Convergence

Let $(a_n)_{n\geq 1}$ be a sequence

1. The sequence $(a_n)_{n\geq 1}$ converges to a limit l in R iff $\forall \epsilon > 0 \; \exists N \in N \; \forall n > N \; |a_n - l| < \epsilon$
   - In this case, we denote this fact as $\lim_{n\to\infty} a_n = l$ or $(a_n)_{n\geq 1} \to l$
2. The sequence $(a_n)_{n\geq 1}$ converges to ∞ (in the extended real line $R \cup \{\infty\}$) iff $\forall r \in R \; \exists N \in N \; \forall n \geq N \; a_n > r$
   - We write $lim_{n\to\infty} a_n = \infty$ or $(a_n)_{n\geq 1} \to \infty$ in that case
3. The sequence $(a_n)_{n\geq 1}$ converges to $-\infty$ (in the extended real line $R \cup \{-\infty\}$) iff $\forall r \in R \; \exists N \in N \; \forall n \geq N \; a_n < r$
   - We write $lim_{n\to\infty} a_n = -\infty$ or $(a_n)_{n\geq 1} \to -\infty$ in that case
4. The sequence $(a_n)_{n\geq 1}$ converges if it converges either to a real number or to ∞ or to $-\infty$
5. The sequence $(a_n)_{n\geq 1}$ diverges if it does not converge

Understanding...

$|a_n - l| < \epsilon$ is called the **limit inequality** and says that the distance from the n^{th} term in the sequence to the limit should be less than ϵ

$\forall n > N \ldots$: There should be some point N after which all the terms a_n in the sequence are within ϵ of the limit

$\forall \epsilon > 0 \ldots$: No matter what value of ϵ we pick, however tiny, we will still be able to find some value of N such that all the terms to the right of a_N in that sequence are within ϵ of the limit

- *Consider the sequence $((-1)^n \cdot n)_{n\geq 1}$. It grows without bounds but it always jumps between negative and positive values in that growth. Therefore, this sequence diverges.*

An Example

$a_n = 1/n$ for $n \geq 1$

$$N(\epsilon) = \left\lceil \frac{1}{\epsilon} \right\rceil$$

Common Convergent Sequences

1. $(a_n)_{n\geq 1} = \left(\frac{1}{n}\right)_{n\geq 1} \to 0$, also $(a_n)_{n\geq 1} = \left(\frac{1}{n^2}\right)_{n\geq 1} \to 0$, $(a_n)_{n\geq 1} = \left(\frac{1}{\sqrt{n}}\right)_{n\geq 1} \to 0$
2. In general, $(a_n)_{n\geq 1} = \left(\frac{1}{n^c}\right)_{n\geq 1} \to 0$ whenever c is a positive real constant $c > 0$
3. $(a_n)_{n\geq 1} = \left(\frac{1}{2^n}\right)_{n\geq 1} \to 0$, also $(a_n)_{n\geq 1} = \left(\frac{1}{3^n}\right)_{n\geq 1} \to 0$, $(a_n)_{n\geq 1} = \left(\frac{1}{e^n}\right)_{n\geq 1} \to 0$
4. In general, $(a_n)_{n\geq 1} = \left(\frac{1}{c^n}\right)_{n\geq 1} \to 0$ whenever c is a real constant with $|c| > 1$;
   or equivalently $(a_n)_{n\geq 1} = \left(c^n\right)_{n\geq 1} \to 0$ whenever c is a real constant with $|c| < 1$
5. $(a_n)_{n\geq 1} = \left(\frac{1}{n!}\right)_{n\geq 1} \to 0$
6. $(a_n)_{n\geq 1} = \left(\frac{1}{\log n}\right)_{n\geq 1} \to 0$

Combinations of Sequences

Given sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ which converge to limits a and b respectively, and a real constant λ, we have:

1. $\lim_{n\to\infty} \lambda a_n = \lambda a$

2. $\lim_{n\to\infty} a_n + b_n = a + b$

3. $\lim_{n\to\infty} a_n - b_n = a - b$

4. $\lim_{n\to\infty} a_n b_n = ab$

5. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$ provided that $b \neq 0$

Triangular Inequality

$$|a+b| \leq |a| + |b|$$

An Example

$$a_n = \frac{3n^2 + 2n}{6n^2 + 7n + 1}$$

$$a_n = \frac{3 + \frac{2}{n}}{6 + \frac{7}{n} + \frac{1}{n^2}} = \frac{b_n}{c_n},$$

$$b_n = 3 + \frac{2}{n} \ \ and \ \ c_n = 6 + \frac{7}{n} + \frac{1}{n^2}.$$

$$if\ (b_n)_{n\geq 1} \to b \ and \ (c_n)_{n\geq 1} \to c \neq 0,$$

$$\left(\frac{b_n}{c_n}\right)_{n\geq 1} \to \frac{b}{c}$$

$$(a_n)_{n\geq 1} \to \frac{3}{6} = \frac{1}{2}.$$

A sequence $(a_n)_{n\geq 1}$ can only have one limit.

Cauchy Sequence

A sequence $(a_n)_{n\geq 1}$ is a Cauchy sequence iff
$\forall \epsilon > 0 \, \exists N \in N \, \forall n, m > N \, |a_n - a_m| < \epsilon$

Every sequence that converges to a real number is a Cauchy sequence

A subset $A \subset R$ is **complete** if any Cauchy sequence in A converges to a limit in A

$(1/n)_{n\geq 1}$ is a Cauchy sequence

The Sandwich Theorem

Let $(l_n)_{n\geq 1}$ and $(u_n)_{n\geq 1}$ be sequences, and l a real number where both $\lim_{n\to\infty} l_n = l$ and $\lim_{n\to\infty} u_n = l$

Suppose that for a third sequence $(a_n)_{n\geq 1}$ we have: $\exists N \in N\ \forall n \geq N\ l_n \leq a_n \leq u_n$

Then $\lim_{n\to\infty} a_n = l$ as well

Ratio tests for Sequences

The Ratio Test For Sequences

Let c in R be such that $0 \leq c < 1$

$$\exists N \in N\ \forall n \geq N\ |\frac{a_{n+1}}{a_n}| \leq c \implies \lim_{n\to\infty} a_n = 0$$

Limit ratio test

$$r = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|\ exists\ \wedge\ r < 1 \implies \lim_{n\to\infty} a_n = 0$$

Subsequences of a Sequence

If $f : N \to R$ is a sequence and $M \subset N$ an infinite subset, then the restriction $f : M \to R$ is a **subsequence of f**

Using the usual notation $(a_n)_{n\geq 1}$ for a sequence, any subsequence of this sequence would be of the form $(a_{n_i})_{i\geq 1}$ where n_i are positive integers with $n_1 < n_2 < n_3 < \ldots.$

Any subsequence of a convergent sequence converges to the limit of the sequence
$(a_n)_{n\geq 1}$ converges to $l \implies (a_{n_i})_{i\geq 1}$ also converges to l

Any sequence of real numbers has a monotonic subsequence

Manipulating absolute values: useful techniques

$$|x| < a \iff -a < x < a$$

$$|x \cdot y| = |x| \cdot |y|.$$

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$$

$|x + y| \leq |x| + |y|$, the triangle inequality.

$|x - y| \geq ||x| - |y||$, the reverse triangle inequality.

Properties of real numbers

Let X be a set of real numbers
Let l and u be real numbers. Then:

- u is an upper bound of X if $\forall x \in X \; (x \leq u)$
- l is a lower bound of X if $\forall x \in X \; (l \leq x)$
- a least upper bound (supremum, $sup(X)$) of X is an upper bound s of X such that $s \leq u$ for all upper bounds u of X
- a greatest lower bound (infimum, $inf(X)$) of X is a lower bound i of X such that $l \leq i$ for all lower bounds l of X

We say that such a set X is:

- Bounded above if X has an upper bound
- Bounded below if X has a lower bound
- Bounded if X has an upper and lower bound

The least upper bound of a set X is unique if it exists

Fundamental Theorem of Analysis

Let $(a_n)_{n \geq 1}$ be a sequence of real numbers that is

- Increasing and
- Bounded above

Then $s = sup\{\, a_n \mid n \geq 1 \,\}$ exists and is the limit of the sequence $(a_n)_{n \geq 1}$

The set of real numbers is complete