Equalities between Relations

$\circ$ is associative: for arbitrary relations $R \subseteq A \times B$ and $S \subseteq B \times C$ and $T \subseteq C \times D$, we have:

$R \circ (S \circ T) = (R \circ S) \circ T$

Proof: Let $\langle x, u \rangle$ be an arbitrary member of $(R \circ S) \circ T$. Then there exists $z \in C$ such that $x \, (R \circ S) \, z$ and $z \, T \, u$; but then there also exists $y \in B$ such that $x \, R \, y$ and $y \, S \, z$.

So there exists $y \in B$ such that $x \, R \, y$ and $y \, S \circ T \, u$, so also $x \, R \circ (S \circ T) \, u$; therefore $(R \circ S) \circ T \subseteq R \circ (S \circ T)$.

The reverse can be proved in a similar way. $\square$

Negative Results

Let R, S be binary relations on A.

Then, in general,

- $R \neq R^{-1}$
- $R \circ S \neq S \circ R$ (composition is not commutative)
- $R \circ R^{-1} \neq Id_A$

Proof The way to prove that a property does not hold is to provide a **counter example**.

A counter example to part 1 is $R = \{\langle a, b \rangle\} \subseteq \{a, b\}^2$.

A counter example to part 2 is $R = \{\langle a, a \rangle\}$ and $S = \{\langle a, b \rangle\}$ where $A = \{a, b\}$. Then $R \circ S \neq S \circ R$.

Exercise: Solve part 3.