Function Composition

Let A, B and C be arbitrary sets, and let $f : A \to B$ and $g : B \to C$ be functions.

The **composition** of f with g, written $g \circ f : A \to C$, is a function defined by $g \circ f(a) \triangleq g(f(a))$ for every element $a \in A$

It is easy to check that $g \circ f$ is indeed a function. The co-domain of f is the same as the domain of g

Relation vs Function Composition

Notice that compositions of relations is different from functional composition: the functional composition $g \circ f$ reads f followed by g, whereas the relational composition $R \circ S$ reads R followed by S; in other words:

$$\langle a, c\rangle \in g \circ f \iff \exists b\,(\langle a, b\rangle \in f \wedge \langle b, c\rangle \in g)$$

whereas

$$\langle a, c\rangle \in R \circ S \iff \exists b\,(\langle a, b\rangle \in R \wedge \langle b, c\rangle \in S)$$

(notice the swap)

This is an overloaded notion; the context will make clear which one is intended.

Example

Let $A = \{1, 2, 3\}$, $B = \{a, b\}$, $C = \{\alpha, \beta, \gamma\}$, and

$$\begin{aligned} f : A \to B &= \{\langle 1, a\rangle, \langle 2, b\rangle, \langle 3, a\rangle\} \\ g : B \to C &= \{\langle a, \gamma\rangle, \langle b, \alpha\rangle\} \end{aligned}$$

Then

$$g \circ f : A \to C = \{\langle 1, \gamma\rangle, \langle 2, \alpha\rangle, \langle 3, \gamma\rangle\}$$