Infinity

Recall that the cardinality of a **finite** set is the number of elements in that set. Let $|A| = n$; then there is a bijection

$$c_A : \{1, 2, \ldots, n\} \to A.$$

Let A and B be two **finite** sets. If A and B have the same number of elements, we can define a bijection $f : A \to B$ by

$$f(a) \quad = \quad (c_B \circ c_A^{-1})(a).$$

Two **finite** sets have the same number of elements when there exists a bijection between then.

Given two arbitrary sets A and B, then A has the same **cardinality** as B, written $|A| = |B|$, when $A \approx B$. Notice that this definition is for all sets, not just the finite ones.

A Set and its Powerset are never equivalent

Cantor showed that no function $f : A \to \wp A$ can be surjective, by showing that every such f misses a subset of A.

For any set A, $A \not\approx \wp A$

Proof: Assume that there exists a function $f : A \rightarrow \wp A$.
Now define $B \triangleq \{x \in A \mid x \notin f(x)\}$, then clearly B is a subset of A. Assume f is surjective, then there exists b such that $f(b) = B$. Then either $b \in B$ or $b \notin B$:

$b \in B$: then $b \notin f(b)$ by definition of B; since $f(b) = B$, we have $b \notin B$; Contradiction.

$b \notin B$: since $f(b) = B$, by definition of B we have $b \notin f(b)$, so $b \in B$. Contradiction.

So there is no b such that $f(b) = B$, so f is not surjective, so f is certainly not bijective. So no bijection between A and $\wp A$ can exist.

So $N \not\approx \wp N \not\approx \wp(\wp N) \not\approx \wp(\wp(\wp N))\ldots$

Countable

A set A is **countable** when **A is finite** or $A \approx N$

The elements of a countable set A can be listed as a **finite** or **infinite** sequence of distinct terms: $A = \{a_0, a_1, a_2, a_3, \ldots\}$

Example

The integers Z are countable, since they can be listed as:

$0, -1, 1, -2, 2, -3, 3, \ldots$

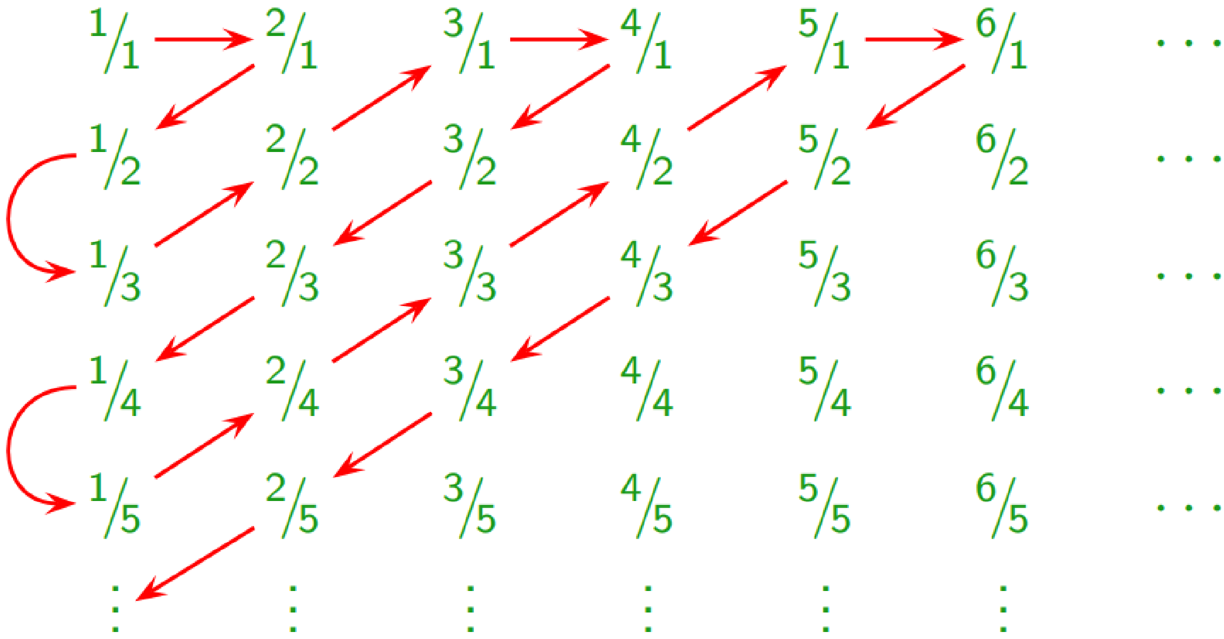
This 'counting' bijection $g : N \rightarrow Z$ is defined formally by

$$g(x) = \begin{cases} x/2, & x \text{ even} \\ -(x+1)/2, & x \text{ odd} \end{cases}$$

Notice that the bijection does not have to preserve the order!

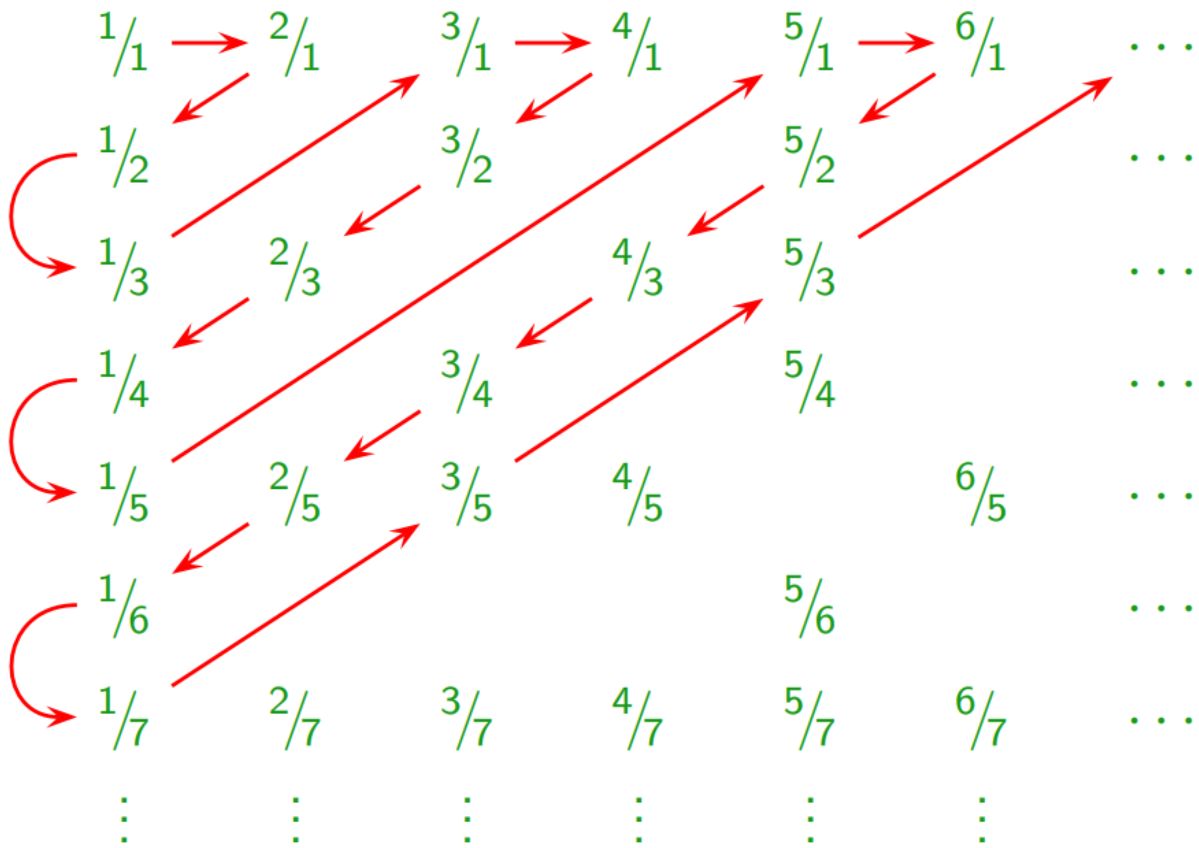
Q is countable

We can use the same approach to show that Q^+ (the set of positive fractions) is countable:



But this mapping is only a **surjection**. . .

Better . . .



So Q^+ is countable

Using the same 'trick' as for Z, so is Q

Useful Statements

Let A be a non-empty set, and B be **infinite** and **countable**

These statements are equivalent:

- A is countable
- There exists a **surjection** from B to A
- There exists an **injection** from A to B

$\wp N$ is Uncountable

We already know that $N \not\approx \wp N$; so $\wp N$ is **uncountable**

Proof:
Through its characteristic function, a subset $V \subseteq N$ can be represented as a list of 0s and 1s as $v_0, v_1, v_2, v_3 ...$ where every $v_i = 1$ if $i \in V$, and $v_i = 0$ if $i \notin V$.

We will show that **any** list of subsets of N is incomplete, so misses a subset of N.

Let $V_0, V_1, V_2, V_3, V_4, \ldots$ be any infinite list of sets. We define the set W through its characteristic function $w_0, w_1, w_2, w_3 \ldots$ so that: $w_i = 1 - v_i^i$; then $W \subseteq N$ and $i \in W \iff i \notin V_i$, so $W \neq V_i$, for all $i \in N$.

So W is **not** in the list, which is therefore **incomplete**

The Diagonalisation Argument

We can represent this through the diagram:

$$
\begin{array}{llllllllll}
V_0 = & v_0^0 & v_1^0 & v_2^0 & v_3^0 & v_4^0 & v_5^0 & v_6^0 & v_7^0 & \cdots \\
V_1 = & v_0^1 & v_1^1 & v_2^1 & v_3^1 & v_4^1 & v_5^1 & v_6^1 & v_7^1 & \cdots \\
V_2 = & v_0^2 & v_1^2 & v_2^2 & v_3^2 & v_4^2 & v_5^2 & v_6^2 & v_7^2 & \cdots \\
V_3 = & v_0^3 & v_1^3 & v_2^3 & v_3^3 & v_4^3 & v_5^3 & v_6^3 & v_7^3 & \cdots \\
V_4 = & v_0^4 & v_1^4 & v_2^4 & v_3^4 & v_4^4 & v_5^4 & v_6^4 & v_7^4 & \cdots \\
V_5 = & v_0^5 & v_1^5 & v_2^5 & v_3^5 & v_4^5 & v_5^5 & v_6^5 & v_7^5 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
W = & 1-v_0^0 & 1-v_1^1 & 1-v_2^2 & 1-v_3^3 & 1-v_4^4 & 1-v_5^5 & 1-v_6^6 & 1-v_7^7 & \cdots
\end{array}
$$

W differs from each V_i *on the diagonal*.

R is not countable

Represent $a \in [0,2] \subseteq I\!R$ via $a_0 a_1 a_2 a_3 \cdots$:

$$a \;=\; a_0 \times 2^0 + a_1 \times 2^{-1} + a_2 \times 2^{-2} + \cdots \;=\; S_{i=0}^{\infty}\, a_i 2^{-i}.$$

Note that $1.5 = 11000\cdots = 101111\cdots$; *dyadic rationals* are of the form $^n/_{2^k}$, and end with a 0-tail (or 1-tail).

Assume $[0,2]$ is countable, and $[0,2] = a^0, a^1, a^2, a^3, \ldots$.

$$\begin{array}{cccccccccc}
a^0 & = & a^0_0 & a^0_1 & a^0_2 & a^0_3 & a^0_4 & a^0_5 & \cdots \\
a^1 & = & a^1_0 & a^1_1 & a^1_2 & a^1_3 & a^1_4 & a^1_5 & \cdots \\
a^2 & = & a^2_0 & a^2_1 & a^2_2 & a^2_3 & a^2_4 & a^2_5 & \cdots \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
b & = & 1-a^0_0 & a^0_0 & 1-a^1_2 & a^1_2 & 1-a^2_4 & a^2_4 & \cdots
\end{array}$$

Note that $b \in [0,2]$ is not in the list; also, if $b_{2i} = 0$, then $b_{2i+1} = 1$, and if $b_{2i} = 1$, then $b_{2i+1} = 0$; so b is not dyadic.

We can also show that $\wp\, I\!N \approx I\!R$ (see notes).

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Comparing Infinities

We will show that Q is insignificant, a **zero set**, in R.

We can count the rationals; let $Q = q_0, q_1, q_2, \ldots$ Cast an interval in R around each element of Q, by defining:

$$\begin{array}{rcl}
V^i_\delta & \triangleq & (q_i - \delta \times 2^{-i},\; q_i + \delta \times 2^{-i}) \\
V_\delta & \triangleq & \cup_{i=0}^{\infty} V^i_\delta
\end{array}$$

Notice that $Q \subseteq V_\delta$. Remark that each V^i_δ is an interval of length $2 \times \delta \times 2^{-i} = 2^{-i+1}\delta$; we write $\| V^i_\delta \|$ for this length.

Now $\;0 \;<\; \Sigma_{i=0}^{\infty} \| V^i_\delta \| \;=\; \Sigma_{i=0}^{\infty} 2^{-i+1}\delta \;=\; 4\delta$

So we can cover Q with a set of size 4δ, for any δ. Since we can make δ as small as we like, this essentially shows that $V = \lim_{\delta \to 0} V_\delta$ is negligible (also called a **null set**) in R. But since $Q \subseteq V$, so is Q.

So $|R| = |\wp N|$ is vastly greater that $|Q| = |N|$