NYU

Virtual and Augmented Reality

CS-GY 9223/CUSP-GX 6004

2023 Fall

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A VR/AR System

Real-time computer graphics

Interfaces

Visual perception

Computational optics and tracking

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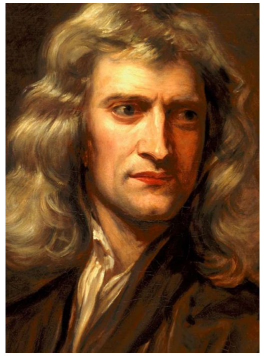
Lecture 1.1 - Basic Optical *Concepts*

Basic Visual Optics – What is Light?



Sir Isaac Newton
1643-1727

Christiaan Hygens
1629-1695

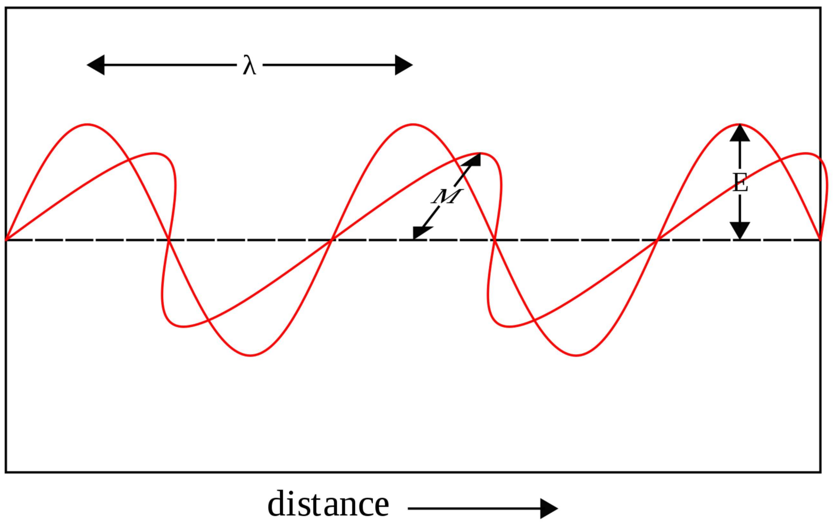
"… very small bodies emitted from shining substances"

– Issac Newton

" "A wave motion" spreading out from the course of all directions"

-- Christian Huygens

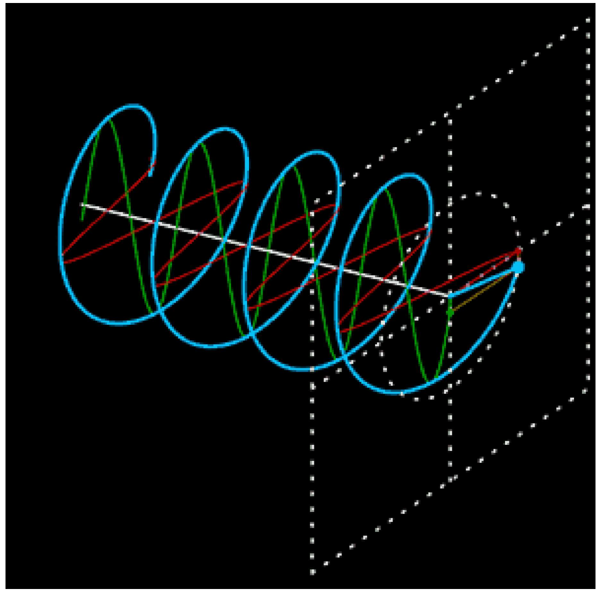
Light wave



λ = wave length

E = amplitude of electric field

M = amplitude of magnetic field



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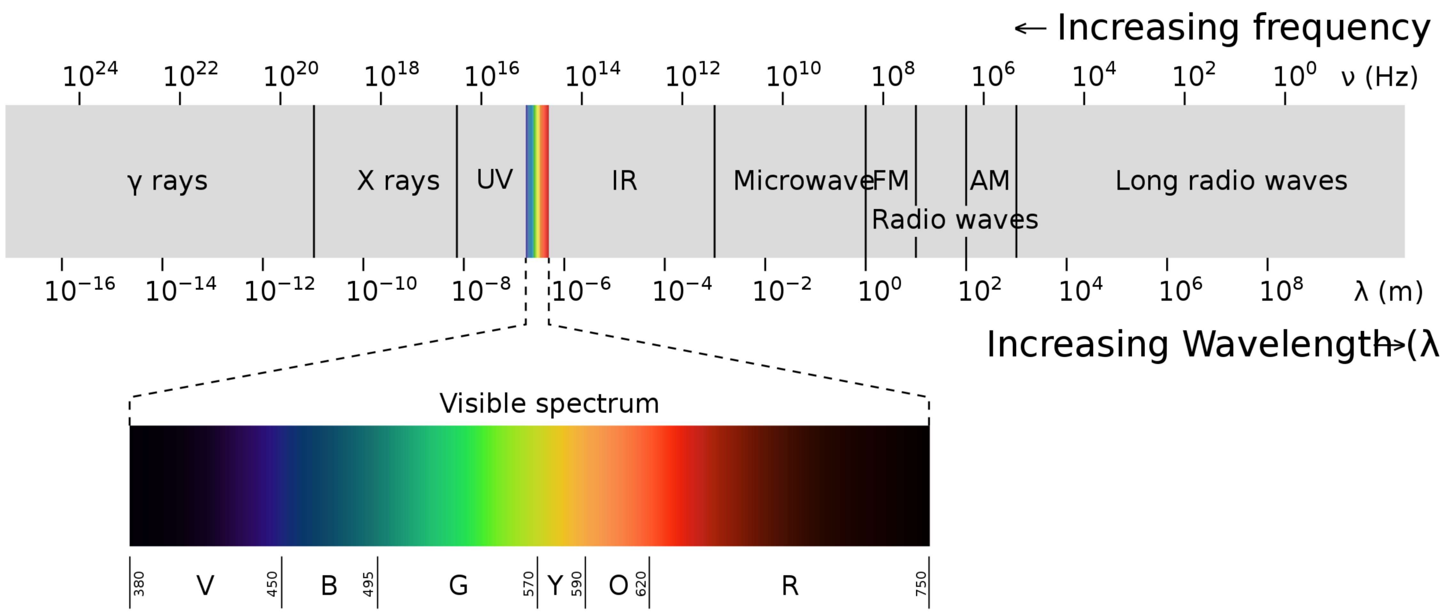
Basic Visual Optics – What is Light?

Visible light is electromagnetic radiation within the portion of the electromagnetic spectrum that is perceived by the human eye.



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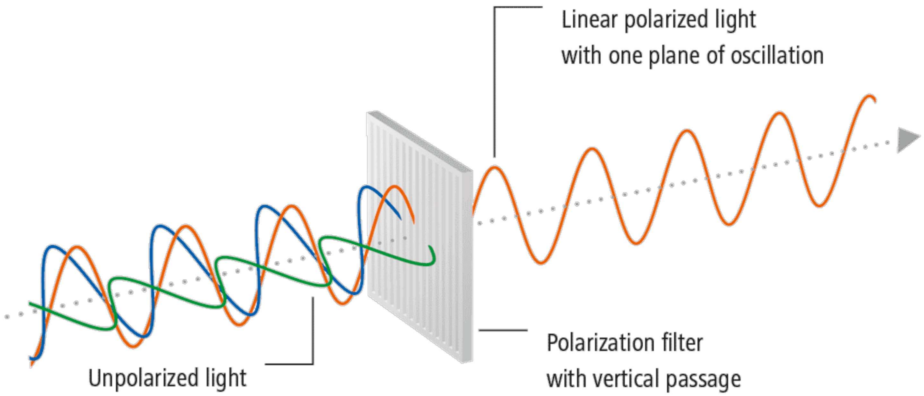
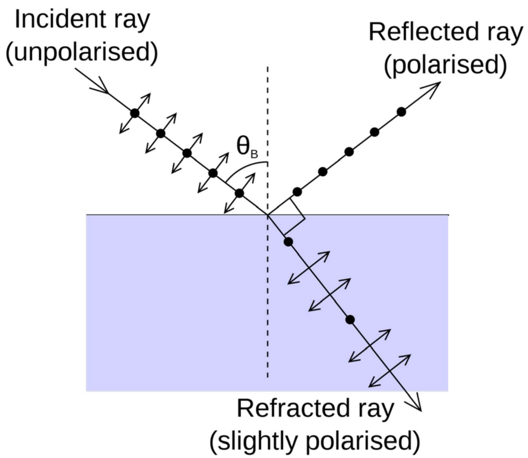
Basic Visual Optics – Polarization

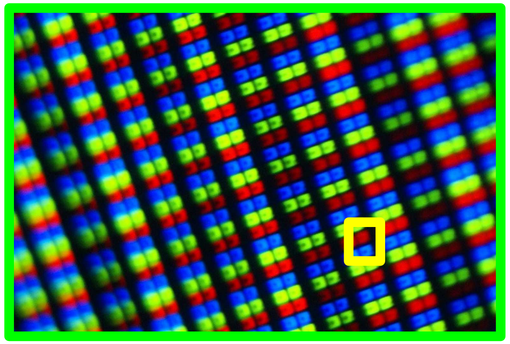


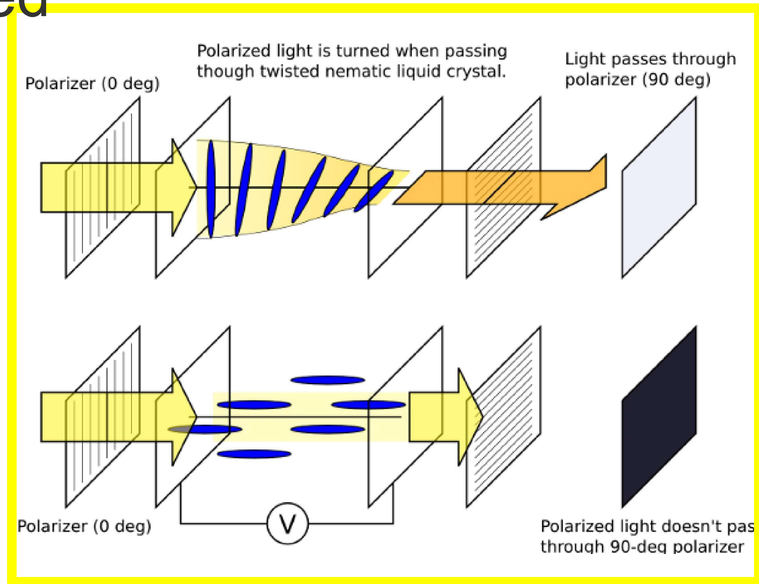


Basic Visual Optics – LCD (Liquid Crystal Display)

- Brighter
- Lower power consumption
- Cheaper

Untethered

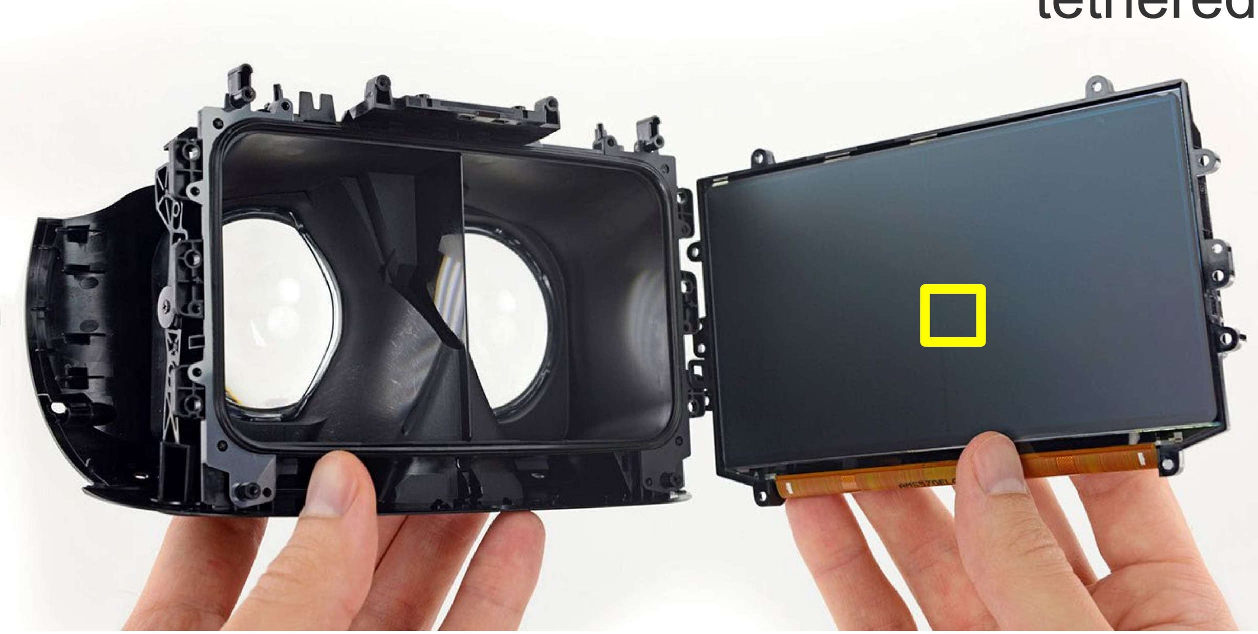


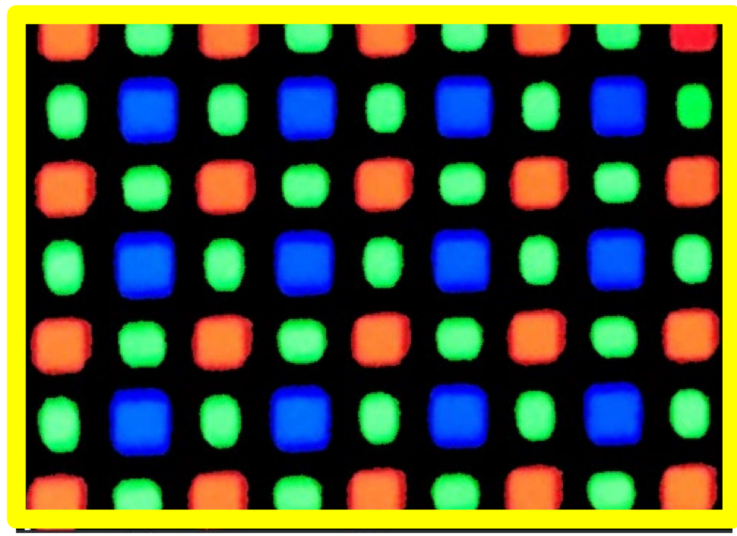


Basic Visual Optics – OLED (Organic Light-Emitting Diode)

- High contrast
- Broader color range
- Better black

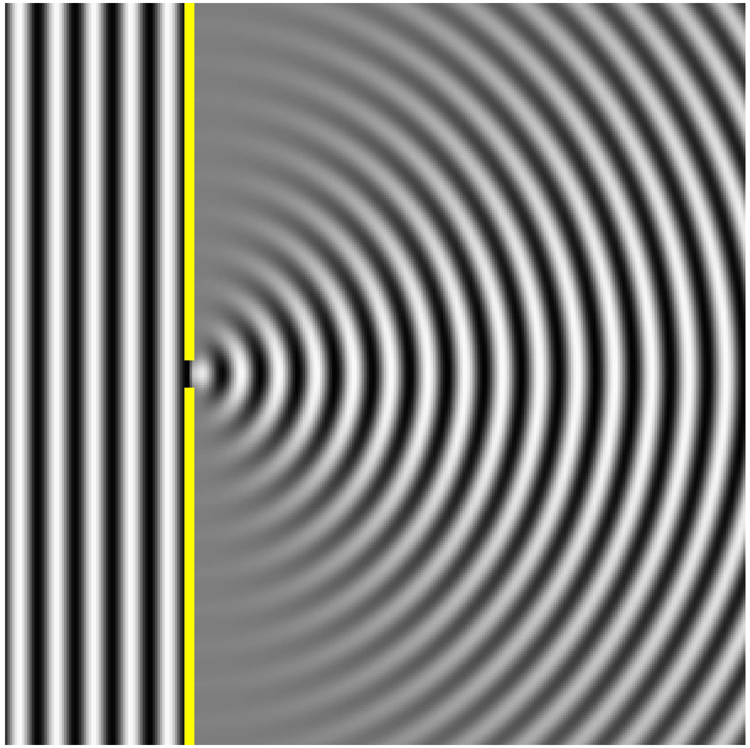
tethered





Basic Visual Optics - Wave Propagation

Diffraction
– superposition of the second wavelet from the same wave



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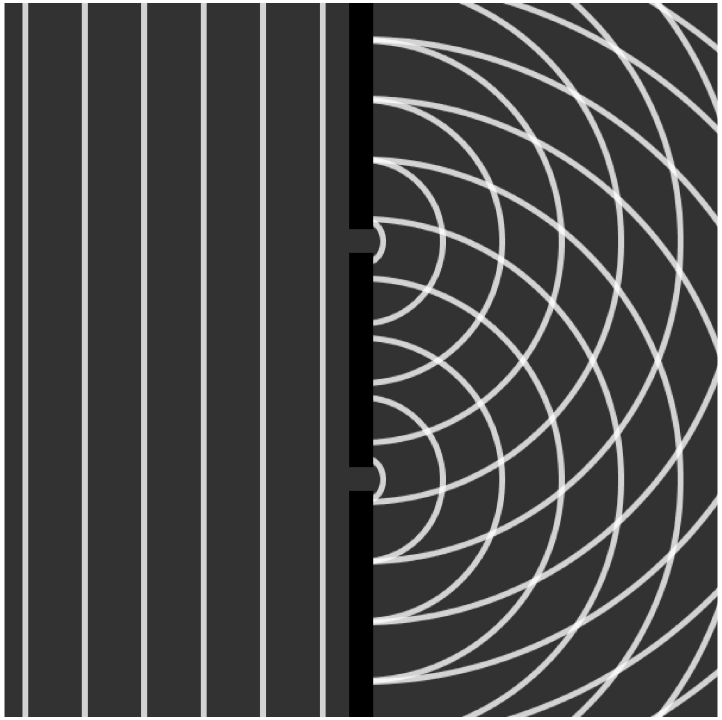
Basic Visual Optics - Wave Propagation

Interference
– superposition of two waves



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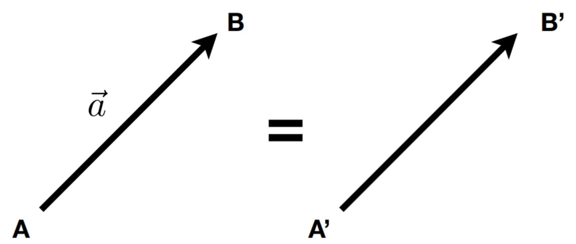
Lecture 1.2
Mathematical Foundations
-- Revisiting Basic Algebra

Vectors

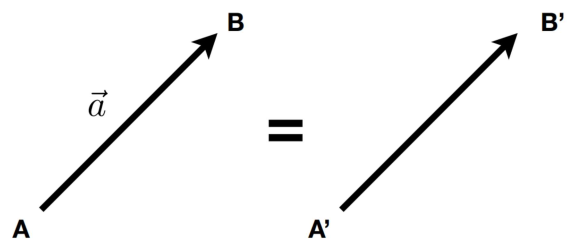


- A vector $\overrightarrow{(x, y, z, \ldots)}$ describes a **direction** and a **length** **without** a starting point
- A vector is NOT a pair {x,y}, NOR a position (x,y)

$$\vec{a} \qquad \boldsymbol{a} \qquad \overrightarrow{AB} = B - A$$

Vector Operation

unit & normalization



- A unit vector is a vector with length = 1 (direction-only)
- Normalization - $\hat{a} = \vec{a}/||\vec{a}||$

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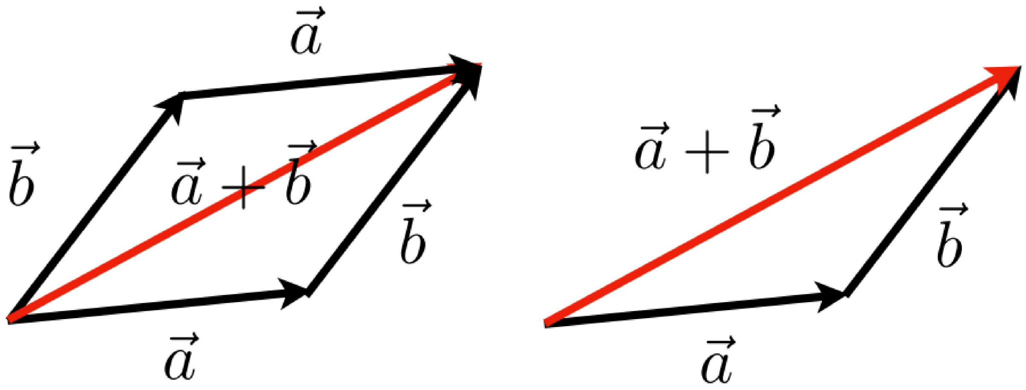
Vector Operation

(positive/negative) add



$$\vec{a}+\vec{b} = \vec{b} + \vec{a}$$

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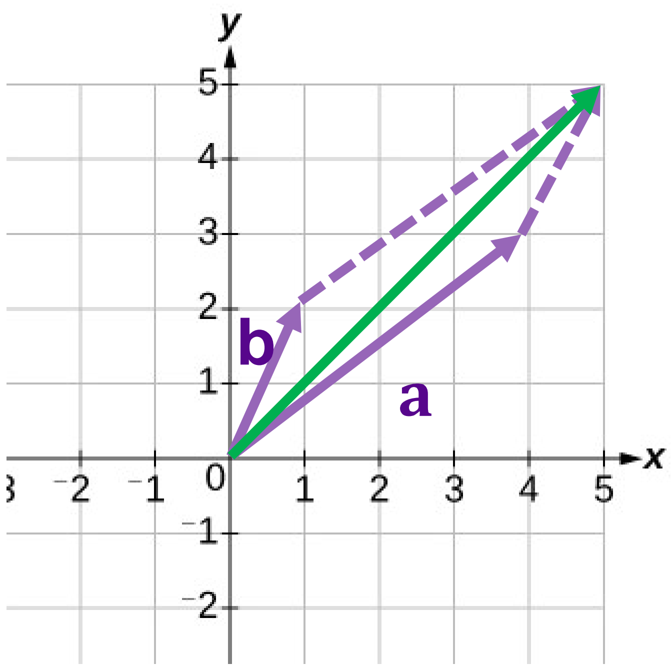
Vector Operation

Cartesian Coordinate



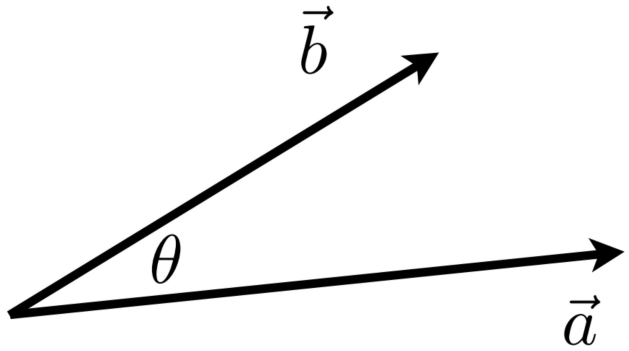
$$\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4\mathbf{x} + 3\mathbf{y}$$

$$\mathbf{a} + \mathbf{b} = (\mathbf{4} + \mathbf{1}, \mathbf{3} + \mathbf{2})$$

$$|\mathbf{a}| = \sqrt{\mathbf{4}^2 + \mathbf{3}^2}$$

Vector Operation

Dot Product



$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}$$

Unit Vector:

$$\cos\theta = \hat{a} \cdot \hat{b}$$

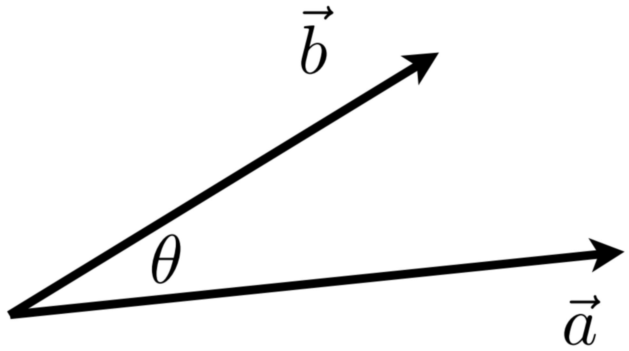
$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

$$(k\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b}) = k(\vec{a} \cdot \vec{b})$$

Vector Operation

Dot Product in Cartesian Coordinate

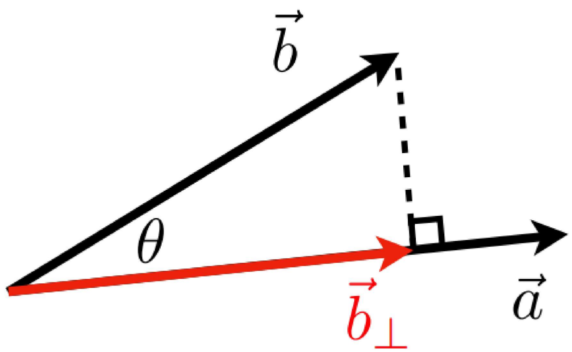


$$\vec{a} \cdot \vec{b} = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \end{pmatrix} = x_a x_b + y_a y_b$$

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = x_a x_b + y_a y_b + z_a z_b$$

Vector Operation

Dot Product in Graphics



Find projected length

$\vec{b}_\perp$: projection of $\vec{b}$ onto $\vec{a}$

- $\vec{b}_\perp$ must be along $\vec{a}$ (or along $\hat{a}$)
  - $\vec{b}_\perp = k\hat{a}$
- What's its magnitude k?
  - $k = \|\vec{b}_\perp\| = \|\vec{b}\| \cos\theta$

Find angles between vectors

$$\theta = \arccos\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|}\right)$$

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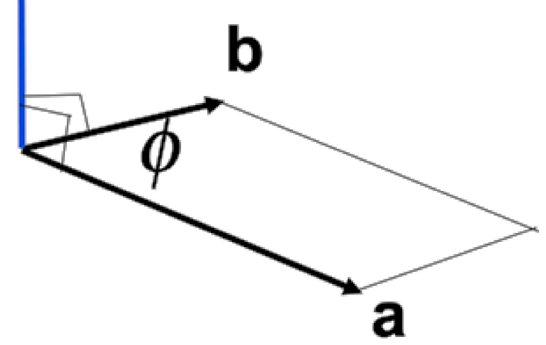
Vector Operation

Cross Product

$$a \times b = -b \times a$$

$$\|a \times b\| = \|a\| \|b\| \sin\phi$$



- Orthogonal to two initial vectors
- Direction determined by right-hand rule
- Useful in constructing coordinate systems (later)

Cross product: Properties

$$\vec{x} \times \vec{y} = +\vec{z}$$

$$\vec{y} \times \vec{x} = -\vec{z}$$

$$\vec{y} \times \vec{z} = +\vec{x}$$

$$\vec{z} \times \vec{y} = -\vec{x}$$

$$\vec{z} \times \vec{x} = +\vec{y}$$

$$\vec{x} \times \vec{z} = -\vec{y}$$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\vec{a} \times \vec{a} = \vec{0}$$

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$\vec{a} \times (k\vec{b}) = k(\vec{a} \times \vec{b})$$

Cross Product in Cartesian Coordinate

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ x_a & y_a & z_a \\ x_b & y_b & z_b \end{vmatrix} = \begin{pmatrix} y_a z_b - y_b z_a \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{pmatrix}$$

determinant

$$\vec{a} \times \vec{b} = A^* b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

dual matrix of vector a

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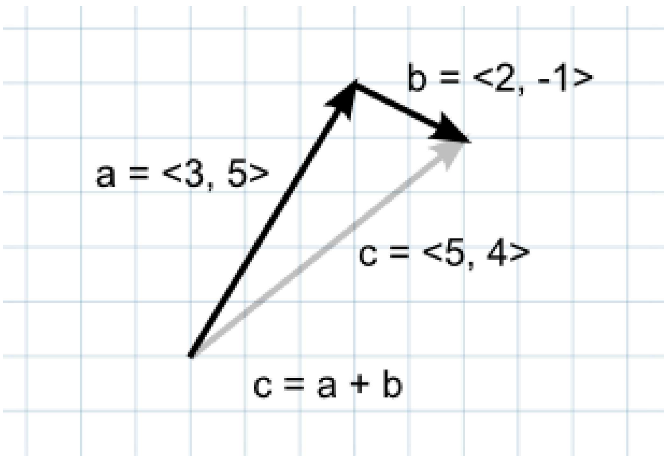
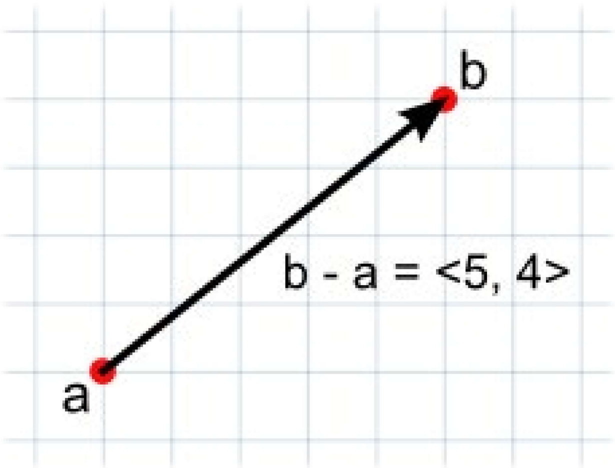
Vectors in Unity

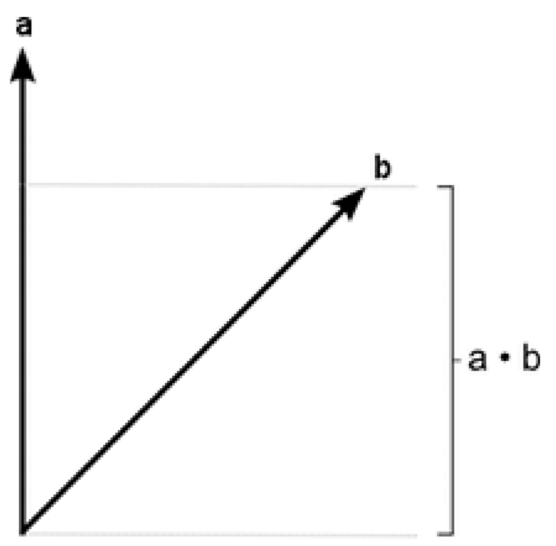



```
var pointInAir = pointOnGround + new Vector3(0, 5, 0);
```

Vectors in Unity



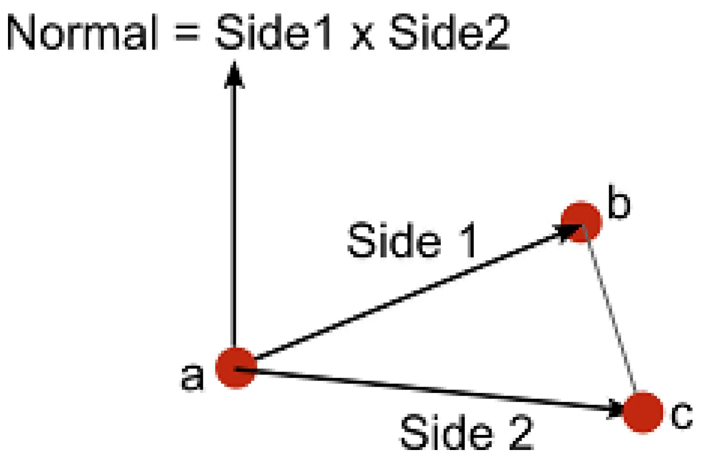

```
// The vector d has the same magnitude as c but points in the opposite direction.
var c = b - a;
var d = a - b;
```




```
Vector3 forward = transform.TransformDirection(Vector3.forward);
Vector3 toOther = other.position - transform.position;

if (Vector3.Dot(forward, toOther) < 0)
```

Vectors in Unity




```
var a: Vector3;
var b: Vector3;
var c: Vector3;

var side1: Vector3 = b - a;
var side2: Vector3 = c - a;
```

```
var perp: Vector3 = Vector3.Cross(side1, side2);
```

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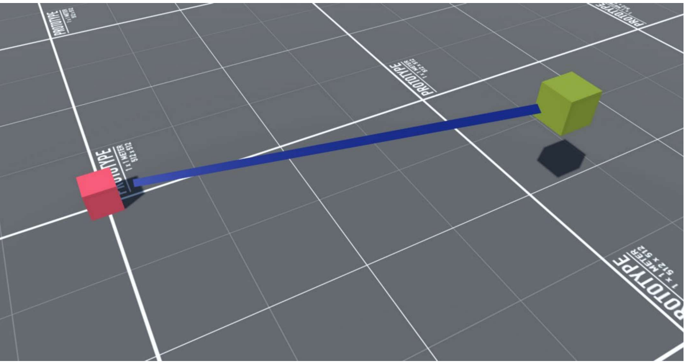
Vectors in Unity




```
// Gets a vector that points from the player's position to the target's.
var heading = target.position - player.position;
```

```
var distance = heading.magnitude;
var direction = heading / distance; // This is now the normalized direction.
```

Coordinates

Orthonormal Bases / Coordinate Frames

- Important for representing points, positions, locations
- Often, many sets of coordinate systems
  - Global, local, world, model, parts of model (head, hands, …)
- Critical issue is transforming between these systems/bases

Orthonormal Coordinate Frames

Any set of 3 vectors (in 3D) that:

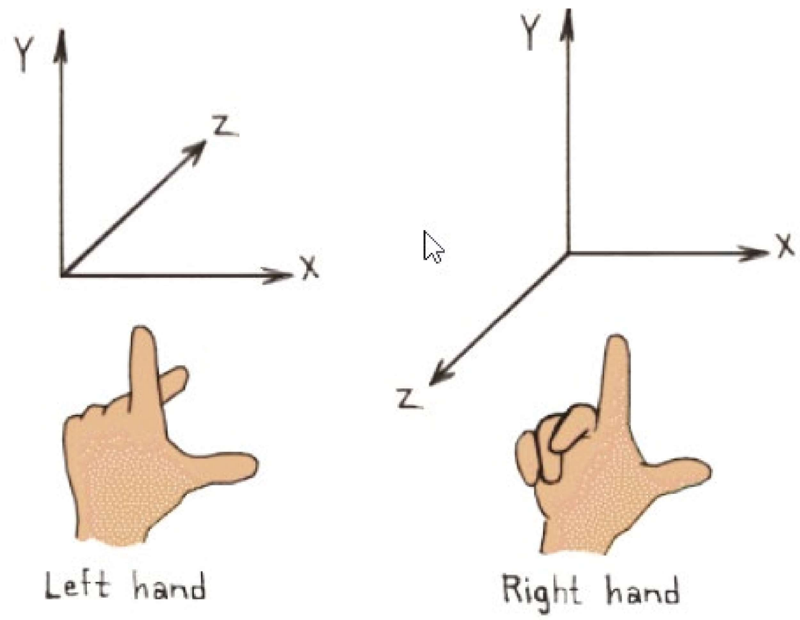
$$\|\vec{u}\| = \|\vec{v}\| = \|\vec{w}\| = 1$$

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{w} = \vec{u} \cdot \vec{w} = 0$$

$$\vec{w} = \vec{u} \times \vec{v}$$ (right-handed)

$$\vec{p} = (\vec{p} \cdot \vec{u})\vec{u} + (\vec{p} \cdot \vec{v})\vec{v} + (\vec{p} \cdot \vec{w})\vec{w}$$

(projection)



Matrix

Matrices (array of numbers)
are essential for data representation and transformation

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix}$$

Matrix Operations

multiplication

- # (number of) columns in A must = # rows in B (M x N) (N x P) = (M x P)
- Element (i, j) in the product is the dot product of row i from A and column j from B

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 & 4 \\ 2 & 7 & 8 & 3 \end{pmatrix} = \begin{pmatrix} 9 & ? & 33 & 13 \\ 19 & 44 & 61 & 26 \\ 8 & 28 & 32 & ? \end{pmatrix}$$

Matrix Operations

Multiplication properties

- Non-commutative
  (AB and BA are different in general)

- Associative and distributive
  - $A(B+C) = AB + AC$
  - $(A+B)C = AC + BC$

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Matrix-Vector Multiplication

- Treat vector as a column matrix (m×1)
- Key for transforming points (next lecture)
- Official spoiler: 2D reflection about y-axis

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

Transpose of a Matrix

- Switch rows and columns (ij -> ji)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

- Property

$$(AB)^T = B^T A^T$$

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Identity Matrix and Inverses

$$I_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$AA^{-1} = A^{-1}A = I$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Vector Multiplication in Matrix Form

- Dot product?

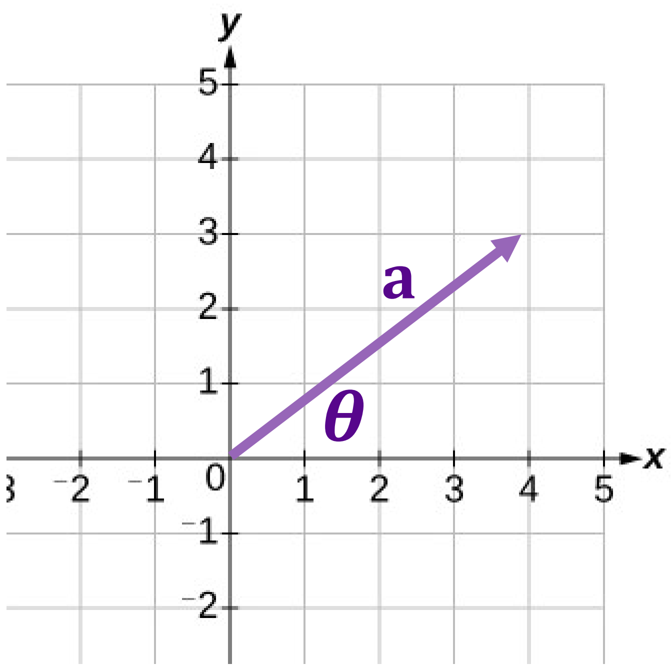
$$\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$$

$$= \begin{pmatrix} x_a & y_a & z_a \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = \begin{pmatrix} x_a x_b + y_a y_b + z_a z_b \end{pmatrix}$$

- Cross product?

$$\vec{a} \times \vec{b} = A^* b = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}$$

Complex Numbers

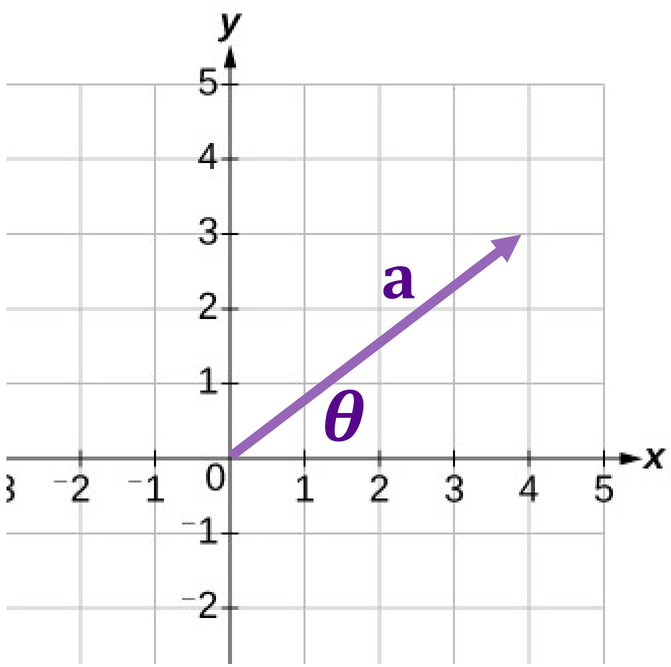


$$\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4\mathbf{x} + 3\mathbf{y} = \mathbf{4 + 3i}$$

$$\frac{\mathbf{a}}{|\mathbf{a}|} = \cos(\theta) + \mathrm{i}\sin(\theta)$$
$$= e^{i\theta}$$

Euler's Formula

Complex Numbers



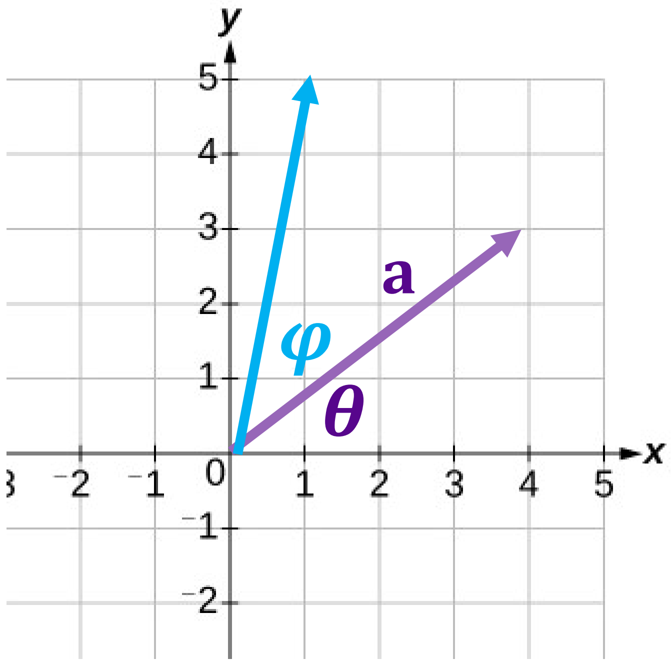
$$(x + yi) + (u + vi) = (x + u) + (y + v)i$$

$$(x + yi)\,(u + vi) = (xu - yv) + (xv + yu)i.$$

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i$$

Complex Numbers

Representing rotations



$$e^{i\theta}\, e^{i\varphi} = e^{i(\theta+\varphi)}$$

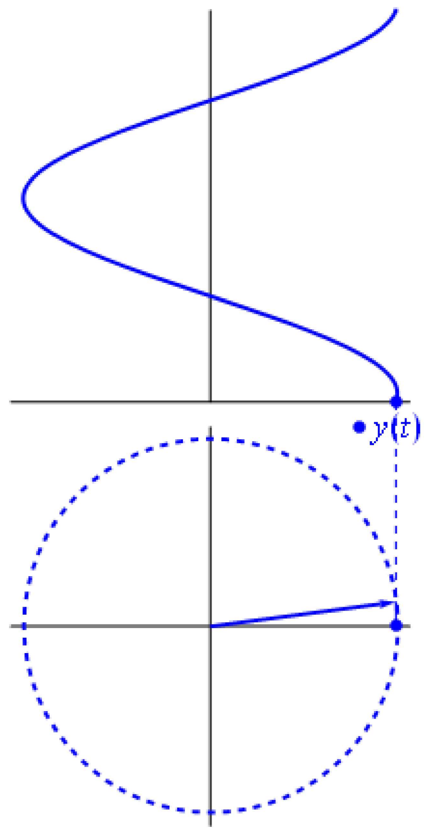
Complex Numbers

Representing light waves

$$A \cdot \cos(\omega t + \theta) = \frac{A \cdot e^{j(\omega t + \theta)}}{2} + \frac{A \cdot e^{-j(\omega t + \theta)}}{2}$$

$$\begin{aligned} A \cdot \cos(\omega t + \theta) &= \mathrm{Re}\Big\{ A \cdot e^{j(\omega t + \theta)} \Big\} \\ &= \mathrm{Re}\{ Ae^{j\theta} \cdot e^{j\omega t} \} \end{aligned}$$

$Ae^{j\theta}$



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Questions?