Continuous Functions

An Example

Let's take a function $f : [0,1] \to \mathbb{R}$ with $f(x) = \sin 1/x$ if $x \neq 0$ and $f(0) = 0$. Intuitively we know that f has a limit at all points in $(0,1]$ but not at 0.

The function $f : [a, b] \to R$ has a limit $l \in R$ at $x_0 \in [a, b]$ if
$\forall \epsilon > 0\ \exists \delta > 0\ (x \in [a, b] \wedge |x - x_0| < \delta \implies |f(x) - l| < \epsilon)$

We write this as $\lim_{x \to x_0} f(x) = l$

The function f in the previous example has a limit for each $x_0 \in (0, 1]$ with $\lim_{x \to x_0} f(x) = \sin 1/x_0$ but f has no limit at $x_0 = 0$

The limit of a function at a point, if it exists, is unique

Suppose the two functions $f, g : [a, b] \to R$ have limits $k \in R$ and $l \in R$ respectively at $x_0 \in [a, b]$

$f \pm g$ has limit $k \pm l$ at x_0.

The product $f \cdot g$ has limit kl at x_0.

If $l \neq 0$, then f/g has limit k/l at x_0.

Continuity

Let $f : [a, b] \to R$ be a function and x in R

- We say f is continuous at $x_0 \in [a, b]$ if $\lim_{x \to x_0} f(x) = f(x_0)$
- We say that f is continuous in $[a, b]$ iff f is continuous at all $x_0 \in [a, b]$

Proposition 19
If the two functions $f, g : [a, b] \to \mathbb{R}$ are continuous at $x_0 \in [a, b]$, then we have:

- *$f \pm g$ is continuous at x_0.*
- *The product $f \cdot g$ is continuous at x_0.*
- *If $g(x_0) \neq 0$, then f/g is continuous at x_0.*

Suppose $f, g : (a, b) \to R$ are continuous functions. Then, we have:

- For any real number λ, the function $\lambda f : (a, b) \to R$ with $(\lambda f)(x) = \lambda f(x)$ is continuous
- $f + g : (a, b) \to R$ with $(f + g)(x) = f(x) + g(x)$ and $fg : (a, b) \to R$ with $(fg)(x) = f(x)g(x)$ are continuous
- If $f(x_0) \neq 0$ for some $x_0 \in (a, b)$ then g/f with $(g/f)(x) = g(x)/f(x)$ is continuous at x_0

Suppose $f : (a, b) \to R$ is a function with $x_0 \in (a, b)$ and $g : (c, d) \to R$ is a function with $Im(f) \subset (c, d)$. If f is continuous at x_0 and g is continuous at $f(x_0)$, then the composition $g \circ f : (a, b) \to R$ with $(g \circ f)(x_0) = g(f(x_0))$ is continuous at x_0

Maxima and Minima

If $f : [a, b] \to R$ is a continuous function, f has a maximum and a minimum in $[a, b]$

In particular a continuous function on a closed interval $[a, b]$ is bounded, i.e., there exists $K > 0$ such that $|f(x)| < K$ for all $x \in [a, b]$

If however f is only continuous in $(a, b]$ then it may not attain its supremum or infimum. For example the function $f : (0, 1] \to R$ with $f(x) = 1/x$ does not have a maximum and, in fact, no supremum over real numbers. On the other hand the identity function g : $[0, 1) \to R$ with $g(x) = x$ has a supremum 1 in $[0, 1)$ but the supremum is not attained in $[0, 1)$

In addition, the theorem is in general false if the interval is unbounded such as $[0, \infty)$

For example the identity map has clearly no maximum or supremum in $[0, \infty)$

Intermediate Value theorem

A continuous function takes all values between any pair of its values

If $f : [a, b] \to R$ is continuous and $s \in R$ is such that $f(a) < s < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = s$

Uniform Continuity

Another key property of a continuous function on a closed bounded interval $[a, b]$ is that it is uniformly continuous. We say that $f : A \to R$ is uniformly continuous on A if $\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x, x_0 \in A \, (|x - x_0| < \delta \to |f(x) - f(x_0)| < \epsilon)$
In other words $\delta > 0$ is independent of $x_0 \in A$

Example 28
The function $f : [a,b] \to \mathbb{R}$ with $f(x) = x^2$ is uniformly continuous. In fact, let $\epsilon > 0$ be given. Then $|f(x)-f(x_0)| = |x^2-x_0^2| = |x-x_0||x+x_0| \leq 2M|x-x_0|$ where $M = \max(|a|,|b|)$. Thus, for $\delta = \epsilon/(2M)$ we get $|x-x_0| < \delta$ implies $|x^2-x_0^2| < \epsilon$.

Theorem 23
If $f : [a,b] \to \mathbb{R}$, for $a, b \in \mathbb{R}$, is continuous, then it is uniformly continuous on $[a,b]$, i.e., for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, x_0 \in [a,b]$ we have $|f(x) - f(x_0)| < \epsilon$ if $|x - x_0| < \delta$.

If a function is continuous **on a closed interval**, then it is uniformly continuous

Intuitively, you can see, by considering the function $f : (0, 1] \to \infty$ with $f(x) = 1/x$, why uniform continuity can fail if the function is continuous on an interval such as $(0, 1]$ that is not closed: As x_0 gets close to 0, the difference $|f(x) - f(x_0)|$ can become arbitrary large no matter how close x is to x_0, i.e., no matter how small a value for $\delta > 0$ you choose in $|x - x_0| < \delta$