Intersection of Subspaces

Let U and W be subspaces of R^n
Then $\mathrm{U} \cap \mathrm{W}$ is also a subspace of R^n

1. $\vec{0} \in \mathrm{U} \cap \mathrm{W}$
2. $\vec{x}, \vec{y} \in \mathrm{U} \cap \mathrm{W}$ then $\vec{x} + \vec{y} \in \mathrm{U} \cap \mathrm{W}$
3. $\lambda \vec{x} \in \mathrm{U} \cap \mathrm{W}$
4. $\mathrm{U} \cap \mathrm{W} \subseteq \mathrm{R}^n$

Example 1

$$\mathrm{U} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right\}$$

$$\mathrm{W} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$$

Find $\mathrm{U} \cap \mathrm{W}$

Let $\vec{x} \in \mathrm{U} \cap \mathrm{W} \iff \vec{x} \in \mathrm{U} \cap \vec{x} \in \mathrm{W}$

$$\vec{x} = \lambda_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = -\lambda_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \lambda_4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 5 & 1 & 1 \\ 3 & 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \vec{0}$$

EROs

$$\begin{bmatrix} 1 & 0 & 0 & 5/3 \\ 0 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\lambda_1 = -(5/3)\lambda_4$
$\lambda_2 = (2/3)\lambda_4$
$\lambda_3 = -\lambda_4$
$\lambda_4 = \lambda_4$

Sub back into $(\vec{x} = \lambda_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = -\lambda_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \lambda_4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix})$

$$\vec{x} = \lambda \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

General Rule

$U, V \subseteq R^n$ (subspaces)

$\dim(U) = k$
$\dim(V) = l$

$$\max(0, k + l - n) \leq \dim(U \cap V) \leq \min(k, l)$$