Inner Product (dot)

In R^n space

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$$

$\vec{a} . \vec{b} = \vec{b} . \vec{a} = \vec{b}^T \vec{a} = \vec{a}^T \vec{b} = \sum_{i=1}^{n} a_i b_i = |\vec{a}||\vec{b}| \cos\theta$

θ = angle from $\vec{a}$ to $\vec{b}$

$\vec{a}$ and $\vec{b}$ are perpendicular($\perp$) $\Leftrightarrow$ $\vec{a}^T \vec{b} = 0$

$(\vec{a}^T \vec{b})^2 \leq |\vec{a}|^2 |\vec{b}|^2$ because $\cos$ is between -1 and 1

$|\vec{a}|^2 = \vec{a} . \vec{a}$

Cauchy-Schwartz Inequality

$(\vec{a} . \vec{b})^2 \leq (\vec{a} . \vec{a})(\vec{b} . \vec{b})$

$(\vec{a} . \vec{b})^2 = (\vec{a} . \vec{a})(\vec{b} . \vec{b})$ when $\vec{a} = k\vec{b}$

Example Use of CSI

Let $K_1, K_2, \dots K_n \geq 0$

To Prove:
$(1/K_1 + 1/K_2 + \dots 1/K_n)(K_1 + K_2 + \dots K_n) \geq n^2$

$$\vec{a} = \begin{bmatrix} 1/\sqrt{K_1} \\ 1/\sqrt{K_2} \\ \dots \\ 1/\sqrt{K_n} \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} \sqrt{K_1} \\ \sqrt{K_2} \\ \dots \\ \sqrt{K_n} \end{bmatrix}$$

Plug into CSI, get what we want to prove

Outer Product

In R^n space:

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$$

$\vec{a}^T \vec{b}$ = scalar

$\vec{b}\vec{a}^T$ = nxn matrix

Orthogonal Projection

Consider the R^2 space

U = span{$\vec{b}$}

$\pi_U(\vec{a})$ = $P_U(\vec{a})$ = Closest point in U to $\vec{a}$

$P_U(\vec{a})$ = $\lambda \vec{b}$

$\vec{b}^T(\vec{a} - \vec{b}\lambda)$ = 0

$\vec{b}^T \vec{a} = \vec{b}^T \vec{b} \lambda$

$\lambda = (\vec{b}^T \vec{b})^{-1} \vec{b}^T \vec{a}$

$\vec{b}\lambda = \vec{b}(\vec{b}^T \vec{b})^{-1} \vec{b}^T \vec{a} = (\vec{b}(\vec{b}^T \vec{b})^{-1} \vec{b}^T)\vec{a}$

Projection Matrix = $\vec{b}(\vec{b}^T\vec{b})^{-1}\vec{b}^T$ = $\vec{b}\vec{b}^T / \vec{b}^T\vec{b}$ = Outer Product / Inner Product

Example 1

U = span{$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$}

$\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Outer Product = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Inner Product = 2

Projection of $\vec{a}$ = (1/2)$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ = $\begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$

In general

U = span{$\vec{b_1}, \vec{b_2}, \ldots \vec{b_k}$} = subspace of R^n

$k < n$ (for now)

$\vec{a} \in R^n$

$\pi_U(\vec{a}) = P_U(\vec{a}) = \lambda_1\vec{b_1} + \lambda_2\vec{b_2} + \ldots + \lambda_k\vec{b_k} = B\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \ldots \\ \lambda_k \end{bmatrix} = B\vec{\lambda}$

($\vec{a} - (\lambda_1\vec{b_1} + \lambda_2\vec{b_2} + \ldots + \lambda_k\vec{b_k})$) is perpendicular $\vec{b_i}\ \forall i$

($\vec{a} - B\vec{\lambda}$) is perpendicular $\vec{b_i}\ \forall i$

B = $[\vec{b_1} \quad \vec{b_2} \quad \ldots \quad \vec{b_k}]$ (of size $n \times k$)

$\vec{b_1}^T(\vec{a} - B\vec{\lambda}) = 0$

$\vec{b_2}^T(\vec{a} - B\vec{\lambda}) = 0$

...

$\vec{b_k}^T(\vec{a} - B\vec{\lambda}) = 0$

$B^T\vec{a} = (B^TB)\vec{\lambda}$

B^TB = of size $k \times k$ = Square Matrix = Invertible

$B\vec{\lambda} = B(B^TB)^{-1}B^T\vec{a}$

Projection Matrix = $B(B^TB)^{-1}B^T$ = BB^T/B^TB = Outer Product / Inner Product

Only works if the columns of B are linearly independent!

Projection Error of $\vec{x}$ = distance between $\vec{x}$ and its projection

Total Error = Sum of projection errors of all points

When k = n...

In R^n space

U = span$\{\vec{b_1}, \vec{b_2}, \ldots, \vec{b_n}\}$ = R^n

B = $[\vec{b_1} \quad \vec{b_2} \quad \ldots \quad \vec{b_n}]$

Little Lemma...

If A and B are square matrices...

$(AB)^{-1} = B^{-1}A^{-1}$

Back to Before...

Sub into Projection Matrix...

Projection Matrix = $B(B^TB)^{-1}B^T$ = $BB^{-1}B^{T^{-1}}B^T$ = II = I