

第二节

二重积分的计算法

一、利用直角坐标计算二重积分

二、利用极坐标计算二重积分

*三、二重积分的换元法



一、利用直角坐标计算二重积分

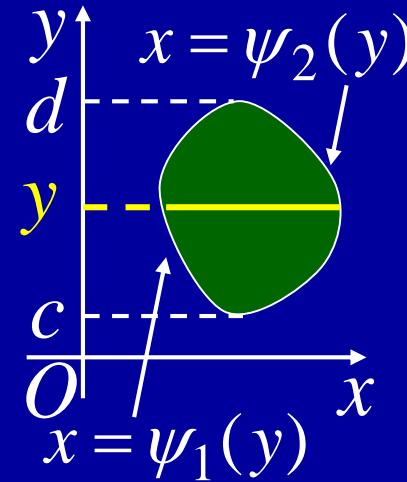
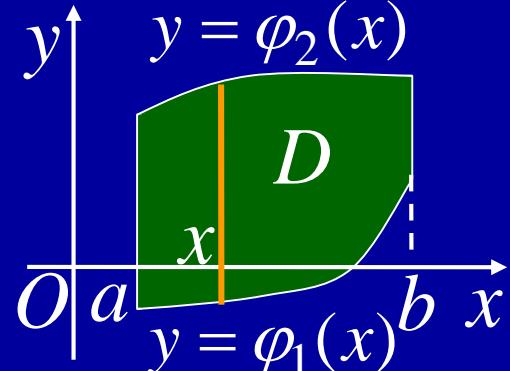
由曲顶柱体体积的计算可知, 当被积函数 $f(x, y) \geq 0$ 且在 D 上连续时, 若 D 为 X -型区域

$$D : \begin{cases} \varphi_1(x) \leq y \leq \varphi_2(x) \\ a \leq x \leq b \end{cases}$$

则 $\iint_D f(x, y) dx dy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy$

若 D 为 Y -型区域 $D : \begin{cases} \psi_1(y) \leq x \leq \psi_2(y) \\ c \leq y \leq d \end{cases}$

则 $\iint_D f(x, y) dx dy = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx$



当被积函数 $f(x, y)$ 在 D 上变号时, 由于

$$f(x, y) = \underbrace{\frac{f(x, y) + |f(x, y)|}{2}}_{f_1(x, y)} - \underbrace{\frac{|f(x, y)| - f(x, y)}{2}}_{f_2(x, y) \text{ 均非负}}$$

$$\begin{aligned} \therefore \iint_D f(x, y) dx dy &= \iint_D f_1(x, y) dx dy \\ &\quad - \iint_D f_2(x, y) dx dy \end{aligned}$$

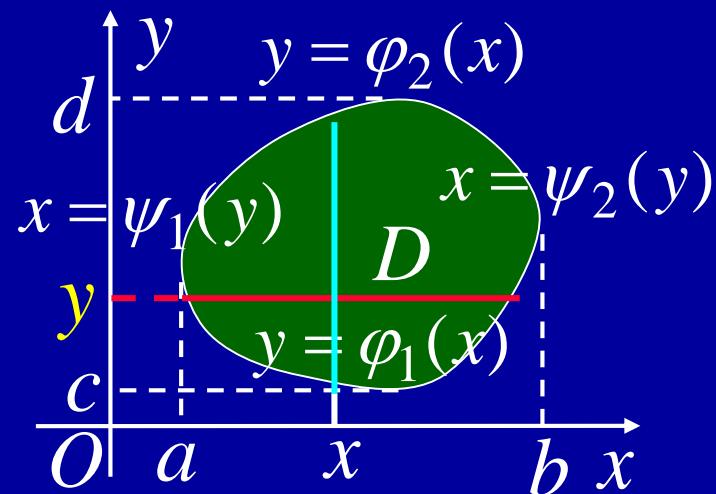
因此上面讨论的累次积分法仍然有效.



说明: (1) 若积分区域既是 X -型区域又是 Y -型区域,

则有

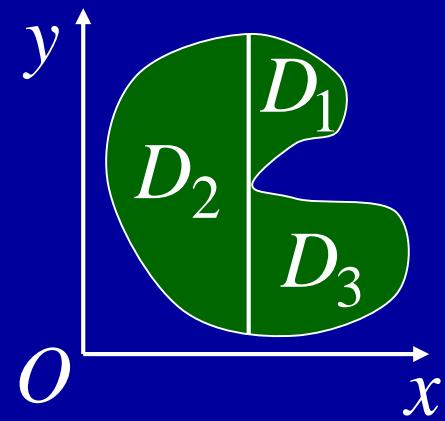
$$\begin{aligned} & \iint_D f(x, y) dx dy \\ &= \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy \\ &= \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx \end{aligned}$$



为计算方便,可选择积分序,必要时还可以交换积分序.

(2) 若积分域较复杂,可将它分成若干 X -型域或 Y -型域,则

$$\iint_D = \iint_{D_1} + \iint_{D_2} + \iint_{D_3}$$

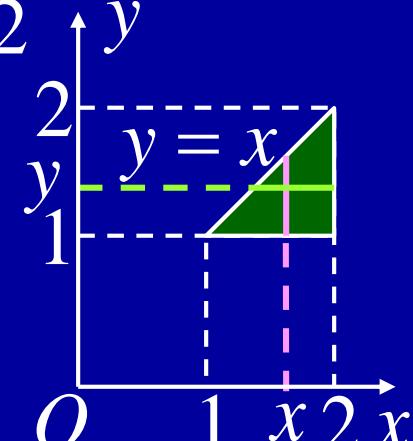


例1. 计算 $I = \iint_D xy \, d\sigma$, 其中 D 是直线 $y=1$, $x=2$, 及 $y=x$ 所围的闭区域.

解法1. 将 D 看作 X -型区域, 则 $D: \begin{cases} 1 \leq y \leq x \\ 1 \leq x \leq 2 \end{cases}$

$$I = \int_1^2 dx \int_1^x xy \, dy = \int_1^2 \left[\frac{1}{2}xy^2 \right]_1^x dx$$

$$= \int_1^2 \left[\frac{1}{2}x^3 - \frac{1}{2}x \right] dx = \frac{9}{8}$$



解法2. 将 D 看作 Y -型区域, 则 $D: \begin{cases} y \leq x \leq 2 \\ 1 \leq y \leq 2 \end{cases}$

$$I = \int_1^2 dy \int_y^2 xy \, dx = \int_1^2 \left[\frac{1}{2}x^2y \right]_y^2 dy = \int_1^2 \left[2y - \frac{1}{2}y^3 \right] dy = \frac{9}{8}$$



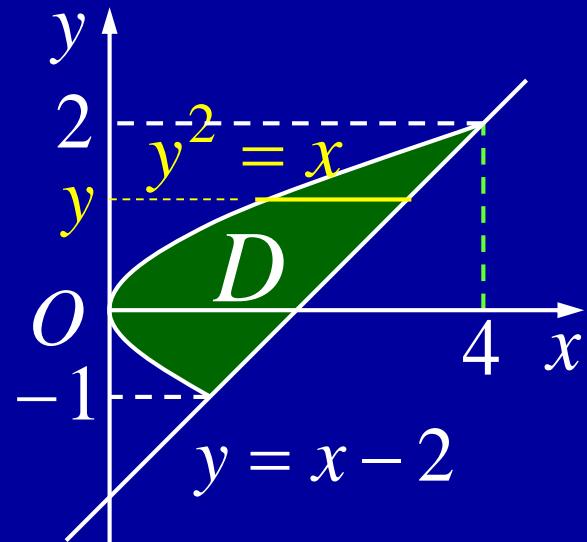
例2. 计算 $\iint_D xy \mathrm{d}\sigma$, 其中 D 是抛物线 $y^2 = x$ 及直线 $y = x - 2$ 所围成的闭区域.

解: 为计算简便, 先对 x 后对 y 积分,

则

$$D : \begin{cases} y^2 \leq x \leq y + 2 \\ -1 \leq y \leq 2 \end{cases}$$

$$\begin{aligned} \therefore \iint_D xy \mathrm{d}\sigma &= \int_{-1}^2 \mathrm{d}y \int_{y^2}^{y+2} xy \mathrm{d}x \\ &= \int_{-1}^2 \left[\frac{1}{2}x^2y \right]_{y^2}^{y+2} \mathrm{d}y = \frac{1}{2} \int_{-1}^2 [y(y+2)^2 - y^5] \mathrm{d}y \\ &= \frac{1}{2} \left[\frac{y^4}{4} + \frac{4}{3}y^3 + 2y^2 - \frac{1}{6}y^6 \right]_{-1}^2 = \frac{45}{8} \end{aligned}$$



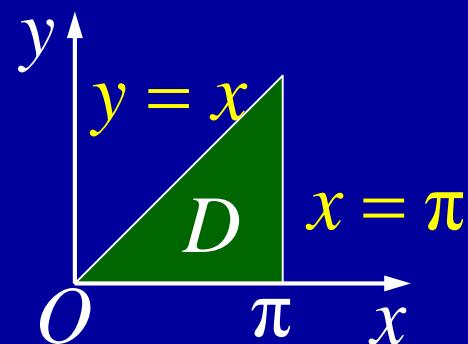
例3. 计算 $\iint_D \frac{\sin x}{x} dx dy$, 其中 D 是直线 $y = x$, $y = 0$, $x = \pi$ 所围成的闭区域.

解: 由被积函数可知, 先对 x 积分不行,
因此取 D 为 X -型域:

$$D: \begin{cases} 0 \leq y \leq x \\ 0 \leq x \leq \pi \end{cases}$$

$$\begin{aligned}\therefore \iint_D \frac{\sin x}{x} dx dy &= \int_0^\pi \frac{\sin x}{x} dx \int_0^x dy \\ &= \int_0^\pi \sin x dx = [-\cos x]_0^\pi = 2\end{aligned}$$

说明: 有些二次积分为了积分方便, 还需交换积分顺序.



例4. 交换下列积分顺序

$$I = \int_0^2 dx \int_0^{\frac{x^2}{2}} f(x, y) dy + \int_2^{2\sqrt{2}} dx \int_0^{\sqrt{8-x^2}} f(x, y) dy$$

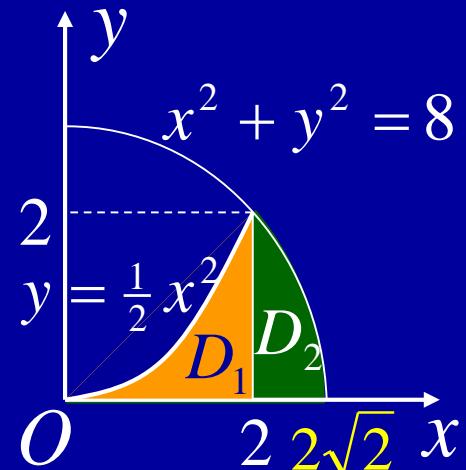
解：积分域由两部分组成：

$$D_1 : \begin{cases} 0 \leq y \leq \frac{1}{2}x^2 \\ 0 \leq x \leq 2 \end{cases}, \quad D_2 : \begin{cases} 0 \leq y \leq \sqrt{8-x^2} \\ 2 \leq x \leq 2\sqrt{2} \end{cases}$$

将 $D = D_1 + D_2$ 视为 Y -型区域，则

$$D : \begin{cases} \sqrt{2y} \leq x \leq \sqrt{8-y^2} \\ 0 \leq y \leq 2 \end{cases}$$

$$I = \iint_D f(x, y) dx dy = \int_0^2 dy \int_{\sqrt{2y}}^{\sqrt{8-y^2}} f(x, y) dx$$



例5. 计算 $I = \iint_D x \ln(y + \sqrt{1 + y^2}) dx dy$, 其中 D 由 $y = 4 - x^2$, $y = -3x$, $x = 1$ 所围成.

解: 令 $f(x, y) = x \ln(y + \sqrt{1 + y^2})$

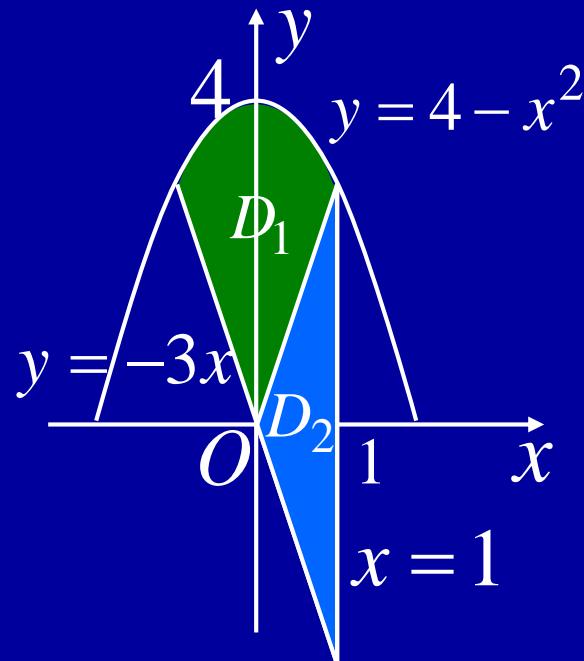
$$D = D_1 \cup D_2 \quad (\text{如图所示})$$

显然, 在 D_1 上, $f(-x, y) = -f(x, y)$

在 D_2 上, $f(x, -y) = -f(x, y)$

$$\therefore I = \iint_{D_1} x \ln(y + \sqrt{1 + y^2}) dx dy$$

$$+ \iint_{D_2} x \ln(y + \sqrt{1 + y^2}) dx dy = 0$$



二、利用极坐标计算二重积分

在极坐标系下,用同心圆 $r = \text{常数}$ 及射线 $\theta = \text{常数}$, 分划区域 D 为

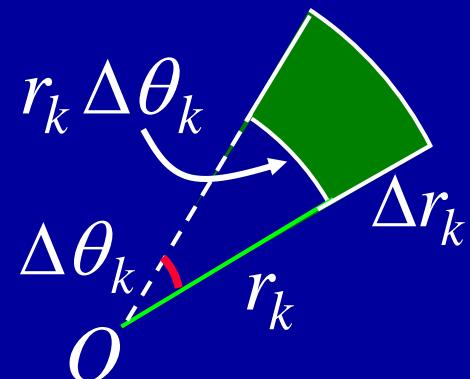
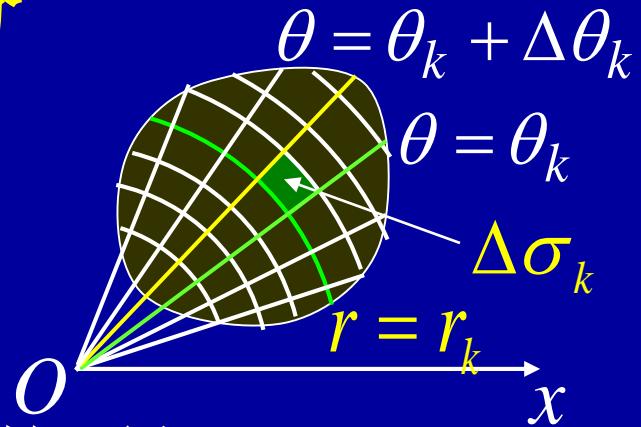
$$\Delta\sigma_k \quad (k=1, 2, \dots, n)$$

则除包含边界点的小区域外,小区域的面积

$$\begin{aligned}\Delta\sigma_k &= \frac{1}{2}(r_k + \Delta r_k)^2 \cdot \Delta\theta_k - \frac{1}{2}r_k^2 \cdot \Delta\theta_k \\ &= \frac{1}{2}[r_k + (r_k + \Delta r_k)]\Delta r_k \cdot \Delta\theta_k \\ &= \bar{r}_k \Delta r_k \cdot \Delta\theta_k\end{aligned}$$

在 $\Delta\sigma_k$ 内取点 $(\bar{r}_k, \bar{\theta}_k)$, 对应有

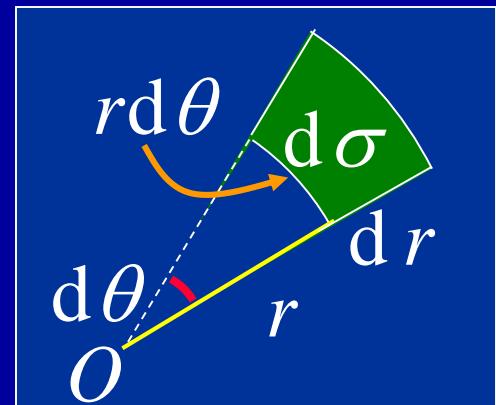
$$\xi_k = \bar{r}_k \cos \bar{\theta}_k, \quad \eta_k = \bar{r}_k \sin \bar{\theta}_k$$



$$\lim_{\lambda \rightarrow 0} \sum_{k=1}^n f(\xi_k, \eta_k) \Delta \sigma_k$$

$$= \lim_{\lambda \rightarrow 0} \sum_{k=1}^n f(\sqrt{r_k} \cos \theta_k, \sqrt{r_k} \sin \theta_k) \sqrt{r_k} \Delta r_k \Delta \theta_k$$

即 $\iint_D f(x, y) d\sigma = \iint_D f(r \cos \theta, r \sin \theta) r dr d\theta$



设 $D: \begin{cases} \varphi_1(\theta) \leq r \leq \varphi_2(\theta), \\ \alpha \leq \theta \leq \beta \end{cases}$, 则

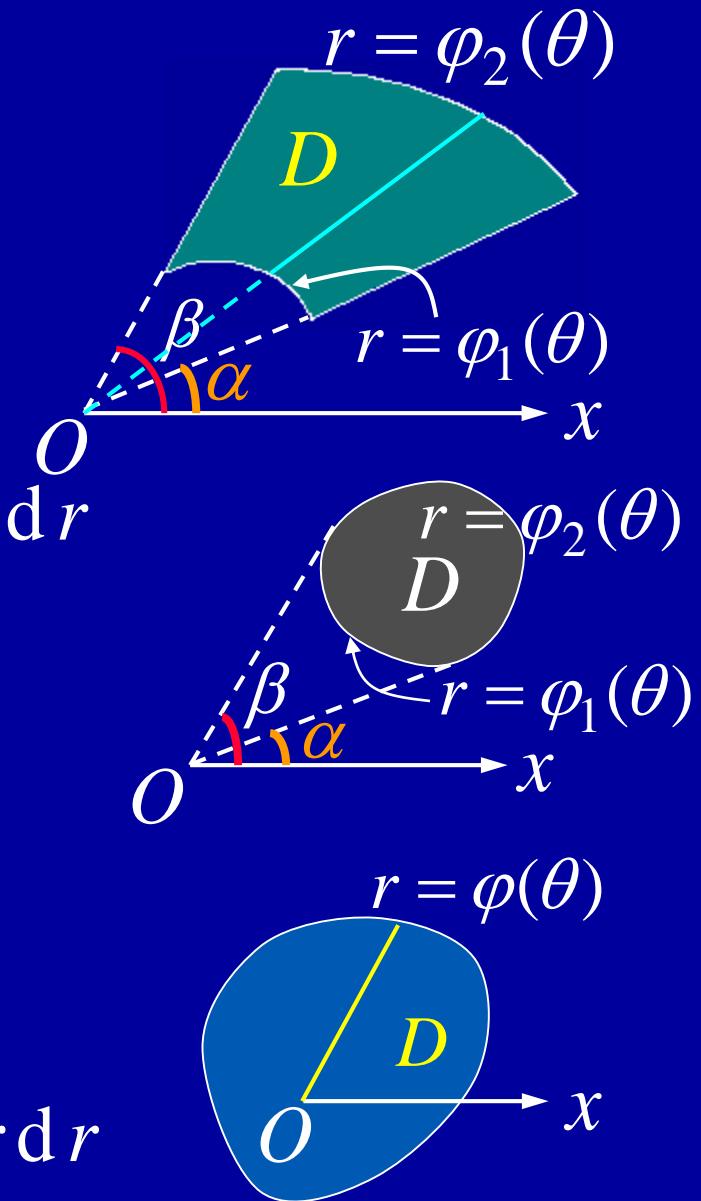
$$\iint_D f(r \cos \theta, r \sin \theta) r dr d\theta$$

$$= \int_{\alpha}^{\beta} d\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(r \cos \theta, r \sin \theta) r dr$$

特别, 对 $D: \begin{cases} 0 \leq r \leq \varphi(\theta) \\ 0 \leq \theta \leq 2\pi \end{cases}$

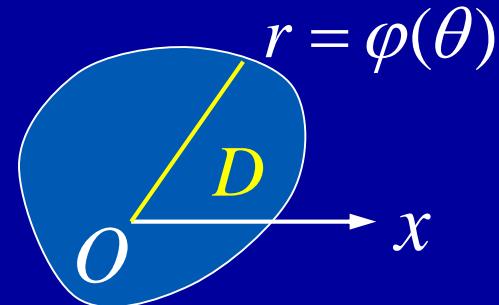
$$\iint_D f(r \cos \theta, r \sin \theta) r dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\varphi(\theta)} f(r \cos \theta, r \sin \theta) r dr$$

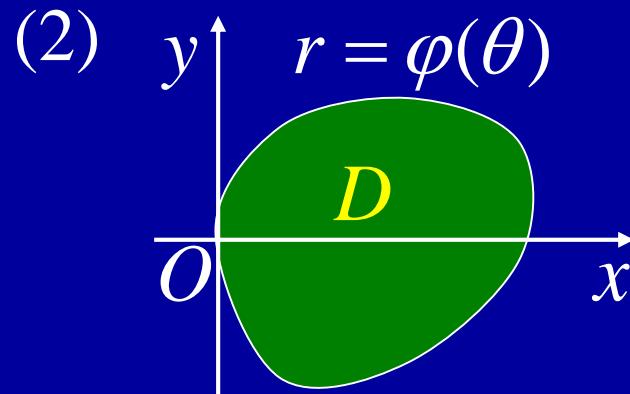
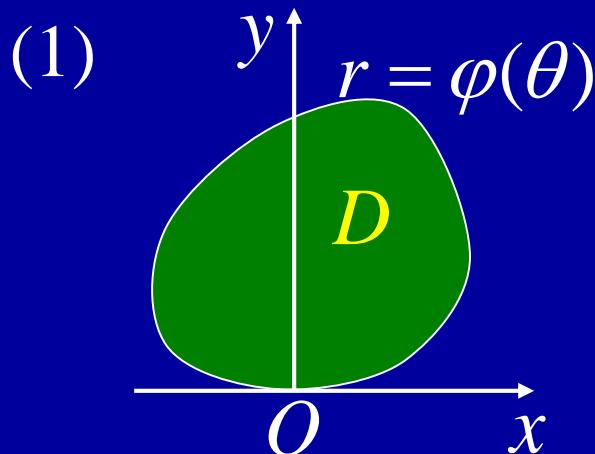


此时若 $f \equiv 1$ 则可求得 D 的面积

$$\sigma = \iint_D d\sigma = \frac{1}{2} \int_0^{2\pi} \varphi^2(\theta) d\theta$$



思考：下列各图中域 D 分别与 x, y 轴相切于原点，试问 θ 的变化范围是什么？



答：(1) $0 \leq \theta \leq \pi$;

(2) $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$



例6. 计算 $\iint_D e^{-x^2-y^2} dx dy$, 其中 $D: x^2 + y^2 \leq a^2$.

解: 在极坐标系下 $D: \begin{cases} 0 \leq r \leq a \\ 0 \leq \theta \leq 2\pi \end{cases}$, 故

$$\begin{aligned} \text{原式} &= \iint_D e^{-r^2} r dr d\theta = \int_0^{2\pi} d\theta \int_0^a r e^{-r^2} dr \\ &= 2\pi \left[\frac{-1}{2} e^{-r^2} \right]_0^a = \pi(1 - e^{-a^2}) \end{aligned}$$

由于 e^{-x^2} 的原函数不是初等函数, 故本题无法用直角坐标计算.



注:利用上题可得一个在概率论与数理统计及工程上非常有用的反常积分公式

$$\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \quad ①$$

事实上,

$$\begin{aligned} \iint_{\mathbb{R}^2} e^{-x^2-y^2} dx dy &= \int_{-\infty}^{+\infty} e^{-x^2} dx \int_{-\infty}^{+\infty} e^{-y^2} dy \\ &= 4 \left(\int_0^{+\infty} e^{-x^2} dx \right)^2 \end{aligned}$$

又

$$\begin{aligned} \iint_{\mathbb{R}^2} e^{-x^2-y^2} dx dy &= \lim_{a \rightarrow +\infty} \iint_{x^2+y^2 \leq a^2} e^{-x^2} dx \\ &= \lim_{a \rightarrow +\infty} \pi (1 - e^{-a^2}) = \pi \end{aligned}$$

故①式成立.

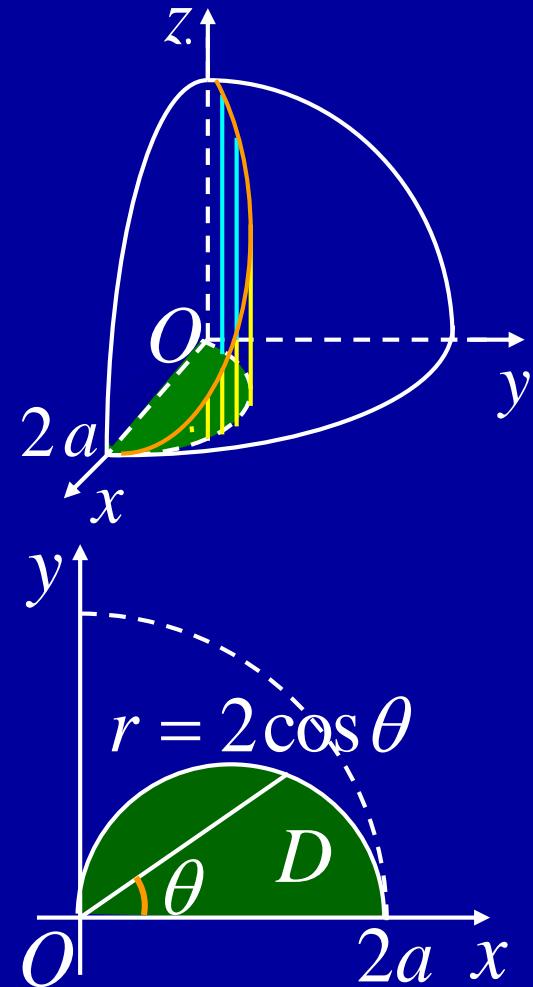


例7. 求球体 $x^2 + y^2 + z^2 \leq 4a^2$ 被圆柱面 $x^2 + y^2 = 2ax$ ($a > 0$) 所截得的(含在柱面内的)立体的体积.

解: 设 $D: 0 \leq r \leq 2a \cos \theta, 0 \leq \theta \leq \frac{\pi}{2}$

由对称性可知

$$\begin{aligned} V &= 4 \iint_D \sqrt{4a^2 - r^2} r dr d\theta \\ &= 4 \int_0^{\pi/2} d\theta \int_0^{2a \cos \theta} \sqrt{4a^2 - r^2} r dr \\ &= \frac{32}{3} a^3 \int_0^{\pi/2} (1 - \sin^3 \theta) d\theta \\ &= \frac{32}{3} a^3 \left(\frac{\pi}{2} - \frac{2}{3} \right) \end{aligned}$$



*三、二重积分换元法

定理: 设 $f(x, y)$ 在闭域 D 上连续, 变换:

$$T : \begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases} \quad (u, v) \in D' \rightarrow D$$

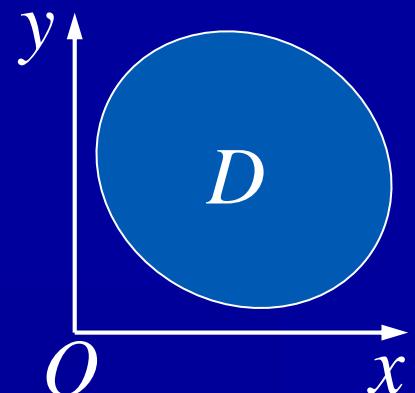
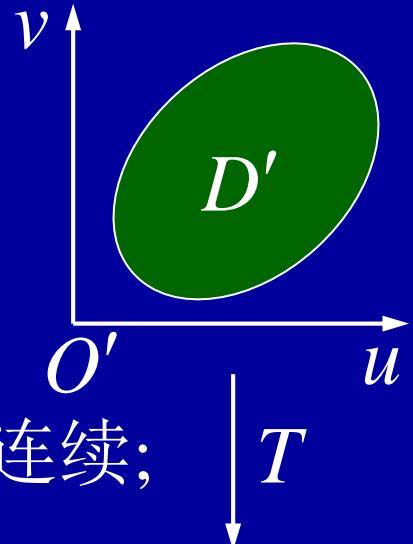
满足 (1) $x(u, v), y(u, v)$ 在 D' 上一阶偏导数连续;

(2) 在 D' 上 雅可比行列式

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} \neq 0;$$

(3) 变换 $T : D' \rightarrow D$ 是一一对应的,

$$\text{则 } \iint_D f(x, y) dx dy = \iint_{D'} f(x(u, v), y(u, v)) |J(u, v)| du dv$$



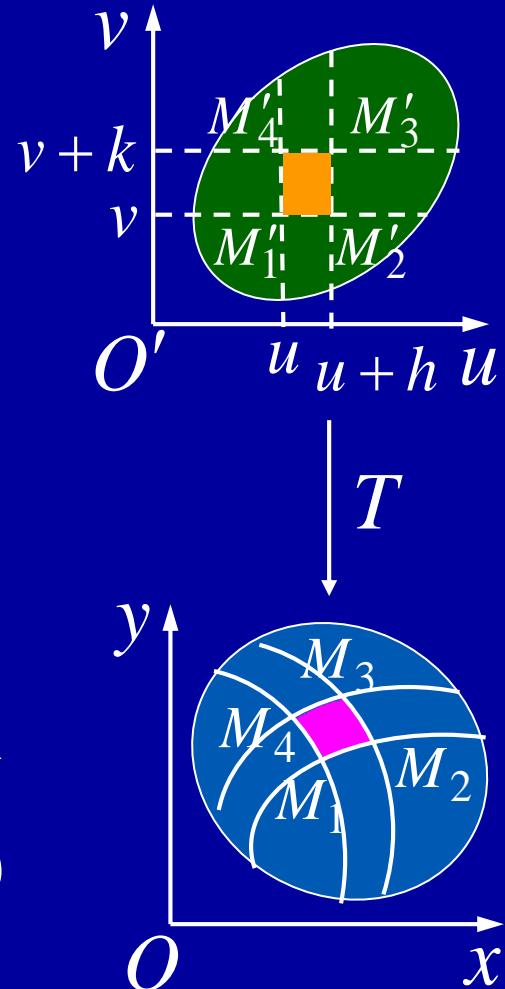
证: 根据定理条件可知变换 T 可逆.
在 $uO'v$ 坐标面上, 用平行于坐标轴的
直线分割区域 D' , 任取其中一个小矩
形, 其顶点为

$$\begin{array}{ll} M'_1(u, v), & M'_2(u+h, v), \\ M'_3(u+h, v+k), & M'_4(u, v+k). \end{array}$$

通过变换 T , 在 xOy 面上得到一个四边
形, 其对应顶点为 $M_i(x_i, y_i)$ ($i = 1, 2, 3, 4$)

令 $\rho = \sqrt{h^2 + k^2}$, 则

$$x_2 - x_1 = x(u+h, v) - x(u, v) = \frac{\partial x}{\partial u} \Big|_{(u, v)} h + o(\rho)$$



$$x_4 - x_1 = x(u, v+k) - x(u, v) = \frac{\partial x}{\partial v} \Big|_{(u, v)} k + o(\rho)$$

同理得 $y_2 - y_1 = \frac{\partial y}{\partial u} \Big|_{(u, v)} h + o(\rho)$

$$y_4 - y_1 = \frac{\partial y}{\partial v} \Big|_{(u, v)} k + o(\rho)$$

当 h, k 充分小时, 曲边四边形 $M_1M_2M_3M_4$ 近似于平行四边形, 故其面积近似为

$$\begin{aligned}\Delta\sigma &\approx \left| \overrightarrow{M_1M_2} \times \overrightarrow{M_1M_4} \right| = \left| \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_4 - x_1 & y_4 - y_1 \end{vmatrix} \right| \\ &\approx \left| \begin{vmatrix} \frac{\partial x}{\partial u} h & \frac{\partial y}{\partial u} k \\ \frac{\partial x}{\partial v} h & \frac{\partial y}{\partial v} k \end{vmatrix} \right| = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| hk = |J(u, v)| hk\end{aligned}$$



因此面积元素的关系为 $d\sigma = |J(u, v)| du dv$

从而得二重积分的换元公式:

$$\begin{aligned} & \iint_D f(x, y) dx dy \\ &= \iint_{D'} f(x(u, v), y(u, v)) |J(u, v)| du dv \end{aligned}$$

例如, 直角坐标转化为极坐标时, $x = r \cos \theta$, $y = r \sin \theta$

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & -r \cos \theta \end{vmatrix} = r$$

$$\begin{aligned} & \therefore \iint_D f(x, y) dx dy \\ &= \iint_{D'} f(r \cos \theta, r \sin \theta) r dr d\theta \end{aligned}$$



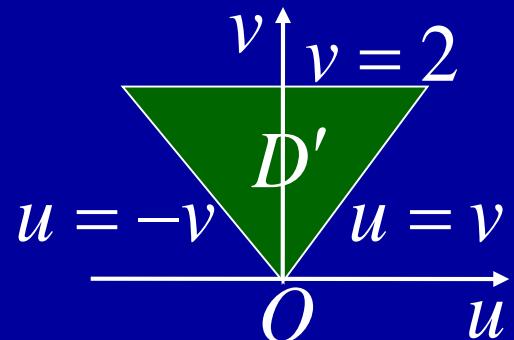
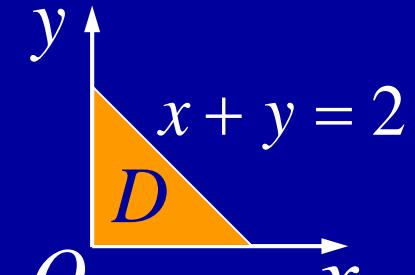
例8. 计算 $\iint_D e^{\frac{y-x}{y+x}} dx dy$, 其中 D 是 x 轴 y 轴和直线 $x + y = 2$ 所围成的闭域.

解: 令 $u = y - x$, $v = y + x$, 则

$$x = \frac{v-u}{2}, y = \frac{v+u}{2} \quad (D' \rightarrow D)$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{-1}{2}$$

$$\begin{aligned} \therefore \iint_D e^{\frac{y-x}{y+x}} dx dy &= \iint_{D'} e^{\frac{u}{v}} \left| \frac{-1}{2} \right| du dv = \frac{1}{2} \int_0^2 dv \int_{-v}^v e^{\frac{u}{v}} du \\ &= \frac{1}{2} \int_0^2 (e - e^{-1}) v dv = e - e^{-1} \end{aligned}$$



例9. 计算由 $y^2 = px$, $y^2 = qx$, $x^2 = ay$, $x^2 = by$ ($0 < p < q$, $0 < a < b$) 所围成的闭区域 D 的面积 S .

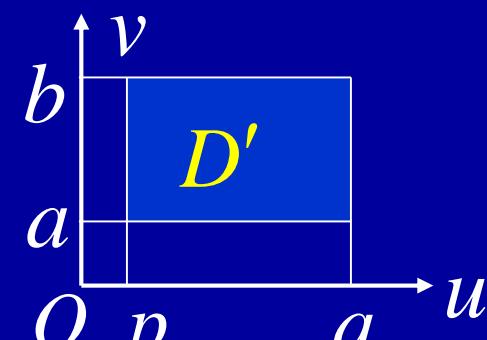
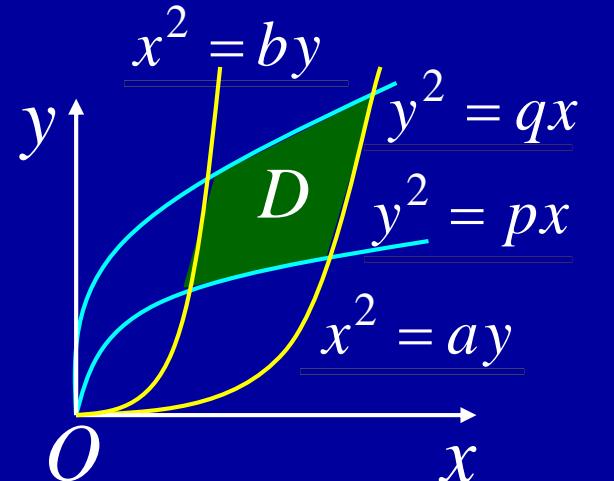
解: 令 $u = \frac{y^2}{x}$, $v = \frac{x^2}{y}$, 则

$$D': \begin{cases} p \leq u \leq q \\ a \leq v \leq b \end{cases} \longrightarrow D$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}} = -\frac{1}{3}$$

$$\therefore S = \iint_D dxdy$$

$$= \iint_{D'} |J| du dv = \frac{1}{3} \int_p^q du \int_a^b dv = \frac{1}{3} (q-p)(b-a)$$



例10. 试计算椭球体 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$ 的体积V.

解: 取 $D: \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$, 由对称性

$$V = 2 \iint_D z \, dx \, dy = 2c \iint_D \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \, dx \, dy$$

令 $x = ar \cos \theta$, $y = br \sin \theta$, 则 D 的原象为

$$D': r \leq 1, 0 \leq \theta \leq 2\pi$$

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} a \cos \theta & -a r \sin \theta \\ b \sin \theta & b r \cos \theta \end{vmatrix} = abr$$

$$\begin{aligned} \therefore V &= 2c \iint_D \sqrt{1 - r^2} abr \, dr \, d\theta \\ &= 2abc \int_0^{2\pi} d\theta \int_0^1 \sqrt{1 - r^2} r \, dr = \frac{4}{3}\pi abc \end{aligned}$$



内容小结

(1) 二重积分化为二次积分的方法

直角坐标系情形：

- 若积分区域为

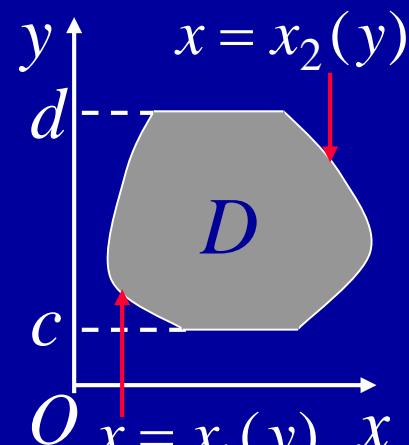
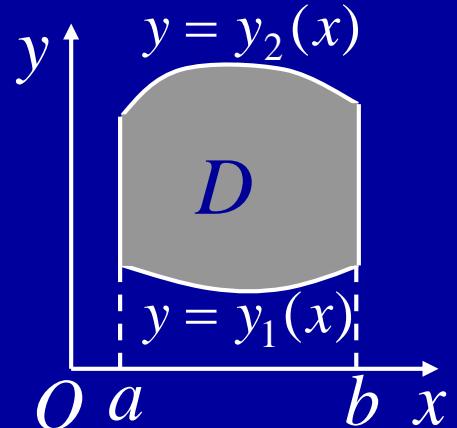
$$D = \{(x, y) \mid a \leq x \leq b, y_1(x) \leq y \leq y_2(x)\}$$

则 $\iint_D f(x, y) d\sigma = \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x, y) dy$

- 若积分区域为

$$D = \{(x, y) \mid c \leq y \leq d, x_1(y) \leq x \leq x_2(y)\}$$

则 $\iint_D f(x, y) d\sigma = \int_c^d dy \int_{x_1(y)}^{x_2(y)} f(x, y) dx$

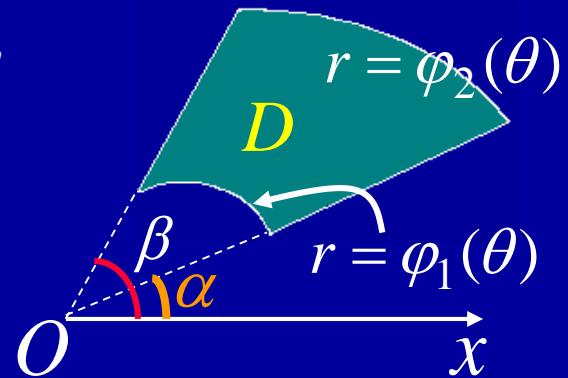


极坐标系情形: 若积分区域为

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, \varphi_1(\theta) \leq r \leq \varphi_2(\theta)\}$$

则 $\iint_D f(x, y) d\sigma = \iint_D f(r \cos \theta, r \sin \theta) r dr d\theta$

$$= \int_{\alpha}^{\beta} d\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(r \cos \theta, r \sin \theta) r dr$$



*(2) 一般换元公式

在变换 $\begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases}$ 下

$$(x, y) \in D \longleftrightarrow (u, v) \in D', \text{ 且 } J = \frac{\partial(x, y)}{\partial(u, v)} \neq 0$$

则 $\iint_D f(x, y) d\sigma = \iint_{D'} f[x(u, v), y(u, v)] |J| du dv$



(3) 计算步骤及注意事项

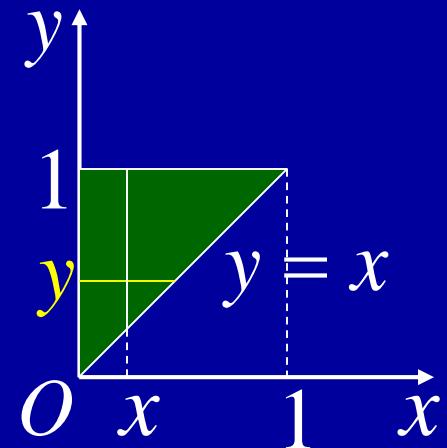
- 画出积分域
- 选择坐标系 $\left\{ \begin{array}{l} \text{域边界应尽量多为坐标线} \\ \text{被积函数关于坐标变量易分离} \end{array} \right.$
- 确定积分序 $\left\{ \begin{array}{l} \text{积分域分块要少} \\ \text{累次积分好算为妙} \end{array} \right.$
- 写出积分限 $\left\{ \begin{array}{l} \text{图示法} \quad (\text{先积一条线, 后扫积分域}) \\ \text{不等式} \end{array} \right.$
- 计算要简便 $\left\{ \begin{array}{l} \text{充分利用对称性} \\ \text{应用换元公式} \end{array} \right.$



思考与练习

1. 设 $f(x) \in C[0,1]$, 且 $\int_0^1 f(x)dx = A$,
求 $I = \int_0^1 dx \int_x^1 f(x)f(y)dy$.

提示: 交换积分顺序后, x, y 互换



$$I = \int_0^1 dy \int_0^y f(x)f(y)dx = \int_0^1 dx \int_0^x f(x)f(y)dy$$

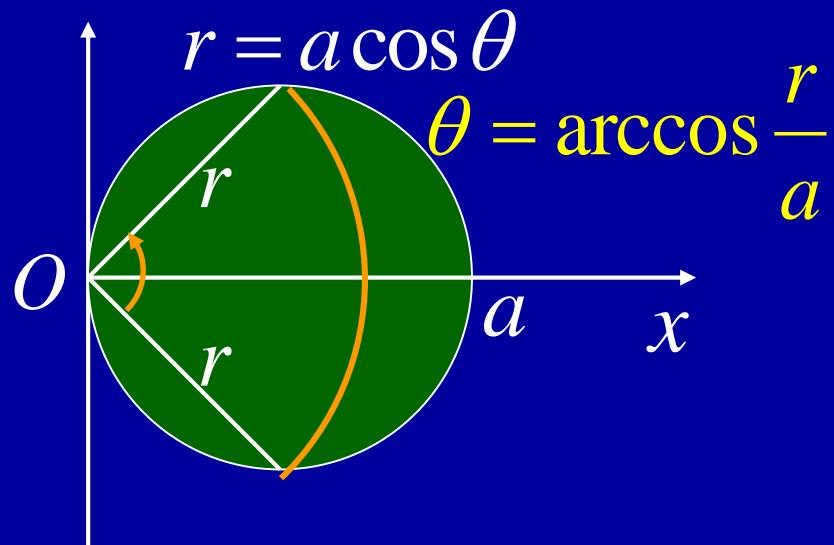
$$\therefore 2I = \int_0^1 dx \int_x^1 f(x)f(y)dy + \int_0^1 dx \int_0^x f(x)f(y)dy$$

$$= \int_0^1 dx \int_0^1 f(x)f(y)dy = \int_0^1 f(x)dx \int_0^1 f(y)dy = A^2$$



2. 交换积分顺序 $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{a \cos \theta} f(r, \theta) dr \quad (a > 0)$

提示：积分域如图



$$I = \int_0^a dr \int_{-\arccos \frac{r}{a}}^{\arccos \frac{r}{a}} f(r, \theta) d\theta$$



备用题 1. 给定 $I = \int_0^{2a} dx \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x, y) dy$ ($a > 0$)

改变积分的次序.

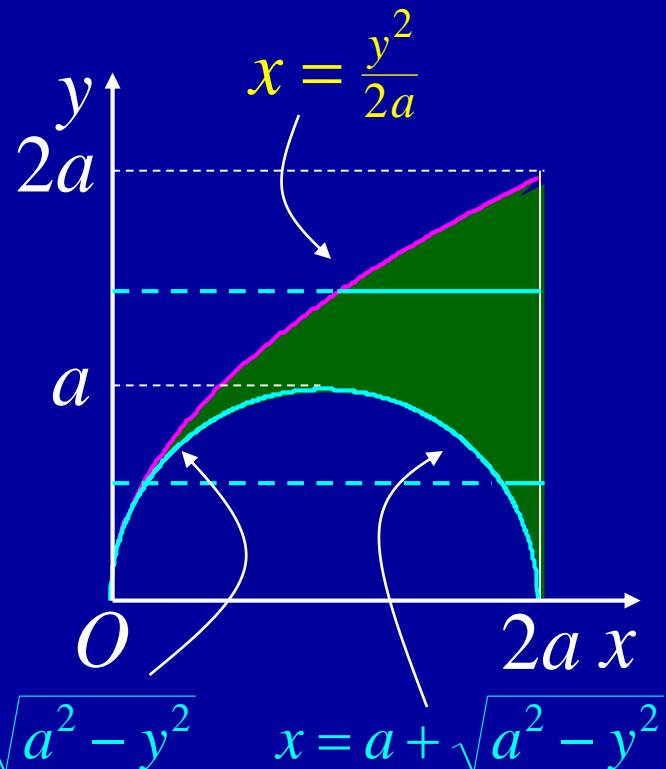
解: $y = \sqrt{2ax} \Rightarrow x = \frac{y^2}{2a}$

$$y = \sqrt{2ax - x^2}$$

$$\Rightarrow x = a \pm \sqrt{a^2 - y^2}$$

$$\text{原式} = \int_0^a dy \int_{\frac{y^2}{2a}}^{a - \sqrt{a^2 - y^2}} f(x, y) dx$$

$$x = a - \sqrt{a^2 - y^2}$$



$$+ \int_0^a dy \int_{a + \sqrt{a^2 - y^2}}^{2a} f(x, y) dx + \int_a^{2a} dy \int_{\frac{y^2}{2a}}^{2a} f(x, y) dx$$



2. 计算 $\iint_D (x^2 + y^2) dx dy$, 其中 D 为由圆 $x^2 + y^2 = 2y$, $x^2 + y^2 = 4y$ 及直线 $x - \sqrt{3}y = 0, y - \sqrt{3}x = 0$ 所围成的平面闭区域.

$$\text{解: } x^2 + y^2 = 2y \Rightarrow r = 2\sin\theta$$

$$x^2 + y^2 = 4y \Rightarrow r = 4\sin\theta$$

$$y - \sqrt{3}x = 0 \Rightarrow \theta_2 = \frac{\pi}{3}$$

$$x - \sqrt{3}y = 0 \Rightarrow \theta_1 = \frac{\pi}{6}$$

$$\therefore \iint_D (x^2 + y^2) dx dy = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta \int_{2\sin\theta}^{4\sin\theta} r^2 \cdot r dr = 15\left(\frac{\pi}{2} - \sqrt{3}\right)$$

