Spacecraft Dynamics and Control

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Lecture 4: The Orbit in Time

Introduction

In this Lecture, you will learn:

Motion of a satellite in time

- Given t, find $f(t)$ and vice-versa.
- New Angles
  - Mean Anomaly
  - True Anomaly
- How to convert between them
  - Kepler's Equation

Problem: Let $a = 25,512km$ and $e = .625$. Find r, v at $t = 4hr$.

Recall the Conic Equation

$$r(t) = \frac{p}{1 + e\cos f(t)}$$

Which we have shown describes elliptic, parabolic or hyperbolic motion.

Question: What is $f(t)$?

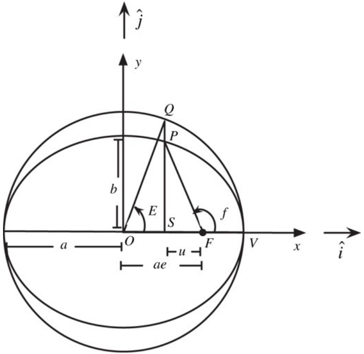
Response: There is no closed-form expression for $f(t)$!

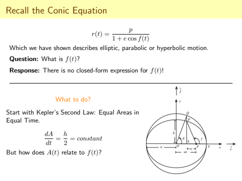
What to do?

Start with Kepler's Second Law: Equal Areas in Equal Time.

$$\frac{dA}{dt} = \frac{h}{2} = constant$$

But how does $A(t)$ relate to $f(t)$?





- While semi-latus rectum and eccentricity define the conic section in polar form, a and b define the conic section in rectilinear coordinates.

A useful geometric tool is to inscribe the ellipse in a circle.

The equation of a conic section is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

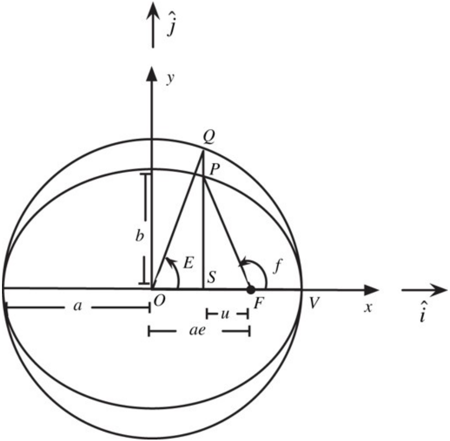
Solving for y,

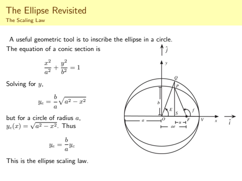
$$y_e = \frac{b}{a}\sqrt{a^2 - x^2}$$

but for a circle of radius a, $y_c(x) = \sqrt{a^2 - x^2}$. Thus

$$y_e = \frac{b}{a} y_c$$

This is the ellipse scaling law.



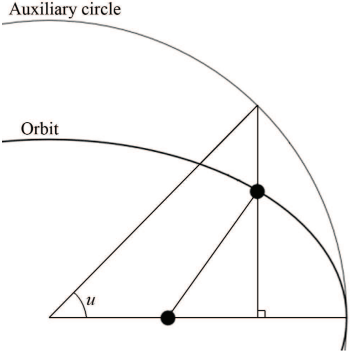


- A circle is defined by $x^2 + y^2 = a^2$. In this case $a = b$. We then solve this for y to get $y = \sqrt{a^2 - x^2}$.
- The ellipse scaling law states that all point on the ellipse are scaled towards the major axis by a factor of b/a. Obviously for an ellipse $b < a$.

The Eccentric Anomaly

The **Eccentric Anomaly** is an artificial angle

- From the *Center* of the ellipse
- To the projection of r on a fictional circular orbit of radius a



- Measured from center of ellipse (not focus).
- No physical interpretation.
- A mathematical convenience

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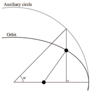

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- No physical interpretation.
- A mathematical convenience

- The eccentric anomaly is convenient because it gives a geometric angle which serves a substitute for time and for which we can compute based on swept area. Since the rate of area sweep is constant, this is significant.
- In the image, u is the eccentric anomaly. However, we typically use E to denote this angle.

For convenience, suppose $t = 0$ at periapse. The area swept out is FVP

Kepler's Second Law say that

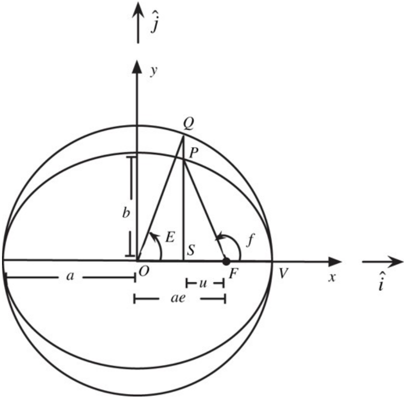
$$\frac{t_P}{T} = \frac{\text{Area of FVP}}{\text{Area of ellipse}} = \frac{A_{FVP}}{\pi ab}$$

But what is A_{FVP}?

$$A_{FVP} = A_{PSV} - A_{PSF}$$

PSF is a triangle, so

$$A_{PSF} = \frac{1}{2}\underbrace{(\overbrace{ae}^{OF} - \overbrace{a\cos E}^{OS})}_{u=SF} \cdot \underbrace{\frac{b}{a}(\overbrace{a\sin E}^{QS})}_{PS}$$



E is the **Eccentric Anomaly**.

The conversion from E to f (or vice-versa) is not difficult.

The Ellipse Revisited

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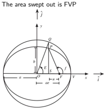

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- This also gives an expression for

$$A_{FVP}(t_P) = \frac{T}{\pi ab} t_P$$

- Of course, what we want is to find $f(t_P)$, or even $E(t_P)$!
  - What we will actually find is $A_{FVP}(E(t_P))$ and try to invert the expression to find a formula for $E(t_P)$
- We will revisit the assumption $t_0 = 0$ at the very end of the lecture.

- FVP, PSV, PSF, et c. denote the triangles or sections with vertices at points F, V, and P, et c.
- A_{FVP} is the area inside section FVP
- t_P is the time at which we reach point P
- line QS is length $a \sin E$. line OS is length $a \cos E$

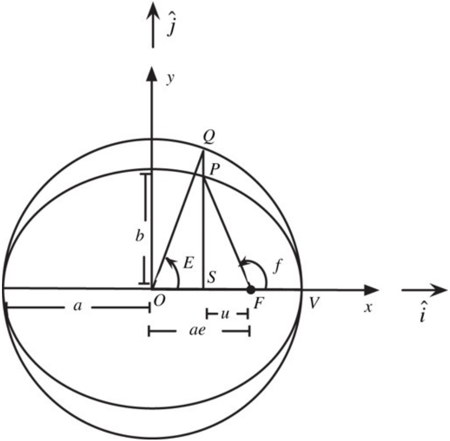
The Ellipse Revisited

It is easy to see by the scaling law that $A_{PSV} = \frac{b}{a} A_{QSV}$. A_{QSV} is easily calculated as

$$\begin{aligned} A_{QSV} &= A_{QOV} - A_{QOS} \\ &= \frac{1}{2}a^2 E - \frac{1}{2}\underbrace{a\cos E}_{OS} \cdot \underbrace{a \sin E}_{QS}. \end{aligned}$$

where E **is in radians**. Thus we conclude

$$\begin{aligned} A_{FVP} &= A_{PSV} - A_{PSF} \\ &= \frac{1}{2}ab(E - \cos E \sin E) \\ &\qquad - \frac{1}{2}ab(e - \cos E)\sin E \\ &= \frac{1}{2}ab(E - e\sin E). \end{aligned}$$



Mean Anomaly

The conclusion is that

$$\frac{t_P}{T} = \frac{A_{FVP}(t_P)}{\pi ab} = \frac{E(t_P) - e \sin E(t_P)}{2\pi}.$$

Since by Kepler's third law,

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

we have

$$\frac{E(t_P) - e \sin E(t_P)}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\mu}{a^3}}t_P.$$

- Thus we have an expression for t_P in terms of $E(t_P)$.
- What we really want is an expression for E in terms of t_P.
- Unfortunately no such analytic solution exists.
  - Equation must be solved numerically for each value of t.
  - Prompted invention of first known numerical algorithm, Newton's Method.

Mean Anomaly

We define some terms

Definition 1.

The mean motion, n is defined as

$$n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}$$

Definition 2.

The mean anomaly, $M(t)$ is defined as

$$M(t) = nt = \sqrt{\frac{\mu}{a^3}}t$$

Neither of these have good physical interpretations.

$$M(t) = E(t) - e \sin E(t)$$

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- Mean Anomaly can be thought of as the fraction of the period of the orbit which has elapsed, but put into radians.
- However, it simplifies the expression for E

$$E(t) - e \sin E(t) = nt = M(t)$$

- We will use Newton's algorithm to solve this equation.

Converting Between E and f

Once we get E from solving Kepler's equation, we still need to find the angle f in order to recover position. Going back to the ellipse...

We express the line OS using both E and f.

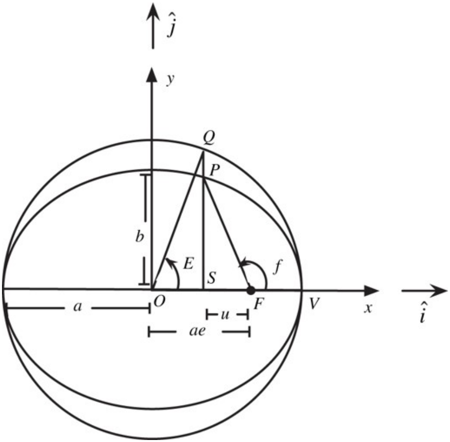
$$\begin{aligned} OS &= a\cos E \\ &= ae + r\cos f \end{aligned}$$

But $r = \frac{a(1-e^2)}{1+e\cos f}$, so

$$\cos E = \frac{e+\cos f}{1+e\cos f}.$$

Using the half-angle formula, we can get the expression

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2}.$$



Given f, we can find E.

Converting Between E and f

Alternatively, given E, we can find f.

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

We can also now directly express the polar equation using E,

$$r(t) = a(1 - e\cos E(t))$$

Example: Use geometry to get time (The Easy Problem)

Problem: Given an Earth orbit with $a = 10,000km$ and $e = .5$, determine the times at which $r = 14,147km$.

Solution: First solve for the true anomaly, f. we have

$$r(t) = \frac{a(1-e^2)}{1+e\cos f(t)}$$

which yields

$$\cos f(t) = \frac{a(1-e^2) - r(t)}{er(t)} = -.9397$$

Solving for f yields two solutions $f = 160°, 200°$.

Now we want to find $E(t)$.

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2} = \pm 3.27$$

This yields

$$E = \pm 146.0337°,$$

Example: Going from E to M is easy

Solving for mean anomaly (*in radians!!*),

$$M(t) = E(t) - e \sin E(t) = 2.2694 rad, 4.0138 rad$$

Now the mean motion is

$$n = \sqrt{\frac{\mu_e}{a^3}} = 6.3135 \cdot 10^{-4}$$

So finally, the times of arrival are

$$t = \frac{M(t)}{n} = 3594s, 6357s$$

Note: In this way, it is easy to find the time between any 2 points in the orbit. e.g. from $f = 160°$ to $f = 200°$ takes time $\Delta t = 6357 - 3594 = 2763s$.

Problem 2: Prediction (The Harder One)

Given t, find r and v

Generally speaking we can follow the previous steps in reverse.

1. Given time, t, solve for Mean Anomaly

$$M(t) = nt$$

2. Given Mean Anomaly, solve for Eccentric Anomaly
   - How???
3. Given eccentric anomaly, solve for true anomaly

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

4. Given true anomaly, solve for r

$$r(t) = \frac{a(1-e^2)}{1+e\cos f(t)}$$

The Missing Piece is how to solve for Eccentric Anomaly, E given Mean Anomaly, M.

Solving the Kepler Equation

Given M, find E

$$M = E - e \sin E$$

- A Transcendental Equation
- No Closed-Form Solution
- However, for any M, there is a unique E.

To Solve Kepler's Equation, Newton had to redefine the meaning of a solution.

Iterative Methods (Algorithms):

Instead of solving a single equation, we solve a sequence of equations until a stopping criterion (usually error tolerance) is met.

- The solution is never exact.
- Perfect for implementation on computers
- Dramatically increased the set of solvable problems.
- Today, most problems are solved via Algorithms.

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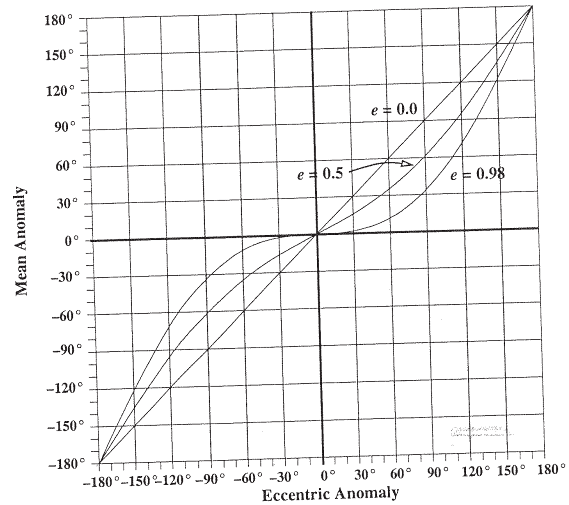
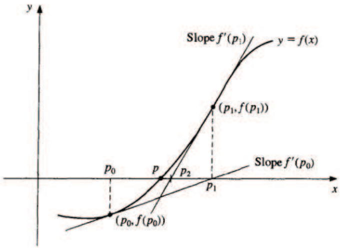


Newton-Raphson Iteration

An Algorithm for solving equations

$$f(x) = 0$$



Start by guessing the solution x_k.

- Approximate $f(x) \cong f(x_k) + f'(x_k)(x - x_k)$.
- Solve $f(x_k) + f'(x_k)(x - x_k) = 0$

$$x = x_k - \frac{f(x_k)}{f'(x_k)}$$

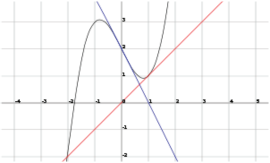
- Update your guess, $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$
- Repeat until $\|f(x_k)\|$ is sufficiently small.

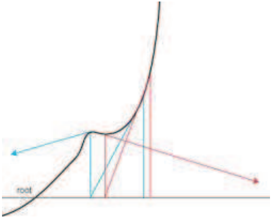
Newton's Method

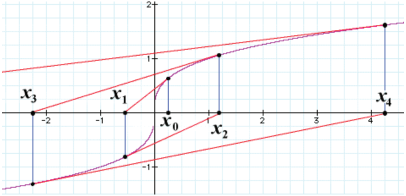
Illustration

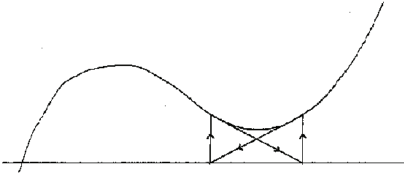
Failure of Newton-Raphson Iteration

When Newton's Method Works, it works well

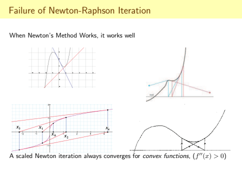






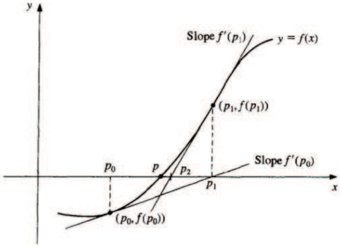


A scaled Newton iteration always converges for *convex functions*, ($f''(x) > 0$)



- By scaled, we mean we use the linear approximation, but only update the guess as
$$x_{k+1} = x_k - \alpha \frac{f(x_k)}{f'(x_k)}$$
where $0 < \alpha < 1$ is some sufficiently small step size.
- For Kepler's equation, we can use $\alpha = 1$.

Applied to Kepler's Equation



Given M, we want to solve

$$f(E) = M - E + e \sin E = 0 \qquad \text{then,} \qquad f'(E) = -1 + e \cos E$$

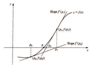
Algorithm: Choose $E_1 = M$.

- Update

$$E_{k+1} = E_k - \frac{M - E_k + e \sin E_k}{e \cos E_k - 1}$$

- If $\|M - E_k + e \sin E_k\| < .001$ or whatever, quit.
- Otherwise repeat.

Applied to Kepler's Equation



Given M, we want to solve

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$$E_{k+1} = E_k - \frac{M - E_k + e\sin E_k}{e\cos E_k - 1}$$
- If $\|M - E_k + e\sin E_k\| < .001$ or whatever, quit.
- Otherwise repeat.

- If N-R converges, it usually only takes 2 or 3 iterations.
- Good for you, as you will do this by hand.
- Balance between convergence rate and stability.
- Scaled N-R always converges but is much slower.

Relationships between M, E, and f

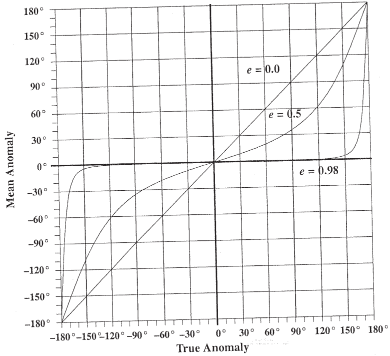

Figure: Elliptic Mean Anomaly vs. True Anomaly

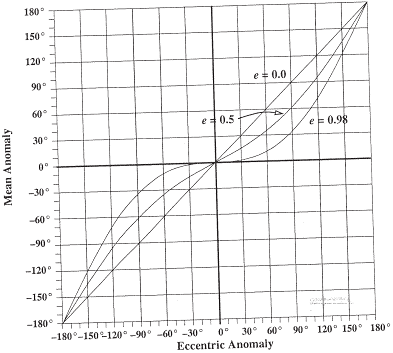

Figure: Mean Anomaly vs. Eccentric Anomaly

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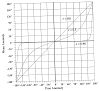

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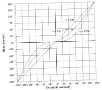


Figure: Mean Anomaly vs. Eccentric Anomaly

- These graphs are from Vallado's book.
- Graphical methods of relating mean and eccentric anomaly are difficult due to dependence on eccentricity.
- The difficulty in using graphical methods is exacerbated for true anomaly, especially for highly elliptic orbits.
- This is one of the reasons why we use solve first for eccentric anomaly and then true anomaly.

Example : Prediction (The Hard Problem)

Problem: Let $a = 25,512km$ and $e = .625$. If $f(0) = 0$, find r, v at $t = 4hr$.

Solution: First, solve for Mean Anomaly.

$$n = \sqrt{\frac{\mu}{a^3}} = 1.549 \cdot 10^{-4} s^{-1}$$

Thus

$$M(t) = nt = 1.549 \cdot 10^{-4} * 4 * 3600 = 2.231 rad$$

Newton Iteration: Now to solve for E, we set $E_1 = M$ and iterate

$$E_2 = E_1 - \frac{2.231 - E_1 + .625 \sin E_1}{.625 \cos E_1 - 1} = 2.588$$

$$f(E_2) = 2.231 - E_2 + .625 \sin E_2 = -.0284$$

We verify that $\|f(E_2)\| = .0284 > .001$, so continue:

$$E_3 = E_2 - \frac{2.231 - E_2 + .625 \sin E_2}{.625 \cos E_2 - 1} = 2.570$$

$$f(E_3) = 2.231 - E_3 + .625 \sin E_3 = -.000892$$

Example

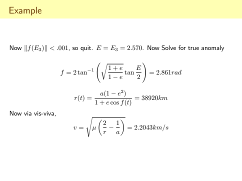
Now $\|f(E_3)\| < .001$, so quit. $E = E_3 = 2.570$. Now Solve for true anomaly

$$f = 2\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right) = 2.861rad$$

$$r(t) = \frac{a(1-e^2)}{1+e\cos f(t)} = 38920km$$

Now via vis-viva,

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)} = 2.2043km/s$$



In the case where $f(t_0) \neq 0$, we must first calculate $M(t_0)$ using

$$\tan\frac{E(t_0)}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{f(t_0)}{2}$$

and

$$M(t_0) = E(t_0) - e\sin E(t_0)$$

Then we calculate the Mean anomaly at the final time as

$$M(t_f) = M(t_0) + n(t_f - t_0)$$

Important! If $M(t_f) > 2\pi$, then subtract off the largest value of 2π before solving Kepler's Equation!!!

Also Important! Don't forget to use radians and NOT degrees when solving Kepler's Equation!!!

Final Note! If you are given $r(t_0)$ and not $f(t_0)$, you can calculate $f(t_0)$ using the polar equation.

$$r(t_0) = \frac{p}{1 + e\cos f(t_0)}$$

However, this has TWO solutions, so you have to be able to resolve the quadrant ambiguity somehow.

Conclusion

In this Lecture, you learned:

- How to predict position given time.
- New Angles
  - Mean Anomaly
  - Eccentric Anomaly
  - True Anomaly
- How to convert between them
  - How to Solve Kepler's Equation

Key Equations:

$$n = \sqrt{\frac{\mu}{a^3}}$$

$$M(t) = nt$$

$$M(t) = E(t) - e\sin E(t)$$

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2}$$

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

Newton Iteration:

$$E_0 = M$$

$$E_{k+1} = E_k - \frac{M - E_k + e\sin E_k}{e\cos E_k - 1}$$