3-3　空間曲線と曲面

3-3-1　曲線とその長さ

線素　$$ds = \left|\frac{d\boldsymbol{r}}{dt}\right| dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{1}$$

単位接線ベクトル　$$\boldsymbol{t} = \frac{\dfrac{d\boldsymbol{r}}{dt}}{\left|\dfrac{d\boldsymbol{r}}{dt}\right|} = \frac{\dfrac{dx}{dt}\boldsymbol{i} + \dfrac{dy}{dt}\boldsymbol{j} + \dfrac{dz}{dt}\boldsymbol{k}}{\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2}} \tag{2}$$

線素ベクトル　$$d\boldsymbol{r} = ds\boldsymbol{t} = \frac{d\boldsymbol{r}}{dt} dt = \left(\frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}\right) dt \tag{3}$$

曲線の長さ（弧長）　$$s = \int_C ds = \int_{C(t)} \left|\frac{d\boldsymbol{r}}{dt}\right| dt = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{4}$$

3-3-2　曲面とその面積

面素　$$dS = \left|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right| du dv \tag{5}$$

単位法線ベクトル　$$\boldsymbol{n} = \frac{\dfrac{\partial \boldsymbol{r}}{\partial u} \times \dfrac{\partial \boldsymbol{r}}{\partial v}}{\left|\dfrac{\partial \boldsymbol{r}}{\partial u} \times \dfrac{\partial \boldsymbol{r}}{\partial v}\right|} \tag{6}$$

面素ベクトル　$$d\boldsymbol{S} = dS\boldsymbol{n} = \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} du dv \tag{7}$$

曲面の面積　$$S = \int_S dS = \iint_{D(u,v)} \left|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right| du dv \tag{8}$$

実践的には、$\boldsymbol{r} = \boldsymbol{r}(u,v) = x(u,v)\boldsymbol{i} + y(u,v)\boldsymbol{j} + z(u,v)\boldsymbol{k} \ \rightarrow\ \boldsymbol{r} = \boldsymbol{r}(x,y) = x\boldsymbol{i} + y\boldsymbol{j} + z(x,y)\boldsymbol{k}$

面素　$$dS = \left|\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y}\right| dx dy = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx dy \tag{5'}$$

単位法線ベクトル　$$\boldsymbol{n} = \frac{\dfrac{\partial \boldsymbol{r}}{\partial x} \times \dfrac{\partial \boldsymbol{r}}{\partial y}}{\left|\dfrac{\partial \boldsymbol{r}}{\partial x} \times \dfrac{\partial \boldsymbol{r}}{\partial y}\right|} = \frac{-\dfrac{\partial z}{\partial x}\boldsymbol{i} - \dfrac{\partial z}{\partial y}\boldsymbol{j} + \boldsymbol{k}}{\sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}} \tag{6'}$$

面素ベクトル　$$d\boldsymbol{S} = dS\boldsymbol{n} = \frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} dx dy = \left(-\frac{\partial z}{\partial x}\boldsymbol{i} - \frac{\partial z}{\partial y}\boldsymbol{j} + \boldsymbol{k}\right) dx dy \tag{7'}$$

曲面の面積　$$S = \int_S dS = \iint_{D(x,y)} \left|\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y}\right| dx dy = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx dy \tag{8'}$$

3-4　線積分と面積分

3-4-1　スカラーの線積分

$$\int_C \varphi ds = \int_{C(t)} \varphi(t) \left| \frac{d\boldsymbol{r}}{dt} \right| dt = \int_{t_1}^{t_2} \varphi(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{9}$$

3-4-2　ベクトルの線積分

$$\begin{aligned}\int_C \boldsymbol{A}\cdot\boldsymbol{t}ds = \int_C \boldsymbol{A}\cdot d\boldsymbol{r} = \int_{C(t)} \boldsymbol{A}\cdot\boldsymbol{t}\left|\frac{d\boldsymbol{r}}{dt}\right| dt = \int_{t_1}^{t_2} \boldsymbol{A}\cdot\boldsymbol{t}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\ = \int_{C(t)} \boldsymbol{A}\cdot\frac{d\boldsymbol{r}}{dt}dt = \int_{t_1}^{t_2} \boldsymbol{A}\cdot\left(\frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}\right)dt\end{aligned} \tag{10}$$

3-4-3　スカラーの面積分

$$\int_S \varphi dS = \iint_{D(u,v)} \varphi(u,v)\left|\frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v}\right| dudv \tag{11}$$

実践的には、

$$\int_S \varphi dS = \iint_{D(x,y)} \varphi(x,y)\left|\frac{\partial\boldsymbol{r}}{\partial x}\times\frac{\partial\boldsymbol{r}}{\partial y}\right| dxdy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} \varphi(x,y)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dxdy \tag{11'}$$

3-4-4　ベクトルの面積分

$$\begin{aligned}\int_S \boldsymbol{A}\cdot\boldsymbol{n}dS = \int_S \boldsymbol{A}\cdot d\boldsymbol{S} = \iint_{D(u,v)} \boldsymbol{A}\cdot\boldsymbol{n}\left|\frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v}\right| dudv \\ = \iint_{D(u,v)} \boldsymbol{A}\cdot\left(\frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v}\right) dudv\end{aligned} \tag{12}$$

実践的には、

$$\begin{aligned}\int_S \boldsymbol{A}\cdot\boldsymbol{n}dS = \int_S \boldsymbol{A}\cdot d\boldsymbol{S} = \iint_{D(x,y)} \boldsymbol{A}\cdot\boldsymbol{n}\left|\frac{\partial\boldsymbol{r}}{\partial x}\times\frac{\partial\boldsymbol{r}}{\partial y}\right| dxdy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} \boldsymbol{A}\cdot\boldsymbol{n}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dxdy \\ = \iint_{D(x,y)} \boldsymbol{A}\cdot\left(\frac{\partial\boldsymbol{r}}{\partial x}\times\frac{\partial\boldsymbol{r}}{\partial y}\right) dxdy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} \boldsymbol{A}\cdot\left(-\frac{\partial z}{\partial x}\boldsymbol{i} - \frac{\partial z}{\partial y}\boldsymbol{j} + \boldsymbol{k}\right) dxdy \\ = \iint_{D(x,y)} \frac{\boldsymbol{A}\cdot\boldsymbol{n}}{\boldsymbol{n}\cdot\boldsymbol{k}} dxdy\end{aligned} \tag{12'}$$