第二节

偏　导　数

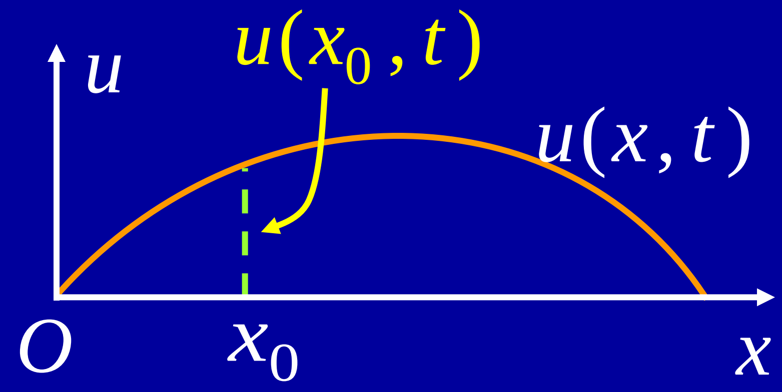
一、 偏导数概念及其计算

二、 高阶偏导数

HIGHER EDUCATION PRESS

一、 偏导数定义及其计算法

引例: 研究弦在点 x_0 处的振动速度与加速度，就是将振幅 $u(x,t)$ 中的 x 固定于 x_0 处，求 $u(x_0,t)$ 关于 t 的一阶导数与二阶导数.



定义1. 设函数 $z=f(x,y)$在点$(x_0\,,y_0)$ 的某邻域内

极限 $$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

存在, 则称此极限为函数 $z=f(x,y)$ 在点(x_0,y_0)对x

的偏导数，记为 $\left.\dfrac{\partial z}{\partial x}\right|_{(x_0,y_0)}$; $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}$; $\left.z_x\right|_{(x_0\,,y_0)}$;

$f_x(x_0,y_0)$; $f_1'(x_0,y_0)$.

注意: $$f_x(x_0,y_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

$$=\frac{\mathrm{d}}{\mathrm{d}x}f(x,y_0)\Big|_{x=x_0}$$

同样可定义对 y 的偏导数

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

$$= \frac{\mathrm{d}}{\mathrm{d}y} f(x_0, y)\Big|_{y=y_0}$$

若函数 $z = f(x, y)$ 在域 D 内每一点 (x, y) 处对 x 或 y 偏导数存在，则该偏导数称为偏导函数, 也简称为**偏导数**，记为

$$\frac{\partial z}{\partial x}, \ \frac{\partial f}{\partial x}, \ z_x, \ f_x(x, y), \ f_1'(x, y)$$

$$\frac{\partial z}{\partial y}, \ \frac{\partial f}{\partial y}, \ z_y, \ f_y(x, y), \ f_2'(x, y)$$

偏导数的概念可以推广到二元以上的函数 .

例如, 三元函数 $u = f(x, y, z)$ 在点 (x, y, z) 处对 x 的偏导数定义为

$$f_x(x,y,z) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x, y, z)-f(x, y, z)}{\Delta x}$$

$$f_y(x,y,z) = ?$$

$$f_z(x,y,z) = ?$$

(请自己写出)

二元函数偏导数的几何意义:

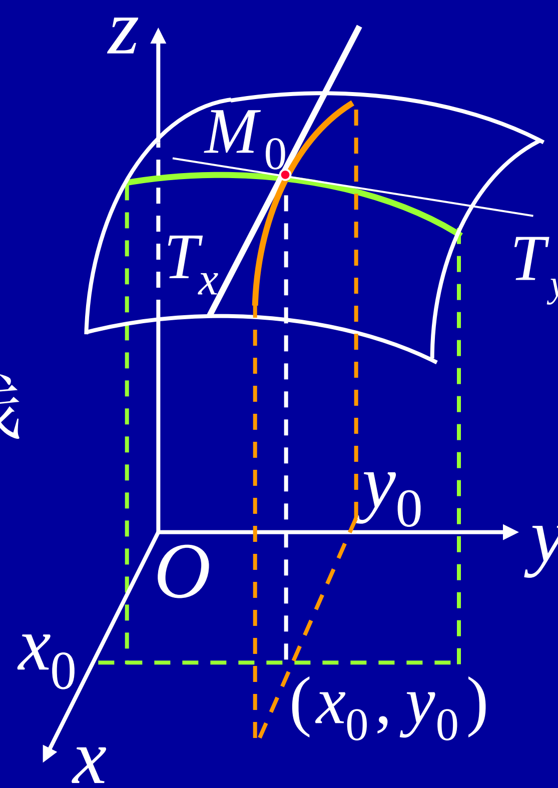
$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}} = \frac{\mathrm{d}}{\mathrm{d}x} f(x, y_0)\Big|_{x=x_0}$$

是曲线 $\begin{cases} z = f(x,y) \\ y = y_0 \end{cases}$ 在点 M_0 处的切线

M_0T_x 对 x 轴的斜率.

$$\left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_0\\ y=y_0}} = \frac{\mathrm{d}}{\mathrm{d}y} f(x_0, y)\Big|_{y=y_0}$$

是曲线 $\begin{cases} z = f(x,y) \\ x = x_0 \end{cases}$ 在点 M_0 处的切线 M_0T_y 对 y 轴的斜率.



注意: 函数在某点各偏导数都存在,

但在该点**不一定连续**.

例如, $$z = f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$$

显然 $$f_x(0, 0) = \frac{\mathrm{d}}{\mathrm{d}x} f(x, 0)\Big|_{x=0} = 0$$

$$f_y(0, 0) = \frac{\mathrm{d}}{\mathrm{d}y} f(0, y)\Big|_{y=0} = 0$$

在上节已证 $f(x, y)$ 在点$(0, 0)$并不连续!

例1. 求 $z = x^2 + 3xy + y^2$ 在点(1 , 2) 处的偏导数.

解法1 $\frac{\partial z}{\partial x} = 2x + 3y, \quad \frac{\partial z}{\partial y} = 3x + 2y$

$\boxed{\text{先求后代}}$

$$\therefore \quad \left.\frac{\partial z}{\partial x}\right|_{(1,2)} = 2\cdot 1 + 3\cdot 2 = 8, \quad \left.\frac{\partial z}{\partial y}\right|_{(1,2)} = 3\cdot 1 + 2\cdot 2 = 7$$

解法2 $z\big|_{y=2} = x^2 + 6x + 4$

$\boxed{\text{先代后求}}$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,\,2)} = (2x+6)\Big|_{x=1} = 8$$

$$z\big|_{x=1} = 1 + 3y + y^2$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,\,2)} = (3+2y)\Big|_{y=2} = 7$$

例2. 设 $z = x^y\ (x>0, 且\ x\neq 1)$, 求证

$$\frac{x}{y}\frac{\partial z}{\partial x}+\frac{1}{\ln x}\frac{\partial z}{\partial y}=2z$$

证: $\because \quad \dfrac{\partial z}{\partial x}=yx^{y-1}, \quad \dfrac{\partial z}{\partial y}=x^y\ln x$

$$\therefore \quad \frac{x}{y}\frac{\partial z}{\partial x}+\frac{1}{\ln x}\frac{\partial z}{\partial y}=x^y+x^y=2z$$

例3. 求 $r=\sqrt{x^2+y^2+z^2}$ 在除了原点之外的偏导数 .

解: $\dfrac{\partial r}{\partial x}=\dfrac{2x}{2\sqrt{x^2+y^2+z^2}}=\dfrac{x}{r}$

$$\frac{\partial r}{\partial y}=\frac{y}{r}, \qquad \frac{\partial r}{\partial z}=\frac{z}{r}$$

例4. 已知理想气体的状态方程 $pV = RT$ (R 为常数)，

求证: $\dfrac{\partial p}{\partial V}\cdot\dfrac{\partial V}{\partial T}\cdot\dfrac{\partial T}{\partial p} = -1$

证: $p = \dfrac{RT}{V}$, $\quad \dfrac{\partial p}{\partial V} = -\dfrac{RT}{V^2}$

$V = \dfrac{RT}{p}$, $\quad \dfrac{\partial V}{\partial T} = \dfrac{R}{p}$

$T = \dfrac{pV}{R}$, $\quad \dfrac{\partial T}{\partial p} = \dfrac{V}{R}$

$$\therefore \frac{\partial p}{\partial V}\cdot\frac{\partial V}{\partial T}\cdot\frac{\partial T}{\partial p} = -\frac{RT}{pV} = -1$$

说明: 此例表明，偏导数记号是一个整体记号，不能看作分子与分母的商！

二、高阶偏导数

设 $z=f(x,y)$在域 D 内存在连续的偏导数

$$\frac{\partial z}{\partial x}=f_x(x,y)\,,\qquad \frac{\partial z}{\partial y}=f_y(x,y)$$

若这两个偏导数仍存在偏导数，则称它们是$z=f(x,y)$的二阶偏导数 . 按求导顺序不同, 有下列四个二阶偏导数:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=f_{xx}(x,y);\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}=f_{xy}(x,y)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y\partial x}=f_{yx}(x,y);\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=f_{yy}(x,y)$$

类似可以定义更高阶的偏导数.

例如， $z = f(x, y)$ 关于 x 的三阶偏导数为

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^3 z}{\partial x^3}$$

$z = f(x, y)$ 关于 x 的 $n-1$ 阶偏导数，再关于 y 的一阶偏导数为

$$\frac{\partial}{\partial y}\left(\frac{\partial^{n-1} z}{\partial x^{n-1}}\right) = \frac{\partial^n z}{\partial x^{n-1}\partial y}$$

例5. 求函数 $z=\mathrm{e}^{x+2y}$ 的二阶偏导数及 $\dfrac{\partial^3 z}{\partial y\partial x^2}$.

解：

$$\frac{\partial z}{\partial x}=\mathrm{e}^{x+2y} \qquad \frac{\partial z}{\partial y}=2\mathrm{e}^{x+2y}$$

$$\frac{\partial^2 z}{\partial x^2}=\mathrm{e}^{x+2y} \qquad \frac{\partial^2 z}{\partial x\partial y}=2\mathrm{e}^{x+2y}$$

$$\frac{\partial^2 z}{\partial y\partial x}=2\mathrm{e}^{x+2y} \qquad \frac{\partial^2 z}{\partial y^2}=4\mathrm{e}^{x+2y}$$

$$\frac{\partial^3 z}{\partial y\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y\partial x}\right)=2\mathrm{e}^{x+2y}$$

注意: 此处 $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial^2 z}{\partial y\partial x}$, 但这一结论并不总成立.

例如， $f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$

$$f_x(x,y)=\begin{cases} y\dfrac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_y(x,y)=\begin{cases} x\dfrac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$\left.\begin{aligned} f_{xy}(0,0)&=\lim_{\Delta y\to 0}\frac{f_x(0,\Delta y)-f_x(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{-\Delta y}{\Delta y}=-1 \\ f_{yx}(0,0)&=\lim_{\Delta x\to 0}\frac{f_y(\Delta x,0)-f_y(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1 \end{aligned}\right\}\text{二者不等}$$

定理. 若 $f_{xy}(x,y)$ 和 $f_{yx}(x,y)$ 都在点 (x_0, y_0) 连续, 则

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$ (证明略)

本定理对 n 元函数的高阶混合导数也成立.

例如, 对三元函数 $u = f(x, y, z)$, 当三阶混合偏导数在点 (x, y, z) **连续**时, 有

$$f_{xyz}(x,y,z) = f_{yzx}(x,y,z) = f_{zxy}(x,y,z)$$
$$= f_{xzy}(x,y,z) = f_{yxz}(x,y,z) = f_{zyx}(x,y,z)$$

说明: 因为初等函数的偏导数仍为初等函数, 而初等函数在其定义区域内是连续的, 故求初等函数的高阶导数可以选择方便的求导顺序.

例6. 证明函数 $u=\frac{1}{r}, r=\sqrt{x^2+y^2+z^2}$ 满足拉普拉斯方程 $\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$

证: $\frac{\partial u}{\partial x}=-\frac{1}{r^2}\frac{\partial r}{\partial x}=-\frac{1}{r^2}\cdot\frac{x}{r}$

$$\frac{\partial^2 u}{\partial x^2}=-\frac{1}{r^3}+\frac{3x}{r^4}\cdot\frac{\partial r}{\partial x}=-\frac{1}{r^3}+\frac{3x^2}{r^5}$$

利用对称性，有 $\frac{\partial^2 u}{\partial y^2}=-\frac{1}{r^3}+\frac{3y^2}{r^5}$， $\frac{\partial^2 u}{\partial z^2}=-\frac{1}{r^3}+\frac{3z^2}{r^5}$

$$\therefore\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-\frac{3}{r^3}+\frac{3(\underbrace{\boxed{x^2+y^2+z^2}}_{=r^2})}{r^5}=0$$

内容小结

1. 偏导数的概念及有关结论

- 定义; 记号; 几何意义
- 函数在一点偏导数存在 $\not\longrightarrow$ 函数在此点连续
- 混合偏导数连续 $\longrightarrow$ 与求导顺序无关

2. 偏导数的计算方法

- 求一点处偏导数的方法 $\begin{cases} \text{先代后求} \\ \text{先求后代} \\ \text{利用定义} \end{cases}$
- 求高阶偏导数的方法 —— 逐次求导法

(与求导顺序无关时, 应选择方便的求导顺序)

备用题 设 $z=f(u)$，方程 $u=\varphi(u)+\int_y^x p(t)\,\mathrm{d}t$ 确定 u 是 x , y 的函数，其中 $f(u),\varphi(u)$ 可微，$p(t),\varphi'(u)$ 连续，且 $\varphi'(u)\neq 1$，求 $p(y)\dfrac{\partial z}{\partial x}+p(x)\dfrac{\partial z}{\partial y}$.

解: $\dfrac{\partial z}{\partial x}=f'(u)\dfrac{\partial u}{\partial x}$, $\quad \dfrac{\partial z}{\partial y}=f'(u)\dfrac{\partial u}{\partial y}$

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\varphi'(u)\frac{\partial u}{\partial x}+p(x)\\ \frac{\partial u}{\partial y}&=\varphi'(u)\frac{\partial u}{\partial y}-p(y)\end{aligned}\right\}\Longrightarrow\left\{\begin{aligned}\frac{\partial u}{\partial x}&=\frac{p(x)}{1-\varphi'(u)}\\ \frac{\partial u}{\partial y}&=\frac{-p(y)}{1-\varphi'(u)}\end{aligned}\right.$$

$$\therefore\ p(y)\frac{\partial z}{\partial x}+p(x)\frac{\partial z}{\partial y}=f'(u)\left[\,p(y)\frac{\partial u}{\partial x}+p(x)\frac{\partial u}{\partial y}\,\right]=0$$