Partial Functions

A **partial function** f from a set A to a set B is a relation $f \subseteq A \times B$ such that just **some** elements of A are related to unique elements of B:

$$\langle a, b_1 \rangle \in f \wedge \langle a, b_2 \rangle \in f \implies b_1 = b_2$$

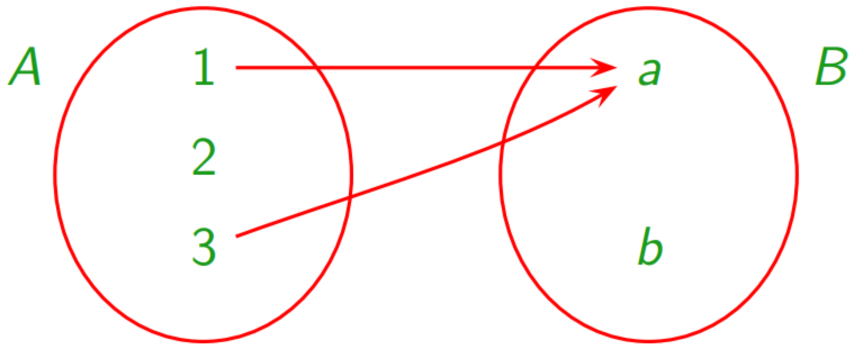
The partial function f is regarded as **undefined** on those elements which do not have an image under f.

We can call this undefined value $\perp$ (pronounced **bottom**). A partial function from A to B is a function from A to $B \cup \{\perp\}$

A partial function does not need to be a function.

Examples of Partial Functions

- The relation $R = \{\langle 1, a\rangle, \langle 3, a\rangle\} \subseteq \{1, 2, 3\} \times \{a, b\}$:

  Not every element in A maps to an element in B.

- The binary relation R on $I\!R$ defined by $R \triangleq \{\langle x, y\rangle \mid \sqrt{x} = y\}$; it is not defined when x is negative.

Properties of Functions

Let $f : A \rightarrow B$ be a function

f is **onto** (**surjective**) when every element of B is in the image of f:

$\forall b \in B\ \exists a \in A\ (f(a) =_B b)$

f is **one-to-one** (**injective**) when for each $b \in B$ there is at most one $a \in A$ with $f(a) = b$:

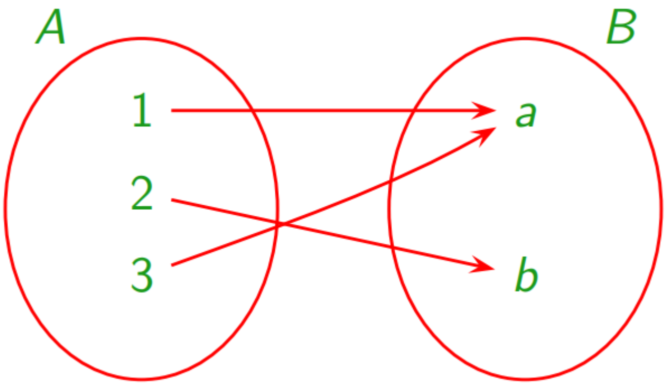
$\forall a, a' \in A\ (f(a) =_B f(a') \implies a =_A a')$

f is a **bijection** when f is **one-to-one** and **onto**

If there exists functions $f : A \to B$ and $g : B \to A$, both injective or both surjective, then there exists a bijection $h : A \to B$

Example 1

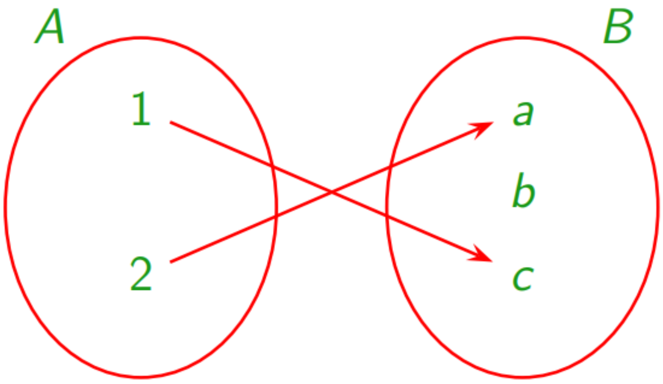
Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. The function $f = \{\langle 1, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle\}$ is *onto*, but *not one-to-one*:



We cannot define a *one-to-one* function from A to B. There are too many elements in A for them to map uniquely to B.

Example 2

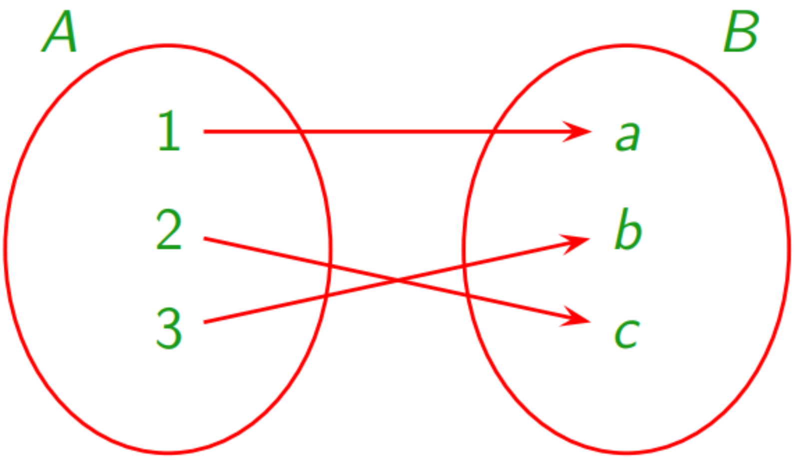
Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. The function $f = \{\langle 1, c\rangle, \langle 2, a\rangle\}$ is *one-to-one*, but not *onto*:



It is not possible to define an *onto* function from A to B; there are not enough elements in A to map to all the elements of B.

Example 3

Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. The function $f = \{\langle 1, a\rangle, \langle 2, c\rangle, \langle 3, b\rangle\}$ is bijective:



The function $f : I\!N^2 \to I\!N$ defined by $f(x, y) = x + y$ is *onto* but not *one-to-one*. To prove that f is *onto*, take an arbitrary $n \in I\!N$. Then $f(n, 0) = n + 0 = n$.

To show that f is not *one-to-one*, we need to produce a counter example: that is, find $\langle m_1, m_2\rangle$, $\langle n_1, n_2\rangle$ such that $\langle m_1, m_2\rangle \neq \langle n_1, n_2\rangle$, but $f(m_1, m_2) = f(n_1, n_2)$. Take, for example, $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$.

The function $f : I\!N \to I\!N$ defined by $f(x) = x^2$ is *one-to-one*. Defined on $Z\!\!\!Z$ it is not.

The function $f : Z\!\!\!Z \to Z\!\!\!Z$ defined by $f(x) = x + 1$ is *onto*. The similar function on $I\!N$ is not.

The function $f : I\!R \to I\!R$ defined by $f(x) = 4x + 3$ is a bijective function.

A Proposition

Let A and B be **finite** sets, and let $f : A \to B$

If f is **one-to-one**, then $|A| \leq |B|$
If f is **onto**, then $|A| \geq |B|$
If f is a **bijection**, then $|A| = |B|$

Proof: (Sketch.) The first part is the contrapositive of the pigeonhole principle.

For the second, notice that if f is *onto* then $f[A] = B$. Hence, $|f[A]| = |B|$. Also $|A| \geq |f[A]|$ by previous proposition. Therefore $|A| \geq |B|$ as required.

The third part follows from the first two parts.