Trees

A rooted graph is a pair (G, x) where G is a graph and $x \in \mathrm{nodes}(G)$. The node x is the root of the tree.

A tree is a rooted, acyclic, connected graph. A nonrooted tree is an acyclic, connected graph. Sometimes the word “tree” is used to refer to nonrooted trees. In this case what we have called a “tree” will be called a “rooted tree”. We use $T, \ldots$ to stand for trees.

In a tree there is a unique (non-repeating) path between any two nodes (otherwise the tree would have a cycle).

The depth of a node x is defined to be the length of the (unique) path from the root to x.

If x is not the root, the parent of x is the unique node which is adjacent to x on the path from x to the root.

The depth of a tree is the maximum of the depths of all its nodes.

Any arc a of a tree T joins a unique non-root node $f(a)$ to its parent. Also, if x is a non-root node then there is a unique arc $g(x)$ which joins x to its parent. Clearly f and g are mutual inverses. So there is a bijection between non-root nodes and arcs. If T has n nodes, then it has $n - 1$ non-root nodes.

Let T be a tree with n nodes. Then T has $n - 1$ arcs.

The same is true for nonrooted trees (make any node into the root).

Let G be a graph. A nonrooted tree T is said to be a spanning tree for G if T spans G (T is a subgraph of G and nodes(T) = nodes(G))

Suppose that G is a connected graph. Then we can obtain a spanning tree as follows:

- If G has a cycle C
  - Remove any arc of C, joining nodes x and y, for example.
  - Now there is still a path from x to y going round the remainder of C.
  - Hence the new graph G_1 is still connected, and we have $\mathrm{nodes}(G_1) = \mathrm{nodes}(G)$

- Continue this process to get graphs $G_1, G_2, \ldots$.
- The process must terminate
- Eventually we must arrive at an acyclic graph
- This will be a spanning tree for G

Let G be a connected graph. Then G has a spanning tree.

Spanning trees are not necessarily unique.

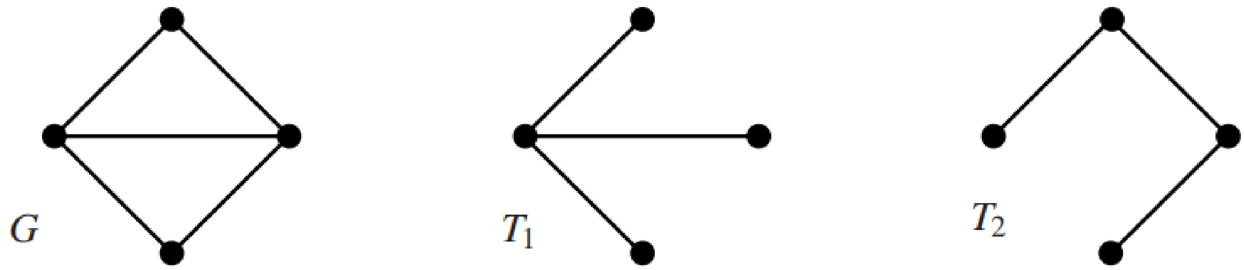

Figure 1.15: A graph and two spanning trees

Graph G has both T_1 and T_2 as spanning trees.

Any two spanning trees for the same graph with n nodes must have the same number of arcs, namely $n - 1$.