Signals and Systems, Midterm Exam Solutions (Draft)

Spring 2006, Edited by bypeng

1. **[10]** A system with output signal $y(t)$ given input signal $x(t)$ as below:

$$y(t)=\int_{-\infty}^{2t} x(\tau)d\tau$$

Is this system memoryless **[2]**, time-invariant **[2]**, linear **[2]**, causal **[2]**, or stable **[2]** ? Justify your answers.

Solution:

(i) Memoryless property: No. *Example*: considering $x_1(t)=u(t)$ and $x_2(t)=u(t-1)$, which satisfies $x_1(t)=x_2(t)$ for any $t>1$. If this system is memoryless, we may find that $y_1(t)=y_2(t)$ for any $t>1$, but $y_1(t)=2t$ for $t>0$ (and thus for $t>1$), and $y_2(t)=2t-1$ for $t>0.5$ (and thus for $t>1$).
[*Another example*: see arguments for causality on the below.]

(ii) Time-Invariance: No. *Example*: considering $x_1(t)=u(t)$ and $x_2(t)=x_1(t-1)=u(t-1)$, and their outputs $y_1(t)=2t$ and $y_2(t)=2t-1$. If the system is time-invariant, we may find that $y_2(t)=y_1(t-1)=2(t-1)=2t-2$, which is a contradiction.

(iii) Linearity: Yes. For any signals $x_1(t)$ and $x_2(t)$, and their outputs $y_1(t)$ and $y_2(t)$, we have that $\int_{-\infty}^{2t}\left[ax_1(\tau)+bx_2(\tau)\right]d\tau=\int_{-\infty}^{2t}ax_1(\tau)d\tau+\int_{-\infty}^{2t}bx_2(\tau)d\tau=a\int_{-\infty}^{2t}x_1(\tau)d\tau+b\int_{-\infty}^{2t}x_2(\tau)d\tau$
$=ay_1(t)+by_2(t)$.

(iv) Causality: No. *Example*: considering $x_1(t)=u(t)$ and $x_2(t)=u(t)u(-t+1)$, which satisfies $x_1(t)=x_2(t)$ for any $t<1$. If this system is causal, we may find that $y_1(t)=y_2(t)$ for any $t<1$, but $y_1(t)=2t$ for $0.5<t<1$, and $y_2(t)=1$ for $0.5<t<1$.

(v) Stability: No. *Example*: consider $x(t)=u(t)$, which is a bounded signal ($B=1$). The output is $y(t)=2tu(t)$, which is not a bounded signal.

2. **[8]** Use the following operational definition of $\delta(t)$,

$$x(t)*\delta(t)=x(t),\qquad \text{any } x(t),$$

show the following properties of $\delta(t)$:

(i) **[4]** $\int_{-\infty}^{\infty}\delta(t)dt=1$

(ii) **[4]** $x(t)\delta(t-t_0)=x(t_0)\delta(t-t_0)$, any $x(t)$

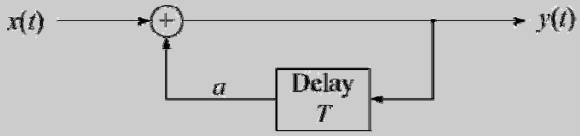
Solution:

(i) Let $x(t)=1$, then $\int_{-\infty}^{\infty}\delta(\tau)d\tau=\int_{-\infty}^{\infty}x(t-\tau)\delta(\tau)d\tau=x(t)*\delta(t)=x(t)=1$, or $\int_{-\infty}^{\infty}\delta(t)dt=1$.

(ii) Since $x(t)*\delta(t)=x(t)$, then $x(t_0-t)=x(t_0-t)*\delta(t)=\int_{-\infty}^{\infty}x(t_0+\tau-t)\delta(\tau)d\tau$. By letting $t=0$, we may find that $x(t_0)=\int_{-\infty}^{\infty}x(t_0+\tau)\delta(\tau)d\tau \overset{\tau_0\triangleq t_0+\tau}{=} \int_{-\infty}^{\infty}x(\tau_0)\delta(\tau_0-t_0)d\tau_0$,
or $x(t_0)=\int_{-\infty}^{\infty}x(t)\delta(t-t_0)dt$. Now

$$\begin{aligned}x(t)\delta(t-t_0)&=x(t)\delta(t-t_0)*\delta(t)\\&=\int_{-\infty}^{\infty}x(\tau)\delta(\tau-t_0)\delta(t-\tau)d\tau\\&=\int_{-\infty}^{\infty}x(\tau)\delta(t-\tau)\delta(\tau-t_0)d\tau\\&=x(t_0)\delta(t-t_0)\end{aligned}$$

3. **[12]** An echo generating environment can be modeled by the following system:



or expressed as:

$$y(t) = x(t) + ay(t-T) \quad .$$

Assuming initially at rest, i.e., $y(t) = 0, t < 0$ if $x(t) = 0, t < 0$.

(i) **[4]** Find the impulse response $h(t)$ of this system.

(ii) **[8]** Does there exist an inverse system which can cancel the echoes? If yes, write down its impulse response $g(t)$ and a block diagram as in the above. If no, explain why.

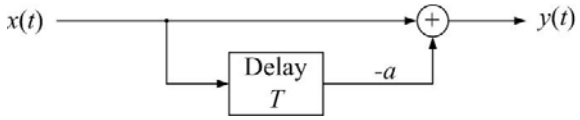
Solution:

(i) We have that $h(t) = \delta(t) + ah(t-T)$. Since the input is an impulse and the output delays the impulse again and again for time multiple of T, the impulse response needs to be $h(t) = \sum_{k=0}^{\infty} h_k \delta(t-kT)$. Now we have $\sum_{k=0}^{\infty} h_k \delta(t-kT) - a\sum_{k=0}^{\infty} h_k \delta(t-(k+1)T) = \delta(t)$, so $h_k = a^k$, and then $h(t) = \sum_{k=0}^{\infty} a^k \delta(t-kT)$.

(ii) To recover $x(t)$, we need to cancel the delayed part. The system is given by $x(t) = y(t) + ax(t-T)$ or $y(t) = x(t) - ax(t-T)$, so the impulse response is given by

$$g(t) = \delta(t) - a\delta(t-T)$$

The block diagram is as follows:



4. **[8]** Consider a discrete-time signal $x[n]$ which is periodic with period N,

$$x[n+N] = x[n],$$

whose Fourier Series coefficients are a_k,

$$x[n] \xleftrightarrow{FS} a_k \quad .$$

Now $g(t)$ is a continuous-time signal,

$$g(t) = \sum_{k=-\infty}^{\infty} x[k]\delta(t-kT) \quad .$$

Can we express $g(t)$ in terms of Fourier Series coefficients b_k? If yes, express b_k in terms of a_k. If no, explain why.

Solution:

Since $x[n]$ is periodic with period N, $g(t)$ is periodic with period NT and then with frequency $\omega_0 = \frac{2\pi}{NT}$. So we can express $g(t)$ in terms of Fourier Series coefficients b_k which is given by

$$\begin{aligned} b_k &= \frac{1}{NT}\int_{NT} g(t) e^{-jk\frac{2\pi}{NT}t} dt = \frac{1}{NT}\int_{0^-}^{NT^-} \sum_{l=0}^{N-1} x[l]\delta(t-lT) e^{-jk\frac{2\pi}{NT}t} dt \\ &= \frac{1}{NT}\sum_{l=0}^{N-1} x[l] \int_{0^-}^{NT^-} \delta(t-lT) e^{-jk\frac{2\pi}{NT}t} dt = \frac{1}{NT}\sum_{l=0}^{N-1} x[l] e^{-jk\frac{2\pi}{NT}lT} \\ &= \frac{1}{T}\frac{1}{N}\sum_{l=0}^{N-1} x[l] e^{-jk\frac{2\pi}{N}l} = \frac{1}{T} a_k \end{aligned}$$

5. **[8]** A discrete-time LTI system has a unit impulse response $h[n]$ which is real and even. Consider a signal

$$x[n] = a\cos(\omega_0 n) \quad .$$

Is $x[n]$ an eigenfunction of the system? If yes, show it and what is the corresponding eigenvalue? If no, explain why.

Solution:

Since the impulse response $h[n]$ is real and even, its Fourier transform $H(e^{j\omega})$ is also real and even. Observing that $x[n] = a\cos(\omega_0 n) = \frac{a}{2}(e^{j\omega_0 n} + e^{-j\omega_0 n})$, and $e^{j\omega_0 n}$ and $e^{-j\omega_0 n}$ are both eigenfunction with eigenvalues $H(e^{j\omega_0})$ and $H(e^{-j\omega_0})$ respectively, and $H(e^{j\omega_0}) = H(e^{-j\omega_0})$ since $H(e^{j\omega})$ is also real and even, we conclude that the output is

$$y[n] = \frac{a}{2}\left[H(e^{j\omega_0})e^{j\omega_0 n} + H(e^{j\omega_0})e^{-j\omega_0 n}\right] = \frac{a}{2}H(e^{j\omega_0})\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right) = H(e^{j\omega_0})a\cos(\omega_0 n)$$

Therefore, $x[n]$ is an eigenfunction with eigenvalue $H(e^{j\omega_0}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega_0 n}$.

Alternative Solution:

We have

$$y[n] = x[n]*h[n] = \sum_{k=-\infty}^{\infty} x[k-n]h[k] = \sum_{k=-\infty}^{\infty} ah[k]\cos(\omega_0 k - \omega_0 n)$$

$$= \sum_{k=-\infty}^{\infty} ah[k]\left[\cos(\omega_0 k)\cos(\omega_0 n) + \sin(\omega_0 k)\sin(\omega_0 n)\right]$$

$$= a\cos(\omega_0 n)\sum_{k=-\infty}^{\infty} h[k]\cos(\omega_0 k) + a\sin(\omega_0 n)\sum_{k=-\infty}^{\infty} h[k]\sin(\omega_0 k)$$

Since $h[n]$ is real and even and $\sin\omega_0 n$ is real and odd, we know $\sum_{k=-\infty}^{\infty} h[k]\sin(\omega_0 k) = 0$, and then

$$y[n] = a\cos(\omega_0 n)\sum_{k=-\infty}^{\infty} h[k]\cos(\omega_0 k) = x[n]\sum_{k=-\infty}^{\infty} h[k]\cos(\omega_0 k)$$

Therefore, $x[n]$ is an eigenfunction with eigenvalue $\sum_{k=-\infty}^{\infty} h[k]\cos(\omega_0 k)$.

Note:

$$\sum_{n=-\infty}^{\infty} h[n]e^{-j\omega_0 n} \overset{\overbrace{h[n]=h[-n]}}{=} h[0] + \sum_{n=1}^{\infty} h[n]\left(e^{-j\omega_0 n} + e^{j\omega_0 n}\right) = h[0] + \sum_{n=1}^{\infty} h[n]\cdot 2\cos(\omega_0 n) \overset{\overbrace{h[n]=h[-n]}}{=} \sum_{n=-\infty}^{\infty} h[n]\cos(\omega_0 n)$$

6. **[8]** $x(t)$ is an impulse train,

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

Find its Fourier Transform $X(j\omega)$ **[4]** and its Fourier Series representation a_k **[4]**,

$$x(t) \xleftrightarrow{FT} X(j\omega), \quad x(t) \xleftrightarrow{FS} a_k \quad .$$

Solution:

The Fourier Series representation of $x(t)$ is

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\frac{2\pi}{T}t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\frac{2\pi}{T}t}dt = \frac{1}{T}$$

And then the Fourier transform of $x(t)$ is

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - k\frac{2\pi}{T}\right) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\delta\left(\omega - k\frac{2\pi}{T}\right) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

7. **[24]** Verify the following properties or relationships. If it is true, prove it. If not, prove it is not.

(i) **[8]** Time expansion for discrete-time signal,

$$x_{(k)}[n] \xleftrightarrow{FT} \tfrac{1}{k} X(e^{jk\omega}),$$

where

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n \text{ is an integer multiple of } k \\ 0 & \text{else} \end{cases}$$

and $X(e^{j\omega})$ is the Fourier Transform of $x[n]$

$$x[n] \xleftrightarrow{FT} X(e^{j\omega}) \quad .$$

(ii) **[8]** (This may be called "generalized Parseval's relation" if it is true, because it reduces to Parseval's relation if $y(t) = x(t)$.)

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) Y^*(j\omega) d\omega,$$

where $x(t)$, $y(t)$ are continuous-time signals, $X(j\omega)$, $Y(j\omega)$ are respectively their Fourier Transforms,

$$x(t) \xleftrightarrow{FT} X(j\omega), \quad y(t) \xleftrightarrow{FT} Y(j\omega)$$

(iii) **[8]** Time reversal

$$x(-t) \xleftrightarrow{FS} a_{-k}^*$$

Where $x(t)$ is a continuous-time periodic signal, a_k are its Fourier Series coefficients,

$$x(t) \xleftrightarrow{FS} a_k$$

Solution:

(i) False. $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{j\omega n}$, so

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(k)}[n] e^{j\omega n} \overset{m \triangleq n/k}{\overbrace{=}} \sum_{m=-\infty}^{\infty} x[m] e^{j\omega km} = X(e^{jk\omega}).$$

(ii) True. Since $x^*(t) \xleftrightarrow{FT} X^*(-j\omega)$,

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t) y^*(t) dt &= \int_{-\infty}^{\infty} x(t) \frac{1}{2\pi} \int_{-\infty}^{\infty} Y^*(-j\omega) e^{j\omega t} d\omega dt \\
&= \int_{-\infty}^{\infty} x(t) \frac{1}{2\pi} \int_{-\infty}^{\infty} Y^*(j\omega) e^{-j\omega t} d\omega dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} Y^*(j\omega) \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} Y^*(j\omega) X(j\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) Y^*(j\omega) d\omega
\end{aligned}$$

(iii) False. $a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$, and then

$$\begin{aligned}
\frac{1}{T} \int_T x(-t) e^{-jk\omega_0 t} dt &= \frac{1}{T} \int_{t_l}^{t_u} x(-t) e^{-jk\omega_0 t} dt \\
&= \frac{1}{T} \int_{-t_l}^{-t_u} -x(t) e^{jk\omega_0 t} dt \\
&= \frac{1}{T} \int_{-t_u}^{-t_l} x(t) e^{jk\omega_0 t} dt \\
&= \frac{1}{T} \int_T x(t) e^{jk\omega_0 t} dt = a_{-k}
\end{aligned}$$

8. **[12]** Consider the time-shift property of continuous-time Fourier Transform, i.e.,

$$x(t) \overset{FT}{\longleftrightarrow} X(j\omega) \quad \text{then} \quad x(t-t_0) \overset{FT}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

Write down all different versions of properties parallel to this one in the family of Fourier Analysis of signals, i.e., for continuous- and discrete-time signals, periodic and aperiodic signals and so on, as well as all their dual properties, if any.

Solution:

Type of Fourier Analysis	Fourier Analysis Pair	Time-Shift Property	Frequency-Shift Property **[*]**
Continuous-Time Fourier Transform	$x(t) \overset{FT}{\longleftrightarrow} X(j\omega)$	$x(t-t_0) \overset{FT}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$	$e^{j\omega_0 t} x(t) \overset{FT}{\longleftrightarrow} X(j(\omega-\omega_0))$
Discrete-Time Fourier Transform	$x[n] \overset{FT}{\longleftrightarrow} X(e^{j\omega})$	$x[n-n_0] \overset{FT}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$	$e^{j\omega_0 n} x[n] \overset{FT}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$ **[*]**
Continuous-Time Fourier Series	$x(t) \underset{\omega_0}{\overset{FS}{\longleftrightarrow}} a_k$	$x(t-t_0) \underset{\omega_0}{\overset{FS}{\longleftrightarrow}} e^{-jk\omega_0 t_0} a_k$	$e^{jM\omega_0 t} x(t) \underset{\omega_0}{\overset{FS}{\longleftrightarrow}} a_{k-M}$ **[*]**
Discrete-Time Fourier Series	$x[n] \underset{\omega_0}{\overset{FS}{\longleftrightarrow}} a_k$	$x[n-n_0] \underset{\omega_0}{\overset{FS}{\longleftrightarrow}} e^{-jk\omega_0 n_0} a_k$	$e^{jM\omega_0 n} x[n] \underset{\omega_0}{\overset{FS}{\longleftrightarrow}} a_{k-M}$

* If the "Frequency-Shift Property" is changed as "Dual Property", then contents between the cell corresponding to "Discrete-Time Fourier Transform" and that corresponding to "Continuous-Time Fourier Series" should be exchanged.

9. **[12]** Consider a system characterized by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t) \quad .$$

(i) **[6]** Find the impulse response $h(t)$ of the system.

(ii) **[6]** Find the output signal $y(t)$ if the input signal is

$$x(t) = e^{-t}u(t)$$

Solution:

Since the system is characterized by a linear constant-coefficient ODE, we claim that the system is a LTI one. If the system is stable,

(i) Applying Fourier Transform to the differential equation, we get

$$(j\omega)^2 Y(j\omega) + 4(j\omega)Y(j\omega) + 3Y(j\omega) = (j\omega)X(j\omega) + 2X(j\omega)$$

And then

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega+2}{(j\omega)^2+4j\omega+3} = \frac{\frac{1}{2}}{j\omega+3} + \frac{\frac{1}{2}}{j\omega+1}$$

Therefore the impulse response is given by

$$h(t) = \left(\tfrac{1}{2}e^{-3t} + \tfrac{1}{2}e^{-t}\right)u(t)$$

(ii) Since the system is LTI,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} e^{-(t-\tau)}u(t-\tau)\left(\tfrac{1}{2}e^{-3\tau} + \tfrac{1}{2}e^{-\tau}\right)u(\tau)d\tau$$

$$= u(t)\int_0^t e^{-(t-\tau)}\left(\tfrac{1}{2}e^{-3\tau} + \tfrac{1}{2}e^{-\tau}\right)d\tau = e^{-t}u(t)\int_0^t \left(\tfrac{1}{2}e^{-2\tau} + \tfrac{1}{2}\right)d\tau$$

$$= e^{-t}u(t)\left(-\tfrac{1}{4}e^{-2t} + \tfrac{1}{2}t - \tfrac{1}{4}\right) = \left(-\tfrac{1}{4}e^{-3t} + \tfrac{1}{2}te^{-t} - \tfrac{1}{4}e^{-t}\right)u(t)$$

(If the system is not stable, then we cannot use the Fourier Transform approach. Since the appropriate approach is not discussed until Chapter 9, the solution above is worth full points in this problem.)

10. **[10]** Answer the following questions.

(i) **[6]** Assume $y[n] = x[n](-1)^n$, and find $Y(e^{j\omega})$ in terms of $X(e^{j\omega})$, where $X(e^{j\omega})$ and $Y(e^{j\omega})$ are respectively the Fourier Transforms of $x[n]$ and $y[n]$.

(ii) **[4]** Explain the meaning of what you obtained above. For example, consider a sequence $z[n] = e^{j\omega n}$ with $\omega = 0$ and $\omega = \pi$.

Solution:

(i) $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}$, so

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{j\omega n} = \sum_{n=-\infty}^{\infty} x[n](-1)^n e^{j\omega n} = \sum_{n=-\infty}^{\infty} x[n](e^{j\pi})^n e^{j\omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{j(\omega+\pi)n} = X(e^{j(\omega+\pi)})$$

(ii) We may find that $(-1)^n = e^{j\pi n} = z[n]$ when $\omega = \pi$. Considering the Fourier Transform of $z[n]$:

$$Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} z[n]e^{j\omega n} = \sum_{n=-\infty}^{\infty} e^{j\pi n}e^{j\omega n} = 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega + \pi - 2\pi l)$$

Then $Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Z(e^{j(\omega-\theta)})d\theta$, which is a witness of the multiplication property.

11. **[8]** Consider a LTI system with unit impulse response $h[n]$,

$$h[n] = a^n u[n], \quad |a| < 1$$

And an input signal $x[n]$,

$$x[n] = b^n u[n], \quad |b| < 1, \quad b \neq a \quad .$$

Find the output signal $y[n]$.

Solution:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$= \sum_{k=-\infty}^{\infty} b^{n-k}u[n-k]a^k u[k] = u[n]\sum_{k=0}^{n} b^{n-k}a^k$$

$$= \frac{b^n(1-(\frac{a}{b})^{n+1})}{1-\frac{a}{b}}u[n] = \frac{b^{n+1} - a^{n+1}}{b-a}u[n]$$