Contents

Part I

Measure Theory And Lebesgue Integration

Chapter 1

Measure Theory

1.1 Measurable Sets (σ-Algebras)

Definition 1.1.1. *A* ***σ-Algebra*** *on a set X is a collection, denote $\mathfrak{A}$, of subsets of X s.t.*

- $\varnothing \in \mathfrak{A}$
- *If $A \in \mathfrak{A}$, then $A^c = X$*
  $A \in \mathfrak{A}$
- *If $\{A_i / i \in \mathbb{N}\}$ is a countable family of sets in $\mathfrak{A}$ then $\cup_{i=1}^{\infty} A_i \in \mathfrak{A}$*

Definition 1.1.2. *A* ***measurable space*** $(X, \mathfrak{A})$ *is a set X with a σ-algebra on X. The elements of that collection are called measurable sets.*

Proposition 1.1.1. *Let a set X and $\mathfrak{A}$ be a σ-Algebra on X. Then $X \in \mathfrak{A}$ and $\mathfrak{A}$ is closed under countable intersections.*

Proof.

- Since $\varnothing \in \mathfrak{A}$ then $X = \varnothing^c \in \mathfrak{A}$
- Let $\{A_i / i \in \mathbb{N}\}$ be a countable family of elements of $\mathfrak{A}$. Be definition, $\forall i \in \mathbb{N}, A_i^c \in \mathfrak{A}$. In other words, $\{A_i^c / i \in \mathbb{N}\}$

$$\therefore, \qquad \cap_{i=1}^{\infty} A_i = (\cup_{i=1}^{\infty} A_i^c)^c \in \mathfrak{A}$$

$\square$

Example 1.1.1. *The smallest σ-Algebra that can be defined over an arbitrary set is the empty set $\{\varnothing, X\}$*

Example 1.1.2. *The largest σ-Algebra that can be defined is the power set, $\mathfrak{P}(x)$. The power set is the collection of all possible sets of X.*

Example 1.1.3. *Let T be the collection of open sets in X ((X,T) is called a **Topological Space**). The σ-Algebra of X generalized by T is called the Borel σ-Algebra on X. We denote this $\mathfrak{B}(x)$. Its elements are called Borel Sets.*

R A closed set is the compliment of an open set. It is possible to be closed and open at the same time.
R By definition, the complement of sets in the σ-Algebra is also in the σ-Algebra. This means that the closed sets are also in the σ-Algebra. For instance, the Borel-Algebra on $\mathbb{R}^n$ is generated by the collection of cubes, C, of the form $C = (a_1, b_1)(a_2, b_2)(\cdots)(a_n, b_n)$.

Definition 1.1.3. *A **measure, $\boldsymbol{\mu}$**, on a set X, is a map $\mu : \mathfrak{A} \to [0, \infty]$ on a σ-Alembra $\mathfrak{A}$ of X s.t.*

- $\mu(\varnothing) = 0$
- *If $\{A_i/i \in \mathbb{N}\}$ is a countable family of mutually disjoint sets of $\mathfrak{A}$, ie $A_i \cap A_j = \varnothing, i \neq j$*

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

The measure is said to be finite if $\mu(x) < \infty$ and said to be σ-finite if $\exists \{A_i \in /i \in \mathbb{N}\}$ of measurable subsets of X s.t. $\forall i \in \mathbb{N}, \mu(A_i) < \infty$ and $X = \cup_{i=1}^{infty} A_i$

Definition 1.1.4. *A measure space is a triple (X, $\mathfrak{A}$, μ)*

Example 1.1.4. *The counting measure, ν, on X is defined by $\nu(x) =$\{The number of elements in X\}. With convention $\nu(x) = \infty$ if x is an infinite set.*

Example 1.1.5. *The **delta measure**, δ_{x_0}, where $x_0 \in \mathbb{R}$ on the Borel-alg of $\mathbb{R}$ is defined by:*

$$\delta_{x_0} = \begin{cases} 1 & if x_0 \in A \\ 0 & if x_0 \notin A \end{cases}$$

Example 1.1.6. *The **Lebesgue Measure**. We saw the Borel Algebra on* $\mathbb{R}$ *is generated by the cubes* $(a_1, b_1)(a_2, b_2)\ldots(a_n, b_n)$*. The Lebesague Measure,* λ *on the Borel Algebra s.t.* $\lambda(c) = volume(c) = (b_1 - a_1)(b_2 - a_2)\ldots(b_n - a_n)$*.*

R The Lebesgue Measure is $\sigma - finite$, indeed $\mathbb{R} = \cup_{i=1}^{\infty}(-i, i)^n$

Theorem 1. *A subset of* $A \in \mathbb{R}^n$ *is Lebesgue Measurable if and only if* $\forall\epsilon > 0$, $\exists$ F *closed,* $\exists$ G *open,* $F \subset A \subset G$, $\lambda(G \setminus F) < \epsilon$*.*

Moreover:

$$\lambda(A) = \inf\{\lambda(u) | u \textit{ is open and } A \subset u\}$$
$$\lambda(A) = \sup\{\lambda(u) | k \textit{ is closed and } k \subset A\}$$

Proposition 1.1.2. *The Lebesgue Measure is translation invariant. ie*

$$\lambda(\mathfrak{T}_h A) = \lambda(A)$$

Where

$$T_h A = \{y \in \mathbb{R}^n | \exists x \in A, y = x + h\}$$

Proposition 1.1.3. *Let* $T : \mathbb{R}^n \to \mathbb{R}^n$ *be a linear mapping. We denote* $TA = \{y \in \mathbb{R}^n | \exists x \in A, y = Tx\}$*. Then* $\lambda(TA) = |det(T)|\lambda(A)$

Corollary 1.1.1. *The Lebesgue Measure is invariant by rotation and has the scaling property. ie* $\forall t > 0, \lambda(tA) = t^n\lambda(A)$

Proof. A rotation R and scaling T are linear transformations. Further, $|det(R)| = 1$ and $|det(T)| = t^n$. □

Definition 1.1.5. *Let* $(X, \mathfrak{A}, \mu)$ *be a measure space. A subset* $A \subset \mathfrak{A}$ *is said to have a measure zero if it is measurable and the measure* $\mu(A) = 0$*.*

Proposition 1.1.4. *A singleton in* $\mathbb{R}$ *has a measure zero. ie* $\forall x \in \mathbb{R}$, $\lambda(\{x\}) = 0$.

Proof. From the previous theorem we know that:

$$\begin{aligned}\lambda(\{x\}) &= \inf\{\lambda(u) | \text{u is open and } \{x\} \subset u\} \\ &= \lim_{\epsilon \to 0^+} \lambda(\{y/|x-y| < \epsilon\}) \\ &= \lim_{\epsilon \to 0^+} 2\epsilon \\ &= 0\end{aligned}$$

□

Corollary 1.1.2. *Every countable subset* $A = \{x_i \in \mathbb{R}/i \in \mathbb{N}\}$ *of* $\mathbb{R}$ *has measure 0.*

Proof. Let $A = \cup_{i=1}^{\infty} A_i$. Then, using the properties of measures, we have:

$$\begin{aligned}\lambda(A) &= \lambda(\bigcup_{i=1}^{\infty} A_i) \\ &= \sum_{i=1}^{infty} \lambda(A_i) \\ &= 0\end{aligned}$$

□