Integration

Lower and Upper Sums, Riemann Sums

A partition P of $[a, b]$ is given by a finite set:

$P = \{r_i : 0 \leq i \leq n-1, a = r_0, b = r_n, r_i < r_{i+1}\}$

of points in $[a, b]$ that includes the end points a and b

We can represent it simply as:

$P : a = r_0 < r_1 < \cdots, < r_i < \cdots, r_{n-1} < r_n = b$

Each closed interval $[r_i, r_{i+1}]$,for $0 \leq i \leq n-1$, is called a **subinterval of P**

The **norm of P** is defined as:

$||P|| = max\{r_{i+1} - r_i : 0 \leq i \leq n-1\}$

i.e., the largest length of the subintervals in P

If P_1 and P_2 are partitions of $[a, b]$, we say P_2 **refines** P_1 if $P_1 \subset P_2$

Given a function $f : [a, b] \rightarrow R$ and a partition of $[a, b]$ given by

$P : a = r_0 < r_1 < \cdots, < r_i < \cdots, r_{n-1} < r_n = b$

the Lower sum $L(f, P)$ and the upper sum $U(f, P)$ of f wrt P are defined as:

$L(f, P) = \sum_{i=0}^{n-1} (r_{i+1} - r_i)\ inf_{x \in [r_i, r_{i+1}]} f(x)$

$U(f, P) = \sum_{i=0}^{n-1} (r_{i+1} - r_i)\ sup_{x \in [r_i, r_{i+1}]} f(x)$

For any choice of $s_i \in [r_i, r_{i+1}]$ for $0 \leq i \leq n-1$, the sum:

$S(f, P, (s_i)_{0 \leq i \leq n-1}) = \sum_{i=0}^{n-1} (r_{i+1} - r_i) f(s_i)$

is called a Riemann sum for P

Note that if f is continuous on $[a, b]$ we can replace sup and inf in the above definition for the upper sum and lower sum by max and min respectively

For any partition we have:

$L(f,P) \leq S(f,P,(s_i)_{0\leq i\leq n-1}) \leq U(f,P)$

In addition, as we refine a partition, the lower sum increases while the upper sum decreases

The **lower and upper integrals** of $f : [a,b] \to R$ are defined as

$\underline{\int_a^b} f(x)dx = \sup_P \ L(f,P)$

$\overline{\int_a^b} f(x)dx = \inf_P \ U(f,P)$

We say f is Riemann integrable if:

$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$

and the common value is called the Riemann integral of f written as $\int_a^b f(x)dx$

Using the definitions of infimum and supremum one can show:

A bounded function $f : [a,b] \to R$ is Riemann integrable with Riemann integral $c \in R$ iff for each $\epsilon > 0$ there exists a partition P of $[a,b]$ with $c - L(f,P) < \epsilon$ and $U(f,P) - c < \epsilon$

A bounded function $f : [a,b] \to R$ is Riemann integrable with Riemann integral $c \in R$ iff for each $\epsilon > 0$ there exists a $\delta > 0$ such that for all partitions P of $[a,b]$ with $||P|| < \delta$ we have $|S(f,P,(s_i)_{0\leq i\leq n-1}) - c| < \epsilon$

Using uniform continuity of continuous functions and the fact that a continuous function attains its supremum and infimum on any closed bounded subinterval, we can show that any **continuous function on** $[a,b]$ **is Riemann integrable**

Let $f : [a,b] \to R$ be a function that is continuous on the interval $[a,b]$

Then the Riemann integral $\int_a^b f(x)dx$ exists

A bounded function with only a countable set of discontinuities on $[a,b]$ is Riemann integrable

Useful properties of the Riemann Integral

$\int_a^b sf(x) + tg(x)\,dx = s\int_a^b f(x)\,dx + t\int_a^b g(x)\,dx$ *if* f *and* g *are integrable.*

$\int_a^b c\,dx = c(b-a)$, *where* $c \in \mathbb{R}$ *is a constant.*

$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ *whenever the integrals exist for* $a < b < c$.

If $f(x) \geq 0$ *for* $x \in [a,b]$ *then* $\int_a^b f(x)\,dx \geq 0$ *if the integral exists.*

Improper Riemann Integral

Consider a function of type $f : [a, \infty) \to R$, respectively $f : [a, b) \to R$

We say that f has improper Riemann integral (or that the integral converges) if the limit $lim_{x\to\infty} \int_a^x f(x)dx$, respectively $lim_{x\to b} \int_a^x f(x)dx$, exists as real numbers

If the limit does not exist or it exists and is $\pm\infty$, then we say that the improper Riemann integral **diverges**

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The improper integral $\int_1^\infty 1/x^2\,dx = 1 - \lim_{x\to\infty} 1/x = 1$ *exists but* $\int_1^\infty 1/x\,dx = \lim_{x\to\infty} \ln x = \infty$ *diverges. Similarly* $\int_0^1 1/\sqrt{x}\,dx = 2 - 2\lim_{x\to 0}\sqrt{x} = 2$ *exists but* $\int_0^1 1/x\,dx$ *diverges.*