Composition of Relations

Given $R \subseteq A{\times}B$ and $S \subseteq B{\times}C$, then the composition of R with S, written $R \circ S$, is defined by:

$$R \circ S \triangleq \{ \langle a, c\rangle \in A{\times}C \mid \exists\, b \in B\ (a\ R\ b \wedge b\ S\ c) \}$$

The notation $R \circ S$ may be read as 'R composed with S'

The relation $R \circ S$ is only defined **if the types of R and S match**

We will write $a\ R\ b\ S\ c$ for $a\ R\ b \wedge b\ S\ c$

Examples

Let R and S be binary relations on $\{1, 2, 3, 4\}$ such that

$$\begin{aligned} R &\triangleq \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,4\rangle, \langle 4,1\rangle\} \\ S &\triangleq \{\langle 1,2\rangle, \langle 2,1\rangle, \langle 3,4\rangle, \langle 4,3\rangle, \langle 4,4\rangle\} \end{aligned}$$

$$\begin{aligned} R \circ S &= \{\langle 1,1\rangle, \langle 2,4\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,2\rangle\} \\ S \circ R &= \{\langle 1,3\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 4,4\rangle, \langle 4,1\rangle\} \\ R^{-1} &= \{\langle 2,1\rangle, \langle 3,2\rangle, \langle 4,3\rangle, \langle 1,4\rangle\} \end{aligned}$$