Contents

Part I

Measure Theory And Lebesgue Integration

Chapter 1

Measure Theory

1.1 Measurable Sets (σ-Algebras)

Definition 1.1.1. *A* ***σ-Algebra*** *on a set X is a collection, denote $\mathfrak{A}$, of subsets of X s.t.*

- $\varnothing \in \mathfrak{A}$
- *If $A \in \mathfrak{A}$, then $A^c = X$*
  $A \in \mathfrak{A}$
- *If $\{A_i / i \in \mathbb{N}\}$ is a countable family of sets in $\mathfrak{A}$ then $\cup_{i=1}^{\infty} A_i \in \mathfrak{A}$*

Definition 1.1.2. *A* ***measurable space*** *$(X, \mathfrak{A})$ is a set X with a σ-algebra on X. The elements of that collection are called measurable sets.*

Proposition 1.1.1. *Let a set X and $\mathfrak{A}$ be a σ-Algebra on X. Then $X \in \mathfrak{A}$ and $\mathfrak{A}$ is closed under countable intersections.*

Proof.

- Since $\varnothing \in \mathfrak{A}$ then $X = \varnothing^c \in \mathfrak{A}$
- Let $\{A_i / i \in \mathbb{N}\}$ be a countable family of elements of $\mathfrak{A}$. Be definition, $\forall i \in \mathbb{N}, A_i^c \in \mathfrak{A}$. In other words, $\{A_i^c / i \in \mathbb{N}\}$

$$\therefore, \qquad \cap_{i=1}^{\infty} A_i = (\cup_{i-1}^{\infty} A_i^c)^c \in \mathfrak{A}$$

$\square$

Example 1.1.1. *The smallest σ-Algebra that can be defined over an arbitrary set is the empty set* $\{\varnothing, X\}$

Example 1.1.2. *The largest σ-Algebra that can be defined is the power set,* $\mathfrak{P}(x)$. *The power set is the collection of all possible sets of X.*

Example 1.1.3. *Let T be the collection of open sets in X ((X,T) is called a* ***Topological Space****). The σ-Algebra of X generalized by T is called the Borel σ-Algebra on X. We denote this* $\mathfrak{B}(x)$. *Its elements are called Borel Sets.*

R A closed set is the compliment of an open set. It is possible to be closed and open at the same time.
R By definition, the complement of sets in the σ-Algebra is also in the σ-Algebra. This means that the closed sets are also in the σ-Algebra. For instance, the Borel-Algebra on $\mathbb{R}^n$ is generated by the collection of cubes, C, of the form $C = (a_1, b_1)(a_2, b_2)(\cdots)(a_n, b_n)$.

Definition 1.1.3. *A* ***measure***, $\boldsymbol{\mu}$, *on a set X, is a map* $\mu : \mathfrak{A} \to [0, \infty]$ *on a σ-Alembra* $\mathfrak{A}$ *of X s.t.*

- $\mu(\varnothing) = 0$
- *If* $\{A_i / i \in \mathbb{N}\}$ *is a countable family of mutually disjoint sets of* $\mathfrak{A}$, *ie* $A_i \cap A_j = \varnothing, i \neq j$

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

The measure is said to be finite if $\mu(x) < \infty$ and said to be σ-finite if $\exists \{A_i \in / i \in \mathbb{N}\}$ of measurable subsets of X s.t. $\forall i \in \mathbb{N}, \mu(A_i) < \infty$ and $X = \cup_{i=1}^{infty} A_i$