

第二节

换元积分法

一、第一类换元法

二、第二类换元法



基本思路

设 $\underline{F'(u) = f(u)}$, $\underline{u = \varphi(x)}$ 可导, 则有

$$dF[\varphi(x)] = f[\varphi(x)]\varphi'(x)dx$$

$$\begin{aligned}\therefore \int f[\varphi(x)]\varphi'(x)dx &= F[\varphi(x)] + C = F(u) + C \Big|_{u=\varphi(x)} \\ &= \int f(u)du \Big|_{u=\varphi(x)}\end{aligned}$$

$$\int f[\varphi(x)]\varphi'(x)dx \begin{array}{c} \xrightarrow{\text{第一类换元法}} \\ \xleftarrow{\text{第二类换元法}} \end{array} \int f(u)du$$



一、第一类换元法

定理1. 设 $f(u)$ 有原函数, $u = \varphi(x)$ 可导, 则有换元公式

$$\int f[\varphi(x)] \underline{\varphi'(x)} dx = \int f(u) du \Big|_{u = \varphi(x)}$$

即
$$\int f[\varphi(x)] \varphi'(x) dx = \int f(\varphi(x)) d\varphi(x)$$

(也称**配元法**, 凑微分法)



例1. 求 $\int (ax + b)^m dx \quad (m \neq -1)$.

解: 令 $u = ax + b$, 则 $du = a dx$, 故

$$\begin{aligned}\text{原式} &= \int u^m \frac{1}{a} du = \frac{1}{a} \cdot \frac{1}{m+1} u^{m+1} + C \\ &= \frac{1}{a(m+1)} (ax + b)^{m+1} + C\end{aligned}$$

注意换回原变量

注: 当 $m = -1$ 时

$$\int \frac{dx}{ax + b} = \frac{1}{a} \ln|ax + b| + C$$



例2. 求 $\int \frac{dx}{a^2 + x^2}$.

解: $\int \frac{dx}{a^2 + x^2} = \frac{1}{a^2} \int \frac{dx}{1 + (\frac{x}{a})^2}$

↓ 令 $u = \frac{x}{a}$, 则 $du = \frac{1}{a} dx$

$$= \frac{1}{a} \int \frac{du}{1 + u^2} = \frac{1}{a} \arctan u + C$$

$$= \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

想到公式

$$\int \frac{du}{1 + u^2} \\ = \arctan u + C$$



例3. 求 $\int \frac{dx}{\sqrt{a^2 - x^2}} \ (a > 0).$

解:
$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{dx}{a \sqrt{1 - (\frac{x}{a})^2}} = \int \frac{d(\frac{x}{a})}{\sqrt{1 - (\frac{x}{a})^2}}$$
$$= \arcsin \frac{x}{a} + C$$

想到 $\int \frac{du}{\sqrt{1 - u^2}} = \arcsin u + C$

$$\int f[\varphi(x)] \varphi'(x) dx = \int f(\varphi(x)) d\varphi(x) \quad (\text{直接配元})$$



例4. 求 $\int \tan x dx$.

解:
$$\begin{aligned}\int \tan x dx &= \int \frac{\sin x}{\cos x} dx = -\int \frac{d\cos x}{\cos x} \\ &= -\ln|\cos x| + C\end{aligned}$$

类似

$$\begin{aligned}\int \cot x dx &= \int \frac{\cos x dx}{\sin x} = \int \frac{d\sin x}{\sin x} \\ &= \ln|\sin x| + C\end{aligned}$$



例5. 求 $\int \frac{dx}{x^2 - a^2}$.

解:

$$\because \frac{1}{x^2 - a^2} = \frac{1}{2a} \frac{(x+a) - (x-a)}{(x-a)(x+a)} = \frac{1}{2a} \left(\frac{1}{x-a} - \frac{1}{x+a} \right)$$

$$\therefore \text{原式} = \frac{1}{2a} \left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a} \right]$$

$$= \frac{1}{2a} \left[\int \frac{d(x-a)}{x-a} - \int \frac{d(x+a)}{x+a} \right]$$

$$= \frac{1}{2a} \left[\ln|x-a| - \ln|x+a| \right] + C = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$



常用的几种配元形式:

$$1) \int f(ax+b) dx = \frac{1}{a} \int f(ax+b) d(ax+b)$$

$$2) \int f(x^n) x^{n-1} dx = \frac{1}{n} \int f(x^n) dx^n$$

$$3) \int f(x^n) \frac{1}{x} dx = \frac{1}{n} \int f(x^n) \frac{1}{x^n} dx^n$$

万能凑幂法

$$4) \int f(\sin x) \cos x dx = \int f(\sin x) d\sin x$$

$$5) \int f(\cos x) \sin x dx = - \int f(\cos x) d\cos x$$



$$6) \int f(\tan x) \sec^2 x dx = \int f(\tan x) d\tan x$$

$$7) \int f(e^x) e^x dx = \int f(e^x) de^x$$

$$8) \int f(\ln x) \frac{1}{x} dx = \int f(\ln x) d\ln x$$

例6. 求 $\int \frac{dx}{x(1+2\ln x)}$.

解: 原式 $= \int \frac{d\ln x}{1+2\ln x} = \frac{1}{2} \int \frac{d(1+2\ln x)}{1+2\ln x}$

$$= \frac{1}{2} \ln|1+2\ln x| + C$$



例7. 求 $\int \frac{e^{3\sqrt{x}}}{\sqrt{x}} dx$.

解: 原式 $= 2 \int e^{3\sqrt{x}} d\sqrt{x} = \frac{2}{3} \int e^{3\sqrt{x}} d(3\sqrt{x})$
 $= \frac{2}{3} e^{3\sqrt{x}} + C$

例8. 求 $\int \sec^6 x dx$.

解: 原式 $= \int (\tan^2 x + 1)^2 d \tan x$
 $= \int (\tan^4 x + 2 \tan^2 x + 1) d \tan x$
 $= \frac{1}{5} \tan^5 x + \frac{2}{3} \tan^3 x + \tan x + C$



例9. 求 $\int \frac{dx}{1+e^x}$.

解法1

$$\begin{aligned}\int \frac{dx}{1+e^x} &= \int \frac{(1+e^x) - e^x}{1+e^x} dx = \int dx - \int \frac{d(1+e^x)}{1+e^x} \\ &= x - \ln(1+e^x) + C\end{aligned}$$

解法2

$$\begin{aligned}\int \frac{dx}{1+e^x} &= \int \frac{e^{-x}}{1+e^{-x}} dx = -\int \frac{d(1+e^{-x})}{1+e^{-x}} \\ &= -\ln(1+e^{-x}) + C\end{aligned}$$

$-\ln(1+e^{-x}) = -\ln[e^{-x}(e^x+1)]$ 两法结果一样



例10. 求 $\int \sec x dx$.

解法1

$$\begin{aligned}\int \sec x dx &= \int \frac{\cos x}{\cos^2 x} dx = \int \frac{d \sin x}{1 - \sin^2 x} \\&= \frac{1}{2} \int \left[\frac{1}{1 + \sin x} + \frac{1}{1 - \sin x} \right] d \sin x \\&= \frac{1}{2} \left[\ln |1 + \sin x| - \ln |1 - \sin x| \right] + C \\&= \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C\end{aligned}$$



$$\begin{aligned}
 \text{解法 2} \quad \int \sec x dx &= \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx \\
 &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\
 &= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} \\
 &= \ln |\sec x + \tan x| + C
 \end{aligned}$$

同样可证

$$\begin{aligned}
 \int \csc x dx &= \ln |\csc x - \cot x| + C \\
 \text{或} \quad \int \csc x dx &= \ln \left| \tan \frac{x}{2} \right| + C
 \end{aligned}$$



例11. 求 $\int \frac{x^3}{(x^2 + a^2)^{3/2}} dx$.

解: 原式 $= \frac{1}{2} \int \frac{x^2 dx^2}{(x^2 + a^2)^{3/2}} = \frac{1}{2} \int \frac{(x^2 + a^2) - a^2}{(x^2 + a^2)^{3/2}} dx^2$

$$= \frac{1}{2} \int (x^2 + a^2)^{-1/2} d(x^2 + a^2)$$
$$- \frac{a^2}{2} \int (x^2 + a^2)^{-3/2} d(x^2 + a^2)$$
$$= \sqrt{x^2 + a^2} + \frac{a^2}{\sqrt{x^2 + a^2}} + C$$



例12. 求 $\int \cos^4 x \, dx$.

解: $\because \cos^4 x = (\cos^2 x)^2 = \left(\frac{1 + \cos 2x}{2}\right)^2$

$$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$
$$= \frac{1}{4}\left(1 + 2\cos 2x + \frac{1 + \cos 4x}{2}\right)$$
$$= \frac{1}{4}\left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right)$$

$$\therefore \int \cos^4 x \, dx = \frac{1}{4} \int \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$
$$= \frac{1}{4} \left[\frac{3}{2} \int dx + \int \cos 2x \, d(2x) + \frac{1}{8} \int \cos 4x \, d(4x) \right]$$
$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$



例13. 求 $\int \sin^2 x \cos^2 3x \, dx$.

解: $\because \sin^2 x \cos^2 3x = [\frac{1}{2}(\sin 4x - \sin 2x)]^2$

$$= \frac{1}{4} \sin^2 4x - \frac{1}{4} \cdot 2 \sin 4x \sin 2x + \frac{1}{4} \sin^2 2x$$
$$= \frac{1}{8} (1 - \cos 8x) - \sin^2 2x \cos 2x + \frac{1}{8} (1 - \cos 4x)$$

$$\therefore \text{原式} = \frac{1}{4} \int dx - \frac{1}{64} \int \cos 8x \, d(8x)$$
$$- \frac{1}{2} \int \sin^2 2x \, d(\sin 2x) - \frac{1}{32} \int \cos 4x \, d(4x)$$
$$= \frac{1}{4} x - \frac{1}{64} \sin 8x - \frac{1}{6} \sin^3 2x - \frac{1}{32} \sin 4x + C$$



例14. 求 $\int \frac{x+1}{x(1+xe^x)} dx$.

解: 原式 $= \int \frac{(x+1)e^x}{xe^x(1+xe^x)} dx = \int \left(\frac{1}{xe^x} - \frac{1}{1+xe^x} \right) d(xe^x)$

$$= \ln|xe^x| - \ln|1+xe^x| + C$$
$$= x + \ln|x| - \ln|1+xe^x| + C$$

分析: $\frac{1}{xe^x(1+xe^x)} = \frac{1+xe^x - xe^x}{xe^x(1+xe^x)} = \frac{1}{xe^x} - \frac{1}{1+xe^x}$

$$(x+1)e^x dx = xe^x dx + e^x dx = d(xe^x)$$



例15. 求 $\int \left[\frac{f(x)}{f'(x)} - \frac{f''(x)f^2(x)}{f'^3(x)} \right] dx$.

解: 原式 $= \int \frac{f(x)}{f'(x)} \left[1 - \frac{f''(x)f(x)}{f'^2(x)} \right] dx$

$$= \int \frac{f(x)}{f'(x)} \cdot \frac{f'^2(x) - f''(x)f(x)}{f'^2(x)} dx$$
$$= \int \frac{f(x)}{f'(x)} d\left(\frac{f(x)}{f'(x)}\right)$$
$$= \frac{1}{2} \left[\frac{f(x)}{f'(x)} \right]^2 + C$$



小结 常用简化技巧:

(1) 分项积分: 利用积化和差; 分式分项;

$$1 = \sin^2 x + \cos^2 x \text{ 等}$$

(2) 降低幂次: 利用倍角公式, 如

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x); \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x);$$

$$\text{万能凑幂法} \begin{cases} \int f(x^n) x^{n-1} dx = \frac{1}{n} \int f(x^n) d x^n \\ \int f(x^n) \frac{1}{x} dx = \frac{1}{n} \int f(x^n) \frac{1}{x^n} d x^n \end{cases}$$

(3) 统一函数: 利用三角公式; 配元方法

(4) 巧妙换元或配元



思考与练习

1. 下列各题求积方法有何不同?

$$(1) \int \frac{dx}{4+x} = \int \frac{d(4+x)}{4+x}$$

$$(2) \int \frac{dx}{4+x^2} = \frac{1}{2} \int \frac{d(\frac{x}{2})}{1+(\frac{x}{2})^2}$$

$$(3) \int \frac{x}{4+x^2} dx = \frac{1}{2} \int \frac{d(4+x^2)}{4+x^2}$$

$$(4) \int \frac{x^2}{4+x^2} dx = \int \left[1 - \frac{4}{4+x^2} \right] dx$$

$$(5) \int \frac{dx}{4-x^2} = \frac{1}{4} \int \left[\frac{1}{2-x} + \frac{1}{2+x} \right] dx$$

$$(6) \int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{d(x-2)}{\sqrt{4-(x-2)^2}}$$



2. 求 $\int \frac{dx}{x(x^{10}+1)}$.

提示:

法1 $\int \frac{dx}{x(x^{10}+1)} = \int \frac{(x^{10}+1) - x^{10}}{x(x^{10}+1)} dx$

法2 $\int \frac{dx}{x(x^{10}+1)} = \frac{1}{10} \int \frac{dx^{10}}{x^{10}(x^{10}+1)}$

法3 $\int \frac{dx}{x(x^{10}+1)} = \int \frac{dx}{x^{11}(1+x^{-10})} = \frac{-1}{10} \int \frac{dx^{-10}}{1+x^{-10}}$



二、第二类换元法

第一类换元法解决的问题

$$\int \underset{\text{难求}}{f[\varphi(x)]\varphi'(x)}dx = \int \underset{\text{易求}}{f(u)}du \Big|_{u=\varphi(x)}$$

若所求积分 $\int f(u)du$ 难求,

$\int f[\varphi(x)]\varphi'(x)dx$ 易求,

则得第二类换元积分法.



定理2. 设 $x = \psi(t)$ 是单调可导函数, 且 $\psi'(t) \neq 0$,
 $f[\psi(t)]\psi'(t)$ 具有原函数, 则有换元公式

$$\int f(x) dx = \int f[\psi(t)]\psi'(t) dt \Big|_{t=\psi^{-1}(x)}$$

其中 $t = \psi^{-1}(x)$ 是 $x = \psi(t)$ 的反函数.

证: 设 $f[\psi(t)]\psi'(t)$ 的原函数为 $\Phi(t)$, 令

$$F(x) = \Phi[\psi^{-1}(x)]$$

则
$$F'(x) = \frac{d\Phi}{dt} \cdot \frac{dt}{dx} = f[\psi(t)]\psi'(t) \cdot \frac{1}{\psi'(t)} = f(x)$$

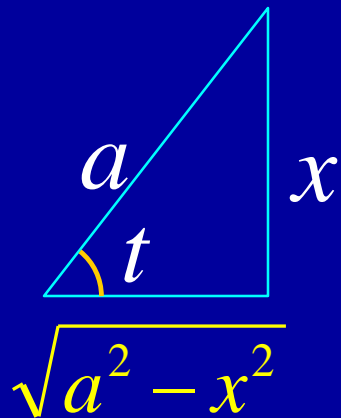
$$\begin{aligned} \therefore \int f(x) dx &= F(x) + C = \Phi[\psi^{-1}(x)] + C \\ &= \int f[\psi(t)]\psi'(t) dt \Big|_{t=\psi^{-1}(x)} \end{aligned}$$



例16. 求 $\int \sqrt{a^2 - x^2} \, dx \quad (a > 0).$

解: 令 $x = a \sin t, \quad t \in (-\frac{\pi}{2}, \frac{\pi}{2}),$ 则

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 t} = a \cos t \\ dx &= a \cos t \, dt\end{aligned}$$



$$\begin{aligned}\therefore \text{原式} &= \int a \cos t \cdot a \cos t \, dt = a^2 \int \cos^2 t \, dt \\ &= a^2 \int \frac{1 + \cos 2t}{2} \, dt = a^2 \left(\frac{t}{2} + \frac{\sin 2t}{4} \right) + C \\ &\quad \left| \begin{array}{l} \sin 2t = 2 \sin t \cos t = 2 \cdot \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a} \\ \downarrow \\ \end{array} \right. \\ &= \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{1}{2} x \sqrt{a^2 - x^2} + C\end{aligned}$$



例17. 求 $\int \frac{dx}{\sqrt{x^2 + a^2}} \quad (a > 0).$

解: 令 $x = a \tan t, t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, 则

$$\sqrt{x^2 + a^2} = \sqrt{a^2 \tan^2 t + a^2} = a \sec t$$

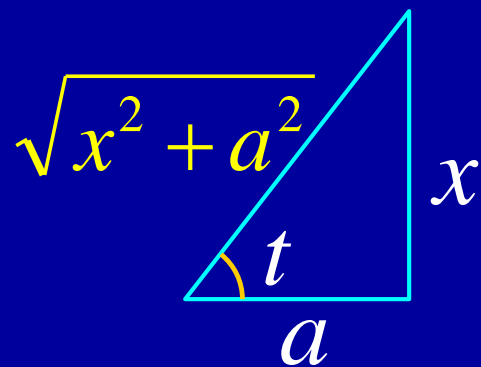
$$dx = a \sec^2 t dt$$

$$\therefore \text{原式} = \int \frac{a \sec^2 t}{a \sec t} dt = \int \sec t dt$$

$$= \ln |\sec t + \tan t| + C_1$$

$$= \ln \left[\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right] + C_1$$

$$= \ln \left[x + \sqrt{x^2 + a^2} \right] + C \quad (C = C_1 - \ln a)$$



例18. 求 $\int \frac{dx}{\sqrt{x^2 - a^2}} \quad (a > 0).$

解: 当 $x > a$ 时, 令 $x = a \sec t, t \in (0, \frac{\pi}{2})$, 则

$$\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 t - a^2} = a \tan t$$

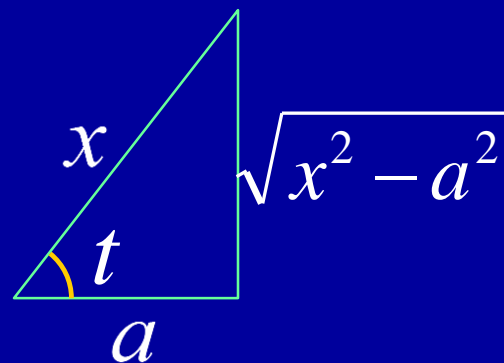
$$dx = a \sec t \tan t dt$$

$$\therefore \text{原式} = \int \frac{a \sec t \tan t}{a \tan t} dt = \int \sec t dt$$

$$= \ln |\sec t + \tan t| + C_1$$

$$= \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C_1$$

$$= \ln \left| x + \sqrt{x^2 - a^2} \right| + C \quad (C = C_1 - \ln a)$$



当 $x < -a$ 时, 令 $x = -u$, 则 $u > a$, 于是

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 - a^2}} &= -\int \frac{du}{\sqrt{u^2 - a^2}} = -\ln \left| u + \sqrt{u^2 - a^2} \right| + C_1 \\&= -\ln \left| -x + \sqrt{x^2 - a^2} \right| + C_1 \\&= -\ln \left| \frac{a^2}{-x - \sqrt{x^2 - a^2}} \right| + C_1 \\&= \ln \left| x + \sqrt{x^2 - a^2} \right| + C \quad (C = C_1 - 2\ln a)\end{aligned}$$

$$x > a \text{ 时, } \int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + C$$



说明:

1. 被积函数含有 $\sqrt{x^2 + a^2}$ 或 $\sqrt{x^2 - a^2}$ 时, 除采用三角代换外, 还可利用公式 $\text{ch}^2 t - \text{sh}^2 t = 1$ 采用双曲代换

$$x = a \text{sh} t \text{ 或 } x = a \text{ch} t$$

消去根式, 所得结果一致.

2. 再补充两个常用双曲函数积分公式

$$(14) \quad \int \text{sh} x dx = \text{ch} x + C$$

$$(15) \quad \int \text{ch} x dx = \text{sh} x + C$$

$$\text{sh} x = \frac{e^x - e^{-x}}{2}$$

$$\text{ch} x = \frac{e^x + e^{-x}}{2}$$



例19. 求 $\int \frac{\sqrt{a^2 - x^2}}{x^4} dx$.

解: 令 $x = \frac{1}{t}$, 则 $dx = \frac{-1}{t^2} dt$

$$\text{原式} = \int \frac{\sqrt{a^2 - \frac{1}{t^2}}}{\frac{1}{t^4}} \cdot \frac{-1}{t^2} dt = -\int (a^2 t^2 - 1)^{\frac{1}{2}} |t| dt$$

当 $x > 0$ 时,

$$\begin{aligned} \text{原式} &= -\frac{1}{2a^2} \int (a^2 t^2 - 1)^{\frac{1}{2}} d(a^2 t^2 - 1) \\ &= -\frac{(a^2 t^2 - 1)^{\frac{3}{2}}}{3a^2} + C = -\frac{(a^2 - x^2)^{\frac{3}{2}}}{3a^2 x^3} + C \end{aligned}$$

当 $x < 0$ 时, 类似可得同样结果.



小结:

1. 第二类换元法常见类型:

$$1) \int f(x, \sqrt[n]{ax+b}) dx, \quad \text{令 } t = \sqrt[n]{ax+b}$$

$$2) \int f(x, \sqrt[n]{\frac{ax+b}{cx+d}}) dx, \quad \text{令 } t = \sqrt[n]{\frac{ax+b}{cx+d}}$$

$$3) \int f(x, \sqrt{a^2 - x^2}) dx, \quad \text{令 } x = a \sin t \text{ 或 } x = a \cos t$$

$$4) \int f(x, \sqrt{a^2 + x^2}) dx, \quad \text{令 } x = a \tan t \text{ 或 } x = a \operatorname{sh} t$$

$$5) \int f(x, \sqrt{x^2 - a^2}) dx, \quad \text{令 } x = a \sec t \text{ 或 } x = a \operatorname{ch} t$$



6) $\int f(a^x)dx$, 令 $t = a^x$

7) 分母中因子次数较高时, 可试用倒代换

2. 常用基本积分公式的补充

$$(16) \quad \int \tan x \, dx = -\ln|\cos x| + C$$

$$(17) \quad \int \cot x \, dx = \ln|\sin x| + C$$

$$(18) \quad \int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$(19) \quad \int \csc x \, dx = \ln|\csc x - \cot x| + C$$



$$(20) \quad \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$(21) \quad \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

$$(22) \quad \int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C$$

$$(23) \quad \int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2}) + C$$

$$(24) \quad \int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left| x + \sqrt{x^2 - a^2} \right| + C$$



例20. 求 $\int \frac{dx}{x^2 + 2x + 3}$.

解: 原式 $= \int \frac{1}{(x+1)^2 + (\sqrt{2})^2} d(x+1)$
 $= \frac{1}{\sqrt{2}} \arctan \frac{x+1}{\sqrt{2}} + C$

例21. 求 $I = \int \frac{dx}{\sqrt{4x^2 + 9}}$.

解: $I = \frac{1}{2} \int \frac{d(2x)}{\sqrt{(2x)^2 + 3^2}} = \frac{1}{2} \ln \left| 2x + \sqrt{4x^2 + 9} \right| + C$



例22. 求 $\int \frac{dx}{\sqrt{1+x-x^2}}.$

解: 原式 $= \int \frac{d(x-\frac{1}{2})}{\sqrt{(\frac{\sqrt{5}}{2})^2 - (x-\frac{1}{2})^2}} = \arcsin \frac{2x-1}{\sqrt{5}} + C$

例23. 求 $\int \frac{dx}{\sqrt{e^{2x}-1}}.$

解: 原式 $= -\int \frac{de^{-x}}{\sqrt{1-e^{-2x}}} = -\operatorname{arcsine}^{-x} + C$



例24. 求 $\int \frac{dx}{x^2 \sqrt{x^2 + a^2}}.$

解: 令 $x = \frac{1}{t}$, 得

$$\begin{aligned}\text{原式} &= -\int \frac{t}{\sqrt{a^2 t^2 + 1}} dt \\&= -\frac{1}{2a^2} \int \frac{d(a^2 t^2 + 1)}{\sqrt{a^2 t^2 + 1}} = -\frac{1}{a^2} \sqrt{a^2 t^2 + 1} + C \\&= -\frac{\sqrt{x^2 + a^2}}{a^2 x} + C\end{aligned}$$



例25. 求 $\int \frac{dx}{(x+1)^3 \sqrt{x^2+2x}}$.

解: 原式 = $\int \frac{dx}{(x+1)^3 \sqrt{(x+1)^2 - 1}}$

令 $x+1 = \frac{1}{t}$

$$= \int \frac{t^3}{\sqrt{\frac{1}{t^2} - 1}} \left(-\frac{1}{t^2}\right) dt = - \int \frac{t^2}{\sqrt{1-t^2}} dt$$

$$= \int \frac{(1-t^2)-1}{\sqrt{1-t^2}} dt = \boxed{\int \sqrt{1-t^2} dt} - \int \frac{1}{\sqrt{1-t^2}} dt$$

$$= \frac{1}{2} t \sqrt{1-t^2} + \frac{1}{2} \arcsin t - \arcsin t + C$$

例16

$$= \frac{1}{2} \frac{\sqrt{x^2+2x}}{(x+1)^2} - \frac{1}{2} \arcsin \frac{1}{x+1} + C$$



思考与练习

1. 下列积分应如何换元才使积分简便？

$$(1) \int \frac{x^5}{\sqrt{1+x^2}} dx$$

$$\text{令 } t = \sqrt{1+x^2}$$

$$(2) \int \frac{dx}{\sqrt{1+e^x}}$$

$$\text{令 } t = \sqrt{1+e^x}$$

$$(3) \int \frac{dx}{x(x^7+2)}$$

$$\text{令 } t = \frac{1}{x}$$



2. 已知 $\int x^5 f(x) dx = \sqrt{x^2 - 1} + C$, 求 $\int f(x) dx$.

解: 两边求导, 得 $x^5 f(x) = \frac{x}{\sqrt{x^2 - 1}}$, 则

$$\begin{aligned}\int f(x) dx &= \int \frac{dx}{x^4 \sqrt{x^2 - 1}} \quad (\text{令 } t = \frac{1}{x}) \\&= \int \frac{-t^3 dt}{\sqrt{1 - t^2}} = \frac{1}{2} \int \frac{(1 - t^2) - 1}{\sqrt{1 - t^2}} dt \\&= \frac{-1}{2} \int (1 - t^2)^{\frac{1}{2}} d(1 - t^2) + \frac{1}{2} \int (1 - t^2)^{-\frac{1}{2}} d(1 - t^2) \\&= \frac{-1}{3} (1 - t^2)^{\frac{3}{2}} + (1 - t^2)^{\frac{1}{2}} + C = \dots\end{aligned}$$

(代回原变量)



作业



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第三节



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结束

备用题 1. 求下列积分:

$$\begin{aligned} 1) \int \underline{x^2} \frac{1}{\sqrt{x^3+1}} dx &= \frac{1}{3} \int \frac{1}{\sqrt{x^3+1}} d(x^3+1) \\ &= \frac{2}{3} \sqrt{x^3+1} + C \end{aligned}$$

$$\begin{aligned} 2) \int \frac{2x+3}{\sqrt{1+2x-x^2}} dx &= \int \frac{-(2-2x)+5}{\sqrt{1+2x-x^2}} dx \\ &= -\int \frac{d(1+2x-x^2)}{\sqrt{1+2x-x^2}} + 5 \int \frac{d(x-1)}{\sqrt{2-(x-1)^2}} \\ &= -2\sqrt{1+2x-x^2} + 5 \arcsin \frac{x-1}{\sqrt{2}} + C \end{aligned}$$



2. 求不定积分 $\int \frac{2 \sin x \cos x \sqrt{1 + \sin^2 x}}{2 + \sin^2 x} dx$.

解: 利用凑微分法, 得

$$\text{原式} = \int \frac{\sqrt{1 + \sin^2 x}}{2 + \sin^2 x} d(1 + \sin^2 x)$$

$$\downarrow \quad \text{令 } t = \sqrt{1 + \sin^2 x}$$

$$= \int \frac{2t^2}{1 + t^2} dt = 2 \int \left(1 - \frac{1}{1 + t^2}\right) dt$$

$$= 2t - 2 \arctan t + C$$

$$= 2 \left[\sqrt{1 + \sin^2 x} - \arctan \sqrt{1 + \sin^2 x} \right] + C$$



3. 求不定积分 $\int \frac{1}{(1+x^2)\sqrt{1-x^2}} dx$.

解: 令 $x = \sin t$, $1+x^2 = 1+\sin^2 t$, $dx = \cos t dt$

$$\text{原式} = \int \frac{\cos t}{(1+\sin^2 t)\cos t} dt = \int \frac{1}{1+\sin^2 t} dt$$

分子分母同除以 $\cos^2 t$

$$= \int \frac{\sec^2 t}{\sec^2 t + \tan^2 t} dt = \int \frac{1}{1+2\tan^2 t} d\tan t$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{1+(\sqrt{2}\tan t)^2} d\sqrt{2}\tan t$$

$$= \frac{1}{\sqrt{2}} \arctan(\sqrt{2}\tan t) + C = \frac{1}{\sqrt{2}} \arctan \frac{\sqrt{2}x}{\sqrt{1-x^2}} + C$$

