Linear Equation Systems

Linear Equation: $a_1x_1 + a_2x_2 + a_3x_3 + \ldots a_nx_n = b$
a_n = coefficients
x_n = variables
b = constant (aka free constant)

System = Collection of equations

System

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots a_{2n}x_n = b_2$
$\ldots$
$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots a_{mn}x_n = b_m$

- m equations
- n variables

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

$A\vec{x} = \vec{b}$

If b = 0 : Homogenous system of LE
Else : Not Homogenous

Task:

Find specific values of $x_1, x_2, \ldots x_n$, such that $A\vec{x} = \vec{b}$ is satisfied

In general:
$m > n \implies$ no solution, unless one equation is a multiple of another
$m < n \implies$ infinite solutions
$m = n \implies$ 1 unique solution

$m = 2, n = 3 \implies$ general solution = 2D plane in 3D space = Flat

Row Echelon Form = each pivot is always to the right of the pivot above
First non-zero coefficient in each row = pivot

Elementary Row Operations

- Order of equations does not matter.
- Multiplying by a non-zero scalar does not change the solution.
- Adding / Subtracting a row does not change the solution.

Example 5

Need augmented form of matrices!

Start:

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Using EROs

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

Reduced Row Echelon Form
x_1, x_2 = have pivots = dependent variables
x_3 = free variable
$x_1 = -2 + x_3$
$x_2 = 3 - 2x_3$
$x_3 = x_3$

$\vec{x} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ = straight line = general solution to original system

This process = Gaussian Elimination Method