Peano Arithmetic

Peano defined a set of axioms for the natural numbers.

The set N is defined as the set satisfying:

- $0 \in N$
- If $n \in N$, then $Succ(n) \in N$
- For all $n \in N$, $Succ(n) \neq 0$ ($Succ$ is not surjective)
- For all $n, m \in N$, if $Succ(n) = Succ(m)$, then $n = m$ ($Succ$ is injective)
- Let V be a set such that $0 \in V$ and, for all $n \in N$, if $n \in V$ then $Succ(n) \in V$, then $N \subseteq V$.

The last point is called the **principle of induction**

Do natural numbers (mathematically) exist?

Accepting that sets are ‘real’, in set theory we can define the natural numbers recursively by:

$$\begin{aligned} [\![0]\!] &\triangleq \emptyset \\ [\![Succ(n)]\!] &\triangleq [\![n]\!] \cup \{[\![n]\!]\} \end{aligned}$$

Then

$$[\![n]\!] = \{[\![0]\!], [\![1]\!], \ldots, [\![n-2]\!], [\![n-1]\!]\}$$

for each natural number n. For example,

$$\begin{aligned} [\![0]\!] &= \emptyset \\ [\![1]\!] &= [\![0]\!] \cup \{[\![0]\!]\} = \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{[\![0]\!]\} \\ [\![2]\!] &= [\![1]\!] \cup \{[\![1]\!]\} = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\} = \{[\![0]\!], [\![1]\!]\} \\ &\vdots \end{aligned}$$

This gives a model of Peano arithmetic.

Integers and Rationals

Having the set N at hand, we can now define Z by:

$$\begin{aligned}
\mathbb{Z} &\triangleq \{0,1\} \times \mathbb{N} \\
=_{\mathbb{Z}} &\triangleq \{\langle\langle i, m\rangle, \langle j, n\rangle\rangle \in \mathbb{Z}^2 \mid i =_{\mathbb{N}} j \wedge m =_{\mathbb{N}} n\} \cup \\
&\quad \{\langle\langle 0,0\rangle, \langle 1,0\rangle\rangle\}
\end{aligned}$$

Notice that $N \subseteq Z$ does not come for free; we can at most embed N in Z. On the other hand, we now can define Q via:

$$\begin{aligned}
\mathbb{Q} &\triangleq \mathbb{Z} \times (\mathbb{N} \setminus \{0\}) \\
=_{\mathbb{Q}} &\triangleq \{\, \langle\langle n_1, m_1\rangle, \langle n_2, m_2\rangle\rangle \in \mathbb{Q}^2 \mid \\
&\qquad n_1 \times m_2 =_{\mathbb{Z}} n_2 \times m_1 \,\}
\end{aligned}$$

Notice we have shown that $=_Q$ is an equivalence relation.

‘Smaller than’ relation on Natural Numbers

We can define a smaller-than relation $<_1$ on numbers through:

$$<_1 \;\triangleq\; \{\langle n, m\rangle \in \mathbb{N}^2 \mid m = Succ(n)\}$$

So this relation contains only the pairs of consecutive numbers. Now the ‘normal’ smaller-than relation on numbers is defined as the transitive closure of this relation.

$$<_{\mathbb{N}} \;\triangleq\; (<_1)^+$$

And

$$\leq_{\mathbb{N}} \triangleq <_{\mathbb{N}} \cup Id_{\mathbb{N}}$$