Orderings

Let R be a binary relation on a set A.

R is a **pre-order**: R is **reflexive** and **transitive**
R is **anti-symmetric**: for all a, b \in A

$$a \; R \; b \wedge b \; R \; a \implies a = b$$

R is a **partial order** (po): R is **reflexive**, **transitive** and **anti-symmetric**
R is **irreflexive**: $\forall a \in A \; (\neg(a \; R \; a))$
R is a **strict partial order**: R is **irreflexive** and **transitive**.
A partial order R is a **total order** if:

$$\forall a, b \in A \; (a \; R \; b \vee b \; R \; a)$$

The numerical orders $\leq$ on $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ are total orders. The orders $<$ are strict partial orders.

Ordering of Products

For any two partially ordered sets $(A, \leq_A)$ and $(B, \leq_B)$, there are two important orders on the product set $A \times B$:

Product Order:

$$\langle a_1, b_1 \rangle \leq_P \langle a_2, b_2 \rangle \triangleq a_1 \leq_A a_2 \wedge b_1 \leq_B b_2$$

Lexicographic order:

$$\langle a_1, b_1 \rangle \leq_L \langle a_2, b_2 \rangle \triangleq a_1 <_A a_2 \vee (a_1 =_A a_2 \wedge b_1 \leq_B b_2)$$

If $(A, \leq)$ and $(B, \leq)$ are total orders, then the lexicographic order on $A \times B$ is total.

In general, the product order is partial.

Analysing Partial Orders

Let $(A, \leq)$ be a partial order, and $a \in A$

a is **minimal** $\triangleq \forall b \in A\ (b \leq a \implies b = a)$

a is **least** $\triangleq \forall b \in A\ (a \leq b)$

a is **maximal** $\triangleq \forall b \in A\ (a \leq b \implies a = b)$

a is **greatest** $\triangleq \forall b \in A\ (b \leq a)$

Well-founded Partial Orders

A partial order $(A, \leq)$ is **well founded** when it has **no infinite decreasing chain of elements**: for every infinite sequence $a_1, a_2, a_3, \ldots$ of elements in A with $a_1 \geq a_2 \geq a_3 \geq \ldots$, there exists $m \in N$ such that $a_n = a_m$ for every $n \geq m$.

A well-founded order need not be total, like the subset relation on the set of finite subsets of N

Proposition

If two partial orders $(A, \leq)$ and $(B, \leq)$ are well founded, then the lexicographical order on $A \times B$ is also well founded.

Proof Suppose $\langle a_1, b_1 \rangle \geq_L \langle a_2, b_2 \rangle \geq_L \langle a_3, b_3 \rangle \geq_L \ldots$.

Then $a_1 \geq_A a_2 \geq_A a_3 \geq_A \ldots$ by the definition of $\geq_L$.

Since $(A, \leq_A)$ is well founded, there exists $m \in I\!N$ such that $a_n = a_m$ for every $n \geq m$.

We also have $b_m \geq_B b_{m+1} \geq_B b_{m+2} \geq_B \ldots$.

This sequence also ends up being constant because $(B, \leq_B)$ is well founded. Thus, the original sequence is ultimately constant.

The Ackermann Function

Take the function $\mathrm{Ack} : N^2 \to N$ defined by:

$$\begin{aligned}
\mathsf{Ack}\,(0, y) &= y + 1 \\
\mathsf{Ack}\,(x + 1, 0) &= \mathsf{Ack}\,(x, 1) \\
\mathsf{Ack}(x + 1, y + 1) &= \mathsf{Ack}\,(x, \mathsf{Ack}\,(x + 1, y))
\end{aligned}$$

We will prove that this function always terminates using a **well-founded partial order**. Consider the **strict** lexicographical order on N^2 by

$\langle x, y\rangle < \langle x', y'\rangle$ when $x < x'$ or ($x = x'$ and $y < y'$)

Notice:

$$\begin{aligned}
\langle x + 1, 0\rangle &> \langle x, 1\rangle \\
\langle x + 1, y + 1\rangle &> \langle x, \mathsf{Ack}(x + 1, y\rangle) \\
\langle x + 1, y + 1\rangle &> \langle x + 1, y\rangle
\end{aligned}$$

Evaluating the Ack function takes us down the order, which is **well founded**. Hence, the Ack program always gives an answer.