}$.

c) (In the case T maximized)



d) An example ("Re" means filtering the real part):



6. **[10]** Assume that the relationship of a continuous-time LTI system is given by

$$\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}+5y(t)=\frac{d^2x(t)}{dt^2}-2\frac{dx(t)}{dt}+x(t)$$

a) **[5]** Find the system function $H(s)$ of the continuous-time LTI system.

b) **[5]** Does this system have a stable and causal inverse? Why or why not?

Solution:

a) $s^2Y(s)+sY(s)+5Y(s)=s^2X(s)-2sX(s)+X(s)$,

$$H(s)=\frac{Y(s)}{X(s)}=\frac{s^2-2s+1}{s^2+s+5}$$

We need the Laplace transform to converge, and the poles are $p=\frac{-1\pm\sqrt{19}j}{2}$, so the ROC is

$$\mathrm{Re}[s]>-\frac{1}{2}$$

b) $H^{-1}(s)=\frac{s^2+s+5}{s^2-2s+1}=\frac{s^2+s+5}{(s-1)^2}$. The poles are in the right-hand side of the imagery axis.

For the case $\mathrm{Re}[s]>1$, the inverse system is causal but not stable, as the ROC is the right-half plane but does not include the imagery axis.

for the case $\mathrm{Re}[s]<1$, the inverse system is stable but not causal, as the ROC is the right-half plane but does not include the imagery axis.

The conclusion is that this system doesn't have a stable AND causal inverse.

7. **[14]** Assume that a discrete-time LTI system has the input and output related by the following difference equation: $y[n]-0.75y[n-1]+0.125y[n-2]=x[n]$.

a) **[8]** Find $y[n]$ by using z-transform when $x[n]=1$ for $n=0$ and $x[n]=0$ for $n\neq 0$.

b) **[6]** Verify the value of $y[0]$ in part a) by using the initial-value theorem.

Solution:

a) $x[n]=\delta[n]$, $X(z)=1$, so

$$Y(z)=H(z)=\frac{1}{1-0.75z^{-1}+0.125z^{-2}}=\frac{-1}{1-\frac{1}{4}z^{-1}}+\frac{2}{1-\frac{1}{2}z^{-1}}$$

Observing the difference equation itself, we may find that the system is causal, so

$$y[n]=-\left(\tfrac{1}{4}\right)^n u[n]+2\left(\tfrac{1}{2}\right)^n u[n]$$

b) $y[0]=-1+2=1$; by initial-value theorem,

$$y[0]=\lim_{z\to\infty}Y(z)=\lim_{z\to\infty}\frac{1}{1-0.75z^{-1}+0.125z^{-2}}=\frac{1}{1}=1$$

8. **[16]** Consider a discrete-time LTI system with transfer function $H(z)=\frac{1-a^*z}{z-a}$, $|a|<1$, where a^* represents the complex conjugate of a.

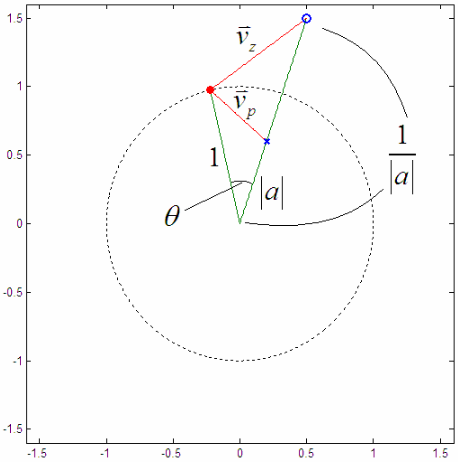
a) **[4]** Sketch the pole-zero plot of $H(z)$ in the z-plane.

b) **[6]** Is $H(z)$ stable and causal? Why?

c) **[6]** Use the graphic method to show what the magnitude response of the system is.

Solution:

a) $H(z)=\dfrac{1-a^{*}z}{z-a}=\dfrac{-a^{*}+z^{-1}}{1-az^{-1}}$, there is one pole $p_1=a$ and one zero $z=\dfrac{1}{a^{*}}=\dfrac{a}{|a|^2}$. The pole-zero diagram is as the right figure.



b) Observing that the only pole is inside the unit circle, $H(z)$ can be stable AND causal.

c) By cosine theorem, we may find that

$$|v_p|^2=1+|a|^2-2|a|\cos\theta$$

$$|v_z|^2=1+\frac{1}{|a|^2}-\frac{2}{|a|}\cos\theta=\frac{1}{|a|^2}|v_p|^2$$

and then $|H(z)|=\dfrac{|v_z|}{|v_p|}=|a^{*}|\dfrac{1}{|a|}=1$, which means the system is an all-pass filter.

9. **[10]** Consider a time-division multiplexing (TDM) system with a sampling rate of 3×10^6 Hz. The baseband signals $x_1(t)$, $x_2(t)$, …, $x_N(t)$ to be multiplexed are converted to pulse-amplitude modulation (PAM) signals, $y_1(t)$, $y_2(t)$, …, $y_N(t)$, respectively.

   a) **[5]** Please state the principle of TDM.

   b) **[5]** Let the pulses be 8×10^{-6} seconds in duration. Determine the maximum value of N to ensure an appropriate TDM operation.

Solution:

a) (See pp. 604 in the textbook.)

b) The sampling period is $\dfrac{1}{3\cdot10^6}=\dfrac{1}{3}\cdot10^{-6}$ seconds, and the duration of the pulses is 8×10^{-6} seconds. Since the duration of the pulses is larger than the sampling period, there is no channel capacity, and then the maximum value of N is $N=0$.
(If the duration of the pulse is changed to be smaller, say, 8×10^{-7} seconds, the channel capacity is given by the quotient, say, $\left\lfloor \frac{1}{3}\cdot10^{-6}\Big/8\cdot10^{-7}\right\rfloor=\left\lfloor\frac{8}{3}\cdot10\right\rfloor=26$.)

10. **[10]** Consider a frequency-division multiplexing (FDM) system of discrete-time signals $x_i[n]$, $i=1,2,3,4$. Let each $x_i[n]$ occupy the entire frequency band ($-\pi\le\omega\le\pi$).

   a) **[5]** Please state the principle of FDM.

   b) **[5]** Let each signal $x_i[n]$ be upsampled and then modulated with $\cos\left(\dfrac{i\pi n}{4}\right)$. Determine the minimum amount of upsampling that can be performed on each $x_i[n]$.

Solution:

a) (See pp. 594 in the textbook.)

b) $\dfrac{2\pi}{\pi/4}=8$.