# Series

Consider a series of the form

$\sum_{i=1}^{\infty} a_i$

We can associate a sequence $(S_n)_{n \geq 1}$ of real numbers to that series where for each $n \geq 1$ the real number $S_n$ is defined as the **partial sum**:

$S_n = \sum_{i=1}^{n} a_i$

of the first N summands in that series. This gives us two transformations, one that maps a sequence $(a_n)_{n \geq 1}$ to its series, and one that maps series to a sequence of its partial sums:

$$(a_n)_{n\geq 1} \quad \mapsto \quad \sum_{i=1}^{\infty} a_i \quad \mapsto \quad (S_n)_{n\geq 1} \qquad \text{where } S_n = \sum_{i=1}^{n} a_i \text{ for all } n \geq 1$$

# Convergence and Divergence of Series

Let $\sum_{i=1}^{\infty} a_i$ be the series determined by the sequence $(a_n)_{n \geq 1}$
Then:

Series $\sum_{i=1}^{\infty} a_i$ has limit $l$ in $R$ or converges to $l$ in $R$ iff its associated sequence of partial sums $(S_n)_{n \geq 1}$ as defined in has limit $l$

Series $\sum_{i=1}^{\infty} a_i$ diverges if it does not converge to some $l$ in $R$

Observe the resulting convention that if the sequence of partial sums converges to $\infty$ or $-\infty$ in the extended real line $R \cup \{\infty, -\infty\}$ then, by definition, the series diverges

# Convergence and being bounded above

When $a_i \geq 0$ for all $i \geq 1$, then the partial sum $(S_n)_{n \geq 1}$ as defined in is increasing, that is:

$S_1 \leq S_2 \leq S_3 \leq \cdots$

We then know that $(S_n)_{n\geq 1}$ (and therefore the associated series $\sum_{i=1}^{\infty} a_i$) has a limit whenever that sequence is bounded above

# Summands of converging series tend to 0

Series converges $\implies$ sequence $(a_n)_{n\geq 1}$ has limit 0

**The converse is false**, the series can diverge even when the sequence $(a_n)_{n\geq 1}$ of summands does converge to 0

# The tail of a series

Since the convergence of a series is defined in terms of the convergence of its partial sums, this convergence does not depend on any initial, finite part of that sequence of partial sums

For Example, a series $S = \sum_{n=1}^{\infty} a_n$ converges iff $S = \sum_{n=10}^{\infty} a_n$ converges or $S = \sum_{n=1000}^{\infty} a_n$ converges or, in general, iff $\sum_{n=N}^{\infty} a_n$ converges for any natural number $N$

This explains why, in comparison tests, we will also only have to take into account later summands in the series from a certain point N onward, rather than from the whole series.

# Geometric Series

**Theorem 27 (Geometric Series Converges)**
*For the convergence behavior of the geometric series, we have that* $\sum_{i=1}^{\infty} x^i$ *converges iff* $|x| < 1$, *in which case its limit equals* $\frac{x}{1-x}$.

# Harmonic Series

The harmonic series is given by:

$$S = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

This series diverges to $\infty$

This implies that the converge of the sequence of summands to 0 is not sufficient for a series to converge

## Series of Inverse Squares

Defined as:

$$S = \sum_{i=1}^{\infty} \frac{1}{i^2}$$

This series converges to:

$$\frac{\pi^2}{6}$$

## Common Series and Convergence

Diverging Series:

1. Harmonic series: $S = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

2. Harmonic primes: $S = \sum_{p\,:\,\text{prime}} \frac{1}{p}$ diverges.

3. Geometric series: $S = \sum_{n=1}^{\infty} x^n$ diverges for $|x| \geq 1$.

Converging series:

1. Geometric series: $S = \sum_{n=1}^{\infty} x^n$ converges to $\frac{x}{1-x}$ for all $x$ in $\mathbb{R}$ with $|x| < 1$.

2. Inverse squares series: $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges to $\frac{\pi^2}{6}$.

3. $\frac{1}{n^c}$ series: $S = \sum_{n=1}^{\infty} \frac{1}{n^c}$ converges for all $c$ in $\mathbb{R}$ with $c > 1$.

## Convergence Tests for Series of Positive Terms

In this section, we assume that the summands of a series are positive so that the sequence of partial sums is strictly increasing

This means that either the sequence $S_n$ of partial sums is bounded above in which case the series is convergent or the sequence $S_n$ is not bounded above in which case $S_n \to \infty$ as $n \to \infty$ and the series diverges. In formulating these comparison tests, we use the following notational conventions:

1. $a_i > 0$ denotes a summand in the series $\sum_{i=1}^{\infty} a_i$ whose convergence or divergence we wish to establish,
2. $\sum_{i=1}^{\infty} c_i$ denotes a series with $c_i > 0$ for which we already have established converge to the sum $c$, and
3. $\sum_{i=1}^{\infty} d_i$ denotes a series with $d_i > 0$ for which we have already established its divergence.

# Comparison Test

**Lemma 3 (Comparison Test)**
*Let $\lambda > 0$ and $N \in \mathbb{N}$. Further, let $\sum_{i=1}^{\infty} c_i$ be a converging series and $\sum_{i=1}^{\infty} d_i$ a diverging series. Then we have:*

1. *If $a_i \leq \lambda c_i$ for all $i > N$, then $\sum_{i=1}^{\infty} a_i$ converges*
2. *If $a_i \geq \lambda d_i$ for all $i > N$, then $\sum_{i=1}^{\infty} a_i$ diverges*

# Limit Comparison Test

**Lemma 4 (Limit Comparison Test)**
*As in the previous lemma, let us suppose that $\sum_{i=1}^{\infty} c_i$ is a converging series and that $\sum_{i=1}^{\infty} d_i$ is a diverging series. Then we have:*

1. *If $\lim_{i\to\infty} \frac{a_i}{c_i} \in \mathbb{R}$ exists, then $\sum_{i=1}^{\infty} a_i$ converges*
2. *If $\lim_{i\to\infty} \frac{d_i}{a_i} \in \mathbb{R}$ exists, then $\sum_{i=1}^{\infty} a_i$ diverges*

# D'Alembert's Ratio Test

**Lemma 5 (D'Alembert's Ratio Test)**
*Let $N$ be in $\mathbb{N}$. Furthermore, we consider a series $\sum_{i=1}^{\infty} a_i$ for which we mean to determine whether it converges or diverges. Then we have:*

1. *If $\frac{a_{i+1}}{a_i} \geq 1$ for all $i \geq N$, then $\sum_{i=1}^{\infty} a_i$ diverges.*
2. *If there exists some $k$ in $\mathbb{R}$ with $k < 1$ such that $\frac{a_{i+1}}{a_i} \leq k$ for all $i \geq N$, then $\sum_{i=1}^{\infty} a_i$ converges.*

# D'Alembert Limit Ratio Test

**Lemma 6 (D'Alembert Limit Ratio Test)**
*As in the previous lemma, we consider a series $\sum_{i=1}^{\infty} a_i$ for which we mean to determine whether it converges or diverges. Suppose that $\lim_{i\to\infty} \frac{a_{i+1}}{a_i}$ exists. Then we have:*

1. *If $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} > 1$, then $\sum_{i=1}^{\infty} a_i$ diverges.*
2. *If $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} = 1$, then $\sum_{i=1}^{\infty} a_i$ may converge or diverge, so the test is inconclusive.*
3. *If $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} < 1$, then $\sum_{i=1}^{\infty} a_i$ converges.*

# Integral Test

**Lemma 7 (Integral Test)**
*Let $f : \mathbb{R} \to \mathbb{R}^+$ be a function that is continuous and decreasing and positive on the interval $[1, \infty)$. Let $a_n = f(n)$ for all $n$ in $\mathbb{N}$. Then we have:*

1. *If $\int_N^{\infty} f(x)\,\mathrm{d}x$ converges, then the series $\sum_{i=1}^{\infty} a_i$ converges as well.*
2. *If $\int_N^{\infty} f(x)\,\mathrm{d}x$ diverges, then the series $\sum_{i=1}^{\infty} a_i$ diverges as well.*

# Absolute Convergence

Previously, we worked with the assumption that all summands of a series are positive. We also remarked that there is then no loss of generality to assume that all such summands are positive. However, series that occur in practice may have negative summands as well.

This gives a problem in that the sequence of partial sums may depend on the order in which summands of the series are listed. This may change the limit of the sequence. This dependency on the order in which summands are listed is bad.

# Unconditional Convergence of Series

A permutation $\pi$ over the natural numbers $N$ is a function $\pi : N \to N$ that has an inverse, i.e. it is injective and surjective

A series $\sum_{i=1}^{\infty} a_i$ is **unconditionally convergent** iff it converges and the permuted series $\sum_{i=1}^{\infty} a_{\pi(i)}$ converges to that same limit, for all permutations $\pi : N \to N$

# Absolute Convergence of Series

A series $\sum_{i=1}^{\infty} a_i$ is **absolutely convergent** iff $\sum_{i=1}^{\infty} |a_i|$ converges

**The previous two concepts are equivalent.**

This means that $\sum_{i=1}^{\infty} |a_i|$ diverges iff the limit behaviour of $\sum_{i=1}^{\infty} a_i$ is sensitive to permutations of summands

# Absolute Value Comparison Test

Suppose $b_i$ is a non-negative sequence such that $\sum_{i=1}^{\infty} b_i$ converges and suppose $a_i$ is a sequence such that $|a_i| \leq b_i$ for all $i \geq 1$

Then $\sum_{i=1}^{\infty} a_i$ converges

If $\sum_{i=1}^{\infty} a_i$ is absolutely convergent then it is convergent

**If a series is absolutely convergent, then it is unconditionally convergent.**

# Limit absolute value ratio Test

Consider the series $\sum_{i=1}^{\infty} a_i$ with $a_i \neq 0$ for $i \geq 1$

1. *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *converges absolutely (and thus also converges).*
2. *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*
3. *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ *then* $\sum_{n=1}^{\infty} a_n$ *may converge or diverge.*

# The nth-root test for convergence

## Limit superior

Let $(a_n)_{n\geq 1}$ be a sequence. This determines another sequence $(b_n)_{n\geq 1}$ where:

$b_n = \sup\{a_m \mid m \geq n\}$

Note that $(b_n)_{n\geq 1}$ is a non-increasing sequence. The **limit superior** of $(a_n)_{n\geq 1}$ is defined to be the ordinary limit of the sequence $(b_n)_{n\geq 1}$ in $R \cup \{-\infty, \infty\}$. We denote this limit as $\limsup_{n\to\infty} a_n$

Note that the limit superior is either a real number, or $\infty$ or $-\infty$

Similarly, we have the definition of limit inferior:

**Definition 26 (Limit inferior)**
*Let $(a_n)_{n\geq 1}$ be a sequence. This determines another sequence $(c_n)_{n\geq 1}$ where*

$$c_n = \inf\{a_m \mid m \geq n\} \tag{8.24}$$

*Note that $(c_n)_{n\geq 1}$ is a non-decreasing sequence. The* **limit inferior** *of $(a_n)_{n\geq 1}$ is defined to be the ordinary limit of the sequence $(c_n)_{n\geq 1}$ in $\mathbb{R} \cup \{-\infty, \infty\}$. We denote this limit as $\liminf_{n\to\infty} a_n$.*

Note that the limit inferior is again either a real number, or $\infty$ or $-\infty$.

# An Example

For the sequence $a_n = (-1)^n$, we have that $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = -1$

**Exercise 20**
*Given a sequence $(a_n)_{n\geq 1}$, show that there exist subsequences $(a_{m_i})_{i\geq 1}$ and $(a_{p_i})_{i\geq 1}$ such that $\limsup_{n\to\infty} a_n = \lim_{i\to\infty} a_{m_i}$ and $\liminf_{n\to\infty} a_n = \lim_{i\to\infty} a_{p_i}$. Show also that if $(a_{q_i})_{i\geq 1}$ is any convergent subsequence of $(a_n)_{n\geq 1}$, then we have:*

$$\liminf_{n\to\infty} a_n \leq \lim_{i\to\infty} a_{q_i} \leq \limsup_{n\to\infty} a_n.$$

It follows from Exercise 20 that the set of limits of all subsequences of a sequence $(a_n)_{n\geq 1}$ is contained in the closed interval of the extended real line:

$$[\liminf_{n\to\infty} a_n, \limsup_{n\to\infty} a_n] \subseteq [-\infty, \infty].$$

In addition, if the sequence does have a limit then $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n$

# $n^{th}$ Root Test

Consider the series $\sum_{n=1}^{\infty} a_n$

If $\limsup_{n\to\infty} |a_n|^{1/n} < 1$ then the series converges absolutely

If $\limsup_{n\to\infty} |a_n|^{1/n} > 1$ then the series diverges

If $\limsup_{n\to\infty} |a_n|^{1/n} = 1$ then the test is inconclusive.

**Example 34**
*Consider the series*

$$1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$$

*with $a_{2n} = 1/2^n$ and $a_{2n+1} = 1/3^n$. Then check that $\limsup_{n\to\infty}(a_n)^{1/n} = 1/\sqrt{2} < 1$. Hence, the series converges. However, the ratio test fails since $\liminf_{n\to\infty} a_{n+1}/a_n = 0$ and $\limsup_{n\to\infty} a_{n+1}/a_n = \infty$ and thus $\lim_{n\to\infty} a_n$ does not exist.*

It can be shown that for any sequence $(a_n)_{n\geq 1}$ of positive terms, we always have the following inequalities

$$\liminf_{n\to\infty} |a_{n+1}/a_n| \leq \liminf_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} |a_{n+1}/a_n|.$$

Thus, if $\lim_{n\to\infty} |a_{n+1}/a_n| = l$ exists then all the four terms above collapse to a single extended real number and the ratio test is a good as the $n$th root test. But if $\lim_{n\to\infty} |a_{n+1}/a_n|$ fails to exist, then the $n$th root test will provide the definite information about the convergence of the series.