Transitive Closure

Let R be a binary relation on A.

We define R^+, the transitive closure of R, as the smallest transitive relation that contains R.

We can construct R^+ from R as follows.

First we define R^i as:

$R^1 \triangleq R$
$R^2 \triangleq R \circ R$
$R^3 \triangleq R \circ R^2 = R^2 \circ R$, since $\circ$ is associative
. . .
$R^n \triangleq R \circ R^{n-1} = R \circ \cdots \circ R$, n times
. . .

and we define

$R^+ \triangleq R \cup R^2 \cup \cdots \cup R^n \cup \cdots \triangleq \cup_{i \geq 1} R^i$

Therefore, we have $a\ R^+\ b \iff \exists n \geq 1\ (a\ R^n\ b)$

Constructing the Transitive Closure

Let R be a **finite binary** relation on A.

If R is already transitive, we are done.

Otherwise, there exists $a, b, c \in A$ such that $a\ R\ b$ and $b\ R\ c$, but not $a\ R\ c$.

We add the pair $\langle a, c \rangle$ to the relation.

Carry on doing this, until there are no more pairs to add.

We now have a **transitive** relation.

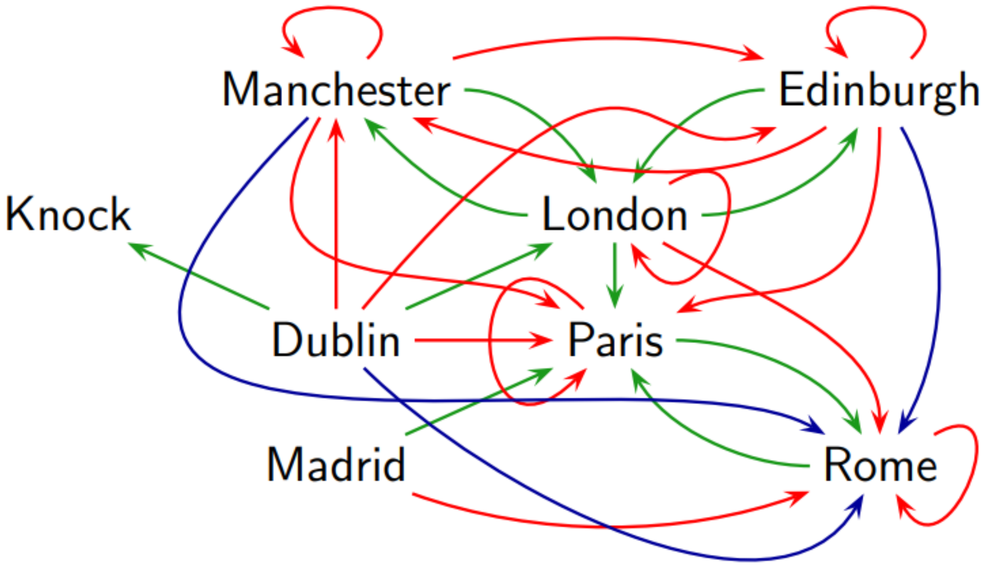
Every added pair was a **necessary requirement of transitivity**, so we have obtained the smallest possible transitive relation containing R.

(Since we accept **infinite** constructions, this procedure even works for infinite relations.)

Transitive Closure: Example

Define a set City of cities and a binary relation R on City such that $a\,R\,b$ when there is a direct flight from a to b.

Define the relation R^+ by $a\ R^+\ b$ when there is a trip from a to b. We will calculate R^+ from R.



$R \quad R^2 \quad R^3$