Spectral Theorem

Special case of **Eigen decomposition / Diagonalisation**

Symmetric: $A = A^T$
If A_{nxn} is a real, symmetric matrix then:

- All its eigenvalues are real
- $\lambda_1, \lambda_2 \in spectrum(A) \cap \lambda_1 \neq \lambda_2 \cap \vec{v_1} \in E_{\lambda_1}, \vec{v_2} \in E_{\lambda_2} \implies \vec{v_1}^T \vec{v_2} = 0$
- $\lambda \in spectrum(A) \implies algebraic\ multiplicity = geometric\ multiplicity$

Algebraic Multiplicity: $(\lambda - \lambda_0)^k \implies k = AM$
Geometric Multiplicity: $dim(E_{\lambda_0})$

Diagonalisation Special Case

$A = BDB^{-1}$
Where B = $[\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}]$
Such that $\vec{v_i}.\vec{v_j} = 0 \; \forall i \neq j \cap |\vec{v_i}| = 1$

$$B^T B = \begin{bmatrix} Row(v_1) \\ Row(v_2) \\ \ldots \\ Row(v_n) \end{bmatrix} [Col(v_1) \quad Col(v_2) \quad \ldots \quad Col(v_n)] = I$$

$B^T = B^{-1}$
Saves computation

An Example

A = $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Characteristic Polynomial = $\lambda^2 - 2\lambda - 3$
$Spectrum(A) = 3, -1$

$E_3 = Ker(A - 3I)$ = span{ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ }

$E_{-1} = Ker(A+I)$ = span{$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$}

B = [$\begin{matrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{matrix}$]

B^T = [$\begin{matrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{matrix}$] = B^{-1}

D = $\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$