Cartesian Product

An **ordered pair** $\langle a, b\rangle$ is a pair of objects a and b where the order of a and b matters (so, if $a \neq b$, then $\langle a, b\rangle \neq \langle b, a\rangle$).

Let A and B be arbitrary sets. The Cartesian product of A and B, written $A \times B$, is defined by:

$$A \times B \triangleq \{ \langle a, b\rangle \mid a \in A \wedge b \in B \}$$

We will write A^2 for $A \times A$

Equality on elements of $A \times B$ is defined as:

$$\forall a, b, c, d \, (\langle a, b\rangle =_{A \times B} \langle c, d\rangle \triangleq a =_A c \wedge b =_B d)$$

Let A and B be finite sets. Then $|A \times B| = |A| \times |B|$

Proof : By counting. Suppose that A and B are arbitrary sets with $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$. To represent the product space, we draw a table with m rows and n columns of the members of $A \times B$:

$$\begin{array}{cccc}
\langle a_1, b_1\rangle & \langle a_1, b_2\rangle & \ldots & \langle a_1, b_n\rangle \\
\langle a_2, b_1\rangle & \langle a_2, b_2\rangle & \ldots & \langle a_2, b_n\rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle a_m, b_1\rangle & \langle a_m, b_2\rangle & \ldots & \langle a_m, b_n\rangle
\end{array}$$

Such a table has $m \times n$ entries.