Basics

Graphs are used to model networks. A graph is a set of points joined by lines. Points are referred to as nodes, and lines are referred to as arcs.

Each arc has two endpoints. Graphs may have the following:

- parallel (multiple) arcs: two arcs with same endpoints
- loops: an arc with both endpoints the same

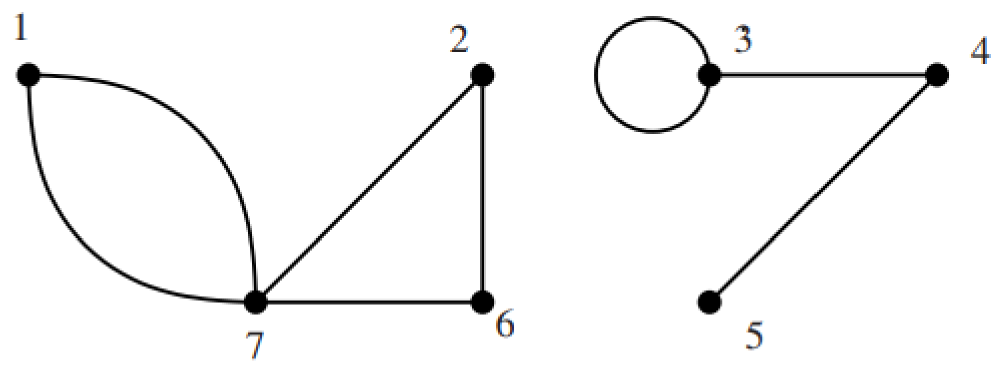

Figure 1.1: A graph

An (undirected) graph is a set N of nodes together with a set A of arcs, such that each $a \in A$ is associated with an unordered pair of nodes (the endpoints of a).

A graph is simple if it has no parallel arcs and no loops.

We will refer to graphs by G.
$\mathrm{nodes}(G)$: The nodes in graph G
$\mathrm{arcs}(G)$: The arcs in graph G

An arc is incident on its endpoints.

A node n is incident on any arc a which has n as one of its endpoints.

An arc joins its two endpoint nodes. Two nodes n, n' are adjacent if they are joined by some arc.

The degree of a node = the number of arcs incident on it, where loops are counted twice.

A node is odd if its degree is odd.
A node is even if its degree is even.

For instance in Figure 1.1, the degrees are as follows:

node	1	2	3	4	5	6	7
degree	2	2	3	2	1	2	4

In any graph, the total of the degrees of all the nodes = twice the number of arcs.

The number of odd nodes is even.

Let G_1, G_2 be graphs. We say that G_1 is a subgraph of G_2 if $\mathrm{nodes}(G_1) \subseteq \mathrm{nodes}(G_2)$ and if $\mathrm{arcs}(G_1) \subseteq \mathrm{arcs}(G_2)$.

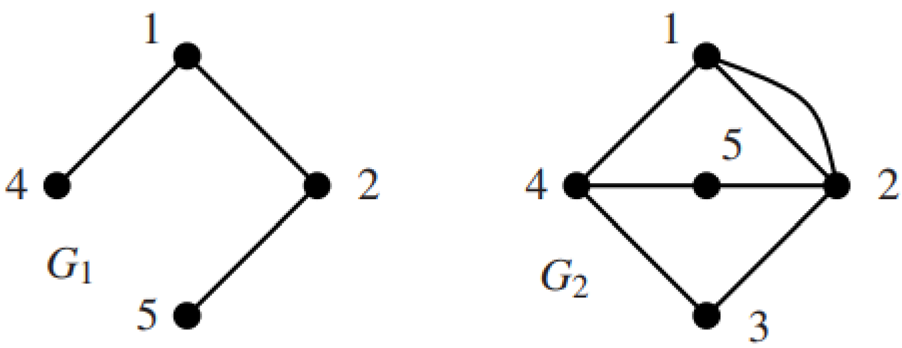

Figure 1.2: G_1 is a subgraph of G_2

Any set $X \subseteq \mathrm{nodes}(G)$ induces a subgraph $G[X]$ with $\mathrm{nodes}(G[X]) = X$ and $G[X]$ inheriting all arcs of G between nodes in X.

A graph G′ is a full (or induced) subgraph of G if $G' = G[X]$ for some $X \subseteq \mathrm{nodes}(G)$. If G′ is a subgraph of G and nodes(G′) = nodes(G), we say that G′ spans G (or is a spanning subgraph of G).

Adjacency Matrix

Suppose that a graph has k nodes $n_1, \ldots, n_k$.

$\mathrm{adj}(i, j)$ = the number of arcs joining n_i to n_j

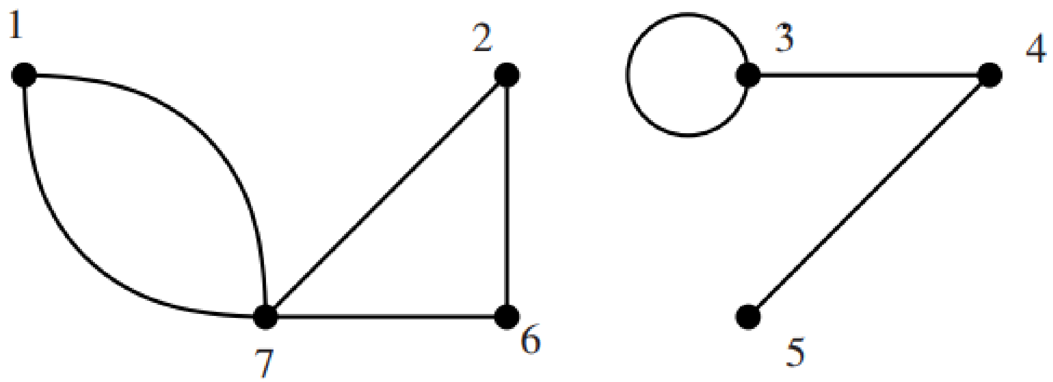

Figure 1.1: A graph

First two rows of Adjacency Matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The matrix is symmetric, since the arcs are undirected (if node i is joined to j then j must be joined to i).

A single graph can have many adjacency matrices, corresponding to different enumerations of the elements (putting the nodes in a different order).

If a graph has n nodes then there are n^2 entries in the adjacency matrix.

A sparse graph: a graph with much fewer than n^2 arcs.

When dealing with a sparse graph, save time and space by using its adjacency list representation instead.

Adjacency List Representation

This consists of an array of n pointers, where the i^{th} pointer points to a linked list $adj[i]$ of the nodes which are adjacent to node i.

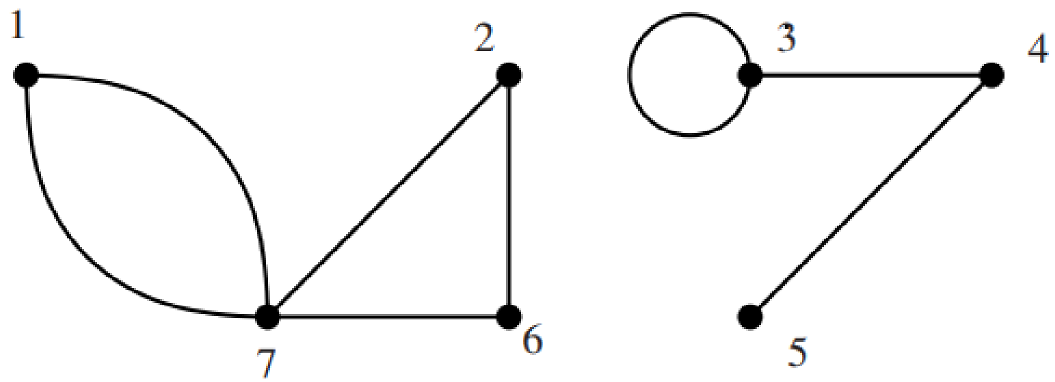

Figure 1.1: A graph

This graph's Adjacency List Representation.

$$\begin{array}{lllllllll}
1 & \to & 7 & \to & 7 \\
2 & \to & 6 & \to & 7 \\
3 & \to & 3 & \to & 4 \\
4 & \to & 3 & \to & 5 \\
5 & \to & 4 \\
6 & \to & 2 & \to & 7 \\
7 & \to & 1 & \to & 1 & \to & 2 & \to & 6
\end{array}$$

Each arc is recorded twice (except for loops, which appear once).

If there are n nodes and m arcs, then the total number of entries in the ALR is bounded by $n + 2m$.

Big-Oh Notation

Suppose that we wish to multiply together two n × n matrices.

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 6 & 1 \\ 2 & 0 & 5 \\ 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 14 & 5 \\ 11 & 18 & 8 \\ 19 & 36 & 20 \end{pmatrix}$$

Each entry in the product takes n multiplications and $n - 1$ additions. There are n^2 entries. Therefore, the total number of multiplications is n^3 , the total number of additions is $n^3 - n^2$, and the total number of arithmetical operations is $2n^3 - n^2$.

We don't need to be so precise. Instead, use 'big-O' notation.

$O(n^k)$ = bounded by a constant factor times n^k.

Therefore, the total number of arithmetical operations is $O(n^3)$.

For some values of k we use special names:
$O(1)$: constant
$O(n)$: linear (in n)
$O(n^2)$: quadratic (in n)