# Eigen Values, Vectors and Spaces

Take arbitrary vector $\vec{x}$
Take arbitrary matrix A and apply to $\vec{x}$

A Changes:

- Magnitude of $\vec{x}$
- direction of $\vec{x}$

Take arbitrary vector $\vec{y}$
Take arbitrary matrix A and apply to $\vec{y}$

If $\vec{y}$ is scaled, but does not change direction:

- $\vec{y}$ is an eigen vector of A
- If $A\vec{y} = \lambda\vec{y}$
    - $\lambda$ = eigen value of $\vec{y}$

**Collection of eigenvalues of A = spectrum of A**

# Example 1

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Any $\delta\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ = eigenvector of A with eigen value of 2

Any $\beta\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ = eigenvector of A with eigen value of 3

# Example 2

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A\vec{x} = \lambda\vec{x}$$
$$A\vec{x} - \lambda I\vec{x} = 0$$

$[A - \lambda I]\cancel{\vec{x}} = \cancel{\vec{0}}$

$M\cancel{\vec{x}} = 0$

$Ker(M) \neq 0$

$\det(M) = 0$

$\det(M)$ = some polynomial in $\lambda$ = $a_0 + a_1\lambda + a_2\lambda$

$[A - \lambda I]$ = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} =$

det($[A - \lambda I]$) = $(1-\lambda)(4-\lambda) - 6$ = $-2 - 5\lambda + \lambda^2$ = 0

$-2 - 5\lambda + \lambda^2$ = Characteristic polynomial of A

Roots of $-2 - 5\lambda + \lambda^2$

$\lambda_1 = (5 + \sqrt{33})/2$

$\lambda_2 = (5 - \sqrt{33})/2$

*Case 1*

$\lambda_1 = (5 + \sqrt{33})/2$

$E_{\lambda_1} = Ker(A - \lambda_1 I)$ = **Eigen Space**

$(A - \lambda_1 I)\cancel{\vec{x}} = 0$

EROs to RREF

$\begin{bmatrix} 1 & 2/(1-\lambda_1) \\ 0 & 0 \end{bmatrix}$

$E_{\lambda_1} = span\{\begin{bmatrix} -2/(1-\lambda_1) \\ 1 \end{bmatrix}\}$

$E_{\lambda_2}$ = same process...

$E_{\lambda_1}$ = Line stretched by scale factor $\lambda_1$

$E_{\lambda_2}$ = Line stretched by scale factor $\lambda_2$

$\lambda_1, \lambda_2$ = Eigen Values

$\begin{bmatrix} -2/(1-\lambda_1) \\ 1 \end{bmatrix}$, other one = Eigen Vectors

$E_{\lambda_1}, E_{\lambda_2}$ = Eigen Spaces

# Example 3

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Same process...

Characteristic Polynomial: $\lambda^2 + 1 = 0$
$\lambda = \pm i$
Spectrum of A = $\{i, -i\}$

**No eigen vectors**

# In general...

if $A_{nxn}$

det$(A - \lambda I)$ = $a_0 + a_1\lambda + a_2\lambda^2 + \ldots + a_n\lambda^n$ = $(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)$
Spectrum = {$\lambda_1, \ldots \lambda_n$}

Some may be joined together...
$(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)$ = $(\lambda - \lambda_1)^2\ldots(\lambda - \lambda_n)$ = $\dim(E_{\lambda_1}) = 2$
$(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)$ = $(\lambda - \lambda_1)^3\ldots(\lambda - \lambda_n)$ = $\dim(E_{\lambda_1}) = 3$
etc

# Example 4

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

Normal Process

Characteristic Polynomial = $(1 - \lambda)(3 - \lambda)^2$
$\lambda = 1, 3$
$\text{spectrum}(A) = \{1, 3\}$

$$E_1 = Ker(A - I) = Ker(\begin{bmatrix} 0 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix})$$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \vec{x} = 0$$

RREF

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1 = \text{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$$

$$E_3 = Ker(A - 3I) = Ker(\begin{bmatrix} -2 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix})$$

$$E_3 = \text{span}\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\}$$

# Interpretation of Eigen Spaces

Take an arbitrary point in the Eigen space
When you apply the matrix A, this point is **stretched by a scale factor of the corresponding eigen value** (and **flipped if the value is negative**), mapped to a new point **in the same eigen space**

# Find magnitude of eigenvector

Find eigenspace
Take arbitrary vector in eigenspace
Divide by magnitude of vector
Magnitude of resulting vector = magnitude of eigenvector