

第三节

第四章

分部积分法

由导数公式 $(uv)' = u'v + uv'$

积分得: $uv = \int u'v dx + \int uv' dx$

$$\begin{aligned} \longrightarrow \int uv' dx &= uv - \int u'v dx \\ \text{或 } \int u dv &= uv - \int v du \end{aligned} \left. \vphantom{\int uv' dx} \right\} \text{分部积分公式}$$



选取 u 及 v' (或 dv) 的原则:

- 1) v 容易求得;
- 2) $\int u'v dx$ 比 $\int uv' dx$ 容易计算.



例1. 求 $\int x \cos x \, dx$.

解: 令 $u = x$, $v' = \cos x$,

则 $u' = 1$, $v = \sin x$

$$\begin{aligned}\therefore \text{原式} &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + C\end{aligned}$$

思考: 如何求 $\int x^2 \sin x \, dx$?

提示: 令 $u = x^2$, $v' = \sin x$, 则

$$\begin{aligned}\text{原式} &= -x^2 \cos x + 2 \int x \cos x \, dx \\ &= \dots\end{aligned}$$



例2. 求 $\int x \ln x \, dx$.

解: 令 $u = \ln x$, $v' = x$

则 $u' = \frac{1}{x}$, $v = \frac{1}{2}x^2$

$$\begin{aligned}\therefore \text{原式} &= \frac{1}{2}x^2 \ln x - \frac{1}{2} \int x \, dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C\end{aligned}$$



例3. 求 $\int x \arctan x \, dx$.

解: 令 $u = \arctan x$, $v' = x$

则 $u' = \frac{1}{1+x^2}$, $v = \frac{1}{2}x^2$

$$\begin{aligned}\therefore \text{原式} &= \frac{1}{2}x^2 \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} \, dx \\ &= \frac{1}{2}x^2 \arctan x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) \, dx \\ &= \frac{1}{2}x^2 \arctan x - \frac{1}{2}(x - \arctan x) + C\end{aligned}$$



例4. 求 $\int e^x \sin x \, dx$.

解: 令 $u = \sin x$, $v' = e^x$, 则

$$u' = \cos x, \quad v = e^x$$

$$\therefore \text{原式} = e^x \sin x - \int e^x \cos x \, dx$$

再令 $u = \cos x$, $v' = e^x$, 则

$$u' = -\sin x, \quad v = e^x$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

故 原式 = $\frac{1}{2} e^x (\sin x - \cos x) + C$

说明: 也可设 $u = e^x$, v' 为三角函数, 但两次所设类型必须一致.



解题技巧: 选取 u 及 v' 的一般方法:

把被积函数视为两个函数之积, 按顺序, 前者为 u 后者为 v' .

“反对幂指三”

反: 反三角函数

对: 对数函数

幂: 幂函数

指: 指数函数

三: 三角函数

例5. 求 $\int \arccos x \, dx$.

解: 令 $u = \arccos x$, $v' = 1$, 则

$$u' = -\frac{1}{\sqrt{1-x^2}}, \quad v = x$$

$$\begin{aligned} \text{原式} &= x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \, dx \\ &= x \arccos x - \frac{1}{2} \int (1-x^2)^{-1/2} d(1-x^2) \\ &= x \arccos x - \sqrt{1-x^2} + C \end{aligned}$$



例6. 求 $\int \frac{\ln \cos x}{\cos^2 x} dx$.

解: 令 $u = \ln \cos x$, $v' = \frac{1}{\cos^2 x}$, 则

$$u' = -\tan x, \quad v = \tan x$$

$$\begin{aligned} \text{原式} &= \tan x \cdot \ln \cos x + \int \tan^2 x dx \\ &= \tan x \cdot \ln \cos x + \int (\sec^2 x - 1) dx \\ &= \tan x \cdot \ln \cos x + \tan x - x + C \end{aligned}$$



例7. 求 $\int e^{\sqrt{x}} dx$.

解: 令 $\sqrt{x} = t$, 则 $x = t^2$, $dx = 2t dt$

$$\text{原式} = 2 \int t e^t dt$$

$$\downarrow \text{令 } u = t, v' = e^t$$

$$= 2 \left(t e^t - \int e^t dt \right)$$

$$= 2(t e^t - e^t) + C$$

$$= 2e^{\sqrt{x}}(\sqrt{x} - 1) + C$$



例8. 求 $\int \sqrt{x^2 + a^2} \, dx \quad (a > 0).$

解: 令 $u = \sqrt{x^2 + a^2}$, $v' = 1$, 则

$$u' = \frac{x}{\sqrt{x^2 + a^2}}, \quad v = x$$

$$\int \sqrt{x^2 + a^2} \, dx = x\sqrt{x^2 + a^2} - \int \frac{x^2}{\sqrt{x^2 + a^2}} \, dx$$

$$= x\sqrt{x^2 + a^2} - \int \frac{(x^2 + a^2) - a^2}{\sqrt{x^2 + a^2}} \, dx$$

$$= x\sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2} \, dx + a^2 \int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$\therefore \text{原式} = \frac{1}{2}x\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln(x + \sqrt{x^2 + a^2}) + C$$



例9. 求 $I_n = \int \frac{dx}{(x^2 + a^2)^n}$.

解: 令 $u = \frac{1}{(x^2 + a^2)^n}$, $v' = 1$, 则 $u' = \frac{-2nx}{(x^2 + a^2)^{n+1}}$, $v = x$

$$\begin{aligned}\therefore I_n &= \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{x^2}{(x^2 + a^2)^{n+1}} dx \\ &= \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{(x^2 + a^2) - a^2}{(x^2 + a^2)^{n+1}} dx \\ &= \frac{x}{(x^2 + a^2)^n} + 2n I_n - 2na^2 I_{n+1}\end{aligned}$$

得递推公式
$$I_{n+1} = \frac{1}{2na^2} \frac{x}{(x^2 + a^2)^n} + \frac{2n-1}{2na^2} I_n$$



$$I_n = \int \frac{dx}{(x^2 + a^2)^n}$$

递推公式
$$I_{n+1} = \frac{1}{2na^2} \frac{x}{(x^2 + a^2)^n} + \frac{2n-1}{2na^2} I_n$$

说明: 已知 $I_1 = \frac{1}{a} \arctan \frac{x}{a} + C$ 利用递推公式可求得 I_n .
例如,

$$\begin{aligned} I_3 &= \frac{1}{4a^2} \frac{x}{(x^2 + a^2)^2} + \frac{3}{4a^2} I_2 \\ &= \frac{1}{4a^2} \frac{x}{(x^2 + a^2)^2} + \frac{3}{4a^2} \left(\frac{1}{2a^2} \frac{x}{x^2 + a^2} + \frac{1}{2a^2} I_1 \right) \\ &= \frac{1}{4a^2} \frac{x}{(x^2 + a^2)^2} + \frac{3}{8a^4} \frac{x}{x^2 + a^2} + \frac{3}{8a^5} \arctan \frac{x}{a} + C \end{aligned}$$



例10. 设 $I_n = \int \sec^n x \, dx$, 证明递推公式:

$$I_n = \frac{1}{n-1} \sec^{n-2} x \cdot \tan x + \frac{n-2}{n-1} I_{n-2} \quad (n \geq 2)$$

证: $I_n = \int \sec^{n-2} x \cdot \sec^2 x \, dx$

$$= \sec^{n-2} x \cdot \tan x$$

$$- (n-2) \int \sec^{n-3} x \cdot \sec x \tan x \cdot \tan x \, dx$$

$$= \sec^{n-2} x \cdot \tan x - (n-2) \int \sec^{n-2} x \cdot (\sec^2 x - 1) \, dx$$

$$= \sec^{n-2} x \cdot \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$\therefore I_n = \frac{1}{n-1} \sec^{n-2} x \cdot \tan x + \frac{n-2}{n-1} I_{n-2} \quad (n \geq 2)$$



说明:

分部积分题目的类型:

- 1) 直接分部化简积分;
- 2) 分部产生循环式, 由此解出积分式;

(注意: 两次分部选择的 u , v 函数类型不变,
解出积分后加 C)

例4

- 3) 对含自然数 n 的积分, 通过分部积分建立递推公式.



例11. 已知 $f(x)$ 的一个原函数是 $\frac{\cos x}{x}$, 求 $\int x f'(x) dx$.

解:
$$\begin{aligned}\int x f'(x) dx &= \int x df(x) \\ &= x f(x) - \int f(x) dx \\ &= x \left(\frac{\cos x}{x} \right)' - \frac{\cos x}{x} + C \\ &= -\sin x - 2 \frac{\cos x}{x} + C\end{aligned}$$

$$\frac{-x \sin x - \cos x}{x^2}$$

说明: 此题若先求出 $f'(x)$ 再求积分反而复杂.

$$\int x f'(x) dx = \int \left(-\cos x + \frac{2 \sin x}{x} + \frac{2 \cos x}{x^2} \right) dx$$



例12. 求 $I = \int \frac{e^{\arctan x}}{(1+x^2)^{3/2}} dx$.

解法1 先换元后分部

令 $t = \arctan x$, 即 $x = \tan t$, 则

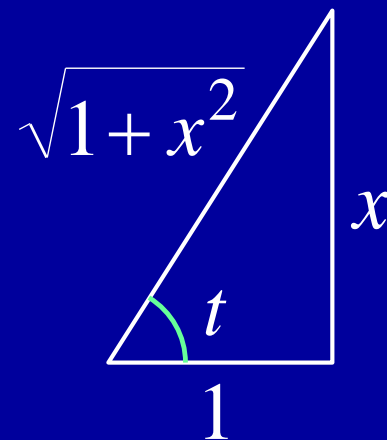
$$I = \int \frac{e^t}{\sec^3 t} \cdot \sec^2 t dt = \int e^t \cos t dt$$

$$= e^t \sin t - \int e^t \sin t dt$$

$$= e^t \sin t + e^t \cos t - \int e^t \cos t dt$$

$$\text{故 } I = \frac{1}{2}(\sin t + \cos t)e^t + C$$

$$= \frac{1}{2} \left[\frac{x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}} \right] e^{\arctan x} + C$$



解法2 直接用分部积分法

$$I = \int \frac{e^{\arctan x}}{(1+x^2)^{3/2}} dx$$

$$I = \int \frac{1}{\sqrt{1+x^2}} d e^{\arctan x}$$

$$= \frac{1}{\sqrt{1+x^2}} e^{\arctan x} + \int \frac{x e^{\arctan x}}{(1+x^2)^{3/2}} dx$$

$$= \frac{1}{\sqrt{1+x^2}} e^{\arctan x} + \int \frac{x}{\sqrt{1+x^2}} d e^{\arctan x}$$

$$= \frac{1}{\sqrt{1+x^2}} e^{\arctan x} (1+x) - I$$

$$\therefore I = \frac{1+x}{2\sqrt{1+x^2}} e^{\arctan x} + C$$



内容小结

分部积分公式 $\int u v' dx = u v - \int u' v dx$

1. 使用原则： v 易求出, $\int u' v dx$ 易积分

2. 使用经验：“反对幂指三”，前 u 后 v'

3. 题目类型：

分部化简； 循环解出； 递推公式

4. 计算格式：

$$\begin{array}{c} u \\ \swarrow \\ v' \end{array} + \begin{array}{c} u' \\ | \\ v \end{array} - \int$$



例13. 求 $I = \int \sin(\ln x) dx$

解: 令 $t = \ln x$, 则 $x = e^t$, $dx = e^t dt$

$$\therefore I = \int e^t \sin t dt \longrightarrow \boxed{= e^t \sin t - \int e^t \cos t dt}$$



$$= e^t (\sin t - \cos t) - I$$

$$\therefore I = \frac{1}{2} e^t (\sin t - \cos t) + C$$

$$= \frac{1}{2} x [\sin(\ln x) - \cos(\ln x)] + C$$

多次分部积分



例14. 求 $\int x^3 (\ln x)^4 dx$.

解: 令 $u = \ln x$, 则 $x = e^u$, $dx = e^u du$

$$\text{原式} = \int e^{3u} u^4 \cdot e^u du = \int u^4 e^{4u} du$$

$$\begin{aligned} \text{原式} &= \frac{1}{4} e^{4u} \left(u^4 - u^3 + \frac{3}{4} u^2 - \frac{3}{8} u + \frac{3}{32} \right) + C \\ &= \frac{1}{4} x^4 \left(\ln^4 x - \ln^3 x + \frac{3}{4} \ln^2 x - \frac{3}{8} \ln x + \frac{3}{32} \right) + C \end{aligned}$$



思考与练习

1. 下述运算错在哪里? 应如何改正?

$$\begin{aligned}\int \frac{\cos x}{\sin x} dx &= \int \frac{d \sin x}{\sin x} = \frac{\sin x}{\sin x} - \int \left(\frac{1}{\sin x}\right)' \sin x dx \\ &= 1 - \int \frac{-\cos x}{\sin^2 x} \sin x dx = 1 + \int \frac{\cos x}{\sin x} dx \\ \therefore \int \frac{\cos x}{\sin x} dx - \int \frac{\cos x}{\sin x} dx &= 1, \quad \text{得 } 0 = 1\end{aligned}$$

$= \ln |\sin x| + C$

答: 不定积分是原函数族, 相减不应为 0.
求此积分的正确作法是用换元法.



2. 设 $F(x)$ 是 $f(x)$ 的一个原函数, $f(x)$ 可微且其反函数 $f^{-1}(x)$ 存在, 证明

$$\int f^{-1}(x) dx = x f^{-1}(x) - F[f^{-1}(x)] + C$$

证:
$$\begin{aligned} \int f^{-1}(x) dx &= x f^{-1}(x) - \int x df^{-1}(x) \\ &= x f^{-1}(x) - \int f[f^{-1}(x)] df^{-1}(x) \\ &= x f^{-1}(x) - F[f^{-1}(x)] + C \end{aligned}$$

注意:

$$x = f[f^{-1}(x)]$$



备用题. 求不定积分 $\int \frac{x e^x}{\sqrt{e^x - 1}} dx$.

解: 方法1 (先分部, 再换元)

$$\begin{aligned} \int \frac{x e^x}{\sqrt{e^x - 1}} dx &= \int \frac{x}{\sqrt{e^x - 1}} d(e^x - 1) \\ &= 2 \int x d\sqrt{e^x - 1} = 2x\sqrt{e^x - 1} - 2 \int \sqrt{e^x - 1} dx \\ &\quad \left| \begin{array}{l} \text{令 } u = \sqrt{e^x - 1}, \text{ 则 } dx = \frac{2u}{1+u^2} du \end{array} \right. \\ &= 2x\sqrt{e^x - 1} - 4 \int \frac{u^2 + 1 - 1}{1 + u^2} du \quad -4(u - \arctan u) + C \\ &= 2x\sqrt{e^x - 1} - 4\sqrt{e^x - 1} + 4 \arctan \sqrt{e^x - 1} + C \end{aligned}$$



方法2 (先换元,再分部)

$$\int \frac{x e^x}{\sqrt{e^x - 1}} dx$$

令 $u = \sqrt{e^x - 1}$, 则 $x = \ln(1 + u^2)$, $dx = \frac{2u}{1 + u^2} du$

故
$$\int \frac{x e^x}{\sqrt{e^x - 1}} dx = \int \frac{(1 + u^2) \ln(1 + u^2)}{u} \cdot \frac{2u}{1 + u^2} du$$

$$= 2 \int \ln(1 + u^2) du$$

$$= 2u \ln(1 + u^2) - 4 \int \frac{1 + u^2 - 1}{1 + u^2} du$$

$$= 2u \ln(1 + u^2) - 4u + 4 \arctan u + C$$

$$= 2x\sqrt{e^x - 1} - 4\sqrt{e^x - 1} + 4 \arctan \sqrt{e^x - 1} + C$$

