# Preliminaries

**See Discrete Maths Folder**

If $S \subseteq T$ but $S \neq T$, then S is a proper subset of T : $S \subset T$.

Because the set theory that we have considered so far admits paradoxes, such as Russell's Paradox, it is called the **naïve set theory**

However, it suffices for everyday use in mathematics

We will often be concerned with multiple sets which are all subsets of some underlying set R, for example, various sets of points on the real line

In this case, given a set S, the difference $R \setminus S$ is called the complement of $S$, denoted by $\bar{S}$

# De Morgan's Laws

$$\overline{\bigcup_{\alpha} S_\alpha} = \bigcap \overline{S_\alpha},$$
$$\overline{\bigcap_{\alpha} S_\alpha} = \bigcup \overline{S_\alpha};$$

as a special case,

$$\overline{S \cup T} = \overline{S} \cap \overline{T},$$
$$\overline{S \cap T} = \overline{S} \cup \overline{T}.$$

The preimage of the union of two sets is the union of the preimages of the sets:

$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

The preimage of the intersection of two sets is the intersection of the preimages of the sets:

$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

The image of the union of two sets equals the union of the images of the sets:

$$f(A \cup B) = f(A) \cup f(B)$$

**Remark 1**

*Surprisingly enough, the image of the intersection of two sets does not necessarily equal the intersection of the images of the sets. For example, suppose the mapping $f$ projects the $xy$-plane onto the $x$-axis, carrying the point $(x, y)$ into the $(x, 0)$. Then the segments $0 \le x \le 1, y = 0$ and $0 \le x \le 1, y = 1$ do not intersect, although their images coincide.*

**Remark 2**

*Theorems 2, 3, and 4 hold for unions and intersections of an* **arbitrary** *number (finite or infinite) of sets $A_\alpha$:*

$$f^{-1}\left(\bigcup_\alpha A_\alpha\right) = \bigcup_\alpha f^{-1}(A_\alpha),$$

$$f^{-1}\left(\bigcap_\alpha A_\alpha\right) = \bigcap_\alpha f^{-1}(A_\alpha),$$

$$f\left(\bigcup_\alpha A_\alpha\right) = \bigcup_\alpha f(A_\alpha).$$

### Every subset of a countable set is countable

**Proof** *Let $S$ be countable, with elements $s_1, s_2, \ldots$, and let $A$ be a subset of $S$. Among the elements $s_1, s_2, \ldots$, let $s_{n_1}, s_{n_2}, ldots$ be those in the set $A$. If the set of numbers $n_1, n_2, \ldots$ has a largest number, then $A$ is finite. Otherwise $A$ is countable (consider the correspondence $i \leftrightarrow a_{n_i}$).* $\square$

The union of a finite or countable number of countable sets $A_1, A_2, \ldots$ is itself countable.

**Proof** *We can assume that no two of the sets $A_1, A_2, \ldots$ have elements in common, since otherwise we could consider the sets*

$$A_1, A_2 \setminus A_1, A_3 \setminus (A_1 \cup A_2), \ldots$$

*instead, which are countable by Theorem 5 and have the same union as the original sets. Suppose we write the elements of $A_1, A_2, \ldots$ in the form of an infinite table*

$$\begin{array}{ccccc}
a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\
a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\
a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\
a_{41} & a_{42} & a_{43} & a_{44} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

*where the elements of the set $A_1$ appear in the first row, the elements of the set $A_2$ appear in the second row, and so on. We now count all the elements in this table "diagonally," i.e., first we choose $a_{11}$, then $a_{12}$, then $a_{21}$, and so on, moving in the way shown in the following table:*

$$\begin{array}{ccccccccc}
a_{11} & \rightarrow & a_{12} & & a_{13} & \rightarrow & a_{14} & & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & & \\
a_{21} & & a_{22} & & a_{23} & & a_{24} & & \cdots \\
\downarrow & \nearrow & & \swarrow & & & & & \\
a_{31} & & a_{32} & & a_{33} & & a_{34} & & \cdots \\
 & \swarrow & & & & & & & \\
a_{41} & & a_{42} & & a_{43} & & a_{44} & & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

*It is clear that this procedure associates a unique number to each element in each of the sets $A_1, A_2, \ldots$, thereby establishing a one-to-one correspondence between the union of the sets $A_1, A_2, \ldots$ and the set $\mathbb{N}$ of all positive integers.* $\square$

The symbol $\exists$ is called the **existential quantifier**

The symbol $\forall$ is called the **universal quantifier**

To prove this:

$$\sqrt{2} + \sqrt{6} < \sqrt{15}.$$

Try this:

$$\sqrt{2}+\sqrt{6} \geq \sqrt{15} \Rightarrow \left(\sqrt{2}+\sqrt{6}\right)^2 \geq 15$$
$$\Rightarrow 8+2\sqrt{12} \geq 15 \Rightarrow 2\sqrt{12} \geq 7 \Rightarrow 48 \geq 49,$$

This is a contradiction, therefore, we have proved what we wanted to prove.

In mathematics we don’t consider functions with side effects. We are solely concerned with the input and output of a function; a mathematical function can have no side effects.