Equivalences

1. $\forall x \forall y \phi$ is logically equivalent to $\forall y \forall x \phi$.
2. $\exists x \exists y \phi$ is (logically) equivalent to $\exists y \exists x \phi$.
3. $\neg \forall x A$ is equivalent to $\exists x \neg \phi$.
4. $\neg \exists x \phi$ is equivalent to $\forall x \neg \phi$.
5. $\forall x(\phi \wedge \psi)$ is equivalent to $\forall x \phi \wedge \forall x \psi$.
6. $\exists x(\phi \vee \psi)$ is equivalent to $\exists x \phi \vee \exists x \psi$.

Equivalences involving variables not occurring free

Suppose that *x doesn't occur free in ϕ* (for example, when x doesn't occur in ϕ at all — see slide 199 for free variables). Then 7–9 below hold. *The restriction is necessary:* see slide 191.

7. $\forall x \phi$ and $\exists x \phi$ are logically equivalent to ϕ.
   E.g., $\forall x \underbrace{\exists x P(x)}_{\phi}$ and $\exists x \underbrace{\exists x P(x)}_{\phi}$ are equivalent to $\underbrace{\exists x P(x)}_{\phi}$.
8. $\exists x(\phi \wedge \psi)$ is equivalent to $\phi \wedge \exists x \psi$, and
   $\forall x(\phi \vee \psi)$ is equivalent to $\phi \vee \forall x \psi$.
9. $\forall x(\phi \rightarrow \psi)$ is equivalent to $\phi \rightarrow \forall x \psi$, and
   $\exists x(\phi \rightarrow \psi)$ is equivalent to $\phi \rightarrow \exists x \psi$.
10. *Note:* if x does not occur free in ψ (x can occur free in ϕ) then
    $\forall x(\phi \rightarrow \psi)$ is equivalent to $\exists x \phi \rightarrow \psi$, and
    $\exists x(\phi \rightarrow \psi)$ is equivalent to $\forall x \phi \rightarrow \psi$.
    The quantifier changes! Watch out!

Renaming bound variables

11. Suppose that x is any variable, y is a variable that does not occur in ϕ, and ψ is got from ϕ by
    - replacing all *bound* occurrences of x in ϕ by y,
    - replacing all $\forall x$ in ϕ by $\forall y$, and
    - replacing all $\exists x$ in ϕ by $\exists y$.

    Then ϕ is equivalent to ψ.

    E.g., $\forall x \exists y\, \texttt{Bought}(x, y)$ is equivalent to $\forall z \exists v\, \texttt{Bought}(z, v)$.
    $\texttt{Human}(x) \wedge \exists x\, \texttt{Lecturer}(x)$ is equivalent to
    $\texttt{Human}(x) \wedge \exists y\, \texttt{Lecturer}(y)$.

Equivalences/validities involving equality

12. $t = t$ is valid (equivalent to $\top$), for any term t.
13. For any terms t, u,
    $t = u$ is equivalent to $u = t$
14. (Leibniz principle) If ϕ is a formula, y doesn't occur in ϕ at all, and ψ is got from ϕ by replacing one or more free occurrences of x by y, then

    $$x = y \to (\phi \leftrightarrow \psi)$$

    is valid.

    Example:
    $x = y \to (\forall z R(x, z) \leftrightarrow \forall z R(y, z))$ is valid.

⚠ **Non-Equivalences**

Depending on ϕ, ψ, the following need *NOT* be logically equivalent (though always, the first $\models$ the second):

- $\forall x(\phi \to \psi)$ and $\forall x \phi \to \forall x \psi$
- $\exists x(\phi \land \psi)$ and $\exists x \phi \land \exists x \psi$.
- $\forall x \phi \lor \forall x \psi$ and $\forall x(\phi \lor \psi)$.

An Example

Using equivalences show that the argument $\forall x(P(x) \to Q(x)) \models \forall x P(x) \to \forall x Q(x)$ is valid.

We take its corresponding implication formula and show that it's equivalent to $\top$.

$$
\begin{aligned}
&\forall x(P(x) \to Q(x)) \to (\forall x P(x) \to \forall x Q(x)) && \text{[the original formula]}\\
&\equiv \neg(\forall x(\neg P(x) \lor Q(x))) \lor \neg \forall x P(x) \lor \forall x Q(x) && \text{[by removing } \to]\\
&\equiv \exists x \neg(\neg P(x) \lor Q(x)) \lor \exists x \neg P(x) \lor \forall x Q(x) && [\text{by } \neg \forall x \phi \equiv \exists x \neg \phi]\\
&\equiv \exists x(\neg\neg P(x) \land \neg Q(x)) \lor \exists x \neg P(x) \lor \forall x Q(x) && \text{[De Morgan laws]}\\
&\equiv \exists x(P(x) \land \neg Q(x)) \lor \exists x \neg P(x) \lor \forall x Q(x) && [\text{by removing } \neg\neg]\\
&\equiv \exists x((P(x) \land \neg Q(x)) \lor \neg P(x)) \lor \forall x Q(x) && [\exists x \phi \lor \exists x \psi \equiv \exists x(\phi \lor \psi)]
\end{aligned}
$$

$$\exists x((P(x) \lor \neg P(x)) \land (\neg Q(x) \lor \neg P(x))) \lor \forall x Q(x)$$
$$\text{[by distributivity of } \lor\text{]}$$

$$\equiv \exists x(\top \land (\neg Q(x) \lor \neg P(x))) \lor \forall x Q(x) \qquad [\text{by } \phi \lor \neg\phi \equiv \top]$$

$$\equiv \exists x(\neg Q(x) \lor \neg P(x)) \lor \forall x Q(x) \qquad [\text{by } \top \land \phi \equiv \phi]$$

$$\equiv \exists x \neg Q(x) \lor \exists x \neg P(x) \lor \forall x Q(x) \qquad [\text{by } \exists x \phi \lor \exists x \psi \equiv \exists x(\phi \lor \psi)]$$

$$\equiv \neg \forall x Q(x) \lor \exists x \neg P(x) \lor \forall x Q(x) \qquad [\text{by } \neg \forall x \phi \equiv \exists x \neg \phi]$$

$$\equiv \neg \forall x Q(x) \lor \forall x Q(x) \lor \exists x \neg P(x) \qquad [\text{by commutativity of } \lor]$$

$$\equiv \top \lor \exists x \neg P(x) \qquad [\text{by } \phi \lor \neg\phi \equiv \top]$$

$$\equiv \top \qquad [\text{by } \top \lor \phi \equiv \top]$$