Metric Spaces

Definition of a metric space

A pair of objects (X, d) consisting of a nonempty set X and a function $d : X \times X \to R$ is called a **metric space** provided that:

- $d(x, y) = 0 \iff x = y$ for all $x, y \in X$
- $d(x, y) = d(y, x)$ for all $x, y \in X$
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

The function d is called a **distance function** or **metric** on X and the set X is called the **underlying set**

Let X be any nonempty set. For $x, y \in X$ define:

$$d(x,y) = \begin{cases} 0, & if\, x = y; \\ 1, & if\, x \neq y. \end{cases}$$

Then (X, d) is a metric space, called a **discrete space** or a **space of isolated points**. In this metric space no distinct pair of points are “close”

If (X, d) is a metric space, and S a subset of X, we may use in S the same distance function d, except that it be restricted to S

S becomes in this way a metric space (S, d), a metric subspace of (X, d)

For the sake of brevity we may refer to the metric space (X, d) as the metric space X when the metric d is clear from the context

The **Euclidean distance** is the usual metric on R^2

The Euclidean distance is not the only metric that can be defined on R^2:

The maximum metric: $d_\infty(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|)$
The taxi cab metric: $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$

Some idea of what these metrics are like can be obtained by plotting in the plane the points at distance 1 from the origin

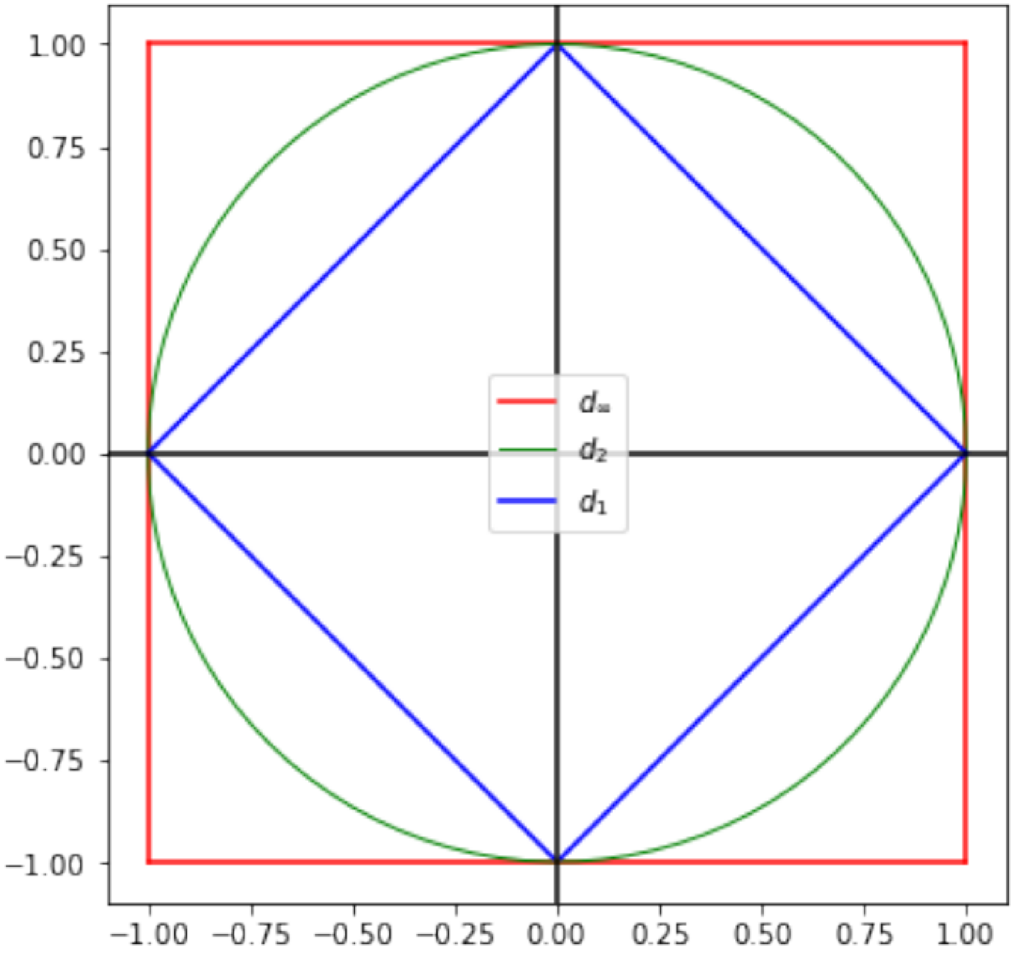

Figure 13.2: Unit circles in $\mathbb{R}^2$ for different metrics.

The above metrics can be extended to R^n by defining, for $x, y \in R^n$:

$$d_2(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

$$d_\infty(x,y) = \max_{1 \le i \le n} |x_i - y_i|$$

These three distance functions are **translation invariant** i.e., for all three of them we have $d(x+z, y+z) = d(x,y)$

They are examples of a **norm** on a vector space, such as R^n

A norm $||\cdot||$ on vectors in R^n is defined as map $||\cdot|| : R^n \to R$ satisfying:

- $||x|| \geq 0$ for all $x \in R^n$
- $||rx|| = |r| ||x||$ for all $x \in R^n$ and $r \in R$
- $||x+y|| \leq ||x|| + ||y||$ for all $x, y \in R^n$

We say two metrics d_1 and d_2 on a space X are **equivalent** if there exist $p > 0$ and $q > 0$ such that:

$\forall x, y \in X .\, d_1(x,y) \leq p d_2(x,y) \wedge d_2(x,y) \leq q d_1(x,y)$

We say two norms on R^n are **equivalent** if they are equivalent as metrics

Exercise 32
Show that we have $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$ for all $x \in \mathbb{R}^n$, by proving that the squares of these expressions satisfy the above inequalities.

Example 56
We define a metric on the space $\{0,1\}^{\mathbb{N}}$ of infinite sequences of bits of the form

$$b = b_0 b_1 b_2 \ldots$$

where $b_i \in \{0,1\}$ for $i \in \mathbb{N}$. For example

$$0000100110011111 \ldots \in \{0,1\}^{\mathbb{N}}$$

Define

$$d(b,b') = \begin{cases} 0 & \text{if } b = b' \\ 1/2^n & \text{if } n \in \mathbb{N} \text{ is the smallest integer with } b_n \neq b'_n \end{cases}$$

Then d clearly satisfies $d(b,b') = 0$ iff $b = b'$, and $d(b,b') = d(b',b)$. Check that d satisfies the triangular inequality and is hence a metric on $\{0,1\}^{\mathbb{N}}$.

Exercise 33
Show that for all $x \in \mathbb{R}^n$

$$\|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty$$

Then show that the three norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ in $\mathbb{R}^n$ are pairwise equivalent by using Exercise 32.

Limits

Definition 39
Let (X,d) be a metric space. Let $x_1, x_2, \ldots$ be a sequence of points of X. A point $x \in X$ is said to be the **limit of the sequence** *$x_1, x_2, \ldots$ if $\lim_{n\to\infty} d(x,x_n) = 0$. In this event, we shall say that the sequence $x_1, x_2, \ldots$* **converges to** *x and write $\lim_{n\to\infty} x_n = x$.*