Arguments, Validity

Valid Arguments

Definition 1.1 (valid argument)

Given formulas $\phi_1, \phi_2, \ldots, \phi_n, \psi$, an argument

$$\text{‘}\phi_1, \phi_2, \ldots, \phi_n, \text{ therefore } \psi\text{’}$$

is *valid* if:

ψ is true in every situation in which $\phi_1, \phi_2, \ldots, \phi_n$ are all true.

If this is so, we write ‘$\phi_1, \phi_2, \ldots, \phi_n \models \psi$’.

The $\models$ is called ‘*double turnstile*’

You can read it as ‘*logically implies*’

Examples

Let ϕ, ψ be arbitrary propositional formulas:

- ‘ϕ, therefore ϕ’ is valid, since in any situation in which ϕ is true, ϕ is true. $\phi \models \phi$.
- ‘$\phi \wedge \psi$, therefore ϕ’ is valid by the semantics of $\wedge$. $\phi \wedge \psi \models \phi$.
- ‘ϕ, therefore $\phi \wedge \psi$’ is not in general valid: depending on ϕ and ψ, there might be situations in which ϕ is true but $\phi \wedge \psi$ is not. Then, $\phi \not\models \phi \wedge \psi$.
- ‘ϕ, $\phi \to \psi$, therefore ψ’ is valid. ϕ, $\phi \to \psi \models \psi$. This argument style has a name: *modus ponens.*
- ‘$\phi \to \psi$, $\neg\psi$, therefore $\neg\phi$’ is also valid: $\phi \to \psi, \neg\psi \models \neg\phi$. This argument is called *modus tollens.*
- ‘$\phi \to \psi$, ψ, therefore ϕ’ is not valid in general, in spite of what lawyers and politicians may say. $\phi \to \psi, \psi \not\models \phi$.

Valid, Satisfiable, Equivalent formulas

Three important ideas are related to valid arguments:

- Valid formulas
- Satisfiable formulas
- Equivalent formulas

Valid Formula

Definition 1.2 (valid formula)

A propositional formula is *(logically) valid* if it is true in every situation.

Valid **propositional** formulas are often called **tautologies**.

Satisfiable formula

Definition 1.3 (satisfiable formula)

A propositional formula is *satisfiable* if it is true in at least one situation.

We typically say a formula is a **contradiction** if it is not satisfiable.

Equivalent formulas

Definition 1.4 (equivalent formulas)

Two propositional formulas ϕ, ψ are *logically equivalent* if they are true in exactly the same situations. Roughly speaking: they mean the same.

Some people write $\phi \equiv \psi$ for this. But $\equiv$ is also used in other ways, so watch out.

Equivalent?	p	$\top$	$\bot$	$p \to q$
$p \wedge p$	yes	no	no	no
$p \wedge \neg p$	yes	yes	no	no
$p \vee \neg p$	no	yes	no	no
$\neg p \vee q$	no	no	no	yes

Relations between the concepts

Valid arguments and **valid, satisfiable, and equivalent formulas** are all definable in terms of each other:

argument validity	formula validity	satisfiability	equivalence
$\phi \models \psi$	$\phi \to \psi$ valid	$\phi \wedge \neg\psi$ unsatisfiable	$(\phi \to \psi) \equiv \top$
$\top \models \phi$	ϕ valid	$\neg\phi$ unsatisfiable	$\phi \equiv \top$
$\phi \not\models \bot$	$\neg\phi$ not valid	ϕ satisfiable	$\phi \not\equiv \bot$
$\phi \models \psi$ and $\psi \models \phi$	$\phi \leftrightarrow \psi$ valid	$\phi \leftrightarrow \neg\psi$ unsatisfiable	$\phi \equiv \psi$

All four statements in each line amount to the same thing.

So we can choose to deal with one of these concepts and get the others for free.

More on argument and formula validity

Definition 1.5

Let $\phi_1, \ldots, \phi_n \models \psi$ be an argument. We call the formula $\phi_1 \wedge \ldots \wedge \phi_n \to \psi$ its *corresponding implication formula*.

For instance, given an argument

$p \to q, \neg q \models \neg p$

its corresponding implication formula is

$((p \to q) \wedge \neg q) \to \neg p$

Theorem 1.6

$\phi_1, \ldots, \phi_n \models \psi$ is a valid argument if and only if its corresponding implication formula $\phi_1 \wedge \ldots \wedge \phi_n \to \psi$ is a valid formula (i.e., tautology).

So we can check the validity of an argument by checking the validity of its corresponding implication formula.

We will allow for arguments to have zero premises and write $\models \psi$, meaning ψ is true in every situation.

When such an argument is valid, the formula ψ is said to be a **tautology** (i.e., a valid formula).

How do we tell whether an argument is valid?

In principle, we know how:

To show $\phi_1, \ldots, \phi_n \models \psi$ is a valid argument, check that the conclusion ψ is true in every situation where the premises $\phi_1, \ldots, \phi_n$ are true.

The same methods work for showing a formula ψ is valid. (ψ is a valid formula if and only if $\models \psi$ is a valid argument.)

Ways to check an argument is propositionally valid

To check whether an argument is propositionally valid, we can:

- Translate all English sentences into propositional logic
- Check whether the resulting argument is valid using:
    - Truth tables
    - Direct 'mathematical' argument

- Equivalences:
  - Make a stock of useful pairs of equivalent formulas
  - Use them to reduce its corresponding implication formula step by step to $\top$
- Natural deduction

For valid formulas, equivalence and satisfiability, we proceed in a similar way

1st method: Checking validity using truth tables

The truth table of a propositional formula ϕ summarises its truth values by considering different truth assignments to the propositional atoms appearing in it (which we called **situations**)

When checking if an argument $\phi_1, \ldots, \phi_n \models \psi$ is valid, we check whether the formula ψ is true in every situation where the formula $\phi_1 \wedge \ldots \wedge \phi_n$ is true

Is $p \to \neg q, q \models \neg p$ a valid argument?

Recall we write tt for the value true and ff for the value false.

p	q	$\neg q$	$p \to \neg q$	$(p \to \neg q) \wedge q$	$\neg p$
tt	tt	ff	ff	ff	ff
tt	ff	tt	tt	ff	ff
ff	tt	ff	tt	tt	tt
ff	ff	tt	tt	ff	tt

So the argument is valid.

p	q	$\neg q$	$p \to \neg q$	$(p \to \neg q) \land q$	$\neg p$	$(p \to \neg q) \land q \to \neg p$
tt	tt	ff	ff	ff	ff	tt
tt	ff	tt	tt	ff	ff	tt
ff	tt	ff	tt	tt	tt	tt
ff	ff	tt	tt	ff	tt	tt

Since the **argument's corresponding implication formula** is valid, hence the **argument** is valid.

Show that the formula $(p \to q) \leftrightarrow (\neg p \lor q)$ is valid

There are two atoms, so 2^2 different truth assignments, hence four rows.

p	q	$p \to q$	$\neg p$	$\neg p \lor q$	$(p \to q) \leftrightarrow (\neg p \lor q)$
tt	tt	tt	ff	tt	tt
tt	ff	ff	ff	ff	tt
ff	tt	tt	tt	tt	tt
ff	ff	tt	tt	tt	tt

The truth value for $(p \to q) \leftrightarrow (\neg p \lor q)$ is tt for every truth value assignment given to p and q

Therefore, it's a **valid formula**

The same table shows that $(p \to q)$ and $(\neg p \lor q)$ are **equivalent**

It also shows that $(p \to q) \leftrightarrow (\neg p \lor q)$ is satisfiable.

Pros and Cons of truth tables

Pros:

- They always work in propositional logic, where only finitely many situations

are relevant to a given formula
- They can be used to show satisfiability and equivalence
- They are easy to implement
- They illustrate how hard deciding argument validity, formula validity, satisfiability and equivalence etc., can be—even for ‘trivial’ propositional logic. (Adding an atom doubles the length of the table.)

Cons:

- They are tedious and error-prone
- No satisfactory way to determine propositional satisfiability is known. The problem is NP-complete

2nd method: Checking validity using direct argument

It involves showing the truth or falsity of a propositional formula by constructing direct logical (or mathematical) arguments from zero or more premises to one or more conclusions.

Is $p \to (p \vee q)$ a valid formula?

Take an arbitrary situation (truth value assignment to p and q).

To be a valid formula, then any situation must be such that, if p is true in that situation then $p \vee q$ is also true in that situation by semantics of $\to$

Well, if p evaluates to true, then so does $p \vee q$ (by semantics of $\vee$)

Done!

Show that $p \wedge (p \to q) \to q$ is a valid formula.

Take any situation. To be a valid formula then the formula must be true in this situation.

For the formula to be true then either $p \wedge (p \to q)$ is false or q is true in that situation (by the semantics of $\to$)

For $p \wedge (p \to q)$ to be true then both p and $(p \to q)$ must be true in that situation (by the semantics of $\wedge$)

For $(p \to q)$ to be true, either p is false, or else p and q are true by semantics of $\to$.

If p is false, then $p \wedge (p \to q)$ is false (by the semantics of $\wedge$), and hence $p \wedge (p \to q) \to q$ is true by the semantics of $\to$.

But if p evaluates to true, then for $(p \to q)$ to be true q must also be true by semantics of $\to$.

Since in any situation where $p \wedge (p \to q)$ is true, q is also true then $p \wedge (p \to q) \to q$ is true in that situation by semantics of $\to$

Show that $(\phi \wedge \psi) \wedge \rho \equiv \phi \wedge (\psi \wedge \rho)$ for arbitrary formulas ϕ, ψ, ρ

> For the two formulas to be semantically equivalent, then they must be true in exactly the same situations.
>
> Take any situation. A formula of the form $(\phi \wedge \psi) \wedge \rho$ is true if and only if both $(\phi \wedge \psi)$ and ρ evaluate to true (by the semantic definition of $\wedge$). This is the case if and only if ϕ and ψ evaluate to true, and ρ evaluates to true— that is, they're all true.
>
> This is so if and only if ϕ is true, and also ψ and ρ are true by semantics of $\wedge$.
>
> This is so if and only if ϕ and $\psi \wedge \rho$ are true by semantics of $\wedge$.
>
> This is so if and only if $\phi \wedge (\psi \wedge \rho)$ is true by semantics of $\wedge$.
>
> So $(\phi \wedge \psi) \wedge \rho$ and $\phi \wedge (\psi \wedge \rho)$ have the same truth value in this situation by semantics of $\wedge$. The situation was arbitrary, so they are logically equivalent.

We could have equally shown that a formula of the form $(\phi \wedge \psi) \wedge \rho \leftrightarrow \phi \wedge (\psi \wedge \rho)$ is a valid formula.

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Show that $((p \to q) \to p) \to p$ (known as '*Peirce's law*') is a valid formula.

Take an arbitrary situation.

- *If p is true in this situation*, then $((p \to q) \to p) \to p$ is true, since any formula of the form $\phi \to \psi$ is true when ψ is true. We are done.
- *If not*, then p must be false in this situation.

  So $p \to q$ is true, because $\phi \to \psi$ is true when ϕ is false by semantics of $\to$.

  So $(p \to q) \to p$ is false by semantics of $\to$, because the antecedent is true and the consequent is false: $\phi \to \psi$ is false when ϕ is true and ψ false.

  So $((p \to q) \to p) \to p$ is true by semantics of $\to$, because $\phi \to \psi$ is true when ϕ is false. We are done again, and finished.

This was an *argument by cases:* p true, or p false. They are exhaustive: this is known as the *'law of excluded middle'*.

3rd Method: Checking validity using equivalences

We know that $\top$ is a valid formula.

Showing an argument is valid using equivalences involves:

- Converting the argument into its corresponding implication formula
- Simplifying the formula to $\top$ (which is a valid formula), always preserving logical equivalence.

Why? Recall this relations . . .

argument validity	formula validity	satisfiability	equivalence
$\phi \models \psi$	$\phi \to \psi$ valid	$\phi \wedge \neg\psi$ unsatisfiable	$(\phi \to \psi) \equiv \top$

Equivalences

In the following, ϕ, ψ, ρ will denote arbitrary formulas.

For short, I will often say '*equivalent*' rather than '*logically equivalent*'

Equivalences involving $\wedge$

1. $\phi \wedge \psi$ is logically equivalent to $\psi \wedge \phi$ (*commutativity of* $\wedge$)
2. $\phi \wedge \phi$ is logically equivalent to ϕ (*idempotence of* $\wedge$)
3. $\phi \wedge \top$ and $\top \wedge \phi$ are logically equivalent to ϕ
4. $\bot \wedge \phi$, $\phi \wedge \bot$, $\phi \wedge \neg\phi$, and $\neg\phi \wedge \phi$ are all equivalent to $\bot$
5. $(\phi \wedge \psi) \wedge \rho$ is equivalent to $\phi \wedge (\psi \wedge \rho)$ (*associativity of* $\wedge$)

Equivalences involving $\vee$

6. $\phi \vee \psi$ is equivalent to $\psi \vee \phi$ (*commutativity of* $\vee$)
7. $\phi \vee \phi$ is equivalent to ϕ (*idempotence of* $\vee$)
8. $\top \vee \phi$, $\phi \vee \top$, $\phi \vee \neg\phi$, and $\neg\phi \vee \phi$ are equivalent to $\top$
9. $\phi \vee \bot$ and $\bot \vee \phi$ are equivalent to ϕ
10. $(\phi \vee \psi) \vee \psi$ is equivalent to $\phi \vee (\phi \vee \psi)$ (*associativity of* $\vee$)

Equivalences involving $\neg$ and $\rightarrow$

11. $\neg\top$ is equivalent to $\bot$
12. $\neg\bot$ is equivalent to $\top$
13. $\neg\neg\phi$ is equivalent to ϕ

14. $\phi \to \phi$ is equivalent to $\top$
15. $\top \to \phi$ is equivalent to ϕ
16. $\phi \to \top$ is equivalent to $\top$
17. $\bot \to \phi$ is equivalent to $\top$
18. $\phi \to \bot$ is equivalent to $\neg\phi$
19. $\phi \to \psi$ is equivalent to $\neg\phi \vee \psi$, and also to $\neg(\phi \wedge \neg\psi)$
20. $\neg(\phi \to \psi)$ is equivalent to $\phi \wedge \neg\psi$

Equivalences involving $\leftrightarrow$

21. $\phi \leftrightarrow \psi$ is equivalent to
    - $(\phi \to \psi) \wedge (\psi \to \phi)$,
    - $(\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi)$,
    - $\neg\phi \leftrightarrow \neg\psi$.

22. $\neg(\phi \leftrightarrow \psi)$ is equivalent to
    - $\phi \leftrightarrow \neg\psi$,
    - $\neg\phi \leftrightarrow \psi$,
    - $(\phi \wedge \neg\psi) \vee (\neg\phi \wedge \psi)$.

    (This is one way to express the *exclusive or* of ϕ, ψ.)

Equivalences—De Morgan laws

23. $\neg(\phi \wedge \psi)$ is equivalent to $\neg\phi \vee \neg\psi$
24. $\neg(\phi \vee \psi)$ is equivalent to $\neg\phi \wedge \neg\psi$

Other Equivalences

Distributivity of $\wedge$, $\vee$

25. $\phi \wedge (\psi \vee \rho)$ is equivalent to $(\phi \wedge \psi) \vee (\phi \wedge \rho)$.
    $(\psi \vee \rho) \wedge \phi$ is equivalent to $(\psi \wedge \phi) \vee (\rho \wedge \phi)$.
26. $\phi \vee (\psi \wedge \rho)$ is equivalent to $(\phi \vee \psi) \wedge (\phi \vee \rho)$
    $(\psi \wedge \rho) \vee \phi$ is equivalent to $(\psi \vee \phi) \wedge (\rho \vee \phi)$

Absorption

27. $\phi \wedge (\phi \vee \psi)$ and $\phi \vee (\phi \wedge \psi)$ are equivalent to ϕ.
    So are $\phi \wedge (\psi \vee \phi)$, $(\phi \wedge \psi) \vee \phi$, etc.

Equivalences—Normal Form

Equivalences can be used to re-write a formula into a **normal form**

These can improve the efficiency of checking validity/satisfiability of a formula which otherwise takes time exponential in the number of atoms.

We consider two common normal forms:

- Disjunctive Normal Form (DNF)
- Conjunctive Normal Form (CNF)

Equivalences—Disjunctive normal form

Definition 2.1 (Normal form DNF)

A formula ϕ is in *disjunctive normal form* if it is a disjunction of conjunctions of literals, and is not further simplifiable by equivalences without leaving this form. (See Def. 1.3 for literals.)

DNF Examples

$$p \lor q \lor \neg r$$

$$(p \land \neg q) \lor r \lor (\neg p \land q \land \neg r)$$

$$q$$

Not DNF Examples

$$(p \land p) \lor (q \land \top \land \neg q)$$

$$\neg(q \lor r)$$

A formula in DNF is unsatisfiable, if and only if each of its conjunctions contains some literal and its negation.

Satisfiability of formulas in DNF can be checked in linear time.

Every formula can be equivalently written as a formula in DNF.

Equivalences—Conjunctive Normal Form

Definition 2.2 (Normal form CNF)

A formula ϕ is in *conjunctive normal form* if it is a conjunction of disjunctions of literals (that is, a conjunction of clauses), and is not further simplifiable by equivalences without leaving this form.

CNF Examples:

$$(p \vee \neg q) \wedge (q \vee r) \wedge (\neg p \vee q)$$

$$p \vee q$$

$$q$$

Not CNF Examples:

$$q \vee (\neg p \wedge r)$$

$$\neg(r \vee s)$$

A formula in CNF is valid, if and only if each of its disjunctions contains some literal and its negation.

Validity of formulas in CNF can be checked in linear time.

Every formula can be equivalently written as a formula in CNF.

Rewriting a formula in Normal Form

1. Remove all occurrences of $\to$ and $\leftrightarrow$ by
   - replacing all subformulas $\phi \to \psi$ by $\neg\phi \vee \psi$
   - replacing all subformulas $\phi \leftrightarrow \psi$ by $(\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi)$

   It's faster to replace $\neg(\phi \to \psi)$ by $\phi \wedge \neg\psi$, and $\neg(\phi \leftrightarrow \psi)$ by $(\phi \wedge \neg\psi) \vee (\neg\phi \wedge \psi)$.

2. Use the De Morgan laws to push negations down next to atoms. Delete all double negations (replace $\neg\neg\phi$ by ϕ).

3. Rearrange using distributivity to get the desired normal form.

4. Simplify:
   - replacing subformulas $p \wedge \neg p$ by $\bot$, and $p \vee \neg p$ by $\top$.
   - replacing subformulas $\top \vee p$ by $\top$, $\top \wedge p$ by p, $\bot \vee p$ by p, and $\bot \wedge p$ by $\bot$.
   - absorption (equivalence 27) is often useful too.
   - repeat till no further progress.

Example 1

Write $\neg(p \to q) \vee (r \to p)$ in DNF

$$\begin{aligned}
& \neg(p \to q) \vee (r \to p) && \text{[the original formula]} \\
\equiv\ & \neg(\neg p \vee q) \vee (\neg r \vee p) && [\phi \to \psi \equiv \neg\phi \vee \psi] \\
\equiv\ & (\neg\neg p \wedge \neg q) \vee (\neg r \vee p) && \text{[by De Morgan laws]} \\
\equiv\ & (p \wedge \neg q) \vee \neg r \vee p && [\neg\neg\phi \equiv \phi] \\
\equiv\ & p \vee (p \wedge \neg q) \vee \neg r && [\text{by commutativity of } \vee] \\
\equiv\ & (p \vee (p \wedge \neg q)) \vee \neg r && [\text{by associativity of } \vee] \\
\equiv\ & p \vee \neg r && \text{[by absorption]}
\end{aligned}$$

Done!

Note that $p \vee \neg r$ is also in CNF

Example 2

Let's write $p \wedge q \to \neg(p \leftrightarrow \neg r)$ in DNF

$$
\begin{aligned}
& p \wedge q \to \neg(p \leftrightarrow \neg r) && \text{[the original formula]} \\
\equiv\ & \neg(p \wedge q) \vee \neg(p \leftrightarrow \neg r) && [\phi \to \psi \equiv \neg\phi \vee \psi] \\
\equiv\ & \neg(p \wedge q) \vee ((p \wedge \neg\neg r) \vee (\neg p \wedge \neg r)) && [\neg\phi \leftrightarrow \psi \equiv (\phi \wedge \neg\psi) \vee (\neg\phi \wedge \psi)] \\
\equiv\ & \neg(p \wedge q) \vee ((p \wedge r) \vee (\neg p \wedge \neg r)) && [\neg\neg\phi \equiv \phi] \\
\equiv\ & \neg p \vee \neg q \vee ((p \wedge r) \vee (\neg p \wedge \neg r)) && \text{[by De Morgan law]} \\
\equiv\ & \neg p \vee \neg q \vee (p \wedge r) \vee (\neg p \wedge \neg r) && [\text{by associativity of } \vee] \\
\equiv\ & \neg q \vee (p \wedge r) \vee (\neg p \wedge \neg r) \vee \neg p && [\text{by commutativity of } \vee] \\
\equiv\ & \neg q \vee (p \wedge r) \vee \neg p && \text{[by absorption]}
\end{aligned}
$$

Consider the last step in the previous slide

$$\equiv\ \neg q \vee (p \wedge r) \vee \neg p \qquad \text{[by absorption]}$$

We can simplify further if we are willing to leave DNF temporarily:

$$
\begin{aligned}
\equiv\ & \neg q \vee (p \vee \neg p) \wedge (r \vee \neg p) && [\text{by distributivity of } \vee] \\
\equiv\ & \neg q \vee \top \wedge (r \vee \neg p) && [\phi \vee \neg\phi \equiv \top] \\
\equiv\ & \neg q \vee (r \vee \neg p) && [\top \wedge \phi \equiv \phi] \\
\equiv\ & \neg q \vee r \vee \neg p && [\text{by associativity of } \vee]
\end{aligned}
$$

Showing formula validity using equivalences

Show that $(p \to q) \vee (q \to p)$ is a valid formula.

$$
\begin{aligned}
& (p \to q) \vee (q \to p) && \text{[the original formula]} \\
\equiv\ & (\neg p \vee q) \vee (\neg q \vee p) && [\phi \to \psi \equiv \neg\phi \vee \psi] \\
\equiv\ & \neg p \vee (q \vee \neg q \vee p) && [\text{by associativity of } \vee] \\
\equiv\ & (q \vee \neg q \vee p) \vee \neg p && [\text{by commutativity of } \vee] \\
\equiv\ & (q \vee \neg q) \vee (p \vee \neg p) && [\text{by associativity of } \vee] \\
\equiv\ & \top \vee \top && [\text{by } \phi \vee \neg\phi \equiv \top] \\
\equiv\ & \top && \text{[by idempotence]}
\end{aligned}
$$

Done!

Things to note when using Equivalences

- Name the equivalence law applied when rewriting, or reference its the overall logical form
- Apply one equivalence law at a time
- You may combine consecutive applications of the same law in one step (e.g., consecutive applications of associativity of $\vee$)
- Reference the equivalence law next to the result
- Reference the correct equivalence law, e.g., $\top \wedge \phi \equiv \phi$ is different from $\phi \wedge \top \equiv \phi$
- Don't forget referencing **associativity** and **commutativity** laws

Writing DNF equivalences from Truth Tables

Given a formula ϕ, we can construct a semantically equivalent formula in DNF from its truth table. The basic principle is based on the fact that every truth assignment (row) can be encoded as a propositional formula which is true just on that assignment and false everywhere else.

To construct this formula, take the conjunction of:

- All propositional atoms that have the value tt in that truth assignment

- the negations of all propositional atoms that have the value ff in that truth assignment

p	q	conjunctive
tt	tt	$p \wedge q$
tt	ff	$p \wedge \neg q$
ff	tt	$\neg p \wedge q$
ff	ff	$\neg p \wedge \neg q$

Now to rewrite a formula ϕ in DNF, we:

- Construct the truth table for ϕ
- Write a conjunctive formula for each assignment in ϕ's truth table in which ϕ evaluates to tt
- Take the disjunction of these conjunctive formulas

Rewrite $(p \to \neg q) \wedge q$ in DNF

p	q	$\neg q$	$p \to \neg q$	$(p \to \neg q) \wedge q$	conjunctive
tt	tt	ff	ff	ff	
tt	ff	tt	tt	ff	
ff	tt	ff	tt	tt	$\neg p \wedge q$
ff	ff	tt	tt	ff	

Therefore, the equivalent DNF formula is $\neg p \wedge q$

Writing CNF equivalences from Truth Tables

Similarly, given a formula ϕ, we can construct a semantically equivalent formula in CNF from its truth table.

The basic principle here instead is that every truth assignment (row) can be encoded as a clause (completion) which is false just on that assignment and true everywhere else.

To construct this clause, take the disjunction of:

- All propositional atoms that have the value ff in that truth assignment
- The negations of all propositional atoms that have the value tt in that truth assignment

p	q	completion
tt	tt	$\neg p \vee \neg q$
tt	ff	$\neg p \vee q$
ff	tt	$p \vee \neg q$
ff	ff	$p \vee q$

Now to rewrite a formula ϕ in CNF, we follow these steps:

- Construct the truth table for ϕ
- Write a clause (completion) for each assignment in ϕ's truth table in which ϕ has value ff
- Take the conjunction of these clauses

Rewrite $(p \to \neg q) \wedge q$ in CNF

p	q	$\neg q$	$p \to \neg q$	$(p \to \neg q) \land q$	completion
tt	tt	ff	ff	ff	$\neg p \lor \neg q$
tt	ff	tt	tt	ff	$\neg p \lor q$
ff	tt	ff	tt	tt	
ff	ff	tt	tt	ff	$p \lor q$

Therefore, the equivalent CNF formula is $(\neg p \lor \neg q) \land (\neg p \lor q) \land (p \lor q)$

4th method Proof Systems: Natural Deduction

We should be able to establish validity of argument by breaking it into smaller arguments and showing the validity of these intermediate ones, i.e., **construct a proof**.

A **proof system** is a way of showing formulas to be valid by using purely **syntactic rules** — not using semantics at all. A computer algorithm should be able to apply the rules.

We will focus on the proof system Natural Deduction

What is Natural Deduction?

A formalisation of 'direct argument'.

Starting perhaps from formulas $\phi_1, \ldots, \phi_n$, we use the rules of the system to reason towards a formula ψ

If we succeed, we can write $\phi_1, \ldots, \phi_n \vdash \psi$

$\phi_1, \ldots, \phi_n$ are called **premises**
ψ is called a **conclusion**
$\phi_1, \ldots, \phi_n \vdash \psi$ is called a **sequent**

Assumptions in Natural Deduction

Proofs in ND are not based on axioms expressed in logical form

They sometimes involve making “temporary” assumptions to prove a conclusion.

An assumption is just a formula, but it’s used in a special way.

We imagine a situation in which a formula is true. Then we derive some additional formulas that help us make progress towards the conclusion.

We need to be careful about how we use these assumptions

How Natural Deduction works

Deduction works by writing down intermediate formulas using **inference rules**

- State the set of premises $\phi_1, \ldots, \phi_n$, and the conclusion ψ
- Intermediate formulas form the **proof** of ψ from the givens $\phi_1, \ldots, \phi_n$
- Once established, they may be usable later
- Each step of the proof is a **valid argument**

Natural deduction inference rules

Mostly, there are two rules for each connective:

- One for *introducing* it a formula
- One for using it a formula

The rules are based on the semantics for the connectives given earlier

Rules for Conjunction

Introduction

To introduce a formula of the form $\phi \wedge \psi$, you have to have already introduced ϕ and ψ

$$
\begin{array}{lll}
1 & \phi & \text{we proved this} \\
2 & \vdots & \text{(other stuff)} \\
3 & \psi & \text{and this} \\
4 & \phi \wedge \psi & \wedge I(1,3)
\end{array}
$$

The line numbers are essential for clarity

ϕ and ψ in the above need not be atomic

Elimination

If you have managed to write down $\phi \wedge \psi$, you can go on to write down ϕ and/or ψ

$$
\begin{array}{lll}
1 & \phi \wedge \psi & \text{we have this somehow} \\
2 & \phi & \wedge E(1) \\
3 & \psi & \wedge E(1)
\end{array}
$$

Example

Prove that the sequent $p \land q, r \vdash q \land r$ *is valid.*

1	$p \land q$	premise
2	r	premise
3	q	$\land E(1)$
4	$q \land r$	$\land I(3, 2)$

Boxes in Natural Deduction

Boxes are used when making additional assumptions.

The first line should always be labelled 'asm' (assumption)

The line immediately following the closed box must match the pattern of the conclusion of the rule that uses the box

Nothing inside the box can be used later

Rules for Implication

Introduction

To introduce a formula of the form $\phi \to \psi$, you *assume* ϕ and then prove ψ

During the proof, you can use ϕ as well as anything already established

But you can't use ϕ or anything from the proof of ψ from ϕ later on (because it's based on an extra assumption)

So we isolate the proof of ψ from ϕ, in a box:

1	ϕ	asm
2	$\ldots$	$\langle$the proof$\rangle$ hard struggle
3	ψ	we made it!

4	$\phi \to \psi$	$\to I(1,3)$

Elimination

If you have managed to write down ϕ and $\phi \to \psi$, in either order, you can go on to write down ψ (*modus ponens*)

1	$\phi \to \psi$		we got this somehow
2	$\vdots$		(other stuff)
3	ϕ		and this too
4	ψ	$\to E(1,3)$	

Examples

Prove that the sequent $p, p \to q, p \to (q \to r) \vdash r$ is valid.

1	p	premise
2	$p \to q$	premise
3	$p \to (q \to r)$	premise
4	$q \to r$	$\to E(3,1)$
5	q	$\to E(2,1)$
6	r	$\to E(4,5)$

Prove that the sequent $\vdash p \to p$ is valid.

1	p	asm
2	p	$\checkmark(1)$
3	$p \to p$	$\to I(1,2)$

you don't need this last step

Using Natural Deduction to prove equivalence

We say two formulas ϕ and ψ are provably equivalent in natural deduction (written as $\phi \dashv\vdash \psi$) iff $\phi \vdash \psi$ and $\psi \vdash \phi$

Prove that $(p \wedge q) \to r \dashv\vdash p \to (q \to r)$

We first show $(p \wedge q) \to r \vdash p \to (q \to r)$

1	$(p \wedge q) \rightarrow r$	premise
2	p	asm
3	q	asm
4	$p \wedge q$	$\wedge I(2,3)$
5	r	$\rightarrow E(1,4)$
6	$q \rightarrow r$	$\rightarrow I(3,5)$
7	$p \rightarrow (q \rightarrow r)$	$\rightarrow I(2,6)$

Now we prove $p \rightarrow (q \rightarrow r) \vdash (p \wedge q) \rightarrow r$

$$
\begin{array}{lll}
1 & p \to (q \to r) & \text{premise} \\
\hline
2 & p \land q & \text{asm} \\
3 & p & \land E(2) \\
4 & q \to r & \to E(1,3) \\
5 & q & \land E(2) \\
6 & r & \to E(4,5) \\
\hline
7 & p \land q \to r & \to I(2,6)
\end{array}
$$

$p \to (q \to r)$ and $p \land q \to r$ are **provably equivalent formulas**

Rules for Disjunction

Introduction

To prove $\phi \lor \psi$, prove ϕ, or (if you prefer) prove ψ

$$
\begin{array}{lll}
\vdots & & \\
3 & \phi & \text{proved this somehow} \\
4 & \phi \lor \psi & \lor I(3)
\end{array}
$$

ψ can be any formula at all!

$$\vdots$$

$$3 \quad \psi \qquad\qquad \text{proved this somehow}$$

$$4 \quad \phi \vee \psi \quad \vee I(3)$$

Similarly ϕ can be any formula at all!

Elimination

To prove something from $\phi \vee \psi$, you have to prove it by assuming ϕ, and prove it by assuming ψ (arguing by cases)

$$\begin{array}{ll}
1 \quad \phi \vee \psi & \text{we got this somehow} \\
\begin{array}{|lll|lll|}
\hline
2 & \phi & \text{ass} & 7 & \psi & \text{ass} \\
 & \vdots & & & \vdots & \\
6 & \rho & & 9 & \rho & \\
\hline
\end{array} & \text{we got } \rho \text{ from both proofs} \\
10 \quad \rho \qquad\qquad \vee E(1, 2-6, 7-9) &
\end{array}$$

The assumptions ϕ, ψ are not usable later, so are put in (side-by-side) boxes.

Both boxes must end with the **same** ρ; ρ can be any formula

Nothing inside the boxes can be used later, or in the other box

Prove that the sequent $q \to r \vdash p \vee q \to p \vee r$ is valid

Example

1	$q \to r$	premise
2	$p \lor q$	asm
3	p	asm
4	$p \lor r$	$\lor I(3)$
5	q	asm
6	r	$\to E(1,5)$
7	$p \lor r$	$\lor I(6)$
8	$p \lor r$	$\lor E(2, 3-4, 5-7)$
9	$p \lor q \to p \lor r$	$\to I(2,8)$

Rules for Single Negation

These rules involve the notion of *contradiction*. They treat $\neg\phi$ like $\phi \to \bot$

The formula $\bot$ stands for the contradiction

An Example of a contradiction is $p \land \neg p$

Introduction

To prove $\neg\phi$, you assume ϕ and prove $\bot$, which stands for the contradiction.

As usual, you can't then use ϕ later on, so enclose the proof of $\bot$ from assumption ϕ in a box:

1	ϕ	asm	
2	$\vdots$		more hard work, oh no
3	$\bot$		we got it!
4	$\neg\phi$	$\neg I(1,3)$	

Elimination

From ϕ and $\neg\phi$, deduce $\perp$

1	$\neg\phi$		proved this somehow
2	$\vdots$		
3	ϕ		and this
4	$\perp$	$\neg E(3,1)$	

Example

Show that $p \to \neg p \vdash \neg p$ is a valid sequent.

1	$p \to \neg p$	premise
2	p	asm
3	$\neg p$	$\to E(1,2)$
4	$\perp$	$\neg E(2,3)$
5	$\neg p$	$\neg I(2,4)$

Rules for Double Negation

Introduction

From ϕ, deduce $\neg\neg\phi$

This is a derived rule.

$$\begin{array}{lll} 1 & \phi & \text{proved this somehow} \\ 2 & \neg\neg\phi & \neg\neg I(1) \end{array}$$

Elimination

From $\neg\neg\phi$, deduce ϕ

$$\begin{array}{lll} 1 & \neg\neg\phi & \text{proved this somehow} \\ 2 & \phi & \neg\neg E(1) \end{array}$$

Rules for Bottom

Elimination

Any formula can be derived from a contradiction

So $\perp$ is a useful formula to aim to prove

$$\begin{array}{lll} 1 & \perp & \text{we got this} \\ 2 & \phi & \perp E(1) \end{array}$$

To prove $\perp$, you must prove ϕ and $\neg\phi$ (for any ϕ you need)

1	ϕ	got this somehow
2	$\vdots$	
3	$\neg\phi$	and this
4	$\bot$	$\bot I(1,3)$

Note that $\bot I$ is the same rule as $\neg E$ (There are two names for this rule!)

Rules for Top

Not useful.

Derived Rules vs. Primitive Rules

$p \vee \neg p$ is the *law of excluded middle* (LEM)

If you haven't got ϕ, you've got $\neg\phi$. This law is true of classical logic, but not of some other logics.

We call it a **derived rule** (DL) because we can infer it from the 'primitive' rules $\vee I$, $\neg I$ and $\neg E$

Derived rules are NOT necessary, but they do help to speed proofs up

DL: Modus Tollens

From $\phi \rightarrow \psi$ and $\neg\psi$, derive $\neg\phi$

1	$\phi \to \psi$	proved this somehow
2	$\vdots$	
3	$\neg\psi$	and this
4	$\neg\phi$	MT(1,3)

DL: Proof by Contradiction

To prove ϕ, assume $\neg\phi$ and prove $\perp$

The effect of PC is to combine applications of $\neg I$ and $\neg\neg$

1	$\neg\phi$	asm
2	$\vdots$	
3	$\perp$	
4	$\neg\neg\phi$	$\neg I(1,3)$
5	ϕ	$\neg\neg E(4)$

replaced by:

1	$\neg\phi$	asm
2	$\vdots$	
3	$\perp$	
4	ϕ	$PC(1,3)$

Using PC cuts out a line

Deduction with Lemmas

A **lemma** is something you prove that helps in proving what you really want

In ND proofs, quote LEM $\phi \vee \neg\phi$ as a lemma, without proving it. Justify by '*Lemma*'.

DL: Rules for Bidirectional Implication

We treat $\phi \leftrightarrow \psi$ as $(\phi \to \psi) \wedge (\psi \to \phi)$

To prove $\phi \leftrightarrow \psi$, prove both $\phi \to \psi$ and $\psi \to \phi$

$$\begin{array}{lllr}
1 & \phi \to \psi & & \text{proved this somehow} \\
 & \vdots & & \\
2 & \psi \to \phi & & \text{and this} \\
3 & \phi \leftrightarrow \psi & \leftrightarrow I(1,2) &
\end{array}$$

From $\phi \leftrightarrow \psi$ and ϕ, you can prove ψ

From $\phi \leftrightarrow \psi$ and ψ, you can prove ϕ

$$\begin{array}{lllr}
1 & \phi \leftrightarrow \psi & & \text{proved this somehow} \\
2 & \phi & & \text{and this} \\
3 & \psi & \leftrightarrow E(1,2) &
\end{array}$$

or

$$\begin{array}{lllr}
1 & \phi \leftrightarrow \psi & & \text{proved this somehow} \\
2 & \psi & & \text{and this} \\
3 & \phi & \leftrightarrow E(1,2) &
\end{array}$$

More on ND

Though ND rules are motivated by the meaning of $\wedge, \vee, \ldots$, they are just **syntactic rules**

A computer could do a ND proof without knowing about ‘meaning’

Definition 2.1 (Natural deduction proof)

Let $\phi_1, \ldots, \phi_n, \psi$ be arbitrary formulas.

$$\phi_1, \ldots, \phi_n \vdash \psi$$

means that there is a (natural deduction) proof of ψ, starting with the formulas $\phi_1, \ldots, \phi_n$ as premises.

You can read $\phi_1, \ldots, \phi_n \vdash \psi$ as 'ψ is provable from $\phi_1, \ldots, \phi_n$'

$\vdash \psi$ means we can prove ψ with no premises at all.

We then say that ψ is a theorem (of natural deduction)

'$\vdash$' is called 'single turnstile'. Do not confuse it with $\models$

$\vdash$ is syntactic and involves proofs
$\models$ is semantic and involves situations

Advice on ND

- Think of a direct argument to prove what you want. Then translate it into ND
- If really stuck in proving ϕ, it can help to:
  - Assume $\neg\phi$ and prove $\perp$
  - Use LEM

Nasty Example

Let's show

$$\phi \vee \psi, \quad \neg\rho \to \neg\phi, \quad \neg(\psi \wedge \neg\rho) \ \vdash \ \rho.$$

is valid.

Well, assume for the sake of argument that you had $\neg\rho$.

Then you'd have $\neg\phi$—but you're given ϕ or ψ, so you get ψ.

Now you've got both ψ and $\neg\rho$, which you're told you don't: contradiction.

SO, you must have ρ.

This is quite easy to translate into ND. (Note the use of the $\vee E$ rule.)

More on box in ND

A box is 'its own little world' with its own assumptions

A box **always** starts with an assumption (the only exception is in $\forall I$ in predicate logic)

An assumption can only occur on the first line of a box

Inside a box, you can use any earlier formulas (except formulas in completed earlier boxes)

The only ways of exporting information from a box are by the rules $\to I$, $\vee E$, $\neg I$, and PC (and also $\exists E$ and $\forall I$ in predicate logic). The first line after a box must be justified by one of these

No formula inside a box can be used outside, except via the above rule

An Example

Show $\neg\phi \vdash \neg\phi$ *(!)*

1st try:

1	$\neg\phi$	premise
2	$\neg\phi$	✓(1) the best proof!

Variants of ND

The ND system we've seen can be varied by:

- Changing the rules (carefully)
- Introducing new connectives and giving rules for them

An Example

From exam 2007: The *IF* connective can be defined as

$$IF(p, q, r) = (p \to q) \land (\neg p \to r).$$

Here's an introduction rule for *IF*, based on the rules →I and ∧I:

1	ϕ	asm	2	$\neg\phi$	asm
	$\vdots$			$\vdots$	
2	ψ	got this	4	ρ	and this

5 $\quad IF(\phi, \psi, \rho) \qquad\qquad IF\ I(1-2, 3-4)$

Exercise: what would a good elimination rule (or rules) be?

Semantic Validity vs. ND

Our main concern is with validity $\models$

Recall $\phi_1, \ldots, \phi_n \models \psi$ if ψ is true in all situations in which $\phi_1, \ldots, \phi_n$ are true

$\vdash$ is useless unless it helps to establish $\models$

Definition 2.2 (Soundness and completeness)

A proof system is *sound* if every theorem is valid, and *complete* if every valid formula is a theorem.

Recall in natural deduction, a theorem is any formula ϕ such that $\vdash \phi$

Soundness of ND rules

It can be shown that ND is sound

Theorem 2.3 (Soundness of natural deduction)

Let $\phi_1, \ldots, \phi_n, \psi$ be any propositional formulas. If $\phi_1, \ldots, \phi_n \vdash \psi$, then $\phi_1, \ldots, \phi_n \models \psi$.

In other words, “*Any provable propositional formula is valid*”

Completeness of ND rules

Theorem 2.4 (Completeness)

Let $\phi_1, \ldots, \phi_n, \psi$ be any propositional formulas. If $\phi_1, \ldots, \phi_n \models \psi$, then $\phi_1, \ldots, \phi_n \vdash \psi$.

In other words, “*Any propositional validity can be proved*”

Bottom line: **We can use natural deduction to check validity**

Satisfiability vs. Consistency

Definition 2.5 (Consistency)

A formula ϕ is said to be *consistent* if $\nvdash \neg\phi$.

A collection $\phi_1, \ldots, \phi_n$ of formulas is said to be *consistent* if $\nvdash \neg \bigwedge_{1 \leq i \leq n} \phi$.

One can extend this definition to infinite collections of formulas too

Recall a propositional formula is *satisfiable* if it is true in at least one situation

By soundness and completeness (Arguments, Validity > ^theorem-2-3, Arguments, Validity > ^theorem-2-4), we get:

Theorem 2.6

A formula ϕ is *consistent* if and only if it is *satisfiable*.

Semantic Equivalence vs. Provable Equivalence

Definition 2.7 (Provable equivalence)

Two propositional formulas ϕ and ψ are *provably equivalent* if and only $\phi \vdash \psi$ and $\psi \vdash \phi$, denoted $\phi \dashv\vdash \psi$.

Recall, two propositional formulas ϕ and ψ are semantically equivalent if they are true in exactly the same situations

Roughly speaking: they mean the same

By soundness and completeness (Arguments, Validity > ^theorem-2-3, Arguments, Validity > ^theorem-2-4), we get:

Theorem 2.8

Two formulas are *provably equivalent* if and only if they are *semantically equivalent*.