Natural deduction for predicate logic

This is quite easy to set up. We keep the old propositional rules e.g., $\phi \vee \neg\phi$ for any first-order sentence ϕ ('lemma')

and add new ones for $\forall, \exists, =$.

You construct natural deduction proofs as for propositional logic: first think of a direct argument, then convert to ND.

This is *even more important than for propositional logic.* There's quite an art to it.

Validating arguments by predicate ND can sometimes be harder than for propositional ones, because the new rules give you wide choices, and at first you may make the wrong ones!

If you find this disconserting, remember, it's a hard problem, there's no computer program to do it (theorem 4.15)!

Some definitions

Definition 4.16 (Substitution)

For an L-formula ϕ, a variable x, and a term t, we define $\phi[t/x]$ to be the formula got from ϕ by replacing all *free* occurrences of x in ϕ by t.

In writing $\phi[t/x]$ we really mean the formula obtained by performing the substitution function $[t/x]$ on ϕ.

Example:

Let ϕ be the L-formula $P(x) \wedge \forall x \exists y R(x, y)$.

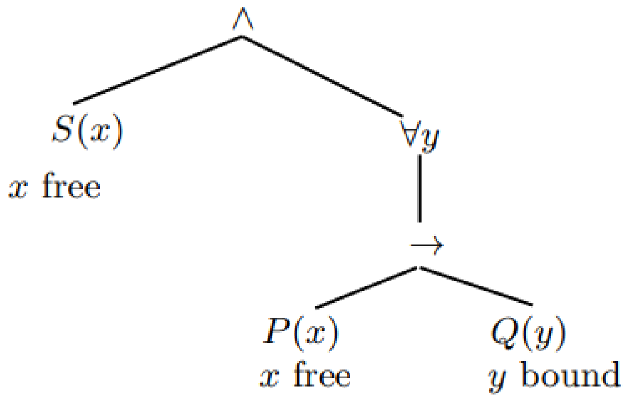
Then $\phi[a/x]$ is $P(a) \wedge \forall x \exists y R(x, y)$.

Notice that the x in $R(x, y)$ is not substituted by t because this x is bound.

Definition 4.17 (Free terms)

For an L-formula ϕ, a variable x, and a term t, we say that t *is free for* x *in* ϕ if there are no free occurrences of x in ϕ in the scope of $\forall y$ or $\exists y$ for any variable y occurring in t.

Let ϕ be the formula $S(x) \wedge \forall y(P(x) \to Q(y))$.



Let t be $f(y, y)$. Let's try to obtain the substitution $\phi[t/x]$

The leftmost x can be substituted; it is not in the scope of any quantifier. The x in $P(x)$ cannot be substituted; it introduces a new variable y in t which would then be bound by $\forall y$.

Therefore, $f(y, y)$ is not free for x in $S(x) \wedge \forall y(P(x) \to Q(y))$.

∃-Introduction, or ∃I

$\exists$-introduction, or $\exists I$: To prove a sentence $\exists x\phi$, you can prove $\phi[t/x]$, for some closed term t of your choice.

$$\begin{array}{lll} & \vdots & \\ 1 & \phi[t/x] & \text{we got this somehow...} \\ 2 & \exists x\phi & \exists I(1) \end{array}$$

Recall a *closed term* (or ground term) is one with no variables.

This rule is reasonable. If in some structure, $\phi[t/x]$ is true, then so is $\exists x\phi$, because there exists an object in M (namely, the value in M of t) making ϕ true.

But choosing the 'right' t can be hard — that's why it's such a good idea to think up a 'direct argument' first!

$\exists$-Elimination, or $\exists$E

$\exists$-elimination, or $\exists E$: Let ϕ be a formula. If you have managed to write down $\exists x\phi$, you can prove a sentence ψ from it by

- assuming $\phi[c/x]$, where c is a *new* constant not used in ψ or in the proof so far,
- proving ψ from this assumption.

During the proof, you can use anything already established.

But once you've proved ψ, you cannot use any part of the proof, *including* c, later on.

So we isolate the proof of ψ from $\phi[c/x]$, in a box:

1	$\exists x \phi$	got this somehow
2	$\phi[c/x]$	asm
	$\langle$the proof$\rangle$	hard struggle
3	ψ	we made it!
4	ψ	$\exists E(1, 2-3)$

The box is controlling the scope of c and the scope of $\phi[c/x]$.

c is often called a Skolem constant. Pandora uses sk1, sk2, ...

Justification

We know $\exists x \phi$ is true, so ϕ is true for at least one value of x.

We do a case analysis over all those possible values, using c as a generic value that represents them all.

If assuming $\phi[c/x]$ allows us to prove ψ which doesn't mention c, then this means that ψ must be true whichever c is chosen to make $\phi[c/x]$ true.

Which is what $\exists x \phi$ allows us to deduce.

Examples

Show $\exists x(P(x) \wedge Q(x)) \vdash \exists x P(x) \wedge \exists x Q(x)$.

1	$\exists x(P(x) \wedge Q(x))$	premise
2	$P(c) \wedge Q(c)$	asm
3	$P(c)$	$\_\_(2)$
4	$\exists x P(x)$	$\exists I(3)$
5	$Q(c)$	$\_\_(2)$
6	$\exists x Q(x)$	$\exists I(5)$
7	$\exists x P(x) \wedge \exists x Q(x)$	$\wedge I(4,6)$
8	$\exists x P(x) \wedge \exists x Q(x)$	$\exists E(1, 2-7)$

Only sentences occur in ND proofs. They should never involve formulas with free variables!

Show $\exists x P(x) \vee \exists x Q(x) \vdash \exists x(P(x) \vee Q(x))$.

1	$\exists x P(x) \vee \exists x Q(x)$	premise
2	$\exists x P(x)$	asm
3	$P(c)$	asm
4	$P(c) \vee Q(c)$	$\vee I(3)$
5	$\exists x(P(x) \vee Q(x))$	$\exists I(4)$
6	$\exists x(P(x) \vee Q(x))$	$\exists E(2, 3-5)$
7	$\exists x Q(x)$	asm
8	$Q(c)$	asm
9	$P(c) \vee Q(c)$	$\vee I(8)$
10	$\exists x(P(x) \vee Q(x))$	$\exists I(9)$
11	$\exists x(P(x) \vee Q(x))$	$\exists E(7, 8-9)$
12	$\exists x(P(x) \vee xQ(x))$	$\vee E(1, 2-6, 7-11)$

Notice that both the formulas in lines 5 (last one in the inner assumption box) and 6 (first one after the inner box) have to be the same when applying the $\exists$ E rule. Similarly for lines 10 and 11.

To apply $\vee$ E to the formula in line 1, the formulas in lines 6 and 11 (last ones in the outer boxes) have to be the same.

Show $\exists x \exists y P(x,y) \vdash \exists y \exists x P(x,y)$.

1	$\exists x \exists y P(x,y)$	premise
2	$\exists y P(c,y)$	asm
3	$P(c,d)$	asm
4	$\exists x P(x,d)$	$\exists I(3)$
5	$\exists y \exists x P(x,y)$	$\exists I(4)$
6	$\exists y \exists x P(x,y)$	$\exists E(2, 3-5)$
7	$\exists y \exists x P(x,y)$	$\exists E(1, 2-6)$

∀-introduction, ∀I

To introduce the sentence $\forall x \phi$, for some $\phi(x)$, you introduce a *new* constant, say c, not used in the proof so far, and prove $\phi[c/x]$.

During the proof, you can use anything already established.

But once you've proved $\phi[c/x]$, you can no longer use the constant c later on.

We isolate the proof of $\phi[c/x]$, in a box:

1	c	$\forall I$ const
	$\langle$the proof$\rangle$	hard struggle
2	$\phi[c/x]$	we made it!
3	$\forall x \phi$	$\forall I(1,2)$

The box here controls the scope of the dummy variable c rather than an assumption.

Note: This is the only time in ND that you write a line (1) containing a term, not a formula.

And it's the *only* time a box doesn't start with a line labelled 'asm'.

Justification

To show $M \models \forall x\phi$, we must show that the formula holds for every object in $dom(M)$ over M

So choose an arbitrary object o, add a new constant c naming o, and prove $\phi[c/x]$.

As o is arbitrary, this shows $\forall x\phi$.

c must be new, because the constants already in use may not name this particular o.

∀-elimination, or ∀E

$\forall$-elimination, or $\forall E$: Let $\phi(x)$ be a formula. If you have managed to write down $\forall x \phi$, you can go on to write down $\phi[t/x]$ for any closed term t. (It's your choice which t!)

$$\begin{array}{lll} & \vdots & \\ 1 & \forall x \phi & \text{we got this somehow...} \\ 2 & \phi[t/x] & \qquad\qquad \forall E(1) \end{array}$$

This is easily justified: if $\forall x \phi$ is true in a structure, then certainly $\phi[t/x]$ is true, for any closed term t.

However, choosing the 'right' t can be hard — that's why it's such a good idea to think up a 'direct argument' first!

Recall that we assume that only x occurs free in A; $\forall x \phi$ is a sentence.

Rules

When deriving $\forall x \phi$ in a proof by applying $\forall I$, then the $\forall I$ const c must not occur in the scope of a $\forall x$ or $\exists x$ already occurring in ϕ.

You cannot keep any occurrences of the $\forall I$ const c in the sentence obtained by applying the $\forall I$ rule.

Note: This restriction does not hold when applying $\exists I$. Going from $P(c, c)$ to $\exists x P(x, c)$ is a valid step.

Examples

Let's show $P \to \forall x Q(x) \vdash \forall x(P \to Q(x))$.

Here, P is a 0-ary relation symbol — that is, a propositional atom.

1	$P \to \forall x Q(x)$	premise
2	c	$\forall I$ const
3	P	asm
4	$\forall x Q(x)$	$\to E(3,1)$
5	$Q(c)$	$\forall E(4)$
6	$P \to Q(c)$	$\to I(3,5)$
7	$\forall x(P \to Q(x))$	$\forall I(2,6)$

Show $\exists x \forall y G(x,y) \vdash \forall y \exists x G(x,y)$.

1	$\exists x \forall y G(x,y)$	premise
2	d	$\forall I$ const
3	$\forall y G(c,y)$	asm
4	$G(c,d)$	$\forall E(3)$
5	$\exists x G(x,d)$	$\exists I(4)$
6	$\exists x G(x,d)$	$\exists E(1,3-5)$
7	$\forall y \exists x G(x,y)$	$\forall I(2,6)$

Show $\neg\forall x Q(x) \vdash \exists x \neg Q(x)$.

1	$\neg\forall x Q(x)$	premise
2	$\neg\exists x \neg Q(x)$	asm
3	c	$\forall c$ const
4	$\neg Q(c)$	asm
5	$\exists x \neg Q(x)$	$\exists I(4)$
6	$\bot$	$\neg E(2,5)$
7	$Q(c)$	$\mathtt{PC}(4-6)$
8	$\forall x Q(x)$	$\forall I(3-7)$
9	$\bot$	$\neg E(1,8)$
10	$\exists x \neg Q(x)$	$\mathtt{PC}(2-9)$

Derived rule $\forall \rightarrow \mathrm{E}$

This is like PC: it collapses two steps into one. Useful, but not essential.

Idea: often we have proved $\forall x(\phi(x) \to \psi(x))$ and $\phi[t/x]$, for some formulas $\phi(x), \psi(x)$ and some closed term t.

We know we can derive $\psi[t/x]$ from this:

1	$\forall x(\phi(x) \to \psi(x))$	(got this somehow)
2	$\phi[t/x]$	(this too)
3	$\phi[t/x] \to \psi[t/x]$	$\forall E(1)$
4	$\psi[t/x]$	$\to E(2,3)$

So let's just do it in 1 step:

1	$\forall x(\phi(x) \to \psi(x))$	(got this somehow)
2	$\phi[t/x]$	(this too)
4	$\psi[t/x]$	$\forall \to E(2,1)$

Example

Show $\quad \forall x \forall y(P(x,y) \to Q(x,y)), \exists x P(x,a) \quad \vdash \quad \exists y Q(y,a)$.

1	$\forall x \forall y(P(x,y) \to Q(x,y))$	premise
2	$\exists x P(x,a)$	premise
3	$P(c,a)$	asm
4	$Q(c,a)$	$\forall \to E(3,1)$
5	$\exists y Q(y,a)$	$\exists I(4)$
6	$\exists y Q(y,a)$	$\exists E(2, 3-5)$

We used $\forall\to E$ on 2 $\forall$s at once. This is even more useful. There is no limit to how many $\forall$s can be covered at once with $\forall\to E$!!

Rules for equality—Reflexivity

There are two: refl and =sub. We also add a derived rule, =sym.

Reflexivity of equality (refl).
Whenever you feel like it, you can introduce the sentence $t = t$, for any closed L-term t and for any L you like.

	$\vdots$	bla bla bla
1	$t = t$	refl

Rules for equality—Substitution

Substitution of equal terms (=sub).
If $\phi(x)$ is a formula, t, u are closed terms, you've proved $\phi[t/x]$, and you've also proved either $t = u$ or $u = t$, you can go on to write down $\phi[u/x]$.

1	$\phi[t/x]$	got this somehow...
2	$\vdots$	yada yada yada
3	$t = u$	...and this
4	$\phi[u/x]$	=sub(1,3)

(Idea: if t, u are equal, there's no harm in replacing t by u as the value of x in ϕ. Compare with the Leibniz principle.)

Rules for equality—Symmetry

Show $c = d \vdash d = c$. (c, d are constants.)

1	$c = d$	premise
2	$c = c$	refl
3	$d = c$	=sub(2,1)

Letting ϕ be $x = c$, then line 2 is $\phi[c/x]$ and line 3 is $\phi(d/x)$.

This is often useful, so make it a derived rule 'symmetry of =':

1	$c = d$	premise
2	$d = c$	=sym(1)

Examples

Show $\exists x \forall y (P(y) \rightarrow y = x), \forall x P(f(x)) \vdash \exists x (x = f(x))$.

1	$\exists x \forall y (P(y) \rightarrow y = x)$	premise
2	$\forall x P(f(x))$	premise
3	$\forall y (P(y) \rightarrow y = c)$	asm
4	$P(f(c))$	$\forall E(2)$
5	$f(c) = c$	$\forall \rightarrow E(4,3)$
6	$c = f(c)$	$= sym(5)$
7	$\exists x (x = f(x))$	$\exists I(6)$
8	$\exists x (x = f(x))$	$\exists E(1, 3-7)$

Show that $\vdash a = b \leftrightarrow \forall x (x = a \rightarrow x = b)$.

1	$a = b$	asm
2	c	$\forall$const
3	$c = a$	asm
4	$c = b$	=sub(1,3)
5	$c = a \rightarrow c = b$	$\rightarrow I(3-4)$
6	$\forall x (x = a \rightarrow x = b)$	$\forall I(2-5)$
7	$a = b \rightarrow \forall x (x = a \rightarrow x = b)$	$\rightarrow I(1-6)$
8	$\forall x (x = a \rightarrow x = b)$	asm
9	$a = a \rightarrow a = b$	$\forall E(8)$
10	$a = a$	refl
	$a = b$	
	$\forall x (x = a \rightarrow x = b) \rightarrow a = b$	
	$a = b \leftrightarrow \forall x (x = a \rightarrow x = b)$	

Soundness and completeness

Natural deduction is also sound and complete for predicate logic:

Theorem 4.18 (soundness)

Let $A_1, \ldots, A_n, B$ be any first-order sentences.
If $A_1, \ldots, A_n \vdash B$, then $A_1, \ldots, A_n \models B$.

'Any provable first-order sentence is valid.'
'Natural deduction never makes mistakes.'

Theorem 4.19 (completeness)

Let $A_1, \ldots, A_n, B$ be any first-order sentences.
If $A_1, \ldots, A_n \models B$, then $A_1, \ldots, A_n \vdash B$.

'Any first-order validity can be proved.' 'Natural deduction powerful enough to prove all valid first-order sentences'.
(We can use natural deduction to check validity.)