Powerset

Let A be any set. Then the power set of A, written $\wp A$, is the set of all subsets of A, i.e.

$$\wp A \triangleq \{ V \mid V \subseteq A \}$$

This is assumed to be a set.

Examples

$$\begin{aligned} \wp\{a, b\} &= \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \\ \wp\,\emptyset &= \{\emptyset\} \\ \wp\,I\!N &= \{\emptyset, \{1\}, \{2\}, \ldots, \{1, 2\}, \{1, 3\}, \ldots, \\ &\qquad \{2, 3\}, \ldots, \{1, 2, 3\}, \ldots\} \end{aligned}$$

The power set of the empty set is not empty!

Let A be a finite set with $|A| = n$. Then $|\wp A| = 2^n$

Proof: Let $A = \{a_1, \ldots, a_n\}$. Form a subset X of A by taking each element a_i in turn and deciding whether or not to include it in X. This gives us n independent choices between two possibilities: in X or not.

We can therefore represent each subset X of A by a binary number $b = b_1 \cdots b_n$, where $b_i = 1$ if $a_i \in X$, and $b_i = 0$ if $a_i \notin X$.

The number of different subsets we can form is therefore the same as the number of binary numbers representable with n bits, which is 2^n.