微分方程期中考

2001/11/29

1. (20%) Solve the differential equations

   (a) $(x^2+1)\dfrac{dy}{dx}+3xy=6x$ (10%)

   (b) $2xy\dfrac{dy}{dx}=4x^2+3y^2$ (10%)

2. (15%) When all the curves in a family $G(x,y,c_1)=0$ intersect orthogonally all the curves in another family $H(x,y,c_2)=0$, the two families are said to be orthogonal trajectories of each other.

   (a) If $\dfrac{dy}{dx}=f(x,y)$ is the differential equation of $G(x,y,c_1)=0$, then what is the differential equation of $H(x,y,c_2)=0$? (5%)

   (b) A one-parameter family of curves is defined by $y=c_1e^x$, find its orthogonal trajectories. (5%)

   (c) Write equations and sketch plots of the curves of both families of (b) which pass through (0,1) (5%)

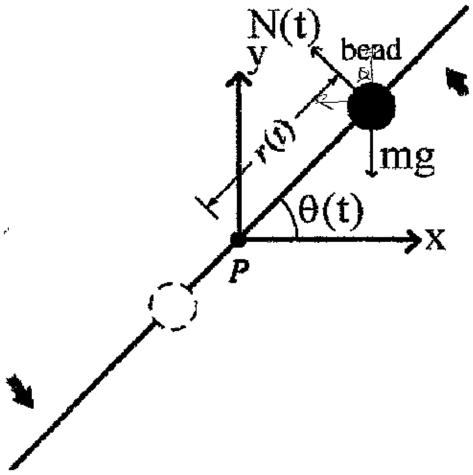
3. (15%) Consider the following differential equation of

   $x^3y^{(3)}-3x^2y''+6xy'-6y=x^6$.

   (a) Please find the general solution. (10%)

   (b) For an initial value problem (IVP) subject to $y''(x_0)=y_2, y'(x_0)=y_1$, and $y(x_0)=y_0$, what is the x_0 value when the solution of the IVP is not unique as given any y_2, y_1, and y_0? Please explain why. (5%)

4. (20%) A bead is constrained to slide along a frictionless rod of Length L which is large enough. The rod is rotating in a vertical plane about a pivot P fixed at the mid point of the rod, but the design of the pivot allows the bead to move along the entire length of the rod. Let r(t) and θ(t) denote the position of the bead relative to this rotating coordinate system as shown in the following figure. There is always a gravity force mg in the negative y direction applied on the bead where m is the mass of the bead and g is the gravity acceleration. In addition, the rod applies a normal force N(t) on the bead which depends on the rotation.



(a) The relationship between the rectangular and rotating coordinate systems is

$$\begin{cases} \hat{r} = \cos\theta(t)\hat{x} + \sin\theta(t)\hat{y} \\ \hat{\theta} = -\sin\theta(t)\hat{x} + \cos\theta(t)\hat{y} \end{cases}.$$

Please prove that

$$\begin{cases} \dfrac{d\hat{r}}{dt} = \dfrac{d\theta}{dt}\hat{\theta} \\ \dfrac{d\hat{\theta}}{dt} = -\dfrac{d\theta}{dt}\hat{r} \end{cases}$$

and use the above result to find what $a_r(t)$ and $a_\theta(t)$ are in

$$\vec{a}(t) = \frac{d^2\vec{r}(t)}{dt^2} = \frac{d^2}{dt^2}[r(t)\hat{r}] = a_r(t)\hat{r} + a_\theta(t)\hat{\theta}$$

where $\vec{r}(t)$ is the position vector of the bead. (5%)

(b) Represent the force in the rotating coordinate system and use Newton's law to prove that the motion of the bead follows the differential equation, (5%)

$$\begin{cases} m\left[\dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2\right] = -mg\sin\theta \\ m\left[r\dfrac{d^2\theta}{dt^2} + 2\dfrac{dr}{dt}\dfrac{d\theta}{dt}\right] = N(t) - mg\cos\theta \end{cases}$$

(c) If the rod is rotating with constant angular velocity ω, i.e., $\theta = \omega t$, and the initial condition of the bead is $r(0) = 0$, $r'(0) = g/(2\omega)$, please find the solution $r(t)$ and the normal force $N(t)$. (10%)

5. (21%) A linear system, as shown by the block diagram in Fig 5.1, can be described by the following equation:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = g(t) \ , \qquad t \geq 0$$

$$y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$$

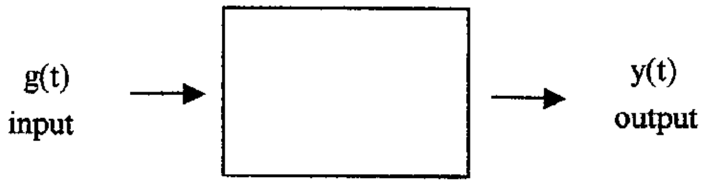


Fig 5.1

When input g(t) is an impulse, i.e. Dirac delta function δ(t), then the output $y(t) = 3e^{-t} + 2e^{-2t} + e^{-3t}$

(a) If input $g(t)$ becomes e^{-t}, then $y(t) = ?$ (7%)

(b) If input $g(t) = u(t) - u(t-2)$, then $y(t) = ?$ (7%)

(c) Find the values of coefficients $a_n, a_{n-1}, \cdots, a_1, a_0$. (7%)

6. (7%) $F(s) = \ln \frac{s-3}{s+1}$. Find the inverse Laplace transform $f(t) = L^{-1}\{F(s)\}$

7. (7%) solve the following differential-integral equation to obtain $f(t)$

$$\frac{df}{dt} + 6f(t) + 9\int_0^t f(t-\tau)d\tau = 1, \qquad f(0) = 0$$