# Cardinality

Let A be a **finite** set.

The cardinality of A, written $|A|$, is the number of distinct elements contained in A

If A and B are disjoint finite sets, then $|A \cup B| = |A| + |B|$

Proof:

Let $|A| = n, |B| = m$

Then there exist distinct $a_1, \ldots, a_n, b_1, \ldots, b_m$ such that $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$.

Since A and B are disjoint, we have $a_i \neq b_j$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$

Then $A \cup B = \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$, which are all distinct, so:

$|A \cup B| = n + m = |A| + |B|$

Let $A$ and $B$ be finite sets. $|A \cup B| = |A| + |B| - |A \cap B|$.

**Proof**: By the previous part, and a previous exercise:

$$\begin{aligned} |A| &= |A \setminus B| + |A \cap B| \\ |B| &= |B \setminus A| + |A \cap B| \\ |A \cup B| &= |A \setminus B| + |A \cap B| + |B \setminus A| \end{aligned}$$

so

$$\begin{aligned} |A \cup B| &= |A \setminus B| + |A \cap B| + |B \setminus A| \\ &= |A| - |A \cap B| + |A \cap B| + |B| - |A \cap B| \\ &= |A| + |B| - |A \cap B| \end{aligned}$$

# Cardinality of Sets

**For any sets $A, B$, we define $A \approx B \triangleq \exists f : A \to B$ (f is a bijection)**

## Proposition

**≈ satisfies the criteria of an equivalence relation.**

**Proof**: We need to show that $\approx$ is reflexive, symmetric and transitive.

The relation $\approx$ is reflexive, as $Id_A : A \to A$ is a *bijection*.

To show that it is symmetric, $A \approx B$ implies that there is a *bijection* $f : A \to B$. By previous proposition, it follows that $f$ has an inverse $f^{-1}$ which is also a *bijection*. Hence $B \approx A$.

The fact that $\approx$ is transitive follows from the fact that a composition of bijections is a bijection, as shown before. $\square$

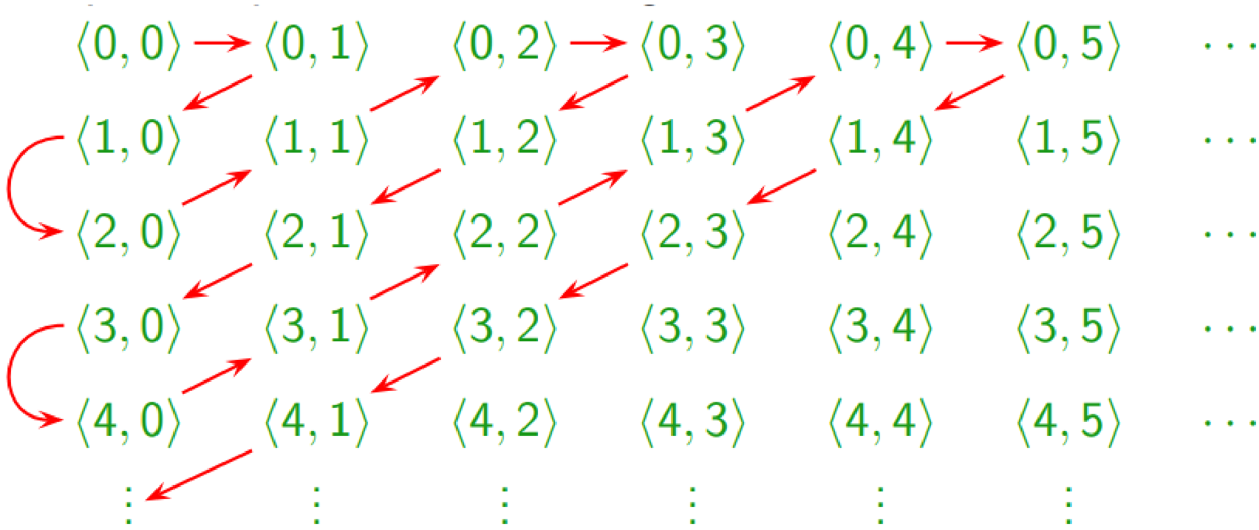
**(Dual) Cantor-Bernstein: If there exists injective (or surjective) functions $f : A \to B$ and $g : B \to A$, then $A \approx B$**

## Important Knowledge

$$
\begin{array}{lllllll}
\mathbb{N} & \approx & \mathbb{Z} & \quad & \mathbb{N} & \approx & \{V \subseteq \mathbb{N} \mid \exists n \in \mathbb{N} \ (|V| = n)\} \\
\mathbb{N} & \approx & \mathbb{N}^2 & & \mathbb{N} & \not\approx & \wp\, \mathbb{N} \\
\mathbb{N} & \approx & \mathbb{Q} & & \mathbb{N} & \not\approx & \mathbb{R}
\end{array}
$$

## Proving $N \approx N^2$

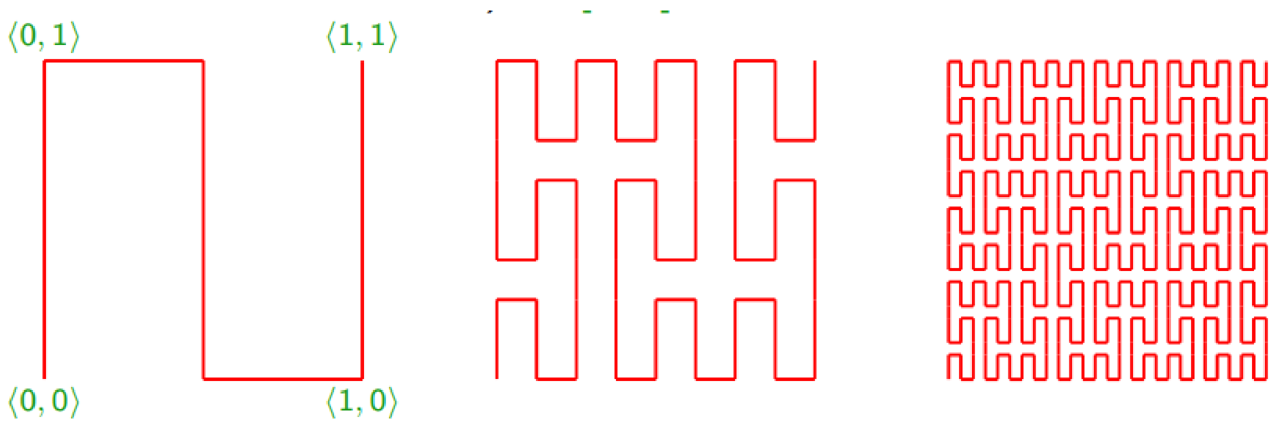
We put the pairs in an infinite grid:




**We visit each pair, and only once.**

## Proving $R \approx R^2$

Peano builds a surjection from $[0,1]$ (the closed interval of reals between 0 and 1) to $[0,1]^2$ in steps:




etc. In the limit, this is a surjection onto $[0,1]^2$

We also know that $g(x,y) = x$ is a surjection from $[0,1]^2$ to $[0,1]$; then by Cantor-Bernstein we get $[0,1]^2 \approx [0,1]$

It is easy to show that $[0,1] \approx R$ and thereby $R \approx R^2$ and also $C \approx R$

## An Example

There is a natural *bijection* $f : (A \times B) \times C \to A \times (B \times C)$:

$$f(\langle\langle a, b\rangle, c\rangle) = \langle a, \langle b, c\rangle\rangle$$

The function $g : A \times (B \times C) \to (A \times B) \times C$:

$$g\,(a, \langle b, c\rangle) = \langle\langle a, b\rangle, c\rangle$$

is the inverse of $f$; so $(A \times B) \times C \approx A \times (B \times C)$.

To be precise:

$$\begin{aligned} \mathit{Left}\,(x, y) &= x \\ \mathit{Right}\,(x, y) &= y \\ f(p, y) &= \langle \mathit{Left}\,(p), \langle \mathit{Right}\,(p), y\rangle\rangle \\ g(x, p) &= \langle\langle x, \mathit{Left}\,(p)\rangle, \mathit{Right}\,(p)\rangle \end{aligned}$$

## Another Example

Consider the set $Even$ of even natural numbers.

There is a **bijection** between Even and $N$ given by $f(n) = 2n$

Not all functions from **Even** to N are bijections.

The function $g : Even \to N$ given by $g(n) = n$ is **one-to-one** but **not onto**.

To show that $Even \approx N$, it is enough to show the existence of such a bijection.