LINEAR ALGEBRA: QUIZ 1

March 8, 2001

Answer ALL questions.

1. (40%) Define the following terms: (a) rank of an $m \times n$ matrix; (b) nullity of an $m \times n$ matrix; (c) span of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$; (d) linearly independent set.

2. Consider the following system of linear equations:

$$\begin{aligned} x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2 \\ -x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\ 2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\ -3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9 \end{aligned}$$

   (a) (20%) Find the reduced row echelon form of the augmented matrix using the Gaussian elimination method.

   (b) (10%) Find a general solution to the system of linear equations.

3. (30%) Determine if the following statements are true or false:

   (a) If a subset of $\mathcal{R}^n$ is linearly dependent, then it must contain at least n vectors.

   (b) If $\text{Span}(\mathcal{S}_1) = \text{Span}(\mathcal{S}_2)$, then $\mathcal{S}_1 = \mathcal{S}_2$.

   (c) No subset of Span $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ that contains more than k vectors is linearly independent.

   (d) If a matrix A can be transformed into a matrix B by performing an elementary row operation, then B can be transformed into A by performing an elementary row operation.

   (e) If A is an $m \times n$ matrix with $m > n$, then the only solution to $A\mathbf{x} = \mathbf{0}$ is the zero vector $\mathbf{x} = \mathbf{0}$.

LINEAR ALGEBRA QUIZ 2

March 26, 2001

Answer ALL questions.

1. (a) (15%)

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -4 \\ -3 & x & 12 \end{bmatrix}$$

(i) Find x, such that nullity $\mathbf{A} = 2$.

(ii) Find x, such that nullity $\mathbf{A} = 1$.

(iii) Can you find x, such that nullity $\mathbf{A} = 0$? Why?

(b) (15%)

$$\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 2 & x \\ 4 & y \\ 8 & z \end{bmatrix}$$

(i) Find x, y, z, such that rank $\mathbf{B} = 1$.

(ii) If $x = 2$, find y, z, such that rank $\mathbf{B} = 2$.

(iii) Can you find x, y, z, such that rank $\mathbf{B} \geq 3$? Why?

2. Suppose that $\mathbf{T}: \Re^2 \to \Re^2$ is a linear transformation such that

$$\mathbf{T}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } \mathbf{T}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

(a) (15%) Find the standard matrix of **T**.

(b) (15%) $\mathbf{T}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = ?$

3. Define $\mathbf{T}: \Re^3 \to \Re^3$ by $\mathbf{T}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x_1 \\ x_2 \end{bmatrix}$.

(a) (10%) What is the null space of **T**?

(b) (10%) Is **T** one-to-one?

(c) (10%) What is the range of **T**?

(d) (10%) Is **T** onto?

Linear Algebra Quiz #3

1. Let $A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 4 & 2 & 0 \\ 4 & 4 & 1 & 0 \\ 2 & 2 & 1 & -1 \end{pmatrix}$

   Let $A' = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 4 & 2 & 0 \\ 2 & 2 & 1 & -1 \\ 4 & 4 & 1 & 0 \end{pmatrix}$

   (i) Show that det A = - det A' . (10%)

   (ii) Please determine the rank of A. (10%)

   (iii) Please show that the row space of A equals the row space of A' . (10%)

   (iv) Please determine the dimension of the null space of A'.(10%)

2. Please determine the following statements as being true or false for an arbitrary nxn matrix A. (6% each)

(a) If all columns vectors of A are linearly independent, then the nullity of A is zero.

(b) If the rank of A^T equals n, then the dimension of the row space of A equals n.

(c) The set of solutions for AX = b , where b is a vector in R^n , is a subspace of R^n.

(d) The determinant of A is zero if the columns of A cannot span R^n.

(e) If AX = 0 has only the zero solution, then the null space of A is an empty set.

3. Let A be an n x n matrix.

(i) Show that if E is an elementary matrix, then EA and A have the same row space. (15%)

(ii) Show that the row space of the reduced row echelon form of matrix A is the same as the row space of matrix A.
(15%)

LINEAR ALGEBRA: QUIZ 4
May 17, 2001

Answer ALL questions.

1. (55%) Consider the linear operator $T : \mathcal{R}^3 \longrightarrow \mathcal{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 \\ x_2 + 2x_3 \\ 2x_2 + x_3 \end{bmatrix}$$

   (a) (5%) Find the characteristic polynomial of T.

   (b) (5%) Find all the eigenvalues of T. What is the multiplicity of each eigenvalue.

   (c) (15%) Find a basis for the eigenspace corresponding to each eigenvalue of T. What is the dimension of each eigenspace?

   (d) (5%) Find a basis $\mathcal{B}$ for $\mathcal{R}^3$ consisting of eigenvectors of T.

   (e) (5%) Using the basis $\mathcal{B}$ you found in (c), find the $\mathcal{B}$-matrix $[T]_{\mathcal{B}}$.

   (f) (5%) Find the standard matrix A of T. How is A related to $[T]_{\mathcal{B}}$?

   (g) (15%) Find A^{100} and A^{-10} ($A^{-10} = (A^{-1})^{10}$).

2. (20%) Let I_n and $\mathbf{0}$ be respectively the $n \times n$ identity matrix and the $n \times n$ zero matrix. What are the eigenvalues of I_n and $\mathbf{0}$? What are the eigenvectors corresponding to these eigenvalues?

3. (25%) Let A be an $n \times n$ matrix. Determine if the following statements are true or false (No explanation is needed.):

   (a) Every eigenvalue of A is also an eigenvalue of A^T.

   (b) If A is diagonalizable, then there is a unique diagonal matrix D such that $A = PDP^{-1}$.

   (c) If A has n distinct eigenvalues, then A is diagonalizable.

   (d) If A is invertible, then 0 is not an eigenvalue of A.

   (e) If A is not invertible, then *any* nonzero vector in Null A is an eigenvector of A.

Linear Algebra: Quiz 5

2001/05/31

1. Let $\mathbf{W} = span\{[0\ 1\ 1\ 1]^T, [1\ 0\ 1\ 1]^T\}$ and $\boldsymbol{v} = [1\ 1\ 0\ -1]^T$. (30%)

   (a) Find a basis for $\mathbf{W}^{\perp}$.

   (b) Find the orthogonal projection of $\boldsymbol{v}$ onto $\mathbf{W}$.

   (c) Find the distance from $\boldsymbol{v}$ to $\mathbf{W}$.

2. Let $\mathbf{W}$ be a subspace of $\Re^n$. Let $\mathbf{P_w}$ be the orthogonal projection matrix for $\mathbf{W}$. Show that (30%)

   (a) $(\mathbf{P_w})^2 = \mathbf{P_w}$

   (b) $\mathbf{P_w}$ is symmetric.

3. Determine the true or false. (40%)

   (1) The norm of the sum of orthogonal vectors is the sum of norms of the vectors.

   (2) The distance between two vectors in $\Re^n$ is the norm of their difference.

   (3) The orthogonal ~~component~~ complement of the row space of a matrix equals the null space of the matrix.

   (4) Any orthogonal subset of $\Re^n$ is linearly independent.

   (5) For any subset S of $\Re^n$, $(S^{\perp})^{\perp} = S$.

   (6) If S is an orthogonal set of n nonzero vectors in $\Re^n$, then S is a basis for $\Re^n$.

   (7) An orthogonal projection matrix is never invertible.

   (8) If $\boldsymbol{v}$ is in $\Re^n$ and $\mathbf{W}$ is a subspace of $\Re^n$, then $P_w\boldsymbol{v}$ is the vector in $\mathbf{W}^{\perp}$.