Differentiation

Differentiation of Real Functions

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and x in $\mathbb{R}$ and let $h > 0$ be given. Then:

The Newton's difference quotient at x for f is given by:

$$\frac{\Delta f(x)}{\Delta (x)} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

Function f is differentiable at x iff the limit:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is a real number. In that case, this limit is the derivative of f at x and denoted as $f'(x)$ or equivalently $\frac{dy}{dx}$ when $y = f(x)$

Newton's difference quotient is a function of the value h for a given x in R

Example 35 (Derivative)
Recall the function **absolute value** $f(x) = |x|$. *We argue that this function has a derivative at all positive and all negative points* x. *For example, consider* $x = -2$. *If* h *represents any sequence that converges to* 0, *then* $x + h$ *represents a corresponding sequence that will consist of negative numbers only, from some point onward. But then Newton's difference quotient will be*

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{|x+h| - x}{h} = \lim_{h \to 0} \frac{(-x-h) - (-x)}{h} = -1 \qquad (9.3)$$

This reasoning remains valid for any negative number x. *A similar analysis shows that this function has derivative* 1 *at positive* x. *But what about* $x = 0$? *Here we see that it is important for the existence of a derivative that the limit is always the same value, regardless of which sequence for* h *is chosen:*

- *Let* h *converge to* 0 *from below, for example as a sequence* $(-1/n)_{n \geq 0}$. *Then the limit* $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ *computes to* -1.
- *Let* h *converge to* 0 *from above, for example as a sequence* $(1/n)_{n \geq 0}$. *Then the limit* $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ *computes to* 1.

Therefore, the limit is not consistently the same value and so the function $|x|$ *does* **not** *have a derivative at* $x = 0$.

The derivative, as a real number, is the slope of the tangent of the curve of function f at point x

Properties of Derivatives

Let $f, g: (a,b) \to \mathbb{R}$ be two functions.

1. *Polynomials have derivatives at all points, given by (9.4).*
2. *If f is differentiable at x, then f is also continuous at x.*

3. *If f is differentiable in (a,b) then $f'(x_0) = 0$ for any point x_0 at which f is maximum or minimum.*

4. *If f and g are differentiable at x, then the product function $f \cdot g$ defined by $(f \cdot g)(x) = f(x) \cdot g(x)$ is also differentiable at x and*

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) \tag{9.5}$$

5. *If g and f are, respectively, differentiable at x and $g(x)$, then the composition function $f \circ g$ defined by $(f \circ g)(x) = f(g(x))$ is also differentiable at x and*

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) \tag{9.6}$$

6. *Differentiation $f \mapsto f'$ is a* **linear function***: for all f and g that are differentiable at x, and for all a and b in $\mathbb{R}$ we have that the function $a \cdot f + b \cdot g$ defined by $(a \cdot f + b \cdot g)(x) = a \cdot f(x) + b \cdot g(x)$ is differentiable at x and*

$$(a \cdot f + b \cdot g)'(x) = a \cdot f'(x) + b \cdot g'(x) \tag{9.7}$$

Property 4 = Product Rule
Property 5 = Chain Rule
Property 6 = Linear Mapping

Mean Value Theorem and Taylor's Theorem

Exercise 24
*(**Rolle's Theorem**) If $f : [a,b] \to \mathbb{R}$ is continuous and $f : (a,b) \to \mathbb{R}$ is differentiable with $f(a) = f(b)$, then there exists $c \in (a,b)$ such that $f'(c) = 0$. (**Hint** We know that f being continuous in $[a,b]$ must have a minimum and a maximum. Consider the two possible cases: (i) Either the minimum or the maximum occurs in (a,b). (ii) Both of them occur at the end points a and b.)*

If $f : [a,b] \to R$ is continuous and $f : (a,b) \to R$ is differentiable, then there exists $c \in (a,b)$ such that:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Rewrite:

$$f(b) = f(a) + (b - a)f'(c)$$

If $f : (a,b) \to R$ is differentiable at $x_0 \in (a,b)$, then the tangent to the graph of f at the point $(x_0, f(x_0))$ is the line $y = f(x_0) + f'(x_0)(x - x_0)$

Thus if x is close to x_0 we expect $f(x) = f(x_0) + f'(x_0)(x - x_0) + E$ where the error E is small

Taylor's theorem formalises the above idea of approximation of a function in the neighbourhood of a point using the derivative and the higher order derivatives of the function at that point, and gives an expression of the error E

Denote the n^{th} derivative of f at a point x, if it exists, by $f^{(n)}(x_0)$

Theorem 33
If f is n times differentiable in (a,b) with $x_0 \in (a,b)$, then for any $x \in (a,b)$ we have:

$$f(x) = f(x_0) + \frac{1}{1!}(x - x_0)f'(x_0) + \frac{1}{2!}(x - x_0)^2 f''(x_0) + \cdots + \frac{1}{(n-1)!}(x - x_0)^{n-1} f^{(n-1)}(x_0) + E_n,$$

where $E_n = \frac{1}{n!}(x - x_0)^n f^{(n)}(x^*)$ for some x^* between x and x_0.

The term

$$\frac{f^{(n+1)}(x^*)}{(n+1)!}(x - x_0)^{n+1}$$

is known as the **Lagrange error term**

Although x^* exists as we have shown, there is unfortunately no easy way to find that value of x^*

So in practice, the bound $x_0 < x^* < x$ or $x < x^* < x_0$ is used to generate a worst-case error for the Lagrange Error Term

For example, if $x_0 < x^* < x$, we may want to compute:

$$\max_{y \in (x_0, x)} \frac{f^{(n+1)}(y)}{(n+1)!}(x-c)^{n+1}$$

using techniques from mathematical optimisation or real analysis.

L'Hopital's Rule

Suppose $f, g : (a, b) \to \mathbb{R}$ have derivatives $f', g' : (a, b) \to \mathbb{R}$ that are continuous in (a, b)

If $f(c) = g(c) = 0$ for some $c \in (a, b)$ and $g'(c) \neq 0$, then:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

The rule can be extended to the case when:

$$\lim_{x \to c} |f(x)| = \lim_{x \to c} |g(x)| = \infty$$

by writing

$$\frac{f(x)}{g(x)} = \frac{1/g(x)}{1/f(x)}$$

It can also be extended to the case when $x \to \infty$ by writing

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{y \to 0} \frac{f(1/y)}{g(1/y)}$$

Where we restrict f and g to the non-negative real numbers, $f, g : [0, \infty) \to \infty)$

Fundamental Theorem of Calculus

If $f : [a, b] \to \mathbb{R}$ is continuous and the function $F : [a, b] \to \mathbb{R}$ is defined by $F(y) = \int_a^y f(x)dx$ then F is uniformly continuous on $[a, b]$ and $F'(x) = f(x)$ for $x \in (a, b)$

Change of Variable Integration

Let $g : [a, b] \to [c, d]$ be a differentiable function with $g' : [a, b] \to R$ continuous, and let $f : [c, d] \to R$ be a continuous function. Let $y = g(x)$. Then:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(y)dy$$