Power Series

Basics of Power Series

A **power series** is a series of form

$$\sum_{i=0}^{\infty} a_i \cdot (x-c)^i$$

where x is a real variable, c is a constant in R, and $(a_n)_{n\geq 0}$ is a sequence of reals

Radius of Convergence

Let c be a constant in R and $(a_n)_{n\geq 0}$ a sequence of reals. The power series $\sum_{i=0}^{\infty} a_i \cdot (x-c)^i$ has a **radius of convergence** r in $[0,\infty) \cup \{\infty\}$ such that:

- If $r \neq \infty$, then:
  - The power series converges for all x in R such that $|x-c| < r$, and
  - The power series diverges for all x in R such that $|x-c| > r$
- If $r = \infty$, then the power series converges for all x in R

Every power series $\sum_{i=0}^{\infty} a_i \cdot (x-c)^i$ has a radius of convergence r which is given by:

$$r^{-1} = \limsup_{n\to\infty} |a_n|^{1/n}$$

Ratio Test for Radius of Convergence

Suppose that the sequence:

$$(\frac{|a_{n+1}|}{|a_n|})_{n\geq 1}$$

has a limit l in R. Then l^{-1} is the radius of convergence of any power series $\sum_{i=0}^{\infty} a_i \cdot (x-c)^i$

Addition and product of power series

Two power series with radii of convergence r_1 and r_2, respectively, can be added term by term to get the sum of the two power series, absolutely convergent with the radius of convergence $\min\{r_1, r_2\}$

Exercise 26
Prove, with the notations as above, that

$$f(x)g(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + \cdots + (a_0b_n + a_1b_{n-1} + \cdots a_{n-1}b_1 + a_nb_0)x^n + \cdots$$

for $|x| < \min\{r_1, r_2\}$.

Maclaurin Series

Smoothness of Function

A function $f : R \to R$ is **smooth** at x_0 if for all $k \geq 1$ the k^{th} derivative of f exists at x_0. These k^{th} derivatives are defined inductively by:

$f^{(1)} = f'$
$f^{(k+1)} = (f^{(k)})'$ for all $k \geq 1$

Given such a function $f : R \to R$ that is smooth at 0, we can develop a power series for f with $c = 0$ that has a positive radius of convergence. If this power series has the same outputs as function f within that radius of convergence, then f is called a **real analytical function**. Not every smooth real function is analytical

The power series:

$$\sum_{i=0}^{\infty} a_i \cdot x^i$$

is called the Maclaurin Series.

Let us see how to compute the Maclaurin series by writing down its formal power series:

$f(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots$

The coefficients a_i are unknown. We must find them

Let us see how we can solve for a_0

$f(0) = a_0$

Therefore, we may define a_0 to be $f(0)$

Next, we differentiate:

$f'(x) = a_1 + 2 \cdot a_2 \cdot x + 3 \cdot a_3 \cdot x^2 + \ldots$
$a_1 = f'(0)$

Repeat:

$a_2 = \frac{f^{(2)}(0)}{2!}$

$a_n = \frac{f^{(n)}(0)}{n!}$

In Summary

Suppose that $f : R \to R$ is infinitely differentiable at $x = 0$ and that f has a formal power series representation, also known as series expansion, of the form:

$$\sum_{i=0}^{\infty} a_i \cdot x^i$$

Maclaurin's series for f:

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$$

The partial sums of a Maclaurin series:

$$f_n(x) = \sum_{i=0}^{n} f^{(i)}(0) \frac{x^i}{i!}$$

for all $n \geq 0$

Taylor Series

The Taylor series generalises the Maclaurin series so that c in the powers $(x - c)^n$ can be non-zero

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n$$

Differentiation and Integration of Power Series

Within the radius of convergence, a power series is continuous and it can be differentiated and integrated term by term.

Theorem 38
Suppose

$$f(x) = \sum_{n=1}^{\infty} a_n (x - x_0)^n$$

with radius of convergence $r > 0$, i.e., the power series converges absolutely for $|x| < r$. Then $f(x)$ is continuous for x with $|x - x_0| < r$ and moreover f is differentiable and integrable with

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1}$$

$$\int_c^x f(t)\, dt = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+1}/(n+1),$$

for $|x - x_0| < r$, i.e., the power series can be differentiated and integrated term by term.

Power Series Solution of ODEs

Consider the differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ky$$

for a constant k in R where $y = f(x)$

We must solve for f, given that $f(0) = 1$ (the **boundary condition**)

Seek the series solution:

$$y = \sum_{i=0}^{\infty} a_i x^i$$

Now, we must find the coefficients.

Since $f(0) = 1$, $a_0 = 1$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \Big(\sum_{i=0}^{\infty} a_i \cdot x^i\Big)' && \text{by (10.17)} \\
&= \sum_{i=1}^{\infty} (i \cdot a_i) \cdot x^{i-1} && \text{differentiating each summand} \\
&= \sum_{i=0}^{\infty} ((i+1) \cdot a_{i+1}) \cdot x^i && \text{changing the index range} \\
&= k \cdot \sum_{i=0}^{\infty} a_i \cdot x^i && \text{by (10.16)} \\
&= \sum_{i=0}^{\infty} (k \cdot a_i) \cdot x^i && \text{moving scalar } k \text{ under the infinite sum}
\end{aligned} \tag{10.18}$$

Matching Coefficients

$$(i+1) \cdot a_{i+1} = k \cdot a_i \qquad \text{for all } i \geq 0$$

This is a recurrence relation

Find a_i in terms of a_0

$$a_i = \frac{k}{i} a_{i-1} = \frac{k}{i} \cdot \frac{k}{i-1} a_{i-2} = \ldots = \frac{k^i}{i!} a_0$$

We already know that $a_0 = 1$

Therefore,

$$y = \sum_{i=0}^{\infty} \frac{(kx)^i}{i!} = e^{kx}$$

The differential equations above are called **ordinary**

ODEs can have more than one solution