Optimization Theory and Applications

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Outline of the Course

- Chapter 1: Introduction to Optimization
  - Overview of optimization
  - Examples of optimization problems
  - Mathematical models and classifications of optimization problems
  - Introduction to Matlab optimization toolbox
- Chapter 2: Preliminary Knowledges
  - Notations
  - Vector Space and Matrix
  - Geometry

Outline of the Course

- Chapter 3: Unconstrained Optimization Problem
    - Introduction
    - Conditions for local minimizers
- Chapter 4: One-dimensional Search Methods
    - Introduction
    - Golden section search
    - Bisection method
    - Secant method
    - Newton's method

Outline of the Course

- Chapter 5: Global Search Method
  - Simulated annealing method
  - Genetic algorithm
  - Particle swarm optimization
- Chapter 6: Linear Programming
  - Brief history of linear programming
  - Simple examples of linear programs
  - Standard form linear programs
  - Basic solutions
  - Application examples of linear programming

Outline of the Course

- Chapter 7: Integer Programming
  - Prototype example
  - Some BIP applications
  - Branch and bound method
  - Knapsack problem
  - Application examples of integer programming
- Chapter 8: Equality Constrained Nonlinear Programming
  - Basics of nonlinear programming
  - Equality constraints
  - The theorem of Lagrange
  - Second-order conditions

Outline of the Course

- Chapter 9: Inequality Constrained Nonlinear Programming
  - The theorem of Karush-Kuhn-Tucker
  - Using KKT conditions
  - Application examples
- Chapter 10: Convex Optimization
  - Introduction to optimization
  - Convex functions
  - Convexity and optimization
  - Application examples of convex optimization

Outline of the Course

- Chapter 11: Duality
  - The Lagrange dual function
  - The Lagrange dual problem
- Chapter 12: Reading Session
  - Presentation

Introduction

- Optimization is a discipline for studying how to choose certain actions for achieving the optimal objective under certain constraints
  - E.g., job hunting
- Optimization is part of Operational Research
- Optimization has been widely applied in different areas
  - Computer science, Communications, Manufacturing, Military
  - Transportation, Management
  - Economics, Finance

Course Objectives

- Introduction to optimization theory and methods, with applications in different areas
- Analysis of optimization problems
- Optimization algorithms and their analysis
- Ability to make precise statements about optimization problems
- Understand and apply optimization techniques for your own research

Benefits

- You get 3 credits
- Read papers more easily
- Possibly publish a paper after this course
- Find good jobs

Benefits

岗位要求：

1. 计算机、数据挖掘、应用数学、统计学、运筹学、计量经济学、量化研究或相关专业。

5、计算机，数学，统计学, 运筹学或相关专业，硕士以上学历；

BASIC QUALIFICATIONS

· Strong analytical, mathematical and statistical skills.

Operations Research Scientist

US, WA, Seattle | 职位 ID: 337975

Amazon Fulfillment Services is looking for a motivated individual with strong numerical optimization and analysis skills and practical inventory modeling experience to join our Modeling and Optimization...阅读更多内容

Course Prerequisites

- Working knowledge of linear algebra (matrix manipulations, vector spaces, bases, eigenvalues, quadratic forms)
- Working knowledge of calculus of several variables (differentiating functions of n variables, gradients, limits)
- Basic state-space systems in discrete time (desirable but not required)
- An appreciation of rigor
- Time and effort

Grading

- Homework: twice, 20%
- Quiz: once, 20%
- Final exam: 60%
- Course Project ?

Bonus

- Additional up to 5 points: if you can write an acceptable paper using any optimization, operational research, and game theory techniques for you own research problems
- Additional up to 5 points: if you can reproduce the results of a good paper using any optimization, operational research, and game theory techniques
    - List the name, institution of the authors
    - List the source, publication year, and citations
    - Write a brief report with results and submit the source code

Textbook and References

- E. K. P. Chong and S. H. Zak, An Introduction to Optimization, Fourth Edition, New York, 2013.
- S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.
- J. Nocedal and S. J. Wright, Numerical Optimization, Springer, 2006.
- F. S. Hiller and G. J. Lieberman, Introduction to Operations Research, Tenth edition, McGraw-Hill Education, 2014.

Research Colleagues

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Degrees

- PhD, Electrical Engineering, Stanford University, 2003
- M.S. Electrical Engineering, Stanford University, 2000
- B.S. (Honors), Electrical Engineering and Mathematics, Stanford University, 1999

Research
Optimization of networks, network utility maximization, network function optimization, Fog networking and the Internet of Things, Smart data pricing and network economics, Social learning networks and online social networks

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Curriculum Vitae

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Dr. Tan is an Associate Professor of Computer Science. He received the M.A and Ph.D. degrees from Princeton University. He was a Senior Fellow of the Institute for Pure and Applied Mathematics (IPAM) for the program on Science at Extreme Scales, and was a Postdoctoral Scholar in the NetLab Group at Caltech. His industrial experience includes consulting for Tencent AI Lab, Qualcomm R&D (QRC), Fraser Research Lab. He is currently serving as an Editor for the IEEE/ACM Transactions on Networking.

His research interests include artificial intelligence, networks and graph analytics, convex optimization theory and algorithms at the interface of statistics and computer science. He is also interested in innovations for CS and math education. His Erdős number is 3.

Dr. Tan received the Princeton University Wu Prize for Excellence and was twice selected to participate at the U.S. National Academy of Engineering China-America Frontiers of Engineering Symposium. His research has been featured in Hong Kong Research Grants Council's Research Frontiers. He has previously served as the Editor of the IEEE Transactions on Communications and the Chairman of the IEEE Information Theory Society Hong Kong Chapter receiving the 2015 IT Society Chapter of the Year award. He organizes the Computer Science Challenge for high school students to learn computer science and advanced mathematics.

Homework 1

- Find a famous research colleague whose work are highly related to optimization
- List 3 of her/his representative papers using optimization
- Find the optimization techniques used in these works
- Write a simple report showing the above items

How to Learn This Course

- Read various books and materials
- Read high quality related papers
- Practice
- Try it for your own problems
- Start from the simple one (Toy Example)

Overview of Optimization Modeling Approach

- In this course, we focus on the mathematical methods of optimization
- Optimization studies are not just mathematical exercises
- Major phases of a typical optimization problem modeling

Overview of Optimization Modeling Approach

- Define the problem of interest and gather relevant data
- Formulate a mathematical model to represent the problem
- Deriving solutions from the model
- Test the model and refine it as needed
- Prepare to apply the model
- Implementation

Define the problem of interest and gather relevant data

- Practical problems are commonly described in a vague, imprecise way
- Problem definition is crucial. Difficult to extract a ”right” answer from the ”wrong” problem
- Determining appropriate objectives (for who, for what, for how long)
- Any constraints?
- Gather data

Formulating a Mathematical Model

- Reformulate the problem in a form that is convenient for analysis (quantitative analysis)
- Define the decision variables
- Define the objective function
- Define the constraints
- Determining the values of parameters (critical)
  - Uncertainty of the parameter
  - How the solution will change if the parameters changed to other plausible values?
  - Sensitivity analysis
- Precision vs Tractability
- Start from the simple version

Deriving Solutions From the Model

- Develop a computer-based procedure for deriving solutions to the problem from the model
- Usually it is a relatively simple step
- Some standard algorithms exist
- Optimizing vs Satisfying
    - Mathematical model vs Reality
      "Optimizing is the science of the ultimate; satisficing is the art of the feasible"
    - Cost for find optimal solution
    - Heuristics approach and suboptimal solution

Testing the Model

- Find bugs and flaws (program)
- Model validation
- Try as many inputs as possible
- A more systematic approach: Retrospective test
    - Using historical data to reconstruct the past
    - Determine how well the model have performed
    - Any drawback?
    - Possible solution?

Focus of this Course

- Mathematical model formulation
- Deriving the solutions

Mathematical Formulation of Optimization Problems

- Optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(\mathbf{x}) \\ \text{subject to} & f_i(\mathbf{x}) \leq b_i, \quad i = 1, \ldots, M \\ \text{variables :} & \mathbf{x}. \end{array}$$

$\mathbf{x} = (x_1, \ldots, x_n)^\top$: optimization variables
$f_0 : \mathbb{R}^n \to \mathbb{R}$: objective function
$f_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, \ldots, M$: constraint functions

- Optimal solution: $\mathbf{x}^\star$ has smallest value of f_0 among all vectors that satisfy the constraints

Solving Optimization Problems

- General optimization problem
  - Very difficult to solve
  - Involve some compromise, e.g., very long computation time, or not always finding the solution
- Exceptions: certain problem classes can be solved efficiently and reliably
  - Linear programming problem
  - least-squares problems
  - Convex optimization problems

Examples

- Portfolio optimization
  - Variables: amounts invested in different assets
  - Constraints: budget, max/min investment per asset, minimum return
  - Objective: overall risk or return variance
- Data fitting
  - Variables: model parameters
  - Constraints: prior information, parameter limits
  - Objective: measure of misfit or prediction error

Linear Programming

- Formulation

$$
\begin{array}{ll}
\text{minimize} & \mathbf{c}^\top \mathbf{x} \\
\text{subject to} & \mathbf{a}_i^\top (\mathbf{x}) \leq b_i, \quad i = 1, \ldots, M \\
\text{variables :} & \mathbf{x}.
\end{array}
$$

- Solving linear programs
  - No analytical formulation for solution
  - Reliable and efficient algorithms
  - Computation time proportional to $n^2 m$
  - A mature technology

An Illustrative Example on Linear Programming Problem

- Problem Setting
  - A company with three plants (Plant 1, Plant 2, and Plant 3) produces windows and doors
  - Plant 1 produces aluminum frames and hardware
  - Plant 2 produces wood frames
  - Plant 3 produces the glass and assembles the products
- Two new products
  - Product 1: A glass door with aluminum framing
  - Product 2: A wood-framed window
  - Product 1 needs Plant 1 and 3
  - Product 2 needs Plant 2 and 3
- Question: which mix of the two products would be most profitable?

Step 1: Define the Problem

- Problem Setting
  - Determine what the production rates should be for the two products in order to maximize their total profit
  - Subject to the restrictions imposed by the limited production capacities available in the three plants

Step 2: Gather the Data

- Data to be gathered
  - Number of hours of production time available per week in each plant
  - Number of hours of production time used in each batch produced of each product
  - Profit per batch produced of each new product

Plant	Production Time per Batch, Hours Product 1	 2	Production Time Available per Week, Hours
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	\$3,000	\$5,000	

Step 3: Mathematical Formulation

- Decision variables
  - Define x_1, x_2 the number of batches of product 1 and product 2 produced per week, respectively
  - Define z the total profit per week
- Linear program formulation

$$
\begin{array}{ll}
\text{minimize} & z = 3x_1 + 5x_2 \\
\text{subject to} & x_1 \leq 4, \\
 & 2x_2 \leq 12, \\
 & 3x_1 + 2x_2 \leq 18, \\
 & x_1 \geq 0, \\
 & x_2 \geq 0. \\
\text{variables :} & x_1, x_2.
\end{array}
$$

Integer Linear Programming Problem

- Formulation

$$
\begin{array}{ll}
\text{minimize} & \mathbf{c}^{\top}\mathbf{x} \\
\text{subject to} & \mathbf{a}_i^{\top}(\mathbf{x}) \leq b_i, \quad i = 1, \ldots, M \\
 & \mathbf{x} \in \mathbb{Z}^n. \\
\text{variables :} & \mathbf{x}.
\end{array}
$$

- Solving linear programs
  - No analytical formulation for solution
  - ILP is NP hard
  - Computation time depends on the algorithms used

An Illustrative Example on Integer Linear Programming Problem

- Problem Setting
  - A company wants to build a new factory in either Los Angeles or San Francisco, or in both cities
  - Also build at most one warehouse, the location will be restricted to the city where a new factory is being built

- Data gathered: net present value, capital required

Decision Number	Yes-or-No Question	Decision Variable	Net Present Value	Capital Required
1	Build factory in Los Angeles?	x_1	\$9 million	\$6 million
2	Build factory in San Francisco?	x_2	\$5 million	\$3 million
3	Build warehouse in Los Angeles?	x_3	\$6 million	\$5 million
4	Build warehouse in San Francisco?	x_4	\$4 million	\$2 million
			Capital available:	\$10 million

An Illustrative Example on Integer Linear Programming Problem

- Objective: find the feasible combination of alternatives to maximize the total net present value
- Decision variables:
  $x_j \in \{0, 1\}$, represents decision j is yes or no.
- Let z denote the total net present value these decisions

An Illustrative Example on Integer Linear Programming Problem

- Formulation

$$
\begin{array}{ll}
\text{minimize} & z = 9x_1 + 5x_2 + 6x_3 + 4x_4 \\
\text{subject to} & 6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10, \\
& x_3 + x_4 \leq 1, \\
& -x_1 + x_3 \leq 0, \\
& -x_2 + x_4 \leq 0, \\
& x_j \leq 1, \\
& x_j \geq 0, \\
& x_j \in \mathbb{Z}^n, \quad j = 1, 2, 3, 4. \\
\text{variables :} & \mathbf{x}.
\end{array}
$$

Least squares

- Formulation

$$\begin{array}{ll} \text{minimize} & \|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2 \\ \text{variables :} & \mathbf{x}. \end{array}$$

- Solving least-squares problems
  - Analytical solution: $\mathbf{x}^\star = (\mathbf{A}^\top \mathbf{A})^{-1}\mathbf{A}^\top \mathbf{b}$
  - Reliable and efficient algorithms
  - Computation time proportional to $n^2 k(\mathbf{A} \in \mathbb{R}^{k\times n})$
  - A mature technology
  - least-squares problems are easy to recognize

Convex Optimization Problem

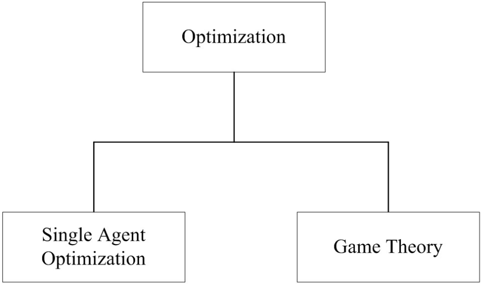
- Formulation

$$\begin{array}{ll} \text{minimize} & f_0(\mathbf{x}) \\ \text{subjectto :} & f_i(\mathbf{x}) \leq b_i, \quad i = 1, \ldots, M. \\ \text{variables :} & \mathbf{x}. \end{array}$$

- Objective and constraint functions are convex
  $f_i(\alpha\mathbf{x} + \beta\mathbf{y}) \leq \alpha f_i(\mathbf{x}) + \beta f_i(\mathbf{y})$, if $\alpha + \beta = 1, \alpha \geq 0, \beta \geq 0$
- least squares problems and linear programs as special cases
- Solving convex optimization problems
  - No analytical solution
  - Reliable and efficient algorithms
  - Computation time proportional to $\max\{n^3, n^2m, F\}$, where F is cost of evaluating the constraints and their first and second derivatives

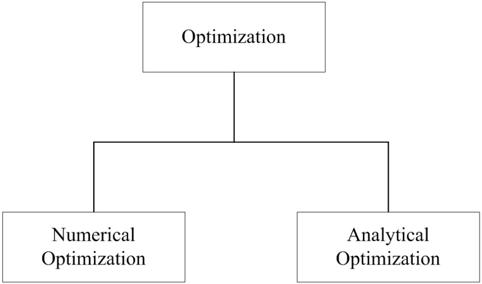
Classification of Optimization Models

- According to agents involved



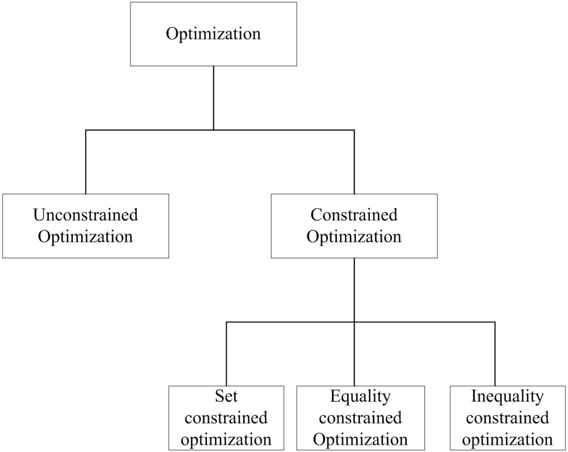
Classification of Optimization Models

- According to solution methods



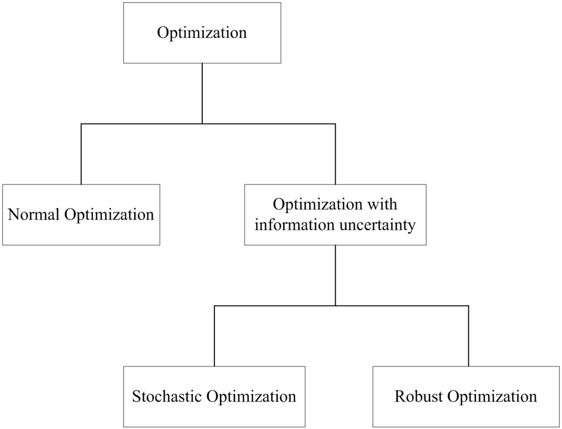
Classification of Optimization Models

- According to constraints involved



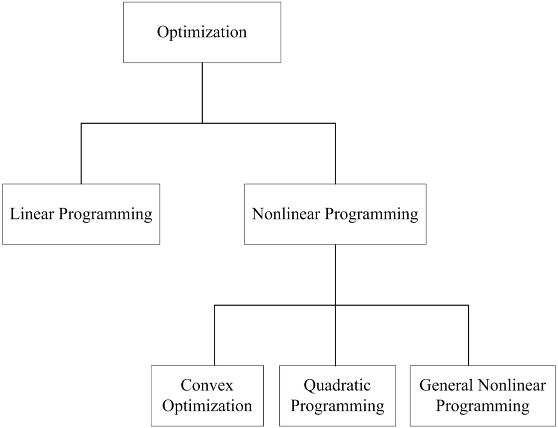
Classification of Optimization Models

- According to information completeness



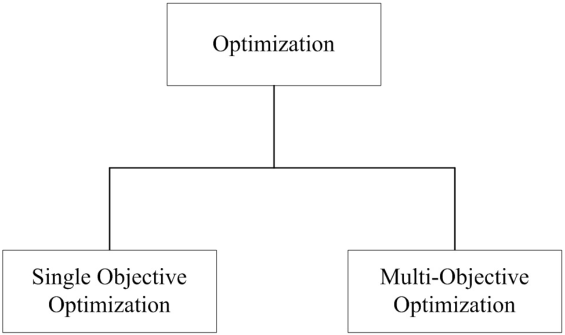
Classification of Optimization Models

- According to property of the problem



Classification of Optimization Models

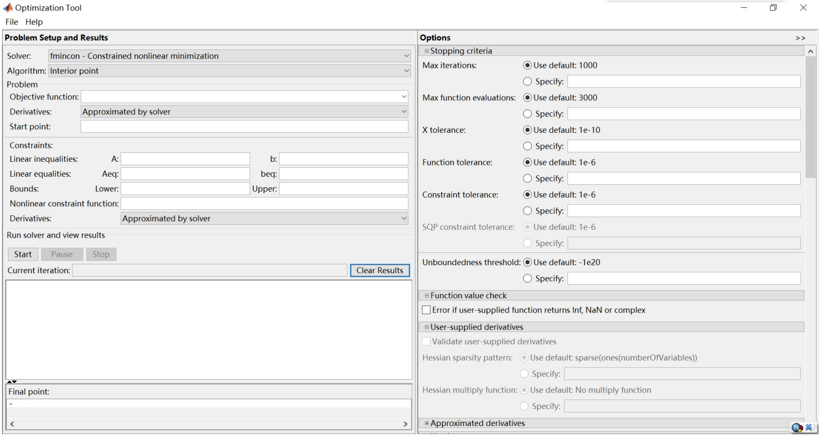
- According to objectives



Related Optimization Softwares

- Excel (with add-ins, e.g., Analytic Solver Platform for Education [ASPE, free for 140 days])
- Matlab Optimization toolbox
- Mathematica
- CVX (free on “http://cvxr.com/cvx/”)

Example of Matlab Optimization Toolbox



Example of Matlab Optimization Toolbox

Consider the problem of finding $[x_1, x_2]$ that solves

$$\min_x f(x) = x_1^2 + x_2^2$$

subject to the constraints

$$\begin{aligned}
0.5 &\le x_1 && \text{(bound)} \\
-x_1 - x_2 + 1 &\le 0 && \text{(linear inequality)} \\
\left.\begin{aligned}
-x_1^2 - x_2^2 + 1 &\le 0 \\
-9x_1^2 - x_2^2 + 9 &\le 0 \\
-x_1^2 + x_2 &\le 0 \\
-x_2^2 + x_1 &\le 0
\end{aligned}\right\} & && \text{(nonlinear inequality)}
\end{aligned}$$

Example of Matlab Optimization Toolbox

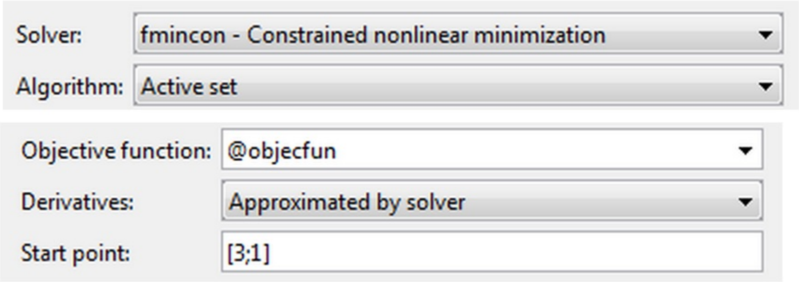
Step 1: Write a file objecfun.m for the objective function.

```
function f = objecfun(x)
f = x(1)^2 + x(2)^2;
```

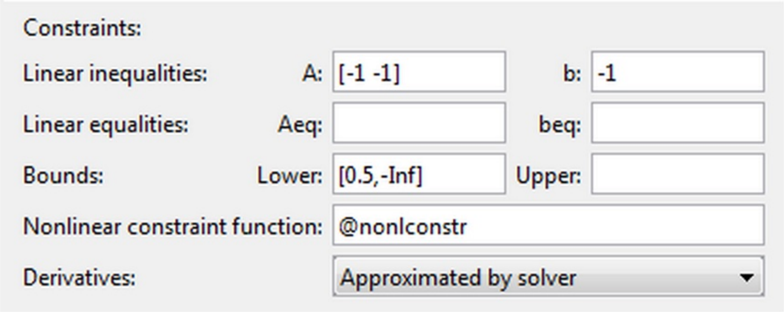
Step 2: Write a file nonlconstr.m for the nonlinear constraints.

```
function [c,ceq] = nonlconstr(x)
c = [-x(1)^2 - x(2)^2 + 1;
     -9*x(1)^2 - x(2)^2 + 9;
     -x(1)^2 + x(2);
     -x(2)^2 + x(1)];
ceq = [];
```

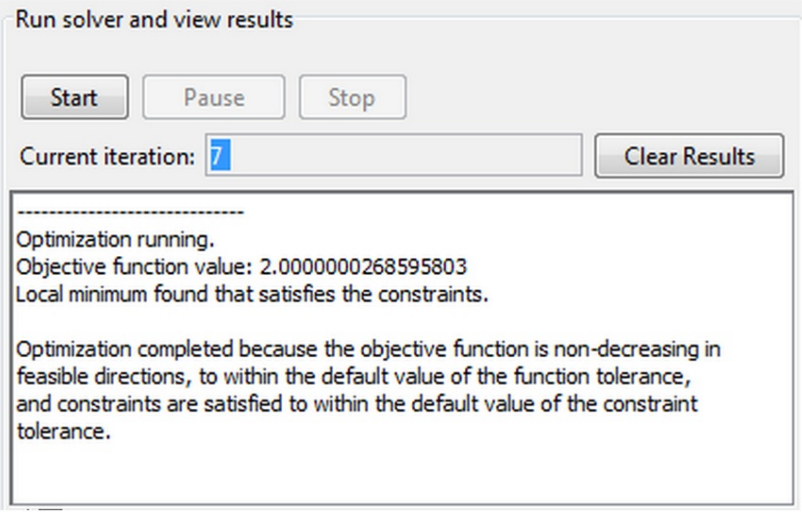
Example of Matlab Optimization Toolbox



Example of Matlab Optimization Toolbox



Example of Matlab Optimization Toolbox



Example of Matlab Optimization Toolbox

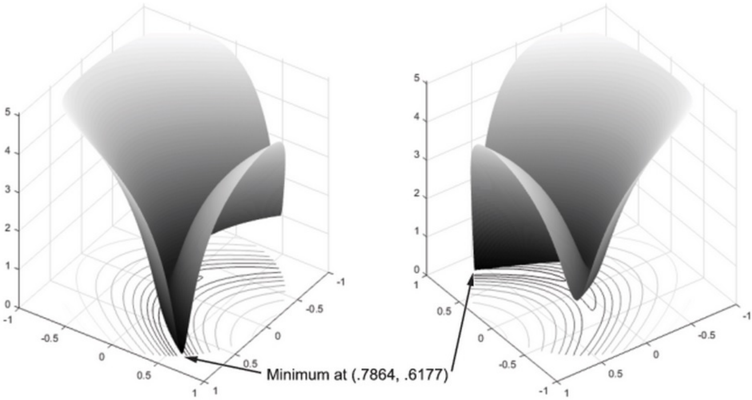
er	F-count	f(x)	Max constraint	Line search steplength	Directional derivative	First-order optimality	Procedure
0	3	10	2				Infeasible start point
1	6	4.84298	-0.1322	1	-5.22	1.74	
2	9	4.0251	-0.01168	1	-4.39	4.08	Hessian modified twice
3	12	2.42704	-0.03214	1	-3.85	1.09	
4	15	2.03615	-0.004728	1	-3.04	0.995	Hessian modified twice
5	18	2.00033	-5.596e-005	1	-2.82	0.0664	Hessian modified twice
6	21	2	-5.327e-009	1	-2.81	0.000522	Hessian modified twice

Example of Matlab Optimization Toolbox

$$f(\mathbf{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

$$x_1^2 + x_2^2 \leq 1$$

Example of Matlab Optimization Toolbox



Thanks for attending!