Functions

A **function** f from a set A to a set B, written $f : A \to B$, is a relation $f \subseteq A \times B$ such that **every element of A is related to exactly one element of B**:

$\forall a \in A, b_1, b_2 \in B\ (\langle a, b_1 \rangle \in f \wedge \langle a, b_2 \rangle \in f \implies b_1 = b_2)$
$\forall a \in A\ \exists b \in B\ (\langle a, b \rangle \in f)$

The set A is called the **domain** and B the **co-domain** of f.

If $a \in A$, then $f(a)$ denotes the **unique** $b \in B$ such that $\langle a, b \rangle \in f$

Function notation

We write B^A for the set of all functions from A to B

We see $f : A \to B$ as shorthand for $f \in B^A$

We define $f =_{A \to B} g \triangleq \forall x \in A\ (f(x) =_B g(x))$

For any $V \subseteq A$, define the **image** of V under f to be
$f[V] \triangleq \{\ b \in B \mid \exists a \in V (f(a) = b)\ \}$

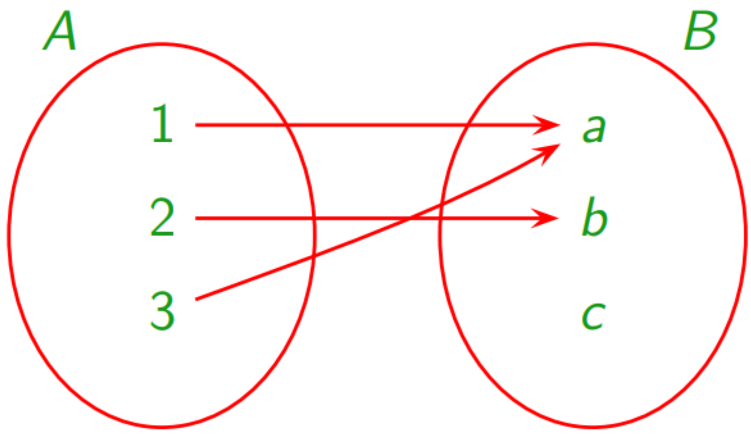
The set $f[A]$ is called the image set of f

If the domain A is the n-ary product $A_1 \times \ldots \times A_n$, then we often write $f(a_1, \ldots, a_n)$ instead of $f(\langle a_1, \ldots, a_n \rangle)$

Example 1

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$.

Let $f \subseteq A \times B$ be defined by $f = \{\langle 1, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle\}$.



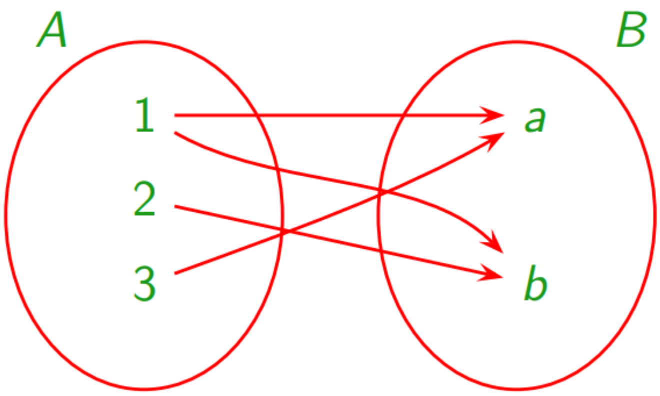
The image set of f, $f[A]$, is $\{a, b\}$.

The image of $\{1, 3\}$ under f, $f[\{1, 3\}]$, is $\{a\}$.

Example 2

Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Let $f \subseteq A \times B$ be defined by $f = \{\langle 1, a\rangle, \langle 1, b\rangle, \langle 2, b\rangle, \langle 3, a\rangle\}$.

This f is not a (well-defined) function.



1 has two images.