A DIRECT PROOF OF THE REFLECTION PRINCIPLE FOR BROWNIAN MOTION

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ABSTRACT. We present a self-contained proof of the reflection principle for Brownian Motion.

1. INTRODUCTION

The reflection principle proved below is one of the most important properties of Brownian Motion. So much so that any treatment of Brownian Motion would be incomplete without mentioning it and some of its many applications (see e.g. [5]). Most notable among these applications, using the hitting time $\tau_x = \inf\{t : B_t = x\}$, is that

$$P(\tau_x \leq t) = 2P(B_t \geq x),$$

which in turn yields that $X_t := \max_{0 \leq s \leq t} B_s$ and $|B_t|$ have the same distribution. This famous result is attributed to Louis Bachelier [1, p. 197], and also, in a later more rigorous treatment, to Paul Lévy [2, p. 293]. In fact it was Bachelier who first introduced the stochastic process, which later on became known as Brownian Motion, as a model for stock prices in his pioneering work in mathematical finance. Remarkably, [1] precedes the rigorous construction of Brownian Motion by almost two decades.

The reflection principle is invariably presented as a consequence of the Strong Markov Property. This approach has pedagogical value as it provides one of the first applications of the Strong Markov Property (see e.g. [4]). However, it has the drawback of being beyond the scope of less specialized texts and consequently the proof of the reflection principle is often omitted. We present here a short and direct proof requiring few prerequisites which is intended to make the reflection principle more accessible.

Recall that a *Standard Brownian Motion* (SBM) on a probability space $(\Omega, \mathcal{F}, P)$ is a gaussian process $(B_t)_{t \geq 0}$ (i.e., the finite-dimensional

distributions are multivariate normal distributions), with $B_0 = 0$, continuous sample paths, $\mathbb{E}[B_t] = 0$, and covariance function $\mathbb{E}[B_s B_t] = \min(s,t)$. The σ-algebra $\mathcal{F}_t$ is the smallest σ-algebra containing all P-null sets for which each B_s $(0 \leq s \leq t)$ is measurable.

A stopping time with respect to the *standard Brownian filtration* $(\mathcal{F}_t)_{t \geq 0}$ is a mapping $T \colon \Omega \to [0,\infty]$ satisfying $\{T \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$. T is allowed to take the value ∞ with positive probability.

A tool that is used in our proof of the reflection principle is the 'uniqueness theorem': the fact that the distribution of an $\mathbb{R}^n$-valued random vector X is determined by its characteristic function $\phi_X(\lambda) := \mathbb{E}[\exp(i\lambda \cdot X)]$ $(\lambda \in \mathbb{R}^n)$ (see e.g. [3, p. 135]). The uniqueness theorem is used in a similar way to prove the Strong Markov Property in [4].

Our proof also uses standard properties of the conditional expectation operator with respect to a sub-σ-algebra $\mathcal{G}$, namely linearity and the fact that $\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}]$ for random variables X, Y when X is $\mathcal{G}$-measurable (see e.g. [3, p. 187]). The 'independence of Brownian increments' is used in the following intuitively obvious but slightly tricky to prove form: if $n \geq 1$ and $s < t_1 < \cdots < t_n$ and $f \colon \mathbb{R}^n \to \mathbb{R}$ is bounded and continuous, then, setting $V := f(B_{t_1} - B_s, \ldots, B_{t_n} - B_s)$,

$$(1) \qquad \mathbb{E}[V|\mathcal{F}_s] = \mathbb{E}[V].$$

For completeness a short proof of this standard fact is given at the end.

2. Reflection Principle

Theorem 2.1. *(Reflection Principle) Let $(B_t)_{t \geq 0}$ be an SBM and let T be a stopping time with respect to $(\mathcal{F}_t)_{t \geq 0}$. Define*

$$B_t^T := \begin{cases} B_t, & 0 \leq t \leq T \\ 2B_T - B_t, & t > T. \end{cases}$$

Then $(B_t^T)_{t \geq 0}$ is an SBM.

Proof. Note that (B_t^T) clearly has continuous sample paths. By the uniqueness theorem, to complete the proof it is enough to show, for each $n \geq 1$ and $0 < t_1 < \cdots < t_n < \infty$, and constants $\lambda_j \in \mathbb{R}$ $(1 \leq j \leq n)$, that $\mathbb{E}[e^{iX^T}] = \mathbb{E}[e^{iX}]$, where

$$X := \sum_{j=1}^n \lambda_j B_{t_j} \quad \text{and} \quad X^T := \sum_{j=1}^n \lambda_j B_{t_j}^T.$$

For notational convenience, set $t_0 := 0$ and $t_{n+1} := \infty$. First, suppose T takes only finitely many values $0 \leq a_1 < \cdots < a_m < \infty$. For each

$1 \leq r \leq m$, choose k_r such that $t_{k_r} \leq a_r < t_{k_r+1}$ and let

$$Y_r := \sum_{j=1}^{k_r} \lambda_j B_{t_j} + \left(\sum_{j=k_r+1}^{n} \lambda_j \right) B_{a_r} \quad \text{and} \quad Z_r := \sum_{j=k_r+1}^{n} \lambda_j (B_{t_j} - B_{a_r}).$$

Note that Y_r is $\mathcal{F}_{a_r}$-measurable, Z_r is independent of $\mathcal{F}_{a_r}$ by (1), and also that

$$X = \sum_{r=1}^{m} (Y_r + Z_r) \mathbb{1}_{\{T=a_r\}} \quad \text{and} \quad X^T = \sum_{r=1}^{m} (Y_r - Z_r) \mathbb{1}_{\{T=a_r\}}.$$

Therefore

$$\mathbb{E}[e^{iX^T}] = \sum_{r=1}^{m} \mathbb{E}[e^{i(Y_r - Z_r)} \mathbb{1}_{\{T=a_r\}}]$$

$$= \sum_{r=1}^{m} \mathbb{E}[e^{iY_r} \mathbb{1}_{\{T=a_r\}} \mathbb{E}[e^{-iZ_r}|\mathcal{F}_{a_r}]]$$

(since $e^{iY_r} \mathbb{1}_{\{T=a_r\}}$ is $\mathcal{F}_{a_r}$-measurable)

$$= \sum_{r=1}^{m} \mathbb{E}[e^{iY_r} \mathbb{1}_{\{T=a_r\}}] \mathbb{E}[e^{-iZ_r}]$$

(by independence of Z_r with respect to $\mathcal{F}_{a_r}$)

$$= \sum_{r=1}^{m} \mathbb{E}[e^{iY_r} \mathbb{1}_{\{T=a_r\}}] \mathbb{E}[e^{iZ_r}]$$

(by symmetry of Z_r)

$$= \sum_{r=1}^{m} \mathbb{E}[e^{i(Y_r + Z_r)} \mathbb{1}_{\{T=a_r\}}] = \mathbb{E}[e^{iX}]$$

(by reversing the steps to get the first equality above). To extend the result to a general stopping time T, we simply approximate T by stopping times T_j which take only finitely many values. To make this precise, let $T_j(\omega) := 2^j$ if $T(\omega) > 2^j$ and $T_j(\omega) := k2^{-j}$ if $(k - 1)2^{-j} < T(\omega) \leq k2^{-j} \leq 2^j$. Then clearly $T_j \to T$ almost surely and, by continuity of the sample paths of (B_t), $X^{T_j} \to X^T$ almost surely. Thus, by the bounded convergence theorem,

$$\mathbb{E}[e^{iX^T}] = \lim_{j \to \infty} \mathbb{E}[e^{iX^{T_j}}] = \mathbb{E}[e^{iX}].$$

$\square$

Finally, we prove (1). By definition of the conditional expectation operator, we have to show that, for all $A \in \mathcal{F}_s$,

$$(2) \qquad \mathbb{E}[V \mathbb{1}_A] = \mathbb{E}[V]P(A).$$

The collection $\mathcal{G}$ of all $A \in \mathcal{F}_s$ for which (2) holds is easily seen to be a *monotone class* (i.e., $\mathcal{G}$ is closed under countable increasing unions and decreasing intersections) containing the P-null sets. Moreover, given $m \geq 1$ and $0 < s_1 < \cdots < s_m \leq s$, $\mathcal{G}$ contains the σ-algebra $\sigma(B_{s_1}, \ldots, B_{s_m})$, the smallest σ-algebra for which each B_{s_j} $(1 \leq j \leq m)$ is measurable: this follows from independence of Brownian increments. The union over all of these σ-algebras as m and $(s_j)_{j=1}^m$ vary is an algebra whose augmentation by the P-null sets generates $\mathcal{F}_s$. The monotone class lemma (see e.g. [3, p. 4]) now gives $\mathcal{G} = \mathcal{F}_s$.

References

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