

## 第七节

## 斯托克斯公式

## \*环流量与旋度

## 一、斯托克斯公式

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# 一、斯托克斯公式

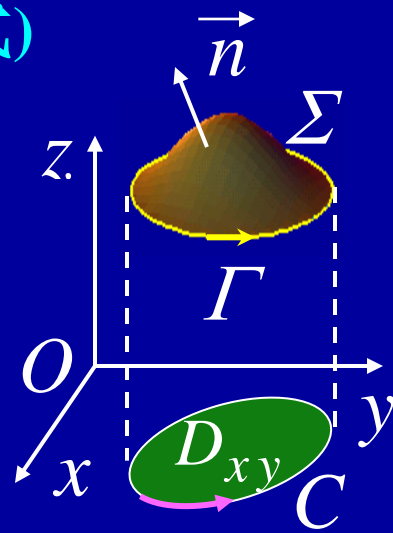
**定理1.** 设光滑曲面 $\Sigma$ 的边界 $\Gamma$ 是分段光滑曲线, $\Sigma$ 的侧与 $\Gamma$ 的正向符合**右手法则**,  $P, Q, R$ 在包含 $\Sigma$ 在内的一个空间域内具有连续一阶偏导数, 则有

$$\iint_{\Sigma} \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ = \oint_{\Gamma} P dx + Q dy + R dz \quad (\text{斯托克斯公式})$$

**证: 情形1.**  $\Sigma$ 与平行 $z$ 轴的直线只交于一点, 设其方程为

$$\Sigma: z = f(x, y), \quad (x, y) \in D_{xy}$$

为确定起见, 不妨设 $\Sigma$ 取上侧 (如图).



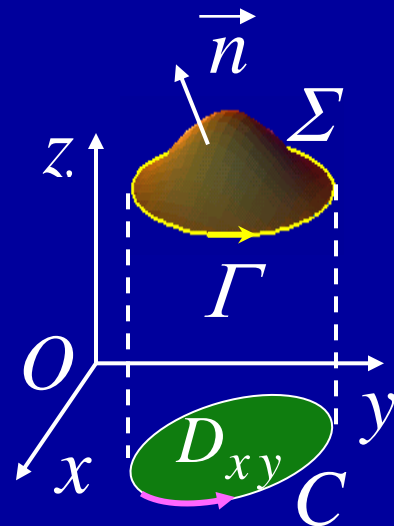
$$\begin{aligned} \text{则 } \oint_{\Gamma} P dx &= \oint_C P(x, y, z(x, y)) dx \\ &= - \iint_{D_{xy}} \frac{\partial}{\partial y} P(x, y, z(x, y)) dx dy \quad (\text{利用格林公式}) \end{aligned}$$

$$= - \iint_{D_{xy}} \left[ \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial y} \right] dx dy$$

$$= - \iint_{\Sigma} \left[ \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} f_y \right] \cos \gamma dS$$

$$\therefore \cos \gamma = \frac{1}{\sqrt{1+f_x^2+f_y^2}}, \quad \cos \beta = \frac{-f_y}{\sqrt{1+f_x^2+f_y^2}},$$

$$\therefore f_y = - \frac{\cos \beta}{\cos \gamma}$$



$$\begin{aligned}
 \text{因此 } \oint_{\Gamma} P \mathrm{d} x &= - \iint_{\Sigma} \left[ \frac{\partial P}{\partial y} - \frac{\partial P}{\partial z} \frac{\cos \beta}{\cos \gamma} \right] \cos \gamma \mathrm{d} S \\
 &= \iint_{\Sigma} \left[ \frac{\partial P}{\partial z} \cos \beta - \frac{\partial P}{\partial y} \cos \gamma \right] \mathrm{d} S \\
 &= \iint_{\Sigma} \frac{\partial P}{\partial z} \mathrm{d} z \mathrm{d} x - \frac{\partial P}{\partial y} \mathrm{d} x \mathrm{d} y
 \end{aligned}$$

$$\text{同理可证 } \oint_{\Gamma} Q \mathrm{d} y = \iint_{\Sigma} \frac{\partial Q}{\partial x} \mathrm{d} x \mathrm{d} y - \frac{\partial Q}{\partial z} \mathrm{d} y \mathrm{d} z$$

$$\oint_{\Gamma} R \mathrm{d} x = \iint_{\Sigma} \frac{\partial R}{\partial y} \mathrm{d} y \mathrm{d} z - \frac{\partial R}{\partial x} \mathrm{d} z \mathrm{d} x$$

三式相加, 即得斯托克斯公式;



**情形2** 曲面 $\Sigma$ 与平行 $z$ 轴的直线交点多于一个, 则可通过作辅助线把 $\Sigma$ 分成与 $z$ 轴只交于一点的几部分, 在每一部分上应用斯托克斯公式, 然后相加, 由于沿辅助曲线方向相反的两个曲线积分相加刚好抵消, 所以对这类曲面斯托克斯公式仍成立. 证毕

**注意:** 如果 $\Sigma$ 是 $xOy$ 面上的一块平面区域, 则斯托克斯公式就是格林公式, 故格林公式是斯托克斯公式的特例.

$$\begin{aligned} \iint_{\Sigma} \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ = \oint_{\Gamma} P dx + Q dy + R dz \end{aligned}$$



为便于记忆, 斯托克斯公式还可写作:

$$\iint_{\Sigma} \begin{vmatrix} dydz & dzdx & dx dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \oint_{\Gamma} P dx + Q dy + R dz$$

或用第一类曲面积分表示:

$$\iint_{\Sigma} \begin{vmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS = \oint_{\Gamma} P dx + Q dy + R dz$$



**例1.** 利用斯托克斯公式计算积分  $\oint_{\Gamma} z dx + x dy + y dz$   
 其中  $\Gamma$  为平面  $x + y + z = 1$  被三坐标面所截三角形的整个边界, 方向如图所示.

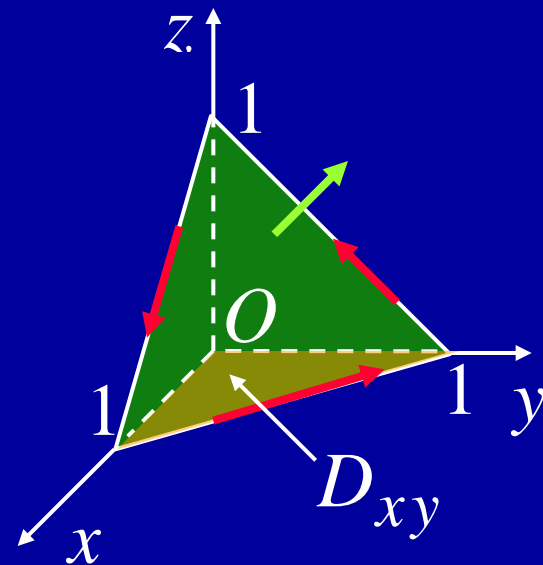
**解:** 记三角形域为  $\Sigma$ , 取上侧, 则

$$\oint_{\Gamma} z dx + x dy + y dz$$

$$= \iint_{\Sigma} \begin{vmatrix} dy dz & dz dx & dx dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

$$= \iint_{\Sigma} dy dz + dz dx + dx dy = 3 \iint_{D_{xy}} dx dy = \frac{3}{2}$$

利用对称性



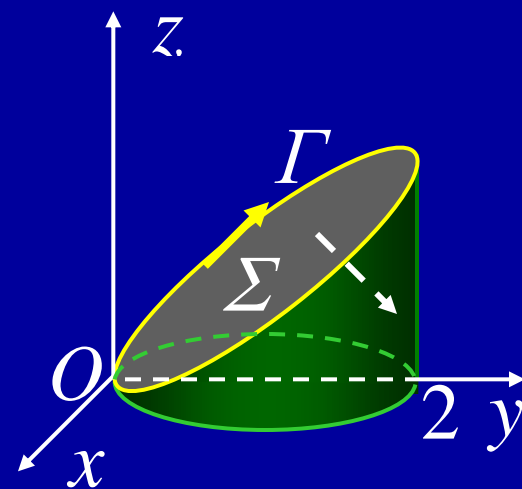
**例2.**  $\Gamma$ 为柱面  $x^2 + y^2 = 2y$  与平面  $y = z$  的交线, 从  $z$  轴正向看为顺时针, 计算  $I = \oint_{\Gamma} y^2 dx + xy dy + xz dz$ .

**解:** 设  $\Sigma$  为平面  $z = y$  上被  $\Gamma$  所围椭圆域, 且取下侧, 则其法线方向余弦

$$\cos \alpha = 0, \cos \beta = \frac{1}{\sqrt{2}}, \cos \gamma = -\frac{1}{\sqrt{2}}$$

利用斯托克斯公式得

$$I = \iint_{\Sigma} \begin{vmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & xy & xz \end{vmatrix} dS = \frac{1}{\sqrt{2}} \iint_{\Sigma} (y - z) dS = 0$$





## \*二、空间曲线积分与路径无关的条件

**定理2.** 设  $G$  是空间一维单连通域, 函数  $P, Q, R$  在  $G$  内具有连续一阶偏导数, 则下列四个条件相互等价:

(1) 对  $G$  内任一分段光滑闭曲线  $\Gamma$ , 有

$$\oint_{\Gamma} P dx + Q dy + R dz = 0$$

(2) 对  $G$  内任一分段光滑曲线  $\Gamma$ ,  $\int_{\Gamma} P dx + Q dy + R dz$  与路径无关

(3) 在  $G$  内存在某一函数  $u$ , 使  $du = P dx + Q dy + R dz$

(4) 在  $G$  内处处有

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$



证: (4)  $\Rightarrow$  (1) 由斯托克斯公式可知结论成立;

(1)  $\Rightarrow$  (2) (自证)

(2)  $\Rightarrow$  (3) 设函数

$$u(x, y, z) = \int_{(x_0, y_0, z_0)}^{(x, y, z)} P dx + Q dy + R dz$$

则

$$\begin{aligned} \frac{\partial u}{\partial x} &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x, y, z) - u(x, y, z)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \int_{(x, y, z)}^{(x + \Delta x, y, z)} P dx + Q dy + R dz \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \int_x^{x + \Delta x} P dx = \lim_{\Delta x \rightarrow 0} p(x + \Delta x, y, z) \\ &= P(x, y, z) \end{aligned}$$



同理可证  $\frac{\partial u}{\partial y} = Q(x, y, z), \quad \frac{\partial u}{\partial z} = R(x, y, z)$

故有  $du = Pdx + Qdy + Rdz$

(3)  $\Rightarrow$  (4) 若(3)成立, 则必有

$$\frac{\partial u}{\partial x} = P, \quad \frac{\partial u}{\partial y} = Q, \quad \frac{\partial u}{\partial z} = R$$

因  $P, Q, R$  一阶偏导数连续, 故有

$$\frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

同理 
$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

证毕



**例3.** 验证曲线积分  $\int_{\Gamma} (y+z)dx + (z+x)dy + (x+y)dz$  与路径无关, 并求函数

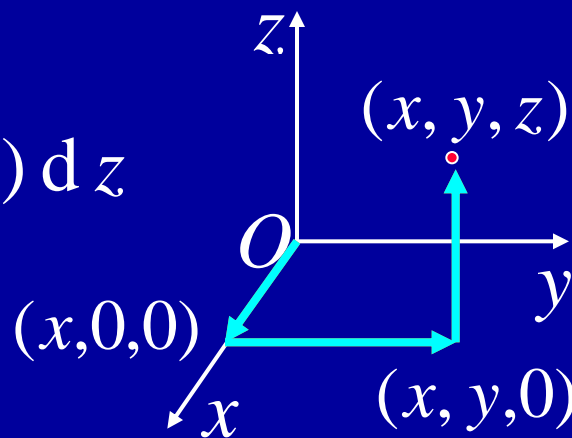
$$u(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} (y+z)dx + (z+x)dy + (x+y)dz$$

**解:** 令  $P = y+z$ ,  $Q = z+x$ ,  $R = x+y$

$$\therefore \frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = 1 = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = 1 = \frac{\partial P}{\partial z}$$

$\therefore$  积分与路径无关, 因此

$$\begin{aligned} u(x, y, z) &= \int_0^x 0 dx + \int_0^y x dy + \int_0^z (x+y) dz \\ &= xy + (x+y)z \\ &= xy + yz + zx \end{aligned}$$



### 三、环流量与旋度

斯托克斯公式

$$\iint_{\Sigma} \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ = \oint_{\Gamma} P dx + Q dy + R dz$$

设曲面  $\Sigma$  的法向量为  $\vec{n} = (\cos \alpha, \cos \beta, \cos \gamma)$

曲线  $\Gamma$  的单位切向量为  $\vec{\tau} = (\cos \lambda, \cos \mu, \cos \nu)$

则斯托克斯公式可写为

$$\iint_{\Sigma} \left[ \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \cos \alpha + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \cos \beta + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cos \gamma \right] dS \\ = \oint_{\Gamma} (P \cos \lambda + Q \cos \mu + R \cos \nu) ds$$



令  $\vec{A} = (P, Q, R)$ , 引进一个向量

$$\left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \nabla \times \vec{A} \quad \underline{\text{记作}} \quad \text{rot} \vec{A}$$

于是得斯托克斯公式的向量形式:

$$\iint_{\Sigma} \text{rot} \vec{A} \cdot \vec{n} dS = \oint_{\Gamma} \vec{A} \cdot \vec{\tau} ds$$

rotation

或 
$$\iint_{\Sigma} (\text{rot} A)_n dS = \oint_{\Gamma} A_{\tau} ds \quad \textcircled{1}$$

**定义:**  $\oint_{\Gamma} P dx + Q dy + R dz = \oint_{\Gamma} A_{\tau} ds$  称为向量场  $\vec{A}$  沿有向闭曲线  $\Gamma$  的**环流量**. 向量  $\text{rot} \vec{A}$  称为向量场  $\vec{A}$  的**旋度**.



## 旋度的力学意义:

设某刚体绕定轴  $l$  转动, 角速度为  $\vec{\omega}$ ,  $M$  为刚体上任一点, 建立坐标系如图, 则

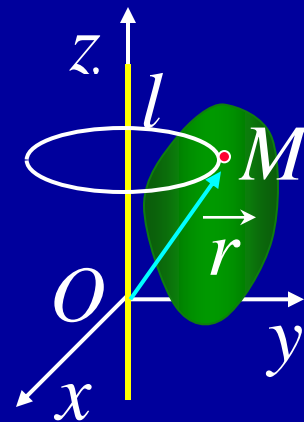
$$\vec{\omega} = (0, 0, \omega), \quad \vec{r} = (x, y, z)$$

点  $M$  的线速度为

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = (-\omega y, \omega x, 0)$$

$$\text{rot} \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = (0, 0, 2\omega) = 2\vec{\omega}$$

(此即“旋度”一词的来源)

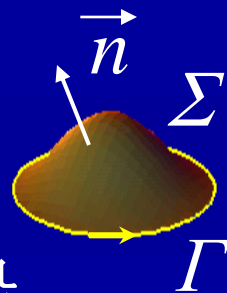


# 斯托克斯公式①的物理意义:

$$\iint_{\Sigma} (\text{rot } \vec{A})_n dS = \oint_{\Gamma} \vec{A}_\tau ds$$

向量场  $\vec{A}$  产生的旋度场  
穿过  $\Sigma$  的通量

向量场  $\vec{A}$  沿  
 $\Gamma$  的环流量



注意  $\Sigma$  与  $\Gamma$  的方向形成右手系!

**例4.** 求电场强度  $\vec{E} = \frac{q}{r^3} \vec{r}$  的旋度.

**解:**  $\text{rot } \vec{E} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{qx}{r^3} & \frac{qy}{r^3} & \frac{qz}{r^3} \end{vmatrix} = (0, 0, 0) \quad (\text{除原点外})$

这说明, 在除点电荷所在原点外, 整个电场无旋.





**例5.** 设  $\vec{A} = (2y, 3x, z^2)$ ,  $\Sigma: x^2 + y^2 + z^2 = 4$ ,  $\vec{n}$  为  $\Sigma$  的外法向量, 计算  $I = \oiint_{\Sigma} \text{rot} \vec{A} \cdot \vec{n} dS$ .

**解:**  $\text{rot} \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & z^2 \end{vmatrix} = (0, 0, 1)$

$$\vec{n} = (\cos \alpha, \cos \beta, \cos \gamma)$$

$$\begin{aligned} \therefore I &= \oiint_{\Sigma} \cos \gamma dS = \iint_{\Sigma_{\text{上}}} dx dy + \iint_{\Sigma_{\text{下}}} dx dy \\ &= \iint_{D_{xy}} dx dy - \iint_{D_{xy}} dx dy = 0 \end{aligned}$$



# 内容小结

## 1. 斯托克斯公式

$$\oint_{\Gamma} P dx + Q dy + R dz$$
$$= \iint_{\Sigma} \begin{vmatrix} dy dz & dz dx & dx dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$
$$= \iint_{\Sigma} \begin{vmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

也可写成:

$$\oint_{\Gamma} A_{\tau} ds = \iint_{\Sigma} (\nabla \times \vec{A})_n dS$$

其中

$$\vec{A} = (P, Q, R)$$

$A_{\tau}$  ——  $\vec{A}$  在  $\Gamma$  的切向量  $\vec{\tau}$  上  
投影

$(\nabla \times \vec{A})_n$  ——  $\vec{A}$  的旋度  $\nabla \times \vec{A}$   
在  $\Sigma$  的法向量  $\vec{n}$  上  
投影



## 2. 空间曲线积分与路径无关的充要条件

设  $P, Q, R$  在  $\Omega$  内具有一阶连续偏导数, 则

$\int_{\Gamma} P dx + Q dy + R dz$  在  $\Omega$  内与路径无关

$\iff$  在  $\Omega$  内处处有

$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}, \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$\iff$  在  $\Omega$  内处处有

$$\text{rot}(P, Q, R) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{0}$$



### 3. 场论中的三个度

设  $u = u(x, y, z)$ ,  $\vec{A} = (P, Q, R)$ ,  $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ , 则

梯度:  $\text{grad } u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}) = \nabla u$

散度:  $\text{div } \vec{A} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \vec{A}$

旋度:  $\text{rot } \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla \times \vec{A}$



## 思考与练习

设  $r = \sqrt{x^2 + y^2 + z^2}$ , 则

$$\operatorname{div}(\operatorname{grad} r) = \underline{\frac{2}{r}}; \quad \operatorname{rot}(\operatorname{grad} r) = \underline{\vec{0}}.$$

提示:  $\operatorname{grad} r = \left( \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right)$

$$\frac{\partial}{\partial x} \left( \frac{x}{r} \right) = \frac{r - x \cdot \frac{x}{r}}{r^2} = \frac{r^2 - x^2}{r^3}, \quad \frac{\partial}{\partial y} \left( \frac{y}{r} \right) = \frac{r^2 - y^2}{r^3}$$

$$\frac{\partial}{\partial z} \left( \frac{z}{r} \right) = \frac{r^2 - z^2}{r^3}$$

三式相加即得  $\operatorname{div}(\operatorname{grad} r)$

$$\operatorname{rot}(\operatorname{grad} r) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{vmatrix} = (0, 0, 0)$$

