# Graph Isomorphism and Planar Graphs

## Example of Isomorphism

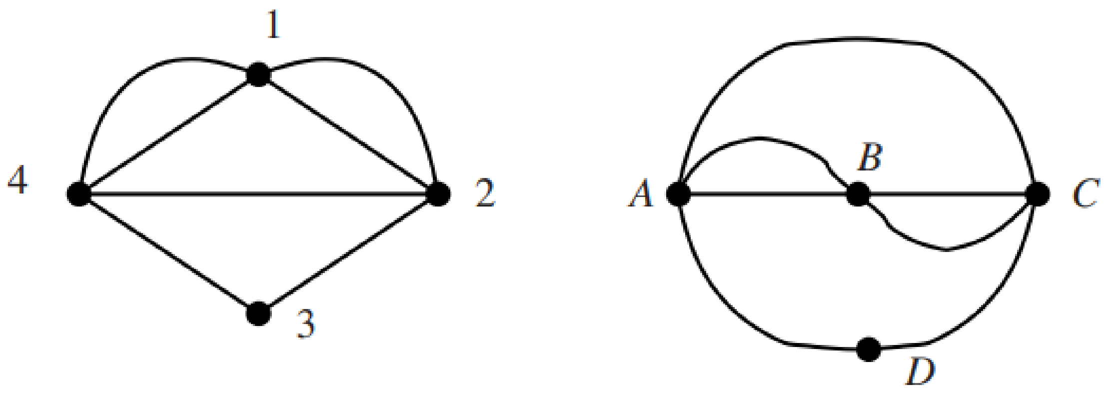



Figure 1.3: Two isomorphic graphs

Although the nodes are labelled differently, they are connected in the same way.

First, 3 matches with D, since they are the only nodes with degree 2. Next, 1 matches with B, since they are not joined to 3, D respectively. Finally 2 matches with C, and 4 matches with A.

We have set up a bijection between the two sets of nodes, which preserves the connectedness of the nodes.

Adjacency Matrix of 1st graph:

$$\begin{pmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

If we order the nodes of the second graph as B, A, D, C we get the same matrix. This bijection is an example of a graph isomorphism.

## Isomorphism Definition

Let G, G′ be graphs.

An isomorphism from G to G′ is a bijection $f : nodes(G) \to nodes(G')$ with a bijection $g : arcs(G) \to arcs(G')$ such that for any arc $a \in arcs(G)$, if the endpoints of a are $n_1, n_2 \in nodes(G)$, then the endpoints of g(a) are $f(n_1)$ and $f(n_2)$.

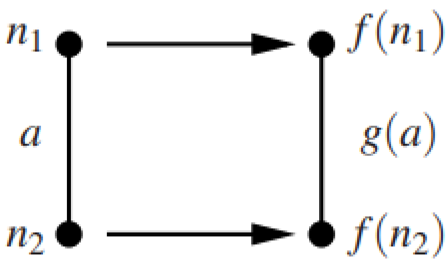



Figure 1.4: Isomorphism

If there is an isomorphism from G to G′ we say that G is isomorphic to G′, or that G and G′ are isomorphic.

# Properties of Isomorphic Graphs

If G is isomorphic to G′ and $n_1, n_2 \in nodes(G)$, then $f(n_1)$ and $f(n_2)$ are connected by the same number of arcs in G′ as $n_1$ and $n_2$ are connected by in G.

- Same number of nodes
- Same number of arcs
- Same adjacency matrices (apart from a possible reordering of rows and columns)
- Same number of loops
- Nodes have the same degrees (possibly reordered)

If they pass the initial tests, then attempt to set up the bijection on nodes, matching nodes with the same degree and preserving connectedness. If we get the adjacency matrices to match then the graphs are isomorphic.

# Automorphism Definition

Let G be a graph. An automorphism on G is an isomorphism from G to itself. The identity is trivial.

# Example of Automorphism

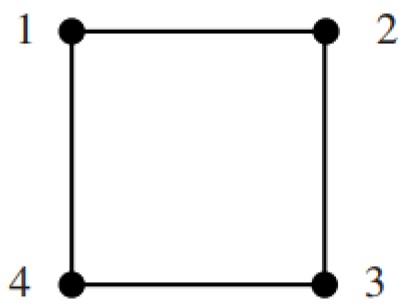



Figure 1.5: A square graph

One possible automorphism maps 1 to 2, 2 to 3, 3 to 4 and 4 to 1. The mapping on arcs is determined once the mapping on nodes is given, since there are no multiple arcs. Another possible automorphism maps 1 to 3, 2 to 2, 3 to 1 and 4 to 4.

Suppose we wish to count the number of automorphisms of a graph.

- Go through the nodes 1 to n in turn
    - See how many possibilities there are for where node i can go, given that nodes 1 to $i-1$ are fixed
- Multiply together the numbers of possibilities

**When you have parallel arcs, rearranging these also creates new possibilities.**

For the square:

- 1 can map to any of the four nodes.
- Given that 1 is fixed, 2 can map to only two of the remaining three nodes (the nodes which are adjacent to where 1 has been assigned)
- Given that 1 and 2 are fixed, the positions of nodes 3 and 4 are determined.
- Therefore, there are $4 \times 2 = 8$ possible automorphisms. One of these is the identity.

# Planar Graphs

Arcs can cross in our diagrams. For Example:

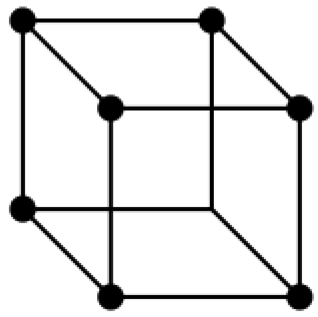


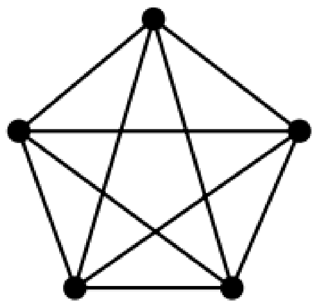
Figure 1.6: A “cube” graph

A graph is planar if it can be embedded in the plane (by rearranging the nodes and arcs). The above graph is planar.

If a (simple) graph is planar, we can always draw the arcs as straight lines which do not cross.

Here are two non-planar graphs:

$K_5$



$K_{3,3}$

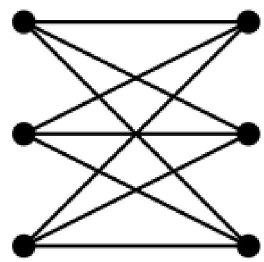


Figure 1.7: Two non-planar graphs

$K_5$ is the complete graph on five nodes (where every node is joined to every other node). $K_{3,3}$ is the graph with all possible arcs between two sets of size 3. There is no way to redraw either of these so that no arcs cross.

Two graphs are homeomorphic if they can be obtained from the same graph by a series of operations replacing an arc $x - y$ by two arcs $x - z - y$.

A graph is planar if and only if it does not contain a subgraph homeomorphic to $K_5$ or $K_{3,3}$.

Any planar graph splits the plane into various regions called “faces”. For instance, the cube graph has 6 faces (the region surrounding the graph counts as a face).

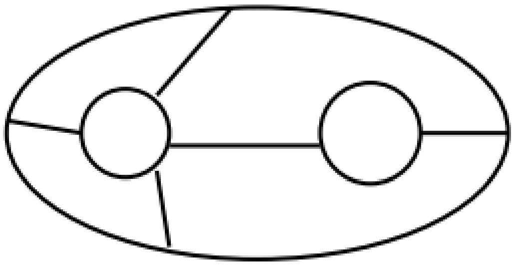


Figure 1.8: A map

Let G be a connected planar graph. Let G have N nodes, A arcs and F faces. Then $F = A - N + 2$.

## Colouring Maps and Graphs

Each map can be turned into a graph by placing a node inside each country (its “capital city”), and joining two nodes precisely if the countries share a border. Call this the dual graph of the map. Clearly the dual graph will be (simple and) planar. Also, any simple planar graph is the dual graph of some map.

Any colouring of a map gives rise to a colouring of the nodes of the dual graph, in such a way that no two adjacent nodes have the same colour.

A graph G is k-colourable if nodes(G) can be coloured using no more than k colours, in such a way that no two adjacent nodes have the same colour.

$K_{3,3}$ is an example of a bipartite graph, as is the cube graph above.

A graph G is bipartite if nodes(G) can be partitioned into two sets X and Y in such a way that no arc of G joins any two members of X, and no arc joins any two members of Y.

A graph is bipartite if and only if it is 2-colourable.

Four Colour Theorem: Every map (equivalently, every simple planar graph) is 4-colourable.