1 (text) Recurrence [10 points] Solve the following recurrence relation using repeated substitution.

$$T(n) = 3T(n/4) + 4n$$

Then solve it by using the Master Method, showing in detail which rule applies

Note: Show your work. You will get 4 points if you identify the pattern, 3 points if you do the proof work necessary to show what it resolves to, and 3 points for solving it using the master theorem.

$T(N) = 3T(n/4) + 4n \qquad T(n/4) = 3T(n/16) + 4(n/4)$

$= 3(3T(n/16) + 4(n/4)) + 4n$

$= 9T(n/16) + 7n \qquad T(n/16) = 3T(n/64) + 4(n/16)$

$= 9(3T(n/64) + 4(n/16)) + 7n$

$= 27T(n/64) + 9n/4 + 7n \qquad T(n/64) = 3T(n/256) + 4(n/64)$

$= 27(3T(n/256) + 4(n/64)) + 7n$

$= 81T(n/256) + 108(n/64) + 7n$

For every k repetitions the coefficient on $T(n/4)$ is 3^k.

The input n/4 is $n/4^k$.

The term added to the end starts at $4n + 3n$.

The next time is $\frac{9n}{4}$, after $\frac{27n}{16}$.

This added term becomes $4n + 3n + \frac{9n}{4} + \frac{27n}{16} + \ldots$

Each term in this sum is being multiplied by $3/4$.

So we can rewrite the added term as $4n(1 + 3/4 + 9/16 + \frac{27}{64} + \ldots)$

We can see that this a geometric series.

To solve the geometric series we know the formula

$\frac{a}{1-r}$ where a is the first term and r is the common multiple.

$a = 4n$ and $r = 3/4$

So the pattern is $3^k T(n/4^k) + 4n(1 + \frac{3}{4} + 9/16 + \frac{27}{64} + \ldots)$

So the pattern is $3^k T(n/4^k) + 4n(1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+\ldots)$

Solving for $4n(1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+\ldots)$

We use the geometric sum formula.

$\frac{a}{1-r}$ where $a = 4n$ and $r = 4/3$.

$\frac{a}{1-r} = \frac{4n}{1-3/4}$ by substitution

$\frac{4n}{1-3/4} = \frac{4n}{1/4} = 4n \cdot \frac{4}{1} = 16n$

Now the pattern is $3^k T(n/4^k) + 16n$

$3^k T(n/4^k)$ can be simplified to $n^{\log_4 3}$ because n is divided by 4 k times and multiplied by 3 k times.

Now the pattern is $n^{\log_4 3} + 16n$.

The slowest growing term is $n^{\log_4(3)}$ so the recurrence solves to $\Theta(n)$.

Using the master theorem,

$3T(n/4) + 4n$

$a = 3$ $b = 4$ $d = 1$

$a < b^d \Rightarrow 3 < 4^1$

$a < b^d$ is the case w the recurrence is $\Theta(n)$.

2 (text) Master Theorem [20 points] Apply the Master method to solve each of the following recurrences, or state that the Master method does not apply. Justify your answers. Note that the Master method covers all the cases

a. $T(n) = 3T(\frac{n}{5}) + n^2$
b. $T(n) = 4T(\frac{n}{3}) + 7n$
c. $T(n) = 5T(\frac{n}{4}) + 10$
d. $T(n) = 9T(\frac{n}{3}) + n^4$
e. $T(n) = 6T(\frac{n}{8}) + n^3$

Let $a \geq 1$ and $b \geq 1$ be constants.
$T(n) = aT(n/b) + f(n)$
If $f(n) = \theta(n^d)$, where $d \geq 0$ then
$T(n) = \theta(n^d)$ if $a < b^d$
$T(n) = \theta(n^d \log n)$ if $a = b^d$
$T(n) = \theta(n^{\log_b a})$ if $a > b^d$

a) $T(n) = 3T(n/5) + n^2$

$a = 3 \quad b = 5 \quad d = 2 \quad a, b, d \geq 0 \quad 3 < 5^2$

a is less than 5^2 so the recurrence is $\theta(n^2)$.

b) $4T(n/3) + 7n$

$a = 4 \quad b = 3 \quad d = 1 \quad a, b, d \geq 0 \quad 4 > 3^1$

a is greater than 3^1 so the recurrence is $\theta(n^{\log_3 4})$

c) $5T(n/4) + 10$

$a = 5 \quad b = 4 \quad d = 1 \quad a, b, d \geq 0 \quad 5 > 4^1$

a is greater than 4^1 so the recurrence is $\theta(n^{\log_4 5})$

d) $9T(n/3) + n^4$

$a = 9 \quad b = 3 \quad d = 4 \quad a, b, d \geq 0 \quad 9 < 3^4$

a is less than 3^4 so the recurrence is $\theta(n^4)$.

e) $6T(n/8) + n^3$

$a = 6 \quad b = 8 \quad d = 3 \quad a, b, d \geq 0 \quad 6 < 8^3$

a is less than 8^3 so the recurrence is $\theta(n^3)$.

3 (text) Radix Sort [10 points] Lexicographical ordering means order of the dictionaries to sequences of ordered symbols therefore (a < b < c < d < e < f < … < m < n < o < … < y < z). The same logic applies to Uppercase letters.

Illustrate the operation of Radix-Sort on the following list of strings using lexicographic ordering.

CAP, COL, USD, SUN, JPY, VEE, ROW, JOB, COX, LOL, RAT, WOW, DOD, CAR, FIG, PIG, VIS, LOW, LOX, VEA, CAD, DOG, TSL

Starting from rightmost character.

VEA, JOB, USD, DOD, CAD, VEE, FIG, PIG, DOG, COL, LOL, TSL, SUN, CAP, CAR, VIS, RAT, ROW, WOW, LOW, COX, LOX, JPY

2nd character from the right.

CAD, CAP, CAR, RAT, VEA, VEE, PIG, VIS, JOB, DOD, DOG, COL, LOL, ROW, WOW, LOW, COX, LOX, JPY, USD, TSL, SUN

3rd character from the right.

CAD, CAP, CAR, COL, COX, DOD, DOG, FIG, JOB, JPY, LOL, LOW, LOX, PIG, RAT, ROW, SUN, TSL, USD, VEA, VEE, VIS, WOW

Step 1

Bucket

Rightmost Largest Letter: Y

A: VEA
B: JOB
C
D: USD, DOD, CAD
E: VEE
F:
G: FIG, PIG, DOG
H
I
J
K
L: COL, LOL, TSL
M
N: SUN
O
P: CAP
Q
R: CAR
S: VIS
T: RAT
U:
V
W: ROW, LOW,
X: COX, LOX
Y: JPY

2nd character from the right.

CAD, CAP, CAR RAT, VEA, VEE, PIG, VIS, JOB, DOD, DOG

COL, LOL, ROW, WOW, LOW, COX, LOX, JPY, USD, TSL, sun

Buckets for step 2: Largest letter value "U"

A: CAD, CAP, CAP, RAT
B
C
D
E: VEA, VEE
F
G
H
I: PIG, VIS
J
K
L
M
N
O: JOB, DOD, DOG, COL, LOL, ROW, WOW, LOW, COX, LOX
P: JPY
Q
R
S: USD, TSL
T
U: sun

3rd character from the right.

CAD, CAP, CAR, COL, COX, DOD, DOG, FIG, JOB, JPY, LOL, LOW, LOX, PIG, RAT, ROW, Sun, TSL, USD, VEA, VEE, VIS WOW

Bucket for Step 3: Larges letter Value: "W"

A:
B
C: CAD, CAP, CAR, COL, COX,
D: DOD, DOG.
E
F FIG
G
H
I
J JOB, JPY
K
L LOL, LOW, LOX,
M
N
O
P PIG
Q
R RAT, ROW
S Sun
T TSL
U USD
V VEA, VEE, VIS
W WOW

Made with Goodnotes

4 (text) Double Hashing [15 points] Consider a hash table consisting of M = 13 slots, and suppose nonnegative integer key values are hashed into the table using the hash function h1() and that collisions are resolved by using double hashing with the secondary hash function Reverse(value), which reverses the digits of v and returns that value; for example, Reverse(3652) = 2563.

```
int h1 (int key) {
    int x = (key + 19) * (key + 11);
    x = x / 15;
    x = x + key;
    x = x % M;
    return x;
```

Add the following items [25, 14, 9, 7, 5, 3, 0, 21, 6, 33, 25, 42, 24, 107] to the HashTable in order.

For each key being inserted to the HashTable, show:
(1) The home slot (the initial hashed slot)
(2) The number of collisions and the probe sequence (if collisions occur)
(3) The final contents of the hash table

Hint: You will have to re-size and rehash once

Hashtable with Collisions:

Index	Element
0	25, 0
1	
2	21
3	33
4	14, 5
5	
6	107
7	9, 24
8	6
9	
10	3, 42
11	
12	7

Result of Hashing:

Items	Hash	
25	0	
14	4	
9	7	
7	12	
5	4	Collision 1
3	10	
0	0	Collision 2
21	2	
6	8	
33	3	
25	0	Already added
42	10	Collision 3
24	7	Collision 4
107	6	Collision 5

Collision 1: Probe = [4, 9]

Item: 5 Reverse(5) = 5

Home Slot: 4 Double Hash = $(4 + 1 \cdot 5) \% 13 = 9$

Collision 2: Probe = [0, 1]

Item: 0 Reverse(0) = $0 \to 1$

Home Slot: 0 Double Hash = $(0 + 1 \cdot 1) \% 13 = 1$

Collision 3: Probe = [10, 11]

Item: 42 Reverse(10) = 1

Home Slot: 10 Double Hash = $(10 + 1 \cdot 1) \% 13 = 11$

Collision 4: Probe: [7, 3, 33, 7, 6]

Item: 24 Reverse(24) = 42 $42 \% 13 = 3$

Home Slot: 7 Double Hash = $(7 + 1 \cdot 3) \% 13 = 10$ taken by 3

$= (10 + 2 \cdot 3) \% 13 = 3$ taken by 33

$= (3 + 3 \cdot 3) \% 13 = 12$ taken by 7

$= (7 + 4 \cdot 3) \% 13 = 6$

Made with Goodnotes

Collision 5:   Probe: [6, 5]

Item 107:   Reverse(107): 701   701 % 13 = 12

Home slot: 6   Double Hash: (6 + 1·(12)) % 13 = 5

Finished Hashtable:

Index	Element
0	25
1	0
2	21
3	33
4	14
5	107
6	24
7	9
8	6
9	5
10	3
11	42
12	7