Orthogonal Basis

Let U be a subspace of R^n

U = span$\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k}\}$

If $\vec{u_i}$ is perpendicular to $\vec{u_j}$ ($\vec{u_i} \cdot \vec{u_j} = 0$) for all i, j
then $\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k}\}$ is an orthogonal basis of U

If **also** $|\vec{u_i}| = 1$ for all i, then $\{\vec{u_1}, \vec{u_2}, \ldots, \vec{u_k}\}$ is orthonormal basis of U

Convert Basis to Orthonormal Basis

V = $\text{span}\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ = span$\{\vec{c_1}, \vec{c_2}, \ldots, \vec{c_k}\}$
such that span$\{\vec{c_1}, \vec{c_2}, \ldots, \vec{c_k}\}$ is orthonormal

Use Gram Schmidt Algorithm

1. $\vec{c_1} = \vec{v_1} / |\vec{v_1}|$
2. $\vec{P_2} = (\vec{v_2} \cdot \vec{c_1})\vec{c_1}$; $\vec{c_2}$= ($\vec{v_2} - \vec{P_2}$) / |$\vec{v_2} - \vec{P_2}$|
3. $\vec{P_3} = (\vec{v_3} \cdot \vec{c_1})\vec{c_1} + (\vec{v_3} \cdot \vec{c_2})\vec{c_2}$; $\vec{c_3}$= ($\vec{v_3} - \vec{P_3}$) / |$\vec{v_3} - \vec{P_3}$|
4. Etc

Example 1

V = span{$\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}3\\4\end{bmatrix}$}

1. $\vec{c_1}$= $\begin{bmatrix}1\\2\end{bmatrix}$ / $\sqrt{5}$
2. $\vec{P_2}$= ($(1/\sqrt{5})[3 \quad 4]\begin{bmatrix}1\\2\end{bmatrix}$)$\begin{bmatrix}1\\2\end{bmatrix}$ = (11/5)$\begin{bmatrix}1\\2\end{bmatrix}$; \vec{c_2} = ... = $(1/\sqrt{5})\begin{bmatrix}2\\-1\end{bmatrix}$

Gaussian Elimination

Construct Matrix A from basis vectors (join them together)
$[A^T A | A^T]$

EROs

[REF | $\begin{bmatrix} \vec{e_1}^T \\ \vec{e_2}^T \end{bmatrix}$]

Normalise Row vectors, then done!

Same Example

A = $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$A^T A$ = $\begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$

A^T = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

[$\begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$ | $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$]

EROs

[$\begin{bmatrix} 1 & 11/5 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1/5 & 2/5 \\ 1 & -1/2 \end{bmatrix}$]

Normalise Row vectors

$\vec{e_1} = (1/\sqrt{5}) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ = Same as before

$\vec{e_2} = (1/\sqrt{5}) \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ = Same as before