Numerical Methods

Iteration of functions

Let $f : R \to R$ be a continuously differentiable map with a fixed point x^* satisfying $f(x^*) = x^*$

Under certain conditions, this fixed point can be obtained by starting with a nearby point x_0 and taking the limit of the sequence of iterates of this point under f:

$x_0, f(x_0), f(f(x_0)), \cdots, f^n(x_0), \cdots$

where $f^n(x)$ denotes n application of f on x

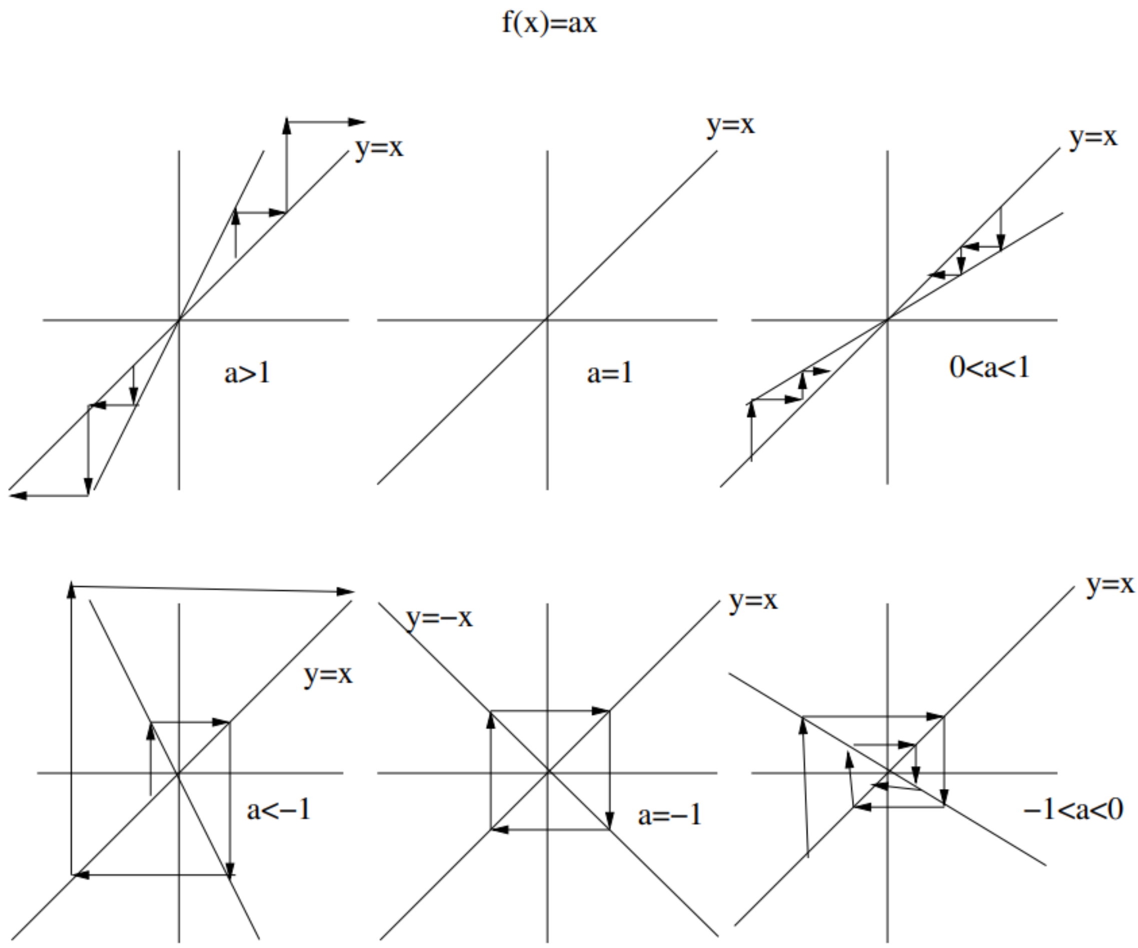
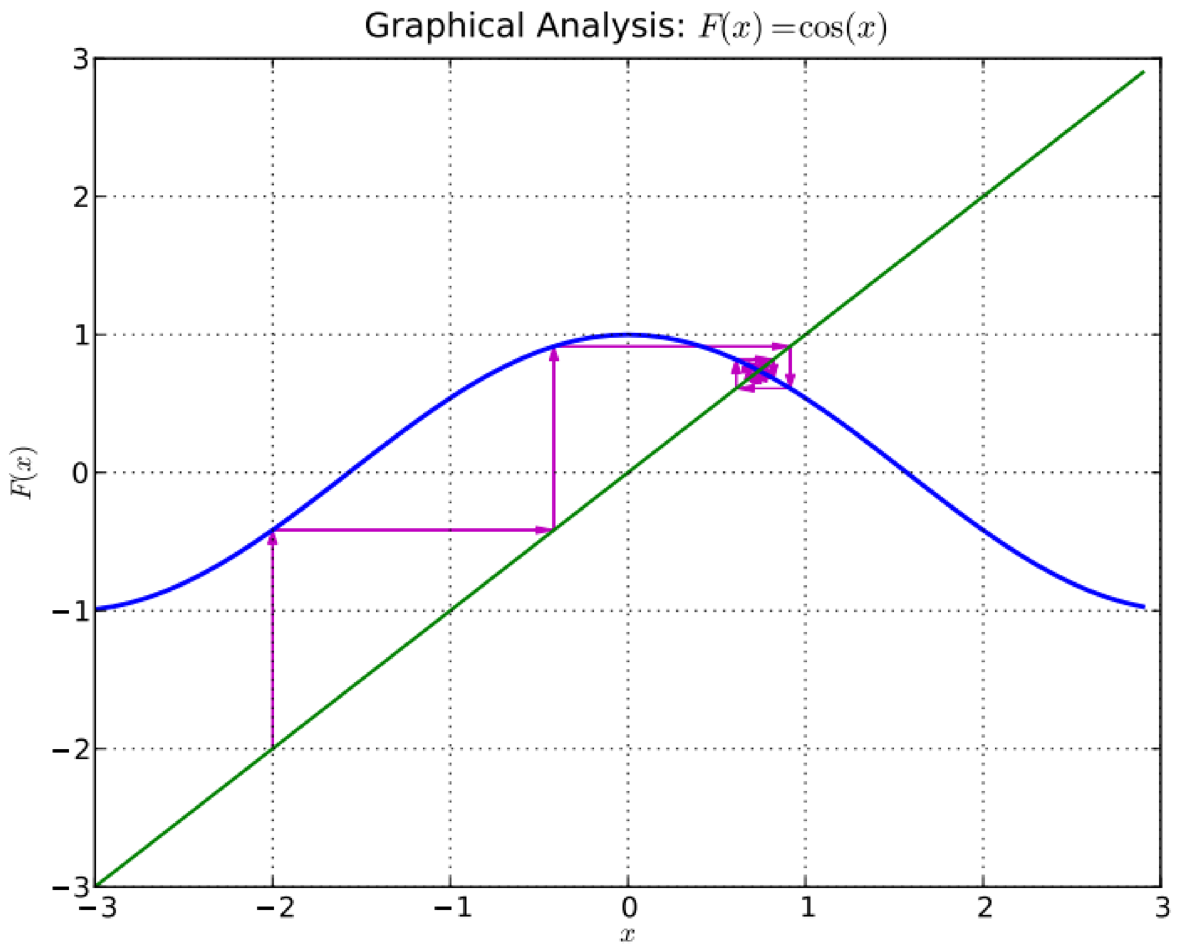


Figure 12.1: Graphical analysis of $x \mapsto ax$ for various ranges of $a \in \mathbb{R}$.

There is a simple graphical way to depict the sequence of iterates of f , called **graphical analysis**

Given the graph of a function F we plot the orbit of a point x_0

- First, superimpose the line $y = x$ on the graph (The points of intersection are the fixed points of F)
- Begin at (x_0, x_0) on the diagonal
- Draw a vertical line to the graph of F, meeting it at $(x_0, F(x_0))$
- From this point draw a horizontal line to the diagonal finishing at $(F(x_0), F(x_0))$. This gives us $F(x_0)$, the next point on the orbit of x_0
- Draw another vertical line to graph of F, intersecting it at $F^2(x_0)$
- From this point draw a horizontal line to the diagonal meeting it at $(F^2(x_0), F^2(x_0))$
- This gives us $F^2(x_0)$, the next point on the orbit of x_0
- Continue this procedure. The resulting “stair-case” visualises the orbit of x_0



Now if x^* is a fixed point of f such that $|f'(x^*)| < 1$ then any sequence of iterates of f starting at a point near x^* will converge to x^*

The fixed point x^* is called a (hyperbolic) attractor

If on the other hand $|f'(x^*)| > 1$ then starting with a point near x^*, the iterates of f will start to run away from x^* and x^* is called a (hyperbolic) repellor

If $|f'(x^*)| = 1$, then iterates of nearby point on either side can either get attracted, repelled by x^*, remain stationary or oscillate

Root finding

Consider $f(x) = 0$

Suppose we have found an initial approximation to a solution of this equation, x_0

The equation of a tangent line to $f(x)$ at x_0 is given by $y = f(x_0) + f'(x_0)(x - x_0)$

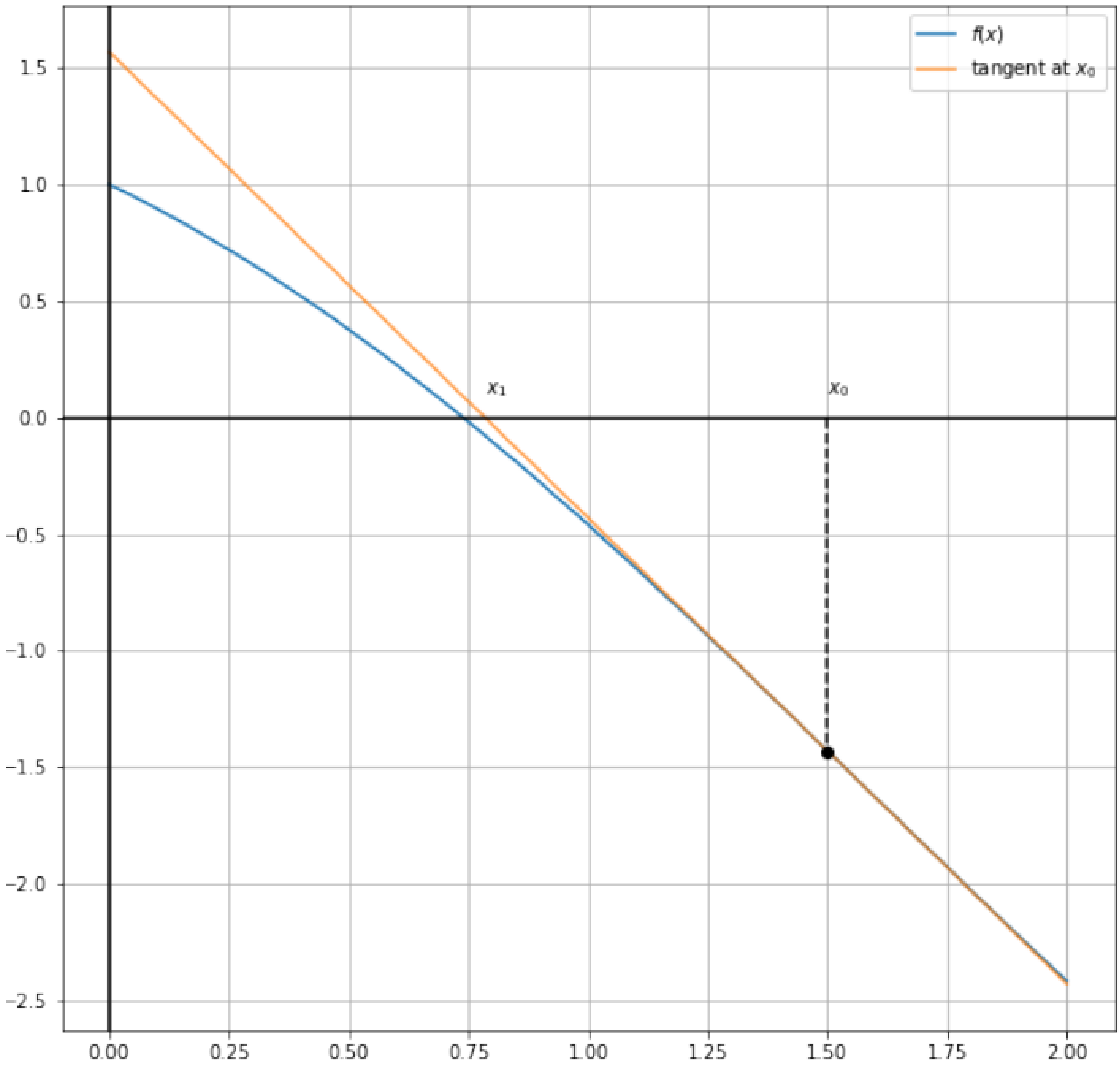


Figure 12.3: Improving on an initial guess.

This line crosses the x-axis closer to the actual solution to the equation than x_0

Call this point x_1, this point is our new approximation to the solution

Find x_1

$$0 = f(x_0) + f'(x_0)(x_1 - x_0)$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Now repeat the process to find an even better approximation:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Repeating this gives us **Newton's Method**:

If x_n is an approximation to a solution of $f(x) = 0$ and if $f'(x_n) \neq 0$ the next approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Rate of Convergence

The rate of convergence is at least quadratic if

- $f'(x) \neq 0$ for all x \in $[\alpha - r, \alpha + r] =: I$ for some $r \geq |\alpha - x_0|$

- $f''(x)$ is continuous for all $x \in I$
- x_0 is “sufficiently” close to the root α

Optimization

Instead of finding the root of $f(x) = 0$, now we wish to minimize $f(x)$

To do this, we must find the root of $f'(x) = 0$

We can now apply Newton's Method:

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}.$$

Gradient Descent

If we use $x \leftarrow x - \eta f'(x)$ to iterate x, the value of the function $f(x)$ might decline

Therefore, in gradient descent we first choose an initial value x and a constant $\eta > 0$ and then use them to continuously iterate x until the stop condition is reached, for example, when the magnitude of the gradient $|f'(x)|$ is small enough or the number of iterations has reached a certain value

We can examine the iterative scheme $x_{n+1} = x_n - \eta f'(x_n)$ as a dynamical system by defining $G_f(x) = x - \eta f'(x)$

The fixed point x^* of G_f is when $f'(x^*) = 0$ and we can determine the nature of this fixed point by looking at $G'_f(x^*) = 1 - \eta f''(x^*)$

Thus, we will have an attractor if $|1 - \eta f''(x^*)| < 1$, equivalently if $f''(x^*) > 0$ and $\eta f''(x^*) < 2$

Higher dimensions

Now consider $f : R^n \to R$ and assume it has a continuous derivative with a minimum at x_0

Thus, the gradient descent recursive scheme is given by

$$x_{n+1} = x_n - \eta \nabla f(x_n)$$