

# 第三节 分部积分方法

由导数公式  $(uv)' = u'v + uv'$

积分得:  $uv = \int u'v dx + \int uv' dx$

$\longrightarrow \left. \begin{array}{l} \int uv' dx = uv - \int u'v dx \\ \text{或 } \int u dv = uv - \int v du \end{array} \right\}$  分部积分公式

选取  $u$  及  $v'$  (或  $dv$ ) 的原则:

- 1)  $v'$  容易求得;
- 2)  $\int u'v dx$  比  $\int uv' dx$  容易计算.



例1. 求  $\int x \cos x \, dx$ .

解: 令  $u = x$ ,  $v' = \cos x$ ,

则  $u' = 1$ ,  $v = \sin x$

$$\begin{aligned}\therefore \text{原式} &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + C\end{aligned}$$

思考: 如何求  $\int x^2 \sin x \, dx$ ?

提示: 令  $u = x^2$ ,  $v' = \sin x$ , 则

$$\begin{aligned}\text{原式} &= -x^2 \cos x + 2 \int x \cos x \, dx \\ &= \dots\end{aligned}$$



**例2.** 求  $\int x \ln x dx$ .

解: 令  $u = \ln x, v' = x$

则  $u' = \frac{1}{x}, v = \frac{1}{2}x^2$

$$\therefore \text{原式} = \frac{1}{2}x^2 \ln x - \frac{1}{2} \int x dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$



例3. 求  $\int x \arctan x \, dx$ .

解: 令  $u = \arctan x, v' = x$

则  $u' = \frac{1}{1+x^2}, \quad v = \frac{1}{2}x^2$

$$\begin{aligned}\therefore \text{原式} &= \frac{1}{2}x^2 \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} \, dx \\&= \frac{1}{2}x^2 \arctan x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) \, dx \\&= \frac{1}{2}x^2 \arctan x - \frac{1}{2}(x - \arctan x) + C\end{aligned}$$



**例4.** 求  $\int e^x \sin x \, dx$ .

解: 令  $u = \sin x, v' = e^x$ , 则

$$u' = \cos x, v = e^x$$

$$\therefore \text{原式} = e^x \sin x - \int e^x \cos x \, dx$$

$$\begin{array}{c} | \\ \text{再令 } u = \cos x, v' = e^x, \text{ 则} \\ \downarrow \\ u' = -\sin x, v = e^x \end{array}$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

故 原式 =  $\frac{1}{2}e^x(\sin x - \cos x) + C$

**说明:** 也可设  $u = e^x, v'$  为三角函数, 但两次所设类型必须一致.



解题技巧：选取  $u$  及  $v'$  的一般方法：

把被积函数视为两个函数之积，按  
顺序，前者为  $u$  后者为  $v'$ .

例5. 求  $\int \arccos x \, dx$ .

解：令  $u = \arccos x$ ,  $v' = 1$ , 则

$$u' = -\frac{1}{\sqrt{1-x^2}}, \quad v = x$$

$$\begin{aligned} \text{原式} &= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \, dx \\ &= x \arccos x - \frac{1}{2} \int (1-x^2)^{-1/2} d(1-x^2) \end{aligned}$$

$$= x \arccos x - \sqrt{1-x^2} + C$$

“反对幂指三”

反：反三角函数  
对：对数函数  
幂：幂函数  
指：指数函数  
三：三角函数



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**例6.** 求  $\int \frac{\ln \cos x}{\cos^2 x} dx$ .

解: 令  $u = \ln \cos x, v' = \frac{1}{\cos^2 x}$ , 则

$$u' = -\tan x, \quad v = \tan x$$

$$\begin{aligned}\text{原式} &= \tan x \cdot \ln \cos x + \int \tan^2 x dx \\&= \tan x \cdot \ln \cos x + \int (\sec^2 x - 1) dx \\&= \tan x \cdot \ln \cos x + \tan x - x + C\end{aligned}$$



例7. 求  $\int e^{\sqrt{x}} dx$ .

解: 令  $\sqrt{x} = t$ , 则  $x = t^2$ ,  $dx = 2t dt$

$$\text{原式} = 2 \int t e^t dt$$

↓ 令  $u = t$ ,  $v' = e^t$

$$= 2(t e^t - \int e^t dt)$$

$$= 2(t e^t - e^t) + C$$

$$= 2e^{\sqrt{x}}(\sqrt{x} - 1) + C$$



**例8.** 求  $\int \sqrt{x^2 + a^2} dx$  ( $a > 0$ ).

**解:** 令  $u = \sqrt{x^2 + a^2}$ ,  $v' = 1$ , 则

$$u' = \frac{x}{\sqrt{x^2 + a^2}}, \quad v = x$$

$$\int \sqrt{x^2 + a^2} dx = x \sqrt{x^2 + a^2} - \int \frac{x^2}{\sqrt{x^2 + a^2}} dx$$

$$= x \sqrt{x^2 + a^2} - \int \frac{(x^2 + a^2) - a^2}{\sqrt{x^2 + a^2}} dx$$

$$= x \sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2} dx + a^2 \int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$\therefore \text{原式} = \frac{1}{2} x \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln(x + \sqrt{x^2 + a^2}) + C$$



例9. 求  $I_n = \int \frac{dx}{(x^2 + a^2)^n}$ .

解: 令  $u = \frac{1}{(x^2 + a^2)^n}, v' = 1$ , 则  $u' = \frac{-2nx}{(x^2 + a^2)^{n+1}}, v = x$

$$\begin{aligned}\therefore I_n &= \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{x^2}{(x^2 + a^2)^{n+1}} dx \\ &= \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{(x^2 + a^2) - a^2}{(x^2 + a^2)^{n+1}} dx \\ &= \frac{x}{(x^2 + a^2)^n} + 2nI_n - 2na^2I_{n+1}\end{aligned}$$

得递推公式  $I_{n+1} = \frac{1}{2na^2} \frac{x}{(x^2 + a^2)^n} + \frac{2n-1}{2na^2} I_n$



$$I_n = \int \frac{dx}{(x^2 + a^2)^n}$$

递推公式  $I_{n+1} = \frac{1}{2na^2} \frac{x}{(x^2 + a^2)^n} + \frac{2n-1}{2na^2} I_n$

说明: 已知  $I_1 = \frac{1}{a} \arctan \frac{x}{a} + C$  利用递推公式可求得  $I_n$ .

例如,

$$I_3 = \frac{1}{4a^2} \frac{x}{(x^2 + a^2)^2} + \frac{3}{4a^2} I_2$$

$$= \frac{1}{4a^2} \frac{x}{(x^2 + a^2)^2} + \frac{3}{4a^2} \left( \frac{1}{2a^2} \frac{x}{x^2 + a^2} + \frac{1}{2a^2} I_1 \right)$$

$$= \frac{1}{4a^2} \frac{x}{(x^2 + a^2)^2} + \frac{3}{8a^4} \frac{x}{x^2 + a^2} + \frac{3}{8a^5} \arctan \frac{x}{a} + C$$



**例10.** 设  $I_n = \int \sec^n x dx$ , 证明递推公式:

$$I_n = \frac{1}{n-1} \sec^{n-2} x \cdot \tan x + \frac{n-2}{n-1} I_{n-2} \quad (n \geq 2)$$

证:  $I_n = \int \sec^{n-2} x \cdot \sec^2 x dx$   
 $= \sec^{n-2} x \cdot \tan x$   
 $- (n-2) \int \sec^{n-3} x \cdot \sec x \tan x \cdot \tan x dx$   
 $= \sec^{n-2} x \cdot \tan x - (n-2) \int \sec^{n-2} x \cdot (\sec^2 x - 1) dx$   
 $= \sec^{n-2} x \cdot \tan x \boxed{-(n-2) I_n} + (n-2) I_{n-2}$

$$\therefore I_n = \frac{1}{n-1} \sec^{n-2} x \cdot \tan x + \frac{n-2}{n-1} I_{n-2} \quad (n \geq 2)$$



# 说明:

分部积分题目的类型:

- 1) 直接分部化简积分；
- 2) 分部产生循环式，由此解出积分式；

(注意: 两次分部选择的  $u, v$  函数类型不变，  
解出积分后加  $C$ )

例4

- 3) 对含自然数  $n$  的积分, 通过分部积分建立递推公式.



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**例11.** 已知  $f(x)$  的一个原函数是  $\frac{\cos x}{x}$ , 求  $\int xf'(x)dx$ .

解:  $\int xf'(x)dx = \int x df(x)$

$$= xf(x) - \int f(x)dx$$

$$\begin{aligned} & \boxed{-\frac{x \sin x - \cos x}{x^2}} \\ &= x \left( \frac{\cos x}{x} \right)' - \frac{\cos x}{x} + C \\ &= -\sin x - 2 \frac{\cos x}{x} + C \end{aligned}$$

**说明:** 此题若先求出  $f'(x)$  再求积分反而复杂.

$$\int xf'(x)dx = \int \left( -\cos x + \frac{2 \sin x}{x} + \frac{2 \cos x}{x^2} \right) dx$$

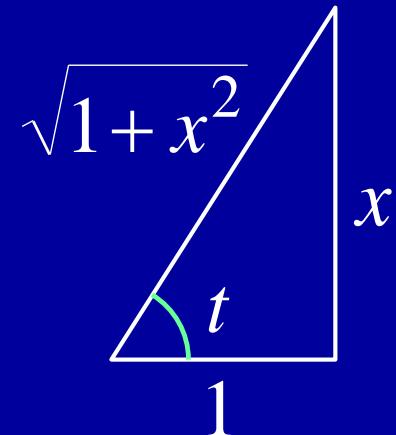


**例12.** 求  $I = \int \frac{e^{\arctan x}}{(1+x^2)^{3/2}} dx$ .

**解法1** 先换元后分部

令  $t = \arctan x$ , 即  $x = \tan t$ , 则

$$\begin{aligned} I &= \int \frac{e^t}{\sec^3 t} \cdot \sec^2 t dt = \int e^t \cos t dt \\ &= e^t \sin t - \int e^t \sin t dt \\ &= e^t \sin t + e^t \cos t - \int e^t \cos t dt \end{aligned}$$



故  $I = \frac{1}{2}(\sin t + \cos t)e^t + C$

$$= \frac{1}{2} \left[ \frac{x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}} \right] e^{\arctan x} + C$$



## 解法2 直接用分部积分法

$$\begin{aligned} I &= \int \frac{1}{\sqrt{1+x^2}} d e^{\arctan x} \\ &= \frac{1}{\sqrt{1+x^2}} e^{\arctan x} + \int \frac{x e^{\arctan x}}{(1+x^2)^{3/2}} dx \\ &= \frac{1}{\sqrt{1+x^2}} e^{\arctan x} + \int \frac{x}{\sqrt{1+x^2}} d e^{\arctan x} \\ &= \frac{1}{\sqrt{1+x^2}} e^{\arctan x} (1+x) - I \\ \therefore I &= \frac{1+x}{2\sqrt{1+x^2}} e^{\arctan x} + C \end{aligned}$$

$$I = \int \frac{e^{\arctan x}}{(1+x^2)^{3/2}} dx$$



## 内容小结

分部积分公式  $\int u v' dx = u v - \int u' v dx$

1. 使用原则：  $v'$  易求出,  $\int u' v dx$  易积分

2. 使用经验：“反对幂指三”，前  $u$  后  $v'$

3. 题目类型：

分部化简； 循环解出； 递推公式

4. 计算格式：

$$\begin{array}{c} u & u' \\ \diagdown & + | - \int \\ v' & v \end{array}$$



**例13.** 求  $I = \int \sin(\ln x) dx$

解: 令  $t = \ln x$ , 则  $x = e^t$ ,  $dx = e^t dt$

$$\therefore I = \int e^t \sin t dt \longrightarrow = e^t \sin t - \int e^t \cos t dt$$



$$= e^t (\sin t - \cos t) - I$$

$$\therefore I = \frac{1}{2} e^t (\sin t - \cos t) + C$$

$$= \frac{1}{2} x [\sin(\ln x) - \cos(\ln x)] + C$$

多次分部积分



**例14.** 求  $\int x^3 (\ln x)^4 dx$ .

解: 令  $u = \ln x$ , 则  $x = e^u$ ,  $dx = e^u du$

$$\text{原式} = \int e^{3u} u^4 \cdot e^u du = \int u^4 e^{4u} du$$

$$\begin{aligned}\text{原式} &= \frac{1}{4} e^{4u} \left( u^4 - u^3 + \frac{3}{4} u^2 - \frac{3}{8} u + \frac{3}{32} \right) + C \\ &= \frac{1}{4} x^4 \left( \ln^4 x - \ln^3 x + \frac{3}{4} \ln^2 x - \frac{3}{8} \ln x + \frac{3}{32} \right) + C\end{aligned}$$



## 思考与练习

1. 下述运算错在哪里? 应如何改正?

$$\int \frac{\cos x}{\sin x} dx = \int \frac{d \sin x}{\sin x} = \frac{\sin x}{\sin x} - \int \left(\frac{1}{\sin x}\right)' \sin x dx$$

$$= 1 - \int \frac{-\cos x}{\sin^2 x} \sin x dx = 1 + \int \frac{\cos x}{\sin x} dx$$

$$\therefore \int \frac{\cos x}{\sin x} dx - \int \frac{\cos x}{\sin x} dx = 1, \quad \text{得 } 0 = 1$$

$$= \ln |\sin x| + C$$

答: 不定积分是原函数族, 相减不应为 0.

求此积分的正确作法是用换元法.



2. 设  $F(x)$  是  $f(x)$  的一个原函数,  $f(x)$  可微且其反函数  $f^{-1}(x)$  存在, 证明

$$\int f^{-1}(x) \, dx = x f^{-1}(x) - F[f^{-1}(x)] + C$$

证:  $\int f^{-1}(x) \, dx = x f^{-1}(x) - \int x \, df^{-1}(x)$

$$= x f^{-1}(x) - \int f[f^{-1}(x)] \, df^{-1}(x)$$
$$= x f^{-1}(x) - F[f^{-1}(x)] + C$$

注意:

$$x = f[f^{-1}(x)]$$



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备用题. 求不定积分  $\int \frac{x e^x}{\sqrt{e^x - 1}} dx$ .

解: 方法1 (先分部, 再换元)

$$\begin{aligned} \int \frac{x e^x}{\sqrt{e^x - 1}} dx &= \int \frac{x}{\sqrt{e^x - 1}} d(e^x - 1) \\ &= 2 \int x d\sqrt{e^x - 1} = 2x\sqrt{e^x - 1} - 2 \int \sqrt{e^x - 1} dx \\ &\quad \left| \text{令 } u = \sqrt{e^x - 1}, \text{ 则 } dx = \frac{2u}{1+u^2} du \right. \\ &= 2x\sqrt{e^x - 1} - 4 \int \frac{u^2 + 1 - 1}{1+u^2} du \quad \boxed{-4(u - \arctan u) + C} \\ &= 2x\sqrt{e^x - 1} - 4\sqrt{e^x - 1} + 4\arctan\sqrt{e^x - 1} + C \end{aligned}$$



## 方法2 (先换元,再分部)

$$\int \frac{x e^x}{\sqrt{e^x - 1}} dx$$

令  $u = \sqrt{e^x - 1}$ , 则  $x = \ln(1 + u^2)$ ,  $dx = \frac{2u}{1 + u^2} du$

故  $\int \frac{x e^x}{\sqrt{e^x - 1}} dx = \int \frac{(1 + u^2) \ln(1 + u^2)}{u} \cdot \frac{2u}{1 + u^2} du$

$$= 2 \int \ln(1 + u^2) du$$

$$= 2u \ln(1 + u^2) - 4 \int \frac{1 + u^2 - 1}{1 + u^2} du$$

$$= 2u \ln(1 + u^2) - 4u + 4 \arctan u + C$$

$$= 2x \sqrt{e^x - 1} - 4 \sqrt{e^x - 1} + 4 \arctan \sqrt{e^x - 1} + C$$

