Multilayer Percetprons

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Deep Learning

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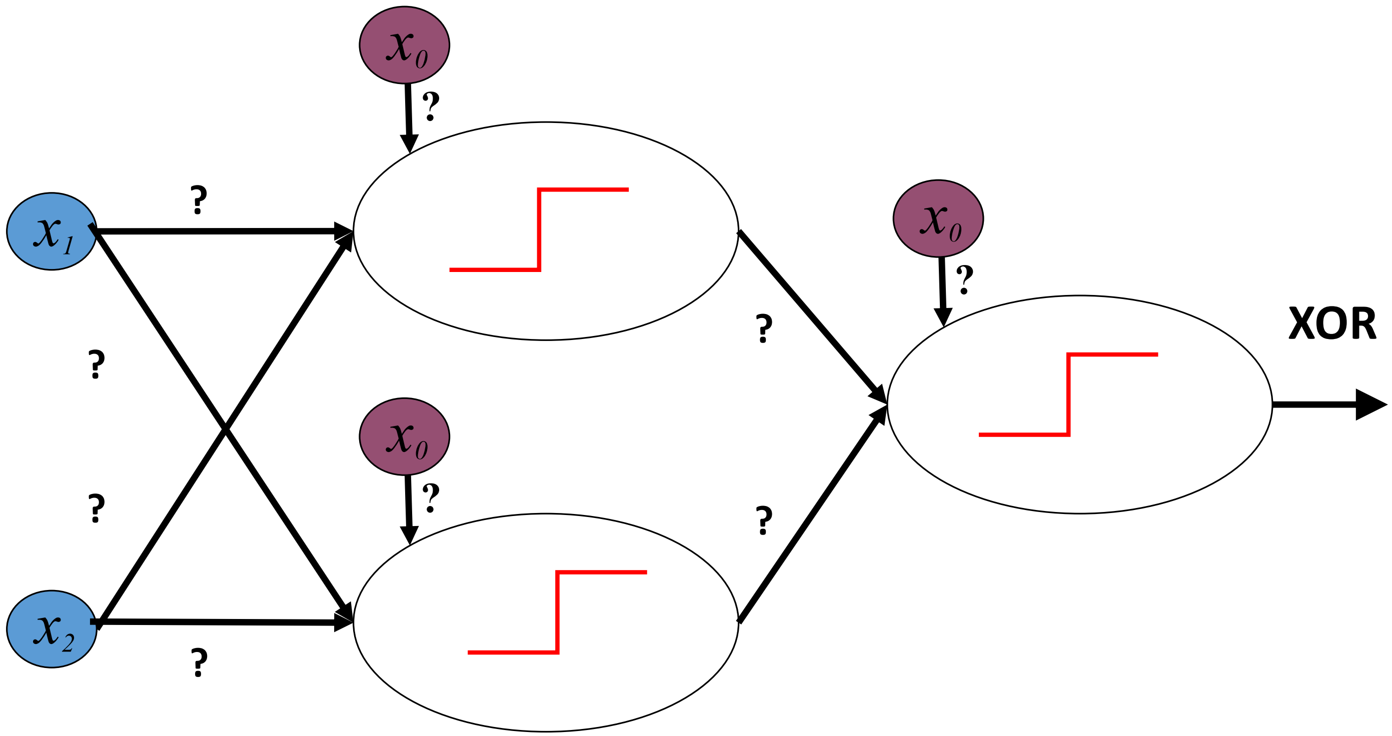
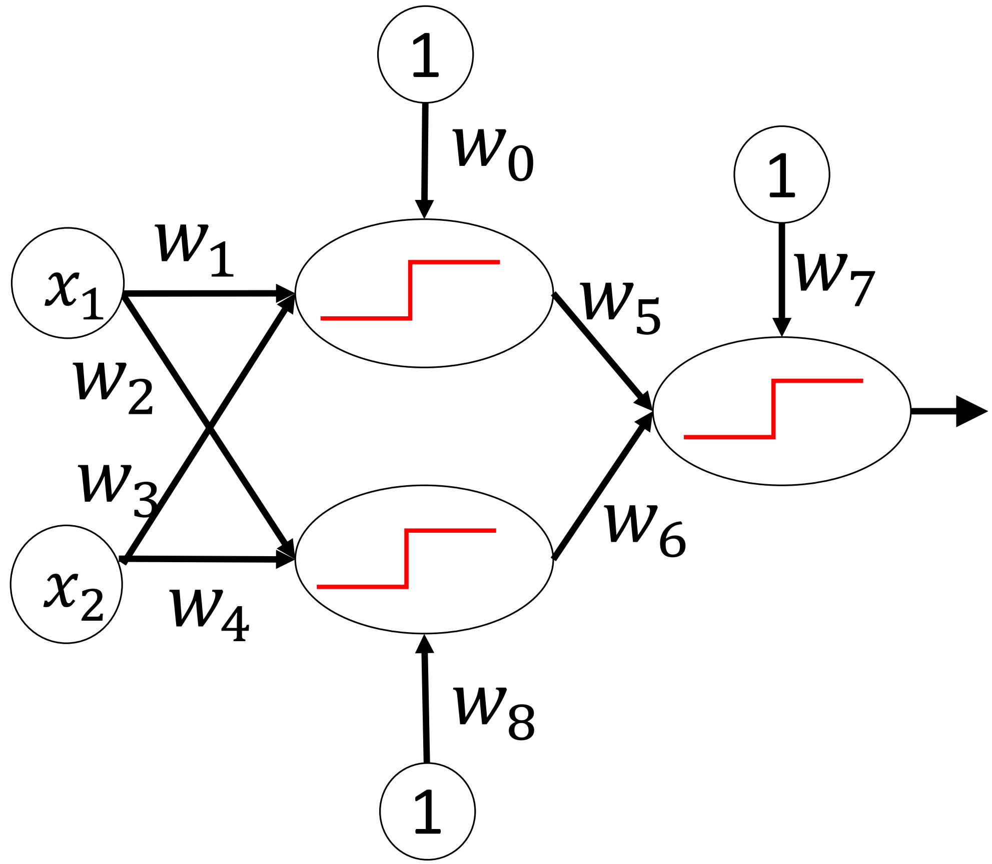
Combining perceptrons can make any Boolean function



…if you can set the weights & connections right

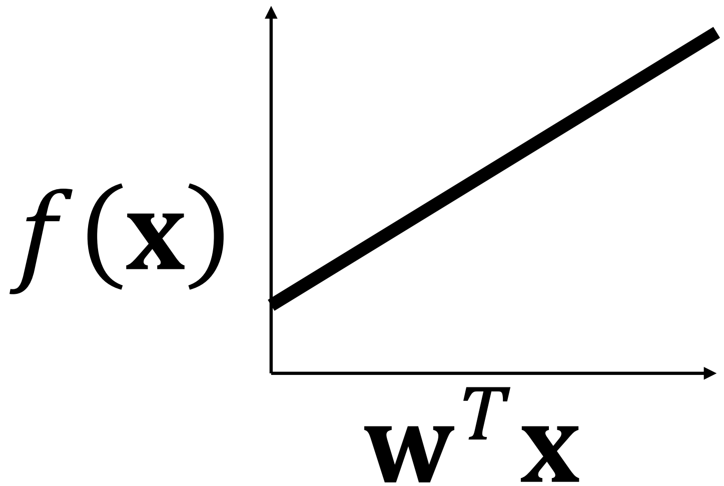
Problem with a step function: Assignment of error

- Stymies multi-layer weight learning
- Limits us to a single layer of units
- Thus, only linear functions
- You can hand-wire XOR perceptrons, but the sytem can't learn XOR with perceptrons



Linear Units & Delta Rule

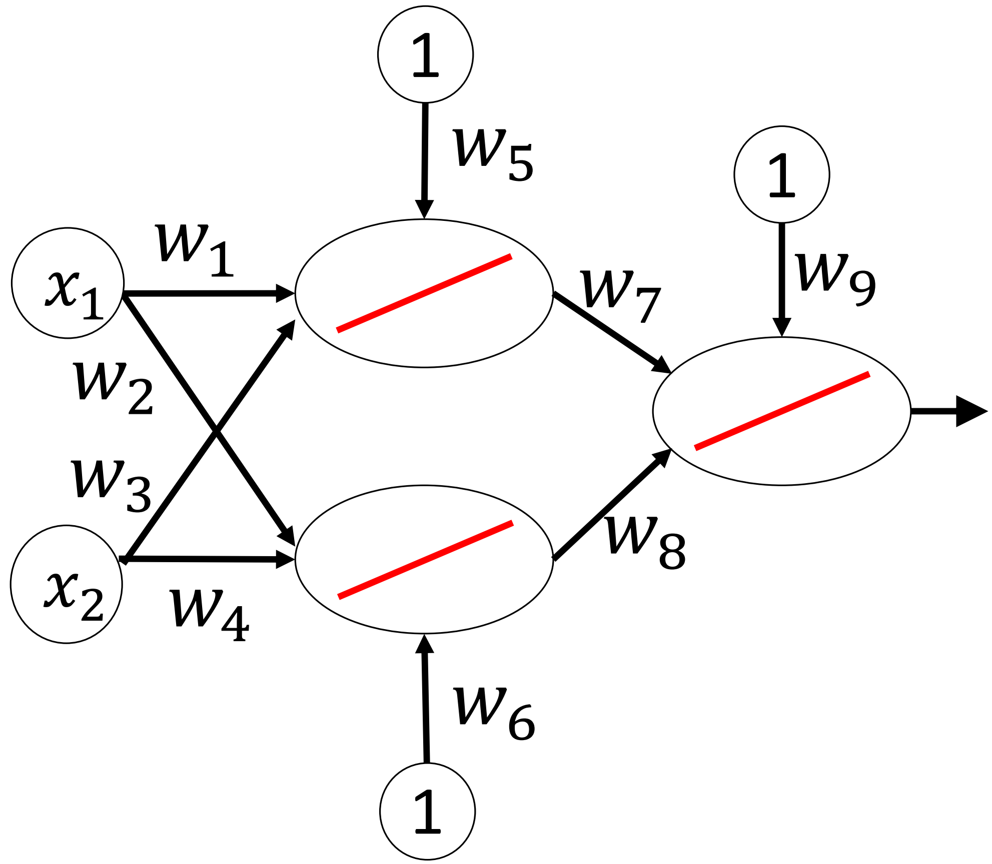
Solution: Remove the step function



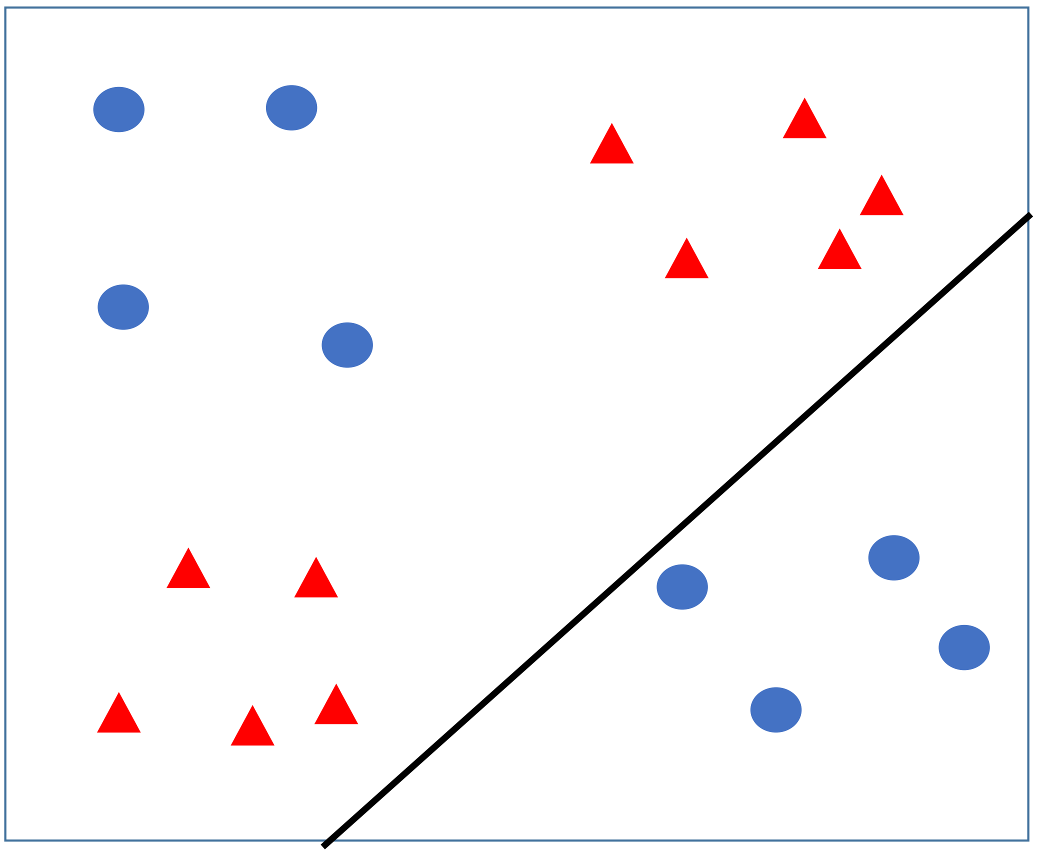
$$f(\mathbf{x}) = \sum_{i=0}^{n} w_i x_i = \mathbf{w}^T\mathbf{x}$$

Better & worse than a perceptron

- All changes in input result in changed output
- This gives us a gradient everywhere
- We can learn multiple layers of weights.
- **Combining linear functions only gives you linear functions**
- you can't represent XOR



Many linear units: Only linear decisions



This is XOR.

A multilayer perceptron with linear units CANNOT learn XOR

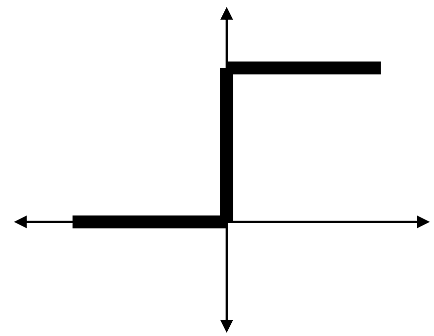
The Sigmoid Unit

Rumelhart, David E., James L. McClelland, and PDP Research Group. Parallel distributed processing. Vol. 1. Cambridge, MA, USA:: MIT press, 1987.

Sigmoid (aka Logistic) function: best of both

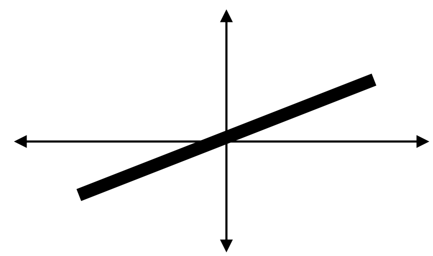
- Perceptron

$$f(x) = \begin{cases} 1 \; if \; 0 < \sum_{i=0}^{n} w_i x_i \\ -1 \; else \end{cases}$$



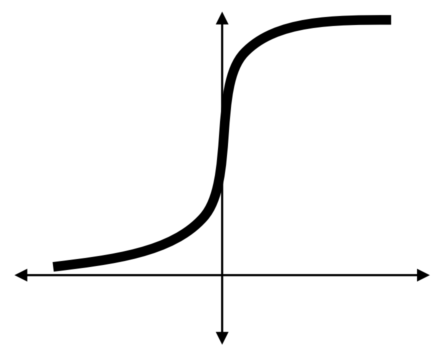
- Linear

$$f(x) = \mathbf{w}^T \mathbf{x} = \sum_{i=0}^{n} w_i x_i$$



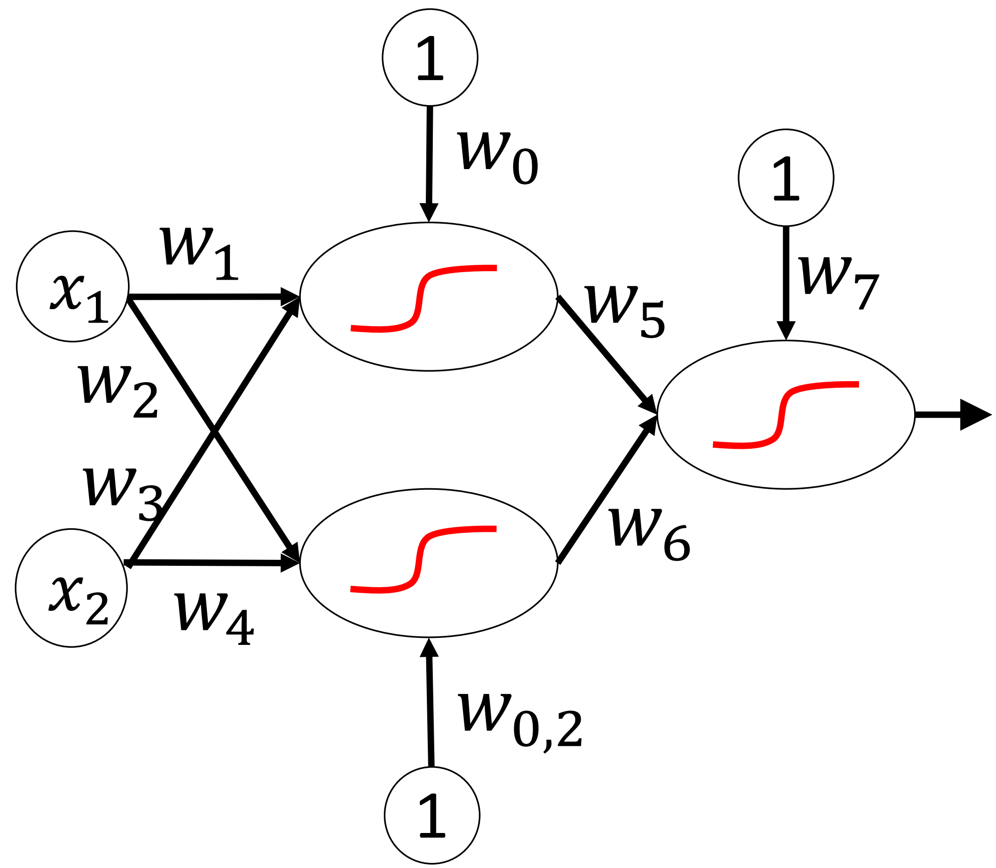
- Sigmoid

$$f(x) = \frac{1}{1 + e^{-(\mathbf{w}^T \mathbf{x})}}$$



A network of sigmoid units

- Small changes in input result in output
- This gives us a gradient everywhere
- We can learn multiple layers of weights.
- Combining layers gives non-linear functions



Sigmoid changes (almost) everything

Easy to differentiate

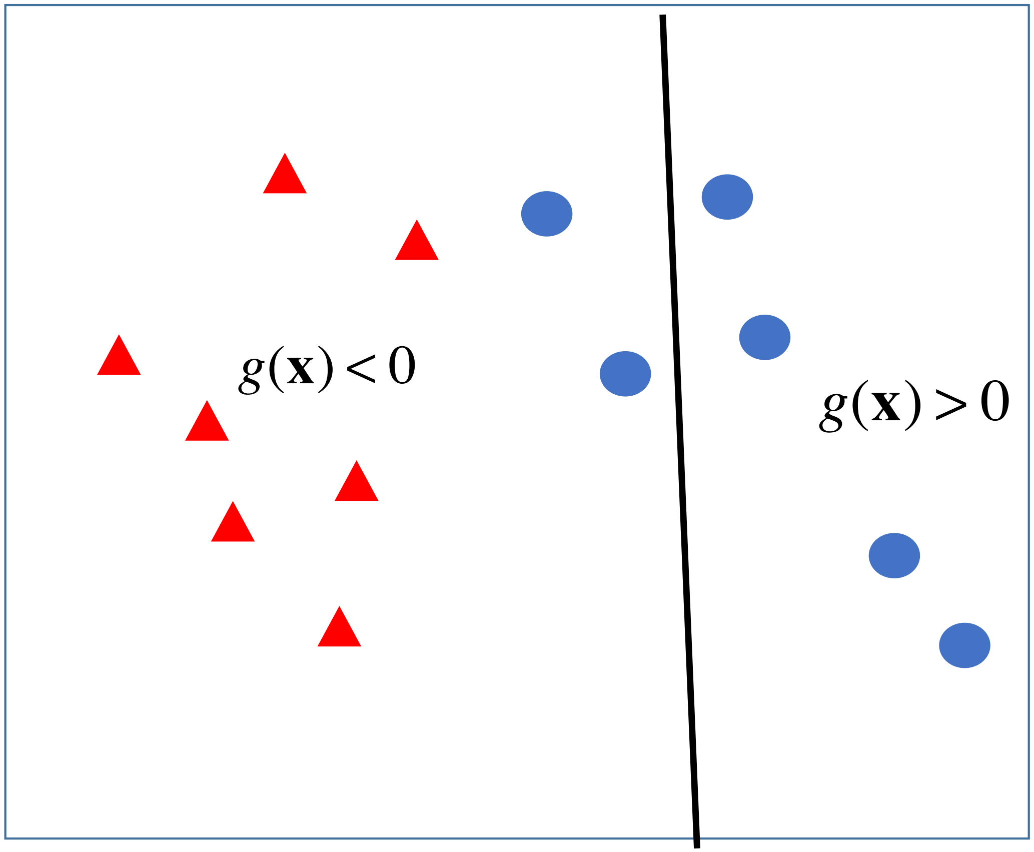
$\sigma'(\mathbf{w}^T\mathbf{x}) = \sigma(\mathbf{w}^T\mathbf{x})$(1- $\sigma(\mathbf{w}^T\mathbf{x})$)

Gradient everywhere

This allows backpropagation of the gradient through multiple layers

Nonlinearity allows arbitrary nonlinear functions to be built by using multiple layers.

Example objective *J* : sum of squared errors

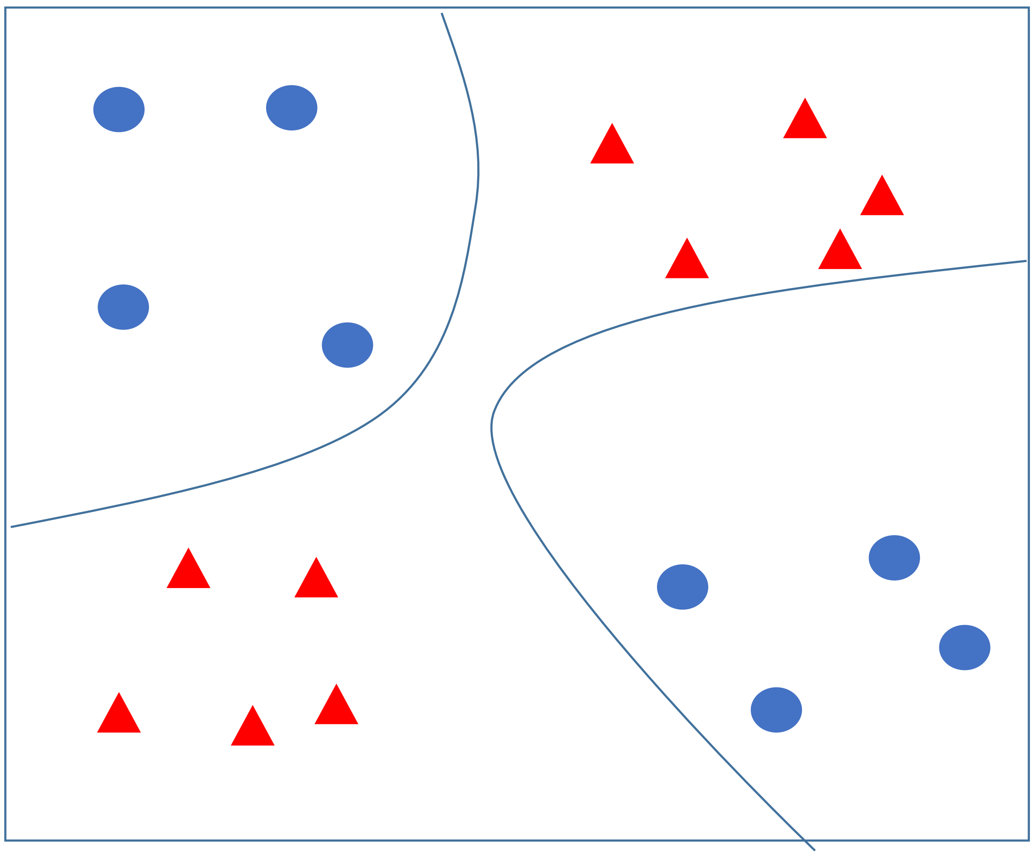


$$h(x) = f(x) = \frac{1}{1 + e^{-(\mathbf{w}^T\mathbf{x})}}$$

$$SSE = \sum_{i}^{n} (y_i - h(\mathbf{x}_i))^2$$

Gradient non-zero everywhere!

Multilayer Perceptron with sigmoid units



This is XOR.

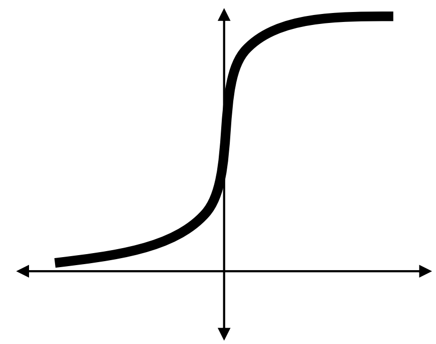
A multilayer perceptron with sigmoid units CAN learn XOR...or any other arbitrary Boolean function.

The promise of many layers

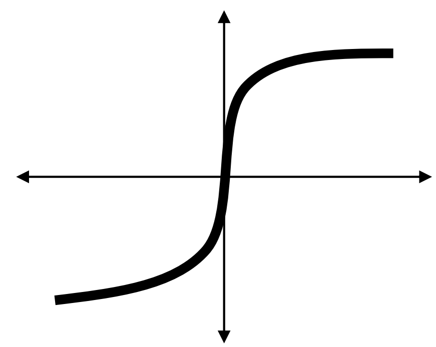
- Each layer learns an abstraction of its input representation (we hope)
-
- As we go up the layers, representations become increasingly abstract
- The hope is that the intermediate abstractions facilitate learning functions that require non-local connections in the input space (recognizing rotated & translated digits in images, for example)
- Modern neural networks are up to 100 layers deep

TanH: A shifted sigmoid

- Sigmoid $$f(x) = \frac{1}{1 + e^{-(\mathbf{w}^T \mathbf{x})}}$$

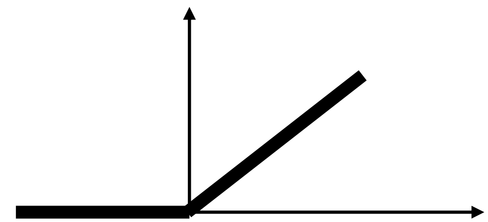


- TanH $$f(x) = \frac{2}{1 + e^{-2(\mathbf{w}^T \mathbf{x})}} - 1$$

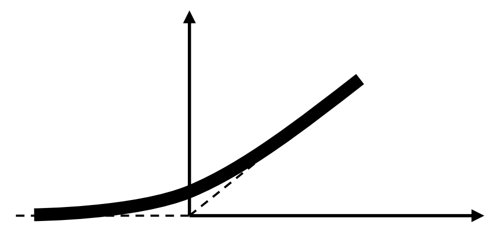


Rectified Linear Unit (ReLU) & Soft Plus :

- ReLU $$f(x) = \max(0, \mathbf{w}^T\mathbf{x})$$



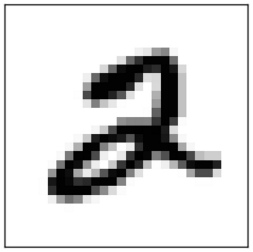
- Soft Plus $$f(x) = \ln(1 + e^{\mathbf{w}^T\mathbf{x}})$$



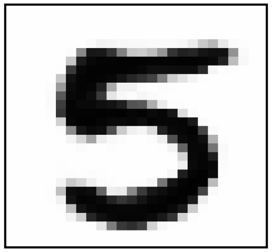
- Both can be combined in layers to make non-linear functions

“One Hot” Encoding

- A vector of values where a single element is 1 and all the rest are 0
- Common way to encode the true label, y, in a multi-class labeling problem
- Can be interpreted as a probability distribution



y = 0 0 1 0 0 0 0 0 0 0



y = 0 0 0 0 0 1 0 0 0 0

Probability distribution

* Discrete random variable X represents some experiment.

* $P(X)$ is the probability distributions over $\{x_1, \ldots, x_n\}$, the set of possible outcomes for X.

* These outcomes are mutually exclusive.

* Their probabilities sum to one: $\sum_{i=1}^{n} P(x_i) = 1$

Soft Max Function

- Turns an N-dimensional vector of real numbers into a probability distribution
- For a deep net, a_i is the output of the ith node in the output layer

$$p_i = \frac{e^{a_i}}{\sum_{j=1}^{N} a_j}$$

Cross Entropy Loss Function

Given: "true" distribution $y = \{y_1, y_2, \dots y_k\}$ <-often a one-hot encoding

and estimated distribution $p = \{p_1, p_2, \dots p_k\}$ <-soft max over the last layer

Define cross entropy loss between 2 distributions as

$$L(y,p) = -\sum_{i=1}^{N} y_i \log(p_i)$$

A common approach...

- Define labels with a one-hot vector encoding
- Make the last layer have n nodes for an n-way classification problem
- Apply soft max to the last layer
- Use a cross-entropy loss function
- The resulting derivative of the loss function is wonderfully simple:

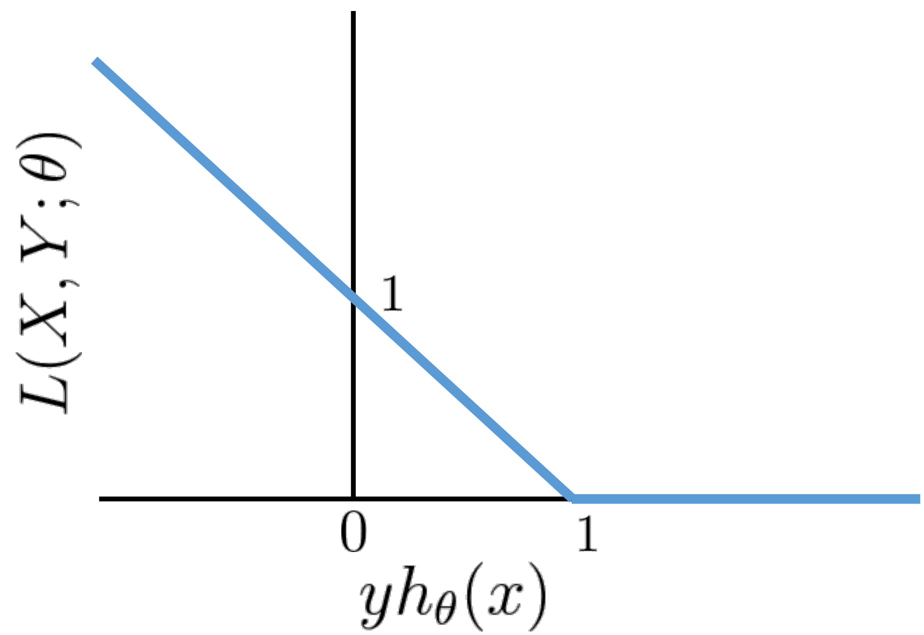
$$\frac{\partial L}{\partial a_i} = p_i - y_i$$

L is the loss, i is the index to a node, a is the output of the last layer, p is the softmax probability and y is the label.

Hinge Loss

$$L_H(X, Y; \theta) = \frac{1}{N} \sum_{i=1}^{N} \max(0, 1 - y_i h_\theta(x_i))$$

- Loss only happens if the data is on the wrong side of the line.



Hinge Loss

The loss function:

$$L_H(X,Y;\theta) = \frac{1}{N}\sum_{i=1}^{N} \max(0, 1 - y_i h_\theta(x_i))$$

The gradient of the loss function is…

$$\nabla_\theta L_H(X,Y;\theta) = \frac{1}{N}\sum_{i=1}^{N} [[1 - y_i h_\theta(x_i) > 0]](-y_i \nabla_\theta h_\theta(x_i))$$

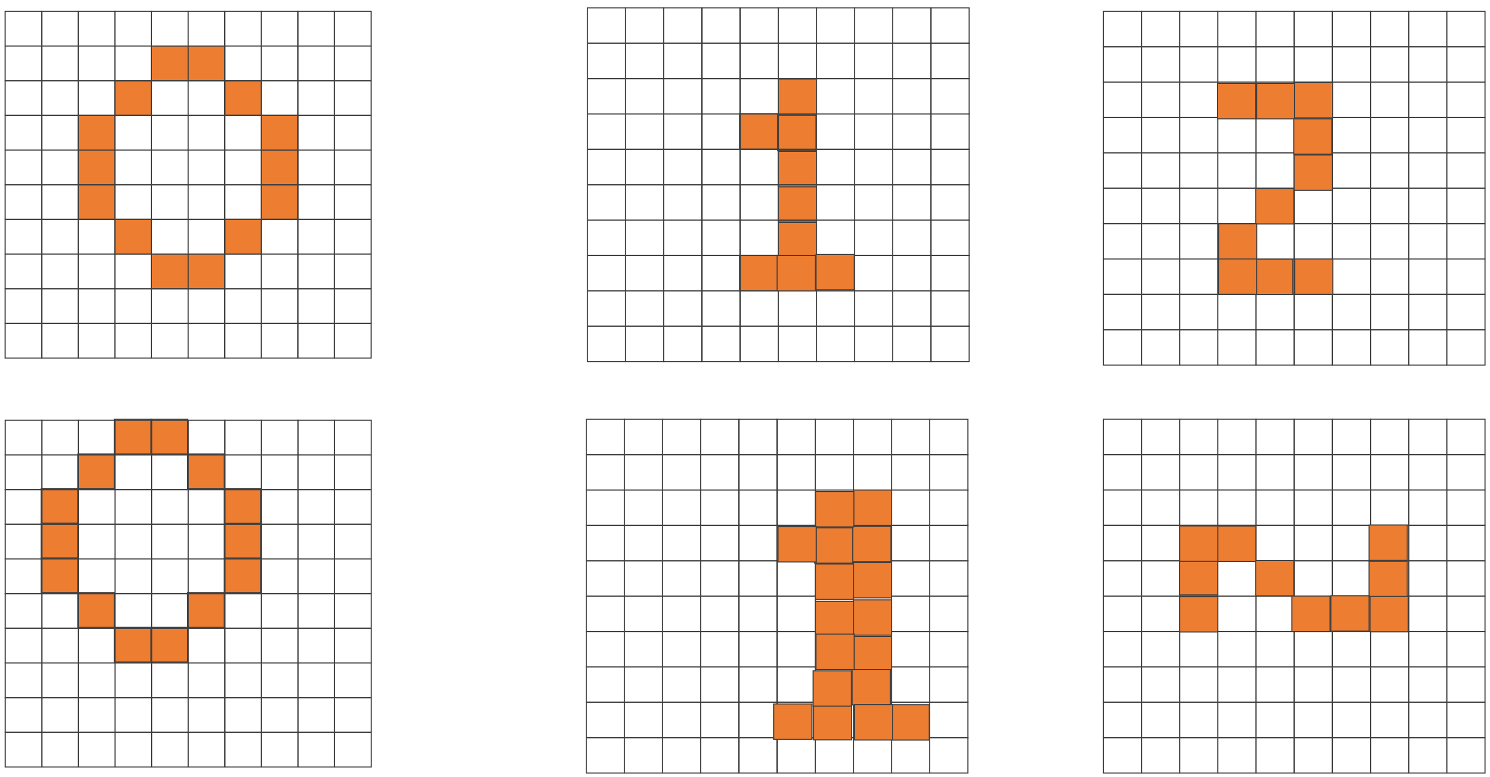
For example, if we use a linear model $h_\theta(x_i) = \theta^T x_i$,

$$\nabla_\theta L_H(X,Y;\theta) = \frac{1}{N}\sum_{i=1}^{N} [[1 - y_i \theta^T x_i > 0]](-y_i x_i)$$

Design choices

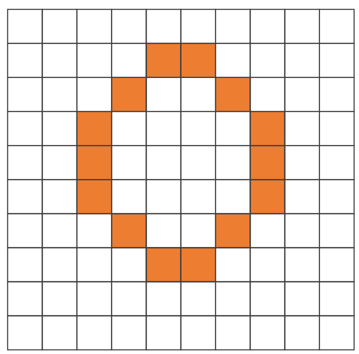
- Define the function you want to learn
- Determine an encoding for the data
- Pick a network architecture
  - Number of layers (between 3 and 100)
  - Activation functions function (tanh,ReLU, linear)
  - Select how units connect within and between layers
- Pick a gradient descent algorithm
- Pick regularization approach (e.g. dropout)

Classifying images of digits



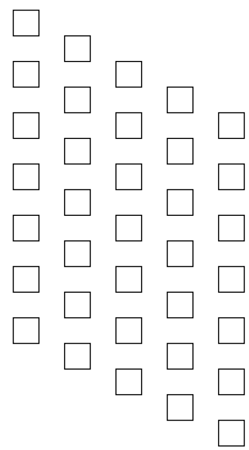
A possibility: one fully-connected hidden layer

INPUT LAYER



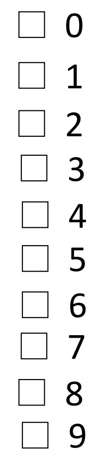
One input
per pixel

HIDDEN LAYER



One hidden node per
potential shifted image
(what activation function?)

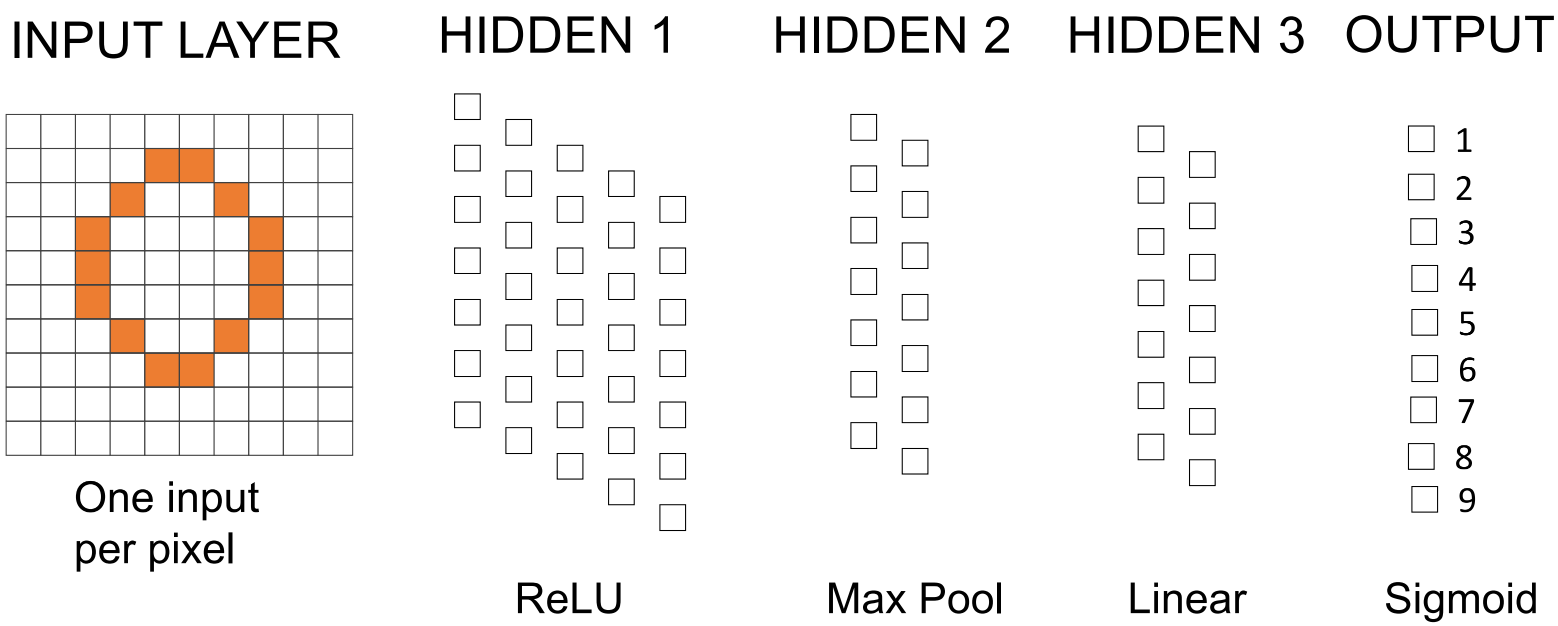
OUTPUT LAYER



One output node
per category
(Sigmoid? ReLU?)

Each node is connected to EVERY node in the prior layer
(it is just too many lines to draw)

Another possibility



HUGE DESIGN SPACE!