第四节 多元复合函数的求导法则

一元复合函数 $y = f(u),\ u = \varphi(x)$

求导法则 $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}$$

微分法则 $$\mathrm{d}y = f'(u)\,\mathrm{d}u = f'(u)\varphi'(x)\,\mathrm{d}x$$

本节内容:

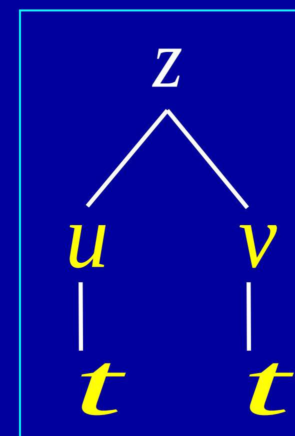
一、多元复合函数求导的链式法则

二、多元复合函数的全微分

一、多元复合函数求导的链式法则

定理. 若函数 $u=\varphi(t), v=\psi(t)$ 在点 t 可导, $z=f(u,v)$ 在点 (u,v) 处偏导连续, 则复合函数 $z=f(\varphi(t),\psi(t))$ 在点 t 可导, 且有链式法则

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}$$



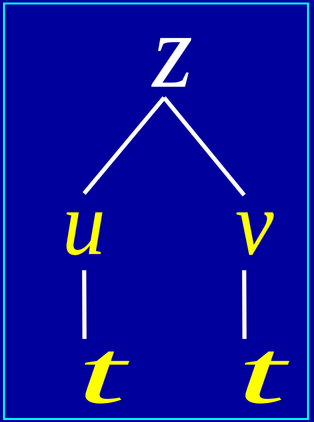
***证:** 设 t 取增量 Δt ,则相应中间变量有增量 Δu , Δv ,

$$\Delta z=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v+o(\rho) \quad \left(\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}\right)$$

$$\frac{\Delta z}{\Delta t}=\frac{\partial z}{\partial u}\frac{\Delta u}{\Delta t}+\frac{\partial z}{\partial v}\frac{\Delta v}{\Delta t}+\frac{o(\rho)}{\Delta t}\quad\left(\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}\right)$$

令 $\Delta t\to 0$, 则有 $\Delta u\to 0$, $\Delta v\to 0$,

$$\frac{\Delta u}{\Delta t}\to\frac{\mathrm{d}u}{\mathrm{d}t},\qquad \frac{\Delta v}{\Delta t}\to\frac{\mathrm{d}v}{\mathrm{d}t}$$



$$\frac{o(\rho)}{\Delta t}=\frac{o(\rho)}{\rho}\sqrt{\left(\frac{\Delta u}{\Delta t}\right)^2+\left(\frac{\Delta v}{\Delta t}\right)^2}\to 0$$

$\boxed{(\triangle t<0\text{ 时, 根式前加 “–”号})}$

$$\boxed{\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}}$$

(全导数公式)

说明: 若定理中 $f(u,v)$ 在点 (u,v) **偏导数连续**减弱为**偏导数存在**, 则定理结论不一定成立.

例如: $z = f(u,v) = \begin{cases} \dfrac{u^2 v}{u^2+v^2}, & u^2+v^2 \neq 0 \\ 0, & u^2+v^2 = 0 \end{cases}$

$u = t, \quad v = t$

易知: $\left.\dfrac{\partial z}{\partial u}\right|_{(0,0)} = f_u(0,0) = 0$, $\left.\dfrac{\partial z}{\partial v}\right|_{(0,0)} = f_v(0,0) = 0$

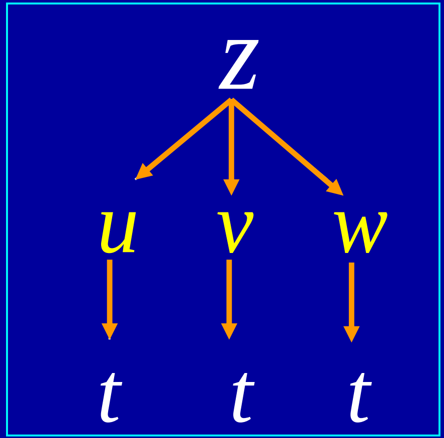
但复合函数 $z = f(t,t) = \dfrac{t}{2}$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{2} \neq \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} = 0\cdot 1 + 0\cdot 1 = 0$$

推广： 设下面所涉及的函数都可微 .

1) 中间变量多于两个的情形. 例如, $z=f(u,v,w)$,

$u=\varphi(t),\ v=\psi(t),\ w=\omega(t)$

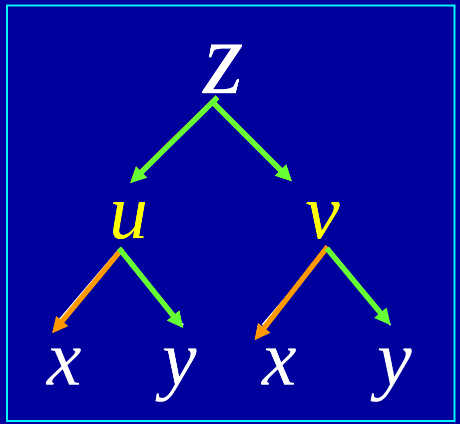


$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{\partial z}{\partial w}\cdot\frac{\mathrm{d}w}{\mathrm{d}t}$$

$$=f_1'\varphi'+f_2'\psi'+f_3'\omega'$$

2) 中间变量是多元函数的情形. 例如,

$z=f(u,v),\quad u=\varphi(x,y),\quad v=\psi(x,y)$



$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=f_1'\varphi_1'+f_2'\psi_1'$$

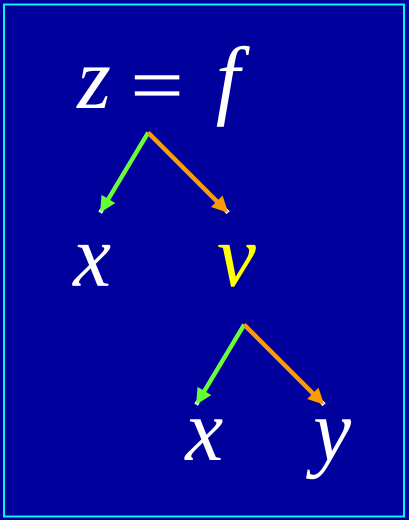
$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=f_1'\varphi_2'+f_2'\psi_2'$$

又如, $z = f(x,v),\ v = \psi(x,y)$

当它们都具有可微条件时, 有

$$\boxed{\frac{\partial z}{\partial x}} = \boxed{\frac{\partial f}{\partial x}} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = f_1' + f_2'\psi_1'$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = f_2'\psi_2'$$



口诀：

分段用乘, 分叉用加,
单路全导, 叉路偏导

注意: 这里 $\frac{\partial z}{\partial x}$ 与 $\frac{\partial f}{\partial x}$ 不同,

$\frac{\partial z}{\partial x}$ 表示 $f(x,\psi(x,y))$ 固定 y， 对 x 求导

$\frac{\partial f}{\partial x}$ 表示 $f(x,v)$ 固定 v， 对 x 求导

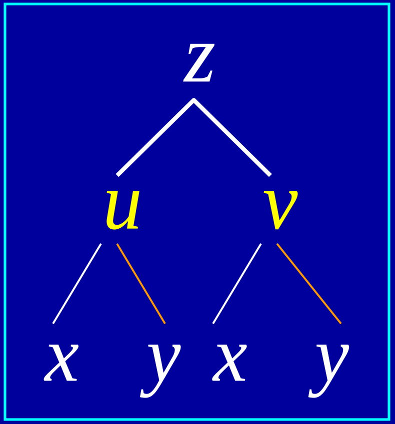
例1. 设 $z = \mathrm{e}^{u}\sin v,\ u = xy\ ,\ v = x + y$, 求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

解:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \mathrm{e}^{u}\sin v\cdot y + \mathrm{e}^{u}\cos v\cdot 1$$

$$= \mathrm{e}^{xy}[y\cdot\sin(x+y) + \cos(x+y)]$$



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \mathrm{e}^{u}\sin v\cdot x + \mathrm{e}^{u}\cos v\cdot 1$$

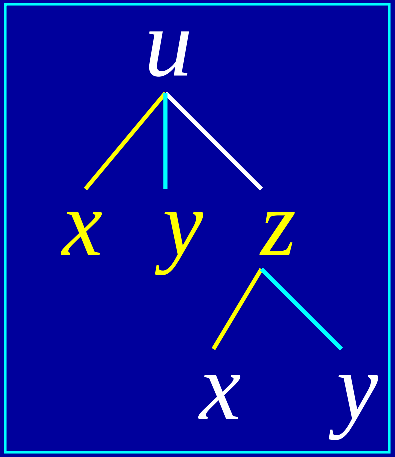
$$= \mathrm{e}^{xy}[x\cdot\sin(x+y) + \cos(x+y)]$$

例2. $u = f(x,y,z) = \mathrm{e}^{x^2+y^2+z^2}$, $z = x^2 \sin y$, 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$

解: $$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$= 2x\mathrm{e}^{x^2+y^2+z^2} + 2z\mathrm{e}^{x^2+y^2+z^2}\cdot 2x\sin y$$

$$= 2x(1+2x^2\sin^2 y)\mathrm{e}^{x^2+y^2+x^4\sin^2 y}$$



$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}$$

$$= 2y\mathrm{e}^{x^2+y^2+z^2} + 2z\mathrm{e}^{x^2+y^2+z^2}\cdot x^2\cos y$$

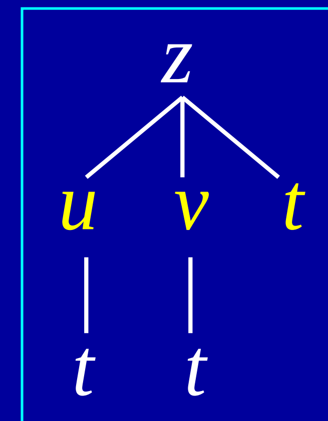
$$= 2(y + x^4\sin y\cos y)\mathrm{e}^{x^2+y^2+x^4\sin^2 y}$$

例3. 设 $z = uv + \sin t\,,\ u = \mathrm{e}^{t}\,,\ v = \cos t\,,$ 求全导数 $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

解: $$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial t}$$

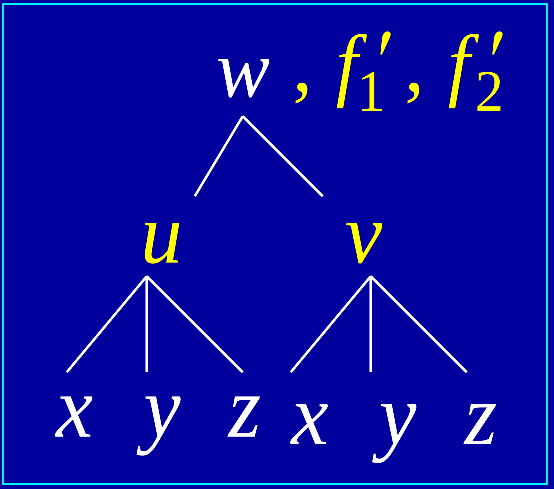
$$= v\,\mathrm{e}^{t} - u\sin t + \cos t$$

$$= \mathrm{e}^{t}(\cos t - \sin t) + \cos t$$



注意：多元抽象复合函数求导在偏微分方程变形与验证解的问题中经常遇到, 下列几个例题有助于掌握这方面问题的求导技巧与常用导数符号.

例4. 设 $w = f(x+y+z, xyz)$, f 具有二阶连续偏导数, 求 $\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x \partial z}$.



解: 令 $u = x+y+z$, $v = xyz$, 则

$$w = f(u, v)$$

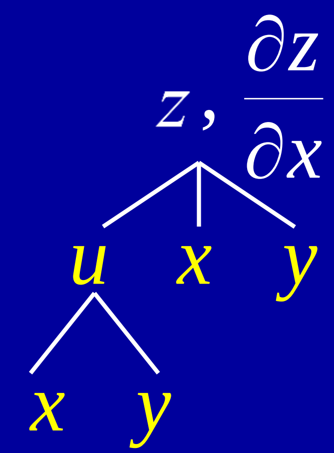
$$\frac{\partial w}{\partial x} = f_1' \cdot 1 + f_2' \cdot yz$$

$$= f_1'(x+y+z, xyz) + \underline{yz\, f_2'(x+y+z, xyz)}$$

$$\frac{\partial^2 w}{\partial x \partial z} = f_{11}'' \cdot 1 + f_{12}'' \cdot xy + y f_2' + yz[\, f_{21}'' \cdot 1 + f_{22}'' \cdot xy]$$

$$= f_{11}'' + y(x+z) f_{12}'' + xy^2 z\, f_{22}'' + y\, f_2'$$

思考题： $z = f(u, x, y),\ u = x\,\mathrm{e}^{y}$



$$\frac{\partial z}{\partial x} = \underline{f_1' \cdot \mathrm{e}^{y}} + \underline{f_2'}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \mathrm{e}^{y} \cdot f_1' + f_{11}'' \cdot x\,\mathrm{e}^{2y} + \mathrm{e}^{y} \cdot f_{13}''$$
$$+ f_{21}'' \cdot x\,\mathrm{e}^{y} + f_{23}''$$

例5. 设 $u=f(x,y)$ 二阶偏导数连续, 求下列表达式在极坐标系下的形式 (1) $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$, (2) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$

解: 已知 $x=r\cos\theta$, $y=r\sin\theta$, 则

$$r=\sqrt{x^2+y^2},\quad \theta=\arctan\frac{y}{x}$$

u

$r \quad \theta$

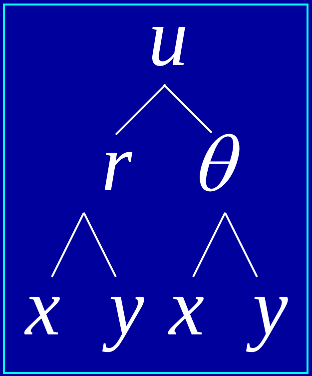
$x \quad y \quad x \quad y$

(1) $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\quad \frac{\partial \theta}{\partial x}=\frac{\frac{-y}{x^2}}{1+\left(\frac{y}{x}\right)^2}=\frac{-y}{x^2+y^2}$$

$$=\frac{\partial u}{\partial r}\frac{x}{r}-\frac{\partial u}{\partial \theta}\frac{y}{r^2}=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$



$$\frac{\partial r}{\partial y}=\frac{y}{r},\qquad \frac{\partial \theta}{\partial y}=\frac{\frac{1}{x}}{1+(\frac{y}{x})^2}=\frac{x}{x^2+y^2}$$

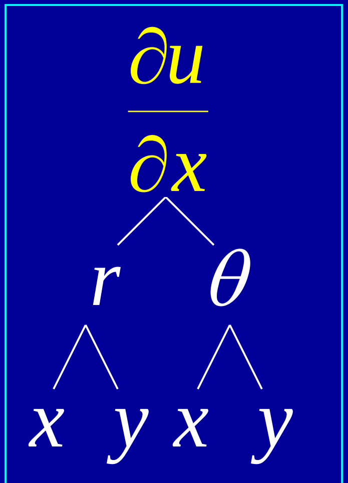
$$=\frac{\partial u}{\partial r}\frac{y}{r}+\frac{\partial u}{\partial \theta}\frac{x}{r^2}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$

$$=\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$$

$$\therefore\quad (\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2=(\frac{\partial u}{\partial r})^2+\frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2$$

已知 $\frac{\partial u}{\partial x} = \boxed{\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}}$



注意利用已有公式

$$(2)\quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\cdot\cos\theta - \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)\frac{\sin\theta}{r}$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\cdot\cos\theta$$

$$- \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\cdot\frac{\sin\theta}{r}$$

$$= \frac{\partial^2 u}{\partial r^2}\cos^2\theta - 2\frac{\partial^2 u}{\partial r\partial \theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial^2 u}{\partial \theta^2}\frac{\sin^2\theta}{r^2}$$

$$+ \frac{\partial u}{\partial \theta}\frac{2\sin\theta\cos\theta}{r^2} + \frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2}\cos^2\theta - 2\cancel{\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}} + \frac{\partial^2 u}{\partial\theta^2}\frac{\sin^2\theta}{r^2}$$

$$+\cancel{\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2}} + \frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}$$

同理可得

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\sin^2\theta + 2\cancel{\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}} + \frac{\partial^2 u}{\partial\theta^2}\frac{\cos^2\theta}{r^2}$$

$$-\cancel{\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2}} + \frac{\partial u}{\partial r}\frac{\cos^2\theta}{r}$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$= \frac{1}{r^2}\left[r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial\theta^2}\right]$$

二、多元复合函数的全微分

设函数 $z=f(u,v)$, $u=\varphi(x,y)$, $v=\psi(x,y)$ 都可微,

则复合函数 $z=f(\varphi(x,y),\psi(x,y))$ 的全微分为

$$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y$$

$$=\Big(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\Big)\mathrm{d}x+\Big(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\Big)\mathrm{d}y$$

$$=\frac{\partial z}{\partial u}\Big(\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y\Big)+\frac{\partial z}{\partial v}\Big(\frac{\partial v}{\partial x}\mathrm{d}x+\frac{\partial v}{\partial y}\mathrm{d}y\Big)$$

$$=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v$$

可见无论 u , v 是自变量还是中间变量, 其全微分表达形式都一样, 这性质叫做**全微分形式不变性.**

例 6. 利用全微分形式不变性再解例1.

解: $\mathrm{d}z = \mathrm{d}(\,\mathrm{e}^u \sin v\,)$

$$= \mathrm{e}^u \sin v\,\mathrm{d}u + \mathrm{e}^u \cos v\,\mathrm{d}v$$

$$= \mathrm{e}^{xy}[\sin(x+y)\,\mathrm{d}(xy) + \cos(x+y)\,\mathrm{d}(x+y)\,]$$

$$= \mathrm{e}^{xy}[\sin(x+y)(y\mathrm{d}x + x\mathrm{d}y) + \cos(x+y)\,(\mathrm{d}x+\mathrm{d}y)\,]$$

$$= \mathrm{e}^{xy}[\,y\sin(x+y)+\cos(x+y)]\mathrm{d}x$$

$$+ \mathrm{e}^{xy}[x\sin(x+y)+\cos(x+y)\,]\mathrm{d}y$$

所以 $$\frac{\partial z}{\partial x} = \mathrm{e}^{xy}[y\cdot\sin(x+y)+\cos(x+y)]$$

$$\frac{\partial z}{\partial y} = \mathrm{e}^{xy}[x\cdot\sin(x+y)+\cos(x+y)]$$

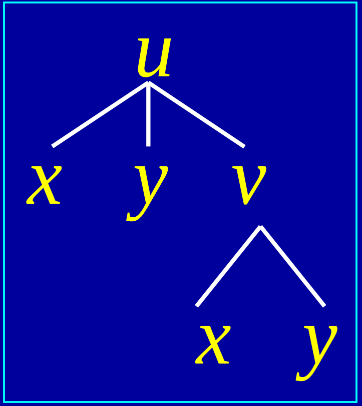
内容小结

1. 复合函数求导的链式法则

“分段用乘，分叉用加，单路全导，叉路偏导”

例如， $u = f(x, y, v),\ v = \varphi(x, y),$

$$\frac{\partial u}{\partial x} = f_1' + f_3' \cdot \varphi_1'\ ; \qquad \frac{\partial u}{\partial y} = f_2' + f_3' \cdot \varphi_2'$$



2. 全微分形式不变性

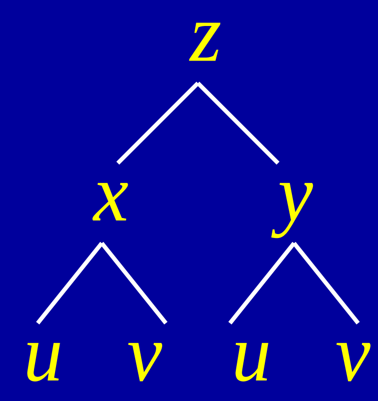
对 $z = f(u, v)$，不论 u, v 是自变量还是中间变量，

$$\mathrm{d}z = f_u(u, v)\,\mathrm{d}u + f_v(u, v)\,\mathrm{d}v$$

思考与练习

解答提示:

P24 题7 $z=\arctan\frac{x}{y},\ x=u+v,\ y=u-v$



$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\,\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\,\frac{\partial y}{\partial v}$$

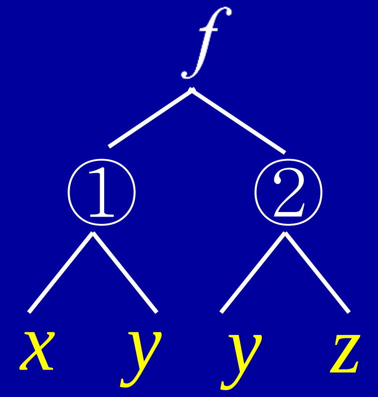
$$=\frac{1}{1+(\frac{x}{y})^2}\cdot\frac{1}{y}\cdot 1+\frac{1}{1+(\frac{x}{y})^2}\cdot(-\frac{x}{y^2})\cdot(-1)$$

$$=\frac{y+x}{x^2+y^2}=\frac{u}{u^2+v^2}$$

……

题

$$u = f\left(\frac{x}{y}, \frac{y}{z}\right)$$



$$\frac{\partial u}{\partial x} = f_1' \cdot \frac{1}{y} = \frac{1}{y} f_1'$$

$$\frac{\partial u}{\partial y} = f_1' \cdot \left(-\frac{x}{y^2}\right) + f_2' \cdot \frac{1}{z} = -\frac{x}{y^2} f_1' + \frac{1}{z} f_2'$$

$$\frac{\partial u}{\partial z} = f_2' \cdot \left(-\frac{y}{z^2}\right) = -\frac{y}{z^2} f_2'$$

备用题

1. 已知 $f(x,y)\big|_{y=x^2}=1$，$f_1'(x,y)\big|_{y=x^2}=2x$，求 $f_2'(x,y)\big|_{y=x^2}$.

解: 由 $f(x,x^2)=1$ 两边对 x 求导, 得

$$f_1'(x,x^2)+f_2'(x,x^2)\cdot 2x=0$$

$$f_1'(x,x^2)=2x$$

$$f_2'(x,x^2)=-1$$

2. 设函数 $z = f(x,y)$ 在点 $(1,1)$ 处可微，且

$$f(1,1)=1,\quad \left.\frac{\partial f}{\partial x}\right|_{(1,1)}=2,\quad \left.\frac{\partial f}{\partial y}\right|_{(1,1)}=3,$$

$\varphi(x) = f(x, f(x,x))$，求 $\left.\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1}$.

解: 由题设 $\varphi(1) = f(1, f(1,1)) = f(1,1) = 1$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1} = 3\varphi^2(x)\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=1}$$

$$= 3\Big[\, f_1'(x, f(x,x))$$

$$\left. + f_2'(x, f(x,x))\big(f_1'(x,x) + f_2'(x,x)\big)\Big]\right|_{x=1}$$

$$= 3\cdot[2 + 3\cdot(2+3)] = 51$$