The Artin-Hasse Exponential and p-adic Numbers

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Introduction & Project Goals

In this project, we provide an original, combinatorial proof that the Artin Hasse exponential, $AH_p(x)$, as a formal power series has coefficients in the subring of p-adic integers: $\mathbb{Z}_{(p)} = \{r \in \mathbb{Q} : |x|_p\}$. Our proof of this result begins with a counting argument on the number homomorphisms, and makes stops in topological group theory, the combinatorics of counting integer partitions.

We then use this result to construct a new, direct proof of a known fact in p-adic number theory: for every p^k root of unity ζ in $\mathbb{C}_p$, there exists α in $\mathbb{C}_p$ such that $AH_p(\alpha) = \zeta$.

The p-adic Numbers

In this section, we provide some definitions and theorems which have been useful to us in gaining intuition for the p-adic numbers and in proving the following results.

Definition. For any prime $p \in \mathbb{Z}$, $\mathbb{Z}_p$, the **p-adic integers**, is the set of all infinite sequences such that: $...a_m a_{m-1}...a_1 a_0$ where each a_m is one of the elements $0, 1, 2, ..., p-1$.

Definition. $\mathbb{Q}_p$, the **p-adic numbers**, is the set of all two-sided sequences such that $...a_2 a_1 a_0.a_{-1} a_{-2}...$ for which $a_i \in \{0, 1, 2, ..., p-1\}$ for each i and such that $a_{-n} = 0$ for large n.

Definition. Fix p prime. For any rational number x, there are some integers n, s, and t with $p \nmid s, t$ so

$$x = \frac{s}{t} p^n.$$

The **p-adic valuation** of x is then $|x|_p := p^{-n}$.

Ostrowski Theorem. (Schikhof, p.22, [2]) Each non-trivial absolute value on $\mathbb{Q}$ is equivalent to either the absolute value, or some p-adic valuation, i.e. the only completions of $\mathbb{Q}$ are $\mathbb{R}$ or some $\mathbb{Q}_p$.

Proposition. (Schikhof, p.61, [2]) Let $\{a_n\}$ be a sequence in $\mathbb{Z}p$. $\sum a_n$ converges if and only if $lim_{n\to\infty} a_n = 0$

Proposition. (Newton Approximation) Let f be a $\mathbb{C}_p$-valued function on a disc $B := B_a(r) \subset \mathbb{C}_p$. Suppose there exists $s \in \mathbb{C}_p$ such that

$$\sup \left\{ \left| \frac{f(x) - f(y)}{x - y} - s \right|_p : x, y \in D, \, x \neq y \right\} < |s|_p$$

Then $s^{-1} f$ is an isometry and f maps discs onto discs; i.e. for all $b \in B$, $r_1 \in (0, r]$, $f : B_b(r_1) \to B_{f(b)}(|s|_p r_1)$ is a surjection.

Definition. $\mathbb{C}_p$, the **p-adic complex numbers**, is the completion of the algebraic closure of $\mathbb{Q}_p$.

Theorem. (Schikhof, p.68, [2]) D open and convex in $\mathbb{C}_p$; let $f : D \to \mathbb{C}_p$ be analytic. For any $v \in D$, there exists a sequence $\{b_i\} \subset \mathbb{C}_p$ so that for all $x \in D$

$$f(x) = \sum_{n=0}^{\infty} b_n (x - v)^n.$$

Theorem. (Hensel's Lemma) f analytic on $B_0(1)$,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad x \in B_0(1)$$

Suppose each $|a_n|_p \leq 1$ and there exists $a \in B_0(1)$ for which $f(a) < 1$ and $|f(a)|_p = 1$. Then there exists $b \in B_0(1)$ such that $|b - a|_p \leq |f(a)|$ and $f(b) = 0$.

Coefficients of $\mathbf{AH_p(x)}$ are in $\mathbb{Z}_{(p)}$

In the following section, let G be a topological group; let S_n be the symmetric group on n letters with the discrete topology. Let t_n be the number of continuous homomorphisms $G \to S_n$; let s_n be the number of such homomorphisms where the image is a transitive subgroup of S_n.

Lemma 1.1 Let G, S_n, t_n, and s_n be as above. Then

$$t_n = s_1 t_{n-1} + \binom{n-1}{1} s_2 t_{n-2} + \ldots + \binom{n-1}{n-2} s_{n-1} t_1 + s_n t_0.$$

Lemma 1.2 Let M_n be the number of open subgroups of G with index n. Then $s_n = M_n \cdot (n-1)!$.

Lemma 1.3 Let G be a topological group such that for every integer n, there are finitely many open subgroups of index n. Then

$$\frac{t_n}{(n-1)!} = t_0 M_n + t_1 \frac{M_{n-1}}{1!} + t_2 \frac{M_{n-2}}{2!} + \ldots + t_{n-1} \frac{M_1}{(n-1)!}.$$

Theorem 1.4 Let G be a topological group so that for every integer n, there are finitely many open subgroups of index n. Then

$$\exp\left(\sum_{H \leq G} \frac{x^{[G:H]}}{[G:H]} \right) = \sum_{n \geq 0} \frac{t_n}{n!} x^n$$

where H runs over the open subgroups of G with finite index.

Theorem 1.5 Let $t_{\mathbb{Z}_p, n}$ denote the number of continuous homomorphisms from $\mathbb{Z}_p$ to S_n. Then the coefficients of AH_p are $t_{\mathbb{Z}_p, n}/n!$, so

$$AH_p(x) = 1 + \sum_{n \geq 1} \frac{t_{\mathbb{Z}_p, n}}{n!} x^n.$$

Theorem 1.6 $AH_p(x) \in \mathbb{Z}_{(p)}[[x]]$.

$\mathbb{C}_\mathbf{p}$ Roots of Unity in the Image of $\mathbf{AH_p}$

Since the domain of convergence of the p-adic exponential is small, the only root of unity in the image of $\exp$ is 1; this leaves us with no way of representing roots of unity as $\exp(2\pi i\theta)$ as we do in $\mathbb{C}$.
This result that the p^k roots of unity in $\mathbb{C}_p$ are in the image of AH_p provides us with an appropriate analogue for the p-adic numbers.

Lemma 2.1 If $x \in m_p := \{\alpha \in \mathbb{C}_p : |\alpha|_p < 1\}$, then $AH_p(x)$ converges.

Lemma 2.2 In particular, $AH_p(x)$ diverges everywhere on the boundary

$$\{\alpha \in \mathbb{C}_p : |\alpha|_p = 1\}$$

thus the disc of convergence for $AH_p(x)$ is exactly m_p.

Lemma 2.3 $AH_p : m_p \to m_p + 1$ is a surjective isometry.

Theorem 2.4 For every p^k root of unity $\zeta \in \mathbb{C}_p$, there is $\alpha \in m_p$ for which $AH_p(\alpha) = \zeta$.

Outlines of Proofs

Lemma 1.1 We use a constructive counting argument

Lemma 1.2 Like Lemma 1.1, using concepts of continuous homomorphism and Orbit-Stabilizer theorem, we could prove it.

Lemma 1.3 Use Lemma 1.1 & 1.2

Theorem 1.4 Use Lemma 1.3 and the complete exponential Bell polynomials, then induction on n.

Theorem 1.5 Follows immediately from 1.4.

Theorem 1.6 Let $T = \{\sigma \in S_n : \sigma^{p^k} = 1, \, k \in \mathbb{N}\}$ be the elements of S_n. Using theorem of Frobenius, $p^{v_p(n!)}$ divides the size of

$$A := \{\sigma \in S_n : \sigma^{p^{v_p(n!)}} = 1\}.$$

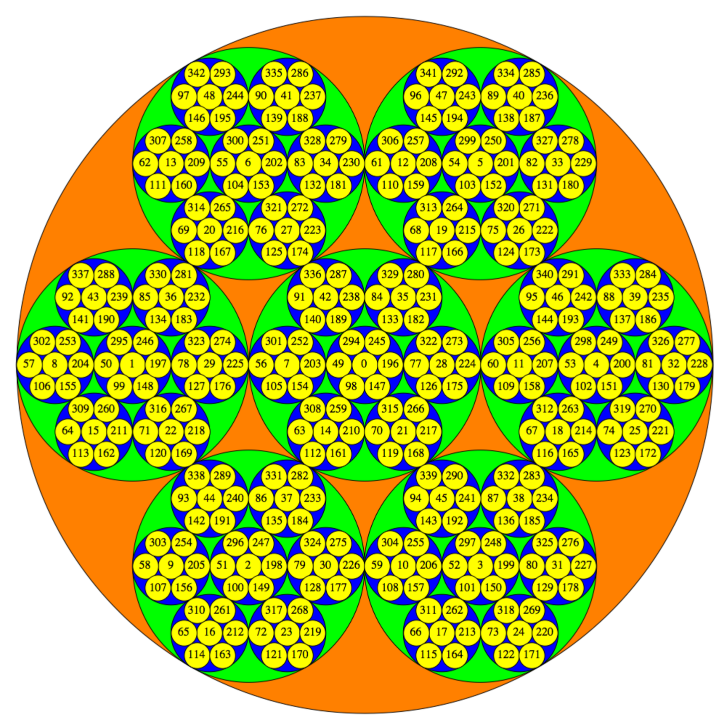
Clearly, $A \subseteq T$, and show that $T \subseteq A$ then done.

Lemma 2.1 From Theorem 1.6, all coefficients of $AH_p(x)$ in $\mathbb{Z}_p$. This means $|\text{coefficients}|_p \leq 1$, and since our domain is m_p. So, $AH_p(x)$ converges in m_p

Lemma 2.2 Theorem 2.10 (Conrad, p.6, [1])

Lemma 2.3 Use Theorem 2.5 (Conrad, p.4, [1]) to show the isometry and Lemma 27.4 (Schikhof, p.78-79, [2]) to show the surjectivity.

Theorem 2.4 As a result based on what we have proved from Theorem 1.1 to Theorem 2.3



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References

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