Gate-level Minimization

CS211 Chapter 3

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Gate-level minimization

- The complexity of digital logic gates to implement a Boolean function is directly related to the complexity of algebraic expression.
- Gate-level minimization is the design task of finding an optimal gate-level implementation of Boolean functions describing a digital circuit.
  - Difficult by hand for more than few inputs.
  - Typically by computer, need to understand the underlying principle.

The map method

- The map method, first proposed by Veitch and slightly improvised by Karnaugh, provides a simple, straightforward procedure for the simplification of Boolean functions.
  - Called *Karnaugh map*.
- The map is a diagram consisting of *square*s. For n variables on a Karnaugh map there are 2^n numbers of squares.
  - Each square or cell represents one of the minterms.
  - Since any Boolean function can be expressed as a sum of minterms, it is possible to recognize a Boolean function graphically in the map from the area enclosed by those squares whose minterms appear in the function.

Two-variable K-map

- A two-variable system can form four minterms

$A \backslash B$	0	1
0	m_0	m_1
1	m_2	m_3

$A \backslash B$	0	1
0	$A'B'$	$A'B$
1	AB'	AB

$A \backslash B$	0	1
0	0	1
1	1	1

- The two-variable Karnaugh map is a useful way to represent any of the 16 Boolean functions.
  - Example:

$$
\begin{aligned}
A + B &= A(B + B') + B(A + A') \\
&= AB + AB' + AB + A'B = AB + AB' + A'B
\end{aligned}
$$

  - So the squares corresponding to AB, AB', and $A'B$ are marked with 1.

Three-variable K-map

- Since there are eight minterms for three variables, the map consists of eight cells or squares.
  - Minterms are arranged, not according to the binary sequence, but according to the sequence similar to the gray code.
  - **Between two consecutive rows or columns, only one single variable changes its logic value from `0` to 1 or from 1 to `0`.**

$A \backslash BC$	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

Three-variable K-map

- To understand the usefulness of the map for simplifying the Boolean functions, we must observe the basic properties of the adjacent squares.
  - Any two adjacent squares in the Karnaugh map differ by only one variable, which is complemented in one square and uncomplemented in one of the adjacent squares.
  - The sum of two minterms can be simplified to a single AND term consisting of less number of literals.
  - $m_1 + m_5 = A'B'C + AB'C = (A' + A)B'C = B'C$

$A \backslash BC$	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

Three-variable K-map

- Example: Simplify the Boolean function $F = A'BC + A'BC' + AB'C' + AB'C$.

$A \backslash BC$	00	01	11	10
0	0	0	1	1
1	1	1	0	0

- The first row: $A'BC + A'BC' = A'B$.
- The second row: $AB'C' + AB'C = AB'$.
- $F = A'B + AB'$.

Three-variable K-map

- Example: Simplify the Boolean function $F = A'BC + AB'C' + ABC + ABC'$.

$A \backslash BC$	00	01	11	10
0	0	0	1	0
1	1	0	1	1

- The third column: $A'BC + ABC = BC$.
- The second row: $AB'C' + ABC' = AC'$.
- $F = BC + AC'$.

Three-variable K-map

- Example: Simplify the Boolean function $F = \sum(1, 2, 3, 5, 7)$.

$A \backslash BC$	00	01	11	10
0	0	1	1	1
1	0	1	1	0

- $F = C + A'B$.

Three-variable K-map

- Example: Simplify the Boolean function $F = \sum(0, 2, 4, 5, 6)$.

$A \backslash BC$	00	01	11	10
0	1	0	0	1
1	1	1	0	1

- $F = C' + AB'$.

Four-variable K-map

- Similar to the method used for two-variable and three-variable Karnaugh maps, four-variable Karnaugh maps may be constructed with 16 squares consisting of 16 minterms.

$AB \backslash CD$	00	01	11	10
00	m_0	m_1	m_3	m_2
01	m_4	m_5	m_7	m_6
11	m_{12}	m_{13}	m_{15}	m_{14}
10	m_8	m_9	m_{11}	m_{10}

Four-variable K-map

- Two, four, or eight adjacent squares can be combined to reduce the number of literals in a function.
- The squares of the top and bottom rows as well as leftmost and rightmost columns may be combined.
  - When two adjacent squares are combined, it is called a *pair* and represents a term with three literals.
  - Four adjacent squares, when combined, are called a *quad* and its number of literals is two.
  - If eight adjacent squares are combined, it is called an *octet* and represents a term with one literal.
  - If, in the case all sixteen squares can be combined, the function will be reduced to 1.

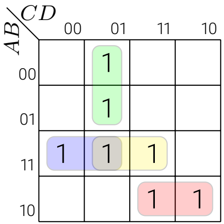
Four-variable K-map

- Example: Simplify the Boolean function
  $F = m_1 + m_5 + m_{10} + m_{11} + m_{12} + m_{13} + m_{15}$.
  - $A'B'C'D + A'BC'D = A'C'D$,
  - $ABC'D' + ABC'D = ABC'$,
- $F = A'C'D + ABC' + ACD + AB'C$.
- This reduced expression is not a unique one.
  - If pairs are formed in different ways, the simplified expression will be different.

$AB \backslash CD$	00	01	11	10
00		1		
01		1		
11	1	1	1	
10			1	1

Four-variable K-map

- Example: Simplify the Boolean function
  $F = m_1 + m_5 + m_{10} + m_{11} + m_{12} + m_{13} + m_{15}$.
- $F = A'C'D + ABC' + ABD + AB'C$.



Four-variable K-map

- Example: Simplify the Boolean function $F = \sum(7, 9, 10, 11, 12, 13, 14, 15)$.
- $F = AB + AC + AD + BCD$.

$AB \backslash CD$	00	01	11	10
00				
01			1	
11	1	1	1	1
10		1	1	1

Four-variable K-map

- Example: Plot the logical expression $F(A,B,C,D) = ABCD + AB'C'D' + AB'C + AB$ on a four-variable Karnaugh map.

$AB \backslash CD$	00	01	11	10
00				
01				
11	1	1	1	1
10	1		1	1

$$
\begin{aligned}
& F(A,B,C,D) \\
&= ABCD + AB'C'D' + AB'C + AB \\
&= ABCD + AB'C'D' + AB'C(D+D') \\
&\quad + AB(C+C')(D+D') \\
&= \dots \\
&= \sum(8,10,11,12,13,14,15) \\
&= AB + AC + AD'
\end{aligned}
$$

Four-variable K-map

- Simplify the expression $F(A,B,C,D) = \sum(0,1,2,4,5,6,8,9,12,13,14)$.

$AB \backslash CD$	00	01	11	10
00	1	1		1
01	1	1		1
11	1	1		1
10	1	1		

Four-variable K-map

- Simplify the expression $F(A,B,C,D) = A'B'C' + B'CD' + A'BCD' + AB'C'$.

$$
\begin{aligned}
&F(A,B,C,D) \\
=& A'B'C'(D+D') + B'CD'(A+A') \\
&+ A'BCD' + AB'C'(D+D') \\
=& A'B'C'D + A'B'C'D' + AB'CD' \\
&+ A'B'CD' + A'BCD' + AB'C'D \\
&+ AB'C'D' \\
=& \sum(0,1,2,6,8,9,10) \\
=& B'C' + B'D' + A'CD'
\end{aligned}
$$

$AB \backslash CD$	00	01	11	10
00	1	1		1
01				1
11				
10	1	1		1

Four-variable K-map

- Simplify the expression $F(A, B, C, D) = \sum(3, 4, 5, 7, 9, 13, 14, 15)$.
  - It may be noted that one quad can also be formed, but it is redundant as the squares contained by the quad are already covered by the pairs which are essential.
- $F = A'BC' + A'CD + AY'D + ABC$.

AB \ CD	00	01	11	10
00			1	
01	1	1	1	
11		1	1	1
10		1		

Four-variable K-map

- Simplify the expression $F(A, B, C, D) = \prod(0, 1, 4, 5, 6, 8, 9, 12, 13, 14)$.
  - The above expression is given in respect to the maxterms.
  - 0's are to placed instead of 1's at the corresponding maxterm squares.
- $F = C(B' + D)$.

$AB \backslash CD$	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	0	0	1	0
10	0	0	1	1

Four-variable K-map

- Simplify the expression $F(A, B, C, D) = \prod(0, 1, 4, 5, 6, 8, 9, 12, 13, 14)$.
  - The other way to achieve the minimized expression is to consider the 1's of the Karnaugh map.
- $F = CD + B'C = C(B' + D)$.

$AB \backslash CD$	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	0	0	1	0
10	0	0	1	1

Five-variable K-map

- Karnaugh maps with more than four variables are not simple to use.
    - The number of cells or squares becomes excessively large and combining the adjacent squares becomes complex.
    - A five-variable Karnaugh map contains 2^5 or 32 cells.

Prime Implicants

- A *prime implicant* is a product term obtained by combining the maximum possible number of adjacent squares in the map.
- The prime implicants of a function can be obtained from the map by combining all possible maximum numbers of squares.
  - If a minterm in a square is covered by only one prime implicant, that prime implicant is said to be *essential*.
- Gate-level minimization:
  - Determine all essential prime implicants.
  - Find other prime implicants that cover remaining minterms.
  - Logical sum all prime implicants.

Don't care conditions

- In practice, Boolean function is not specified for certain combinations of input variables.
  - Input combinations never occur during the process of a normal operation.
  - Those input conditions are guaranteed never to occur.
- Such input combinations are called *don't-care conditions*.
- These input combinations can be plotted on the Karnaugh map for further simplification.
  - The don't care conditions are represented by d or X in a K-map.
  - They can be either `1` or `0` upon needed.

Don’t care conditions

- Simplify the expression $F(A,B,C,D) = \sum(1,3,7,11,15), d = \sum(0,2,5)$.
- $F = A'B' + CD$.

$AB \backslash CD$	00	01	11	10
00	X	1	1	X
01		X	1	
11			1	
10			1	

Don't care conditions

- Simplify the expression $F(A,B,C,D) = \sum(1,3,7,11,15), d = \sum(0,2,5)$.
- $F = A'D + CD$.

$AB \backslash CD$	00	01	11	10
00	X	1	1	X
01		X	1	
11			1	
10			1	

More examples

- Using the Karnaugh map method obtain the minimal sum of the products expression for the function $F(A, B, C, D) = \sum(0, 2, 3, 6, 7) + d(8, 10, 11, 15)$.