Problem Reduction

Since P $\subseteq$ NP, if we show that a problem such as the Hamiltonian Path Problem belongs to NP we do not know whether it is tractable (in P) or not.

We want to identify the hard (high complexity) problems in NP.

We start by saying what it means for one problem to be harder than another, using the concept of **reduction**.

Suppose that D and D' are two decision problems. We say that D (many-one) reduces to D' ($D \leq D'$) if there is a p-time computable function f such that

$$D(x) \text{ iff } D'(f(x))$$

Note that f can be a many-one function.

The idea is that we reduce a question about D (the easier problem) to a question about D' (the harder problem).

Suppose that we have an algorithm A' which decides D' in time $p'(n)$. Then if $D \leq D'$ via the reduction function f running in time $p(n)$ we have an algorithm A to decide D:

Algorithm A (input x):

1. Compute $f(x)$
2. Run A' on input $f(x)$ and return the answer (yes/no)

Now A runs in p-time, as can be seen using the same argument as when composing p-time functions:

- Step 1 takes $p(n)$ steps.
- $|f(x)| \leq q(n)$ for some polynomial q.
- Step 2 takes $p'(q(n))$ steps.

Hence:
Suppose $D \leq D'$ and $D' \in P$. Then $D \in P$

Similarly, for NP:

Suppose $D \leq D'$ and $D' \in NP$. Then $D \in NP$

Proof. Assume that $D \leq D'$ via reduction function f, and $D' \in \mathsf{NP}$. Then there is $E'(x,y) \in \mathsf{P}$ and a polynomial $p'(n)$ such that $D'(x)$ iff $\exists y.E'(x,y)$ and if $E'(x,y)$ then $|y| \leq p'(|x|)$. Also we have $D(x)$ iff $D'(f(x))$ (property of reduction). Combining: $D(x)$ iff $\exists y.E'(f(x),y)$

Define $E(x,y)$ iff $E'(f(x),y)$. Then $D(x)$ iff $\exists y.E(x,y)$. Also $E \in \mathsf{P}$ (composition of p-time functions/relations).

We check that E is p-balanced: Suppose $E(x,y)$. Then $E'(f(x),y)$, so that $|y| \leq p'(|f(x)|)$ As before we have $|f(x)| \leq q(n)$. Hence $|y| \leq p'(q(|x|))$.

Hence $D \in \mathsf{NP}$.

The reduction order is reflexive and transitive:

- $D \leq D$
- $D \leq D' \leq D'' \implies D \leq D''$

If both $D \leq D'$ and $D' \leq D$ we write $D \sim D'$. Here D and D' are as hard as each other.