Arguments and validity

Consider the following argument

p: Chariots of Fire is an Oscar Winning film.

q: All Oscar winning films are great films.

r: Chariots of Fire is a great film.

The argument is not *propositionally* valid... Why?

$$p, q \not\models r$$

Our experience with propositional logic tells us how to define 'valid argument' etc.

Definition 4.11 (valid argument)

Let L be a signature and $A_1, \ldots, A_n, B$ be L-formulas.
An argument '$A_1, \ldots, A_n$, therefore B' is *valid* if for any L-structure M and assignment h over M,
if $M, h \models A_1$, $M, h \models A_2$, $\ldots$, and $M, h \models A_n$, then $M, h \models B$.
We write $A_1, \ldots, A_n \models B$ in this case.

This says: in any situation (structure + assignment) in which $A_1, \ldots, A_n$ are all true, B must be true too.

Special case: $n = 0$. Then we write just $\models B$. It means that B is true in every L-structure under every assignment over it.

Validity, satisfiability, equivalence

These are defined as in propositional logic. Let L be a signature.

Definition 4.12 (valid formula)

An L-formula A is *(logically) valid* if for every L-structure M and assignment h over M, we have $M, h \models A$.
We write '$\models A$' (as above) if A is valid.

Definition 4.13 (satisfiable formula)

An L-formula A is *satisfiable* if for some L-structure M and assignment h over M, we have $M, h \models A$.

Definition 4.14 (equivalent formulas)

L-formulas A, B are *logically equivalent* if for every L-structure M and assignment h over M, we have $M, h \models A$ if and only if $M, h \models B$.

The links between these definitions that we have seen in propositional logic, also hold for predicate logic.

So (e.g.) the notions of valid/satisfiable formula, and equivalence, can all be expressed in terms of valid arguments.

Which arguments are valid?

Some examples of valid arguments:

- $\forall x(horse(x) \to animal(x)) \models \forall x[\exists y(headof(x,y) \wedge horse(y)) \to \exists y(headof(x,y) \wedge animal(y))]$.

  A horse is an animal $\models$ the head of a horse is the head of an animal.

Deciding if an argument $A_1, \ldots, A_n \models B$ is valid is extremely hard in general. We can't just check that all L-structures + assignments that make $A_1, \ldots, A_n$ true also make B true (like truth tables), because there are infinitely many L-structures (some are infinite!)

Theorem 4.15 (Church, 1936)

No computer program can be written to identify precisely the valid arguments of predicate logic.

Useful ways of validating arguments

In spite of Theorem 4.15, we can often verify in practice that a particular argument in predicate logic is valid. Ways to do it include:

- direct argument (the easiest)
- equivalences
- proof systems (like natural deduction)

The same methods work for showing a formula is valid. (A is valid if and only if $\models A$) Truth tables no longer work (you can't tabulate all structures — there are infinitely many).

Direct argument Examples

Let's show

$$\forall x(\texttt{Human}(x) \to \texttt{Lecturer}(x)), \forall x(\texttt{PC}(x) \to \texttt{Lecturer}(x)),$$
$$\forall x(\texttt{Human}(x) \vee \texttt{PC}(x)) \models \forall x\, \texttt{Lecturer}(x)$$

Let's show
$$\forall x(horse(x) \to animal(x)) \models \forall x[\exists y(headof(x,y) \land horse(y)) \to \exists y(headof(x,y) \land animal(y))].$$

Let's show $\forall x \forall y(x = y \land \exists z R(x,z) \to \exists v R(y,v))$ is valid.

Take any structure M, and objects a, b in $dom(M)$.
We need to show

$$M \models a = b \land \exists z R(a,z) \to \exists v R(b,v).$$

So we need to show that

$$\text{IF } M \models a = b \land \exists z R(a,z) \text{ THEN } M \models \exists v R(b,v).$$

But IF $M \models a = b \land \exists z R(a,z)$, then a, b are the same object.
So $M \models \exists z R(b,z)$.
So there is an object c in $dom(M)$ such that $M \models R(b,c)$.
Therefore, $M \models \exists v R(b,v)$.
We're done!

Using direct argument, show that $f(x, f(y, z)) = f(f(x, y), z)$ is satisfiable.

All we need to do is find a structure M and assignment h over M in which the above formula is true.

Take an M where dom(M) is the set of natural numbers $\mathbb{N}$ and the value of f in M is addition $+$.

Let h be an assignment in which the valiues of x, y, z are $2, 4, 9$ respectively.

Then $M, h \models f(x, f(y, z)) = f(f(x, y), z)$

Exercise:
Using direct argument, show that $\neg\forall x(\exists y\ R(x, y) \to R(x, f(x, x)))$ is satisfiable.