Properties of Relations

Let R be a binary relation on A.

R is reflexive $\triangleq \forall x \in A\ (x\ R\ x)$
R is symmetric $\triangleq \forall x, y \in A\ (x\ R\ y \implies y\ R\ x)$
R is transitive $\triangleq \forall x, z \in A\ (\exists y \in A\ (x\ R\ y \wedge y\ R\ z) \implies x\ R\ z)$

Let R, $S \subseteq A^2$. Prove that the following statements are true:

$$\begin{aligned} R \subseteq S &\Rightarrow R^{-1} \subseteq S^{-1} \\ (R \cap S)^{-1} &= R^{-1} \cap S^{-1} \\ (R \cup S)^{-1} &= R^{-1} \cup S^{-1} \\ (\overline{R})^{-1} &= \overline{R^{-1}} \\ (R \circ S)^{-1} &= S^{-1} \circ R^{-1} \end{aligned}$$

If $R \subseteq A \times B$, then $Id_A \circ R = R = R \circ Id_B$.

Let R, S be binary relations on A.

1. R is reflexive if and only if $Id_A \subseteq R$.
2. R is symmetric if and only if $R = R^{-1}$.
3. R is transitive if and only if $R \circ R \subseteq R$.
4. Show that $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$
5. Now suppose that $R \subseteq S$ and S are symmetric. Show that $R \cup R^{-1} \subseteq S$.
6. Use part 4 to show that R symmetric implies $R \circ R$ is symmetric.