Floating point

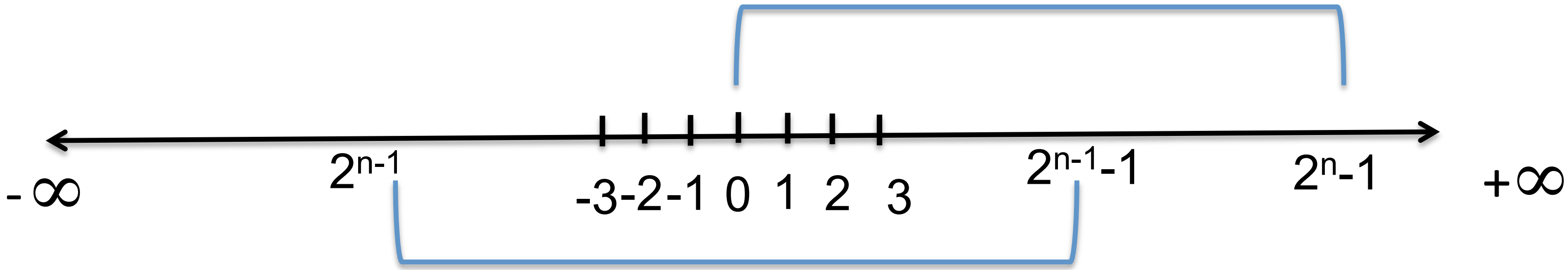
Jinyang Li

Some are based on Tiger Wang's slides

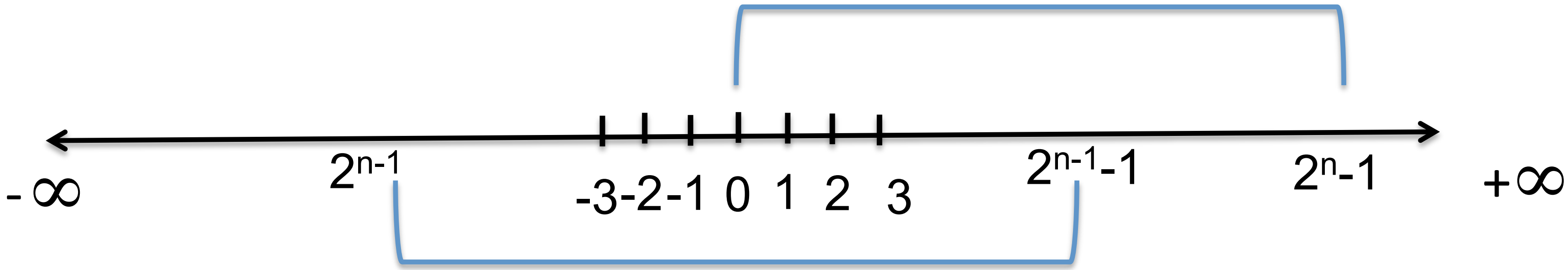
Representing Real Numbers using bits

Representing Numbers in bits

What we have studied



Representing Numbers in bits



Today: How to represent fractional numbers?

Representing real numbers: decimal

Real Numbers	Decimal Representation (Expansion)
11 / 2	$(5.5)_{10}$
1 / 3	$(0.3333333...)_{10}$
$\sqrt{2}$	$(1.4128...)_{10}$

Representing real numbers: decimal

Real Numbers	Decimal Representation (Expansion)
11 / 2	$(5.5)_{10}$
1 / 3	$(0.3333333...)_{10}$
√2	$(1.4128...)_{10}$

$(5.5)_{10} = 5 * 10^{0} + 5 * 10^{-1}$

$(0.333333...)_{10} = 3 * 10^{-1} + 3 * 10^{-2} + 3 * 10^{-3} + ...$

$(1.4128...)_{10} = 1 * 10^{0} + 4 * 10^{-1} + 1 * 10^{-2} + 2 * 10^{-3} + ...$

Representing real numbers: decimal

Real Numbers	Decimal Representation (Expansion)
11 / 2	$(5.5)_{10}$
1 / 3	$(0.3333333...)_{10}$
√2	$(1.4128...)_{10}$

$(5.5)_{10} = 5 * 10^0 + 5 * 10^{-1}$

$(0.333333...)_{10} = 3 * 10^{-1} + 3 * 10^{-2} + 3 * 10^{-3} + ...$

$(1.4128...)_{10} = 1 * 10^0 + 4 * 10^{-1} + 1 * 10^{-2} + 2 * 10^{-3} + ...$

$$r_{10} = (d_m d_{m-1} \ldots d_1 d_0 \,.\, d_{-1} d_{-2} \ldots d_{-n})_{10}$$

$$= \sum_{i=-n}^{m} 10^i \times d_i$$

Binary Representation

$(5.5)_{10} = 4 + 1 + 1 / 2$

$= 1 * 2^2 + 0 * 2^1 + 1 * 2^0 + 1 * 2^{-1}$

Binary Representation

$$(5.5)_{10} = 4 + 1 + 1 / 2$$

$$= 1 * 2^2 + 0 * 2^1 + 1 * 2^0 + 1 * 2^{-1}$$

$$= (101.1)_2$$

Binary Representation

$$(5.5)_{10} = 4 + 1 + 1 / 2$$

$$= 1 * 2^2 + 0 * 2^1 + 1 * 2^0 + 1 * 2^{-1}$$

$$= (101.1)_2$$

$$(0.333333...)_{10} = 1 / 4 + 1 / 16 + 1 / 64 + ...$$

$$= (0.01010101...)_2$$

Binary Representation

$$r_{10} = (d_m d_{m-1} d_1 d_0 \,.\, d_{-1} d_{-2} \ldots d_{-n})_{10}$$

$$= (b_p b_{p-1} b_1 b_0 \,.\, b_{-1} b_{-2} \ldots b_{-q})_2$$

$$\overset{2^p}{b_p}\,\overset{2^{p-1}}{b_{p-1}} \cdots \overset{2}{b_1}\,\overset{1}{b_0} \,.\, \underset{1/2}{b_{-1}}\,\underset{1/4}{b_{-2}} \cdots \underset{1/2^q}{b_{-q}} = \sum_{i=-q}^{p} 2^i \times b_i$$

Exercise

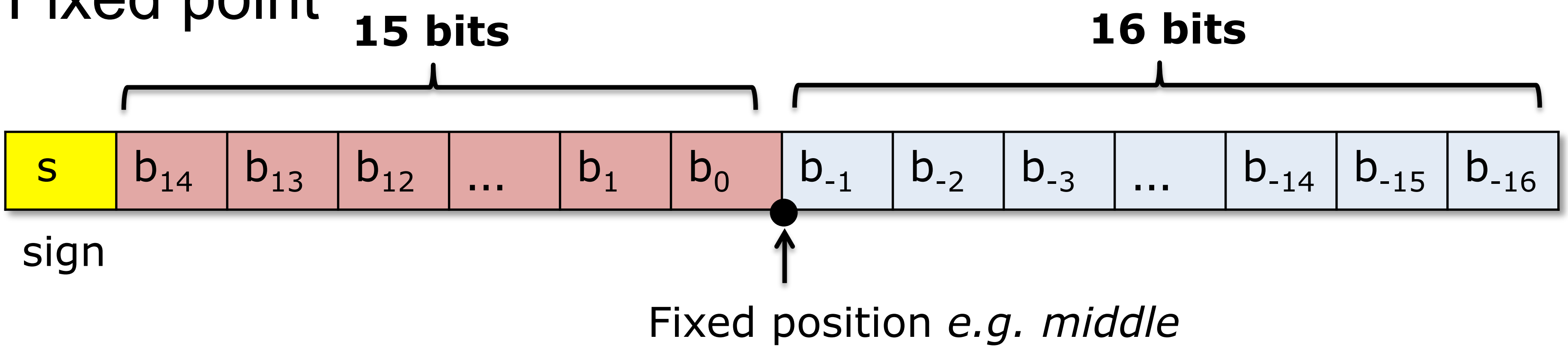
Binary Expansion	Formula	Decimal
10.011_2		
	$2^{-3} + 2^{-4} + 2^{-6}$	
	$2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}$	

Exercise

Binary Expansion	Formula	Decimal
10.011_2	$2^1 + 2^{-2} + 2^{-3}$	2.375_{10}
0.001101_2	$2^{-3} + 2^{-4} + 2^{-6}$	0.203125_{10}
0.1111_2	$2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}$	0.9375_{10}

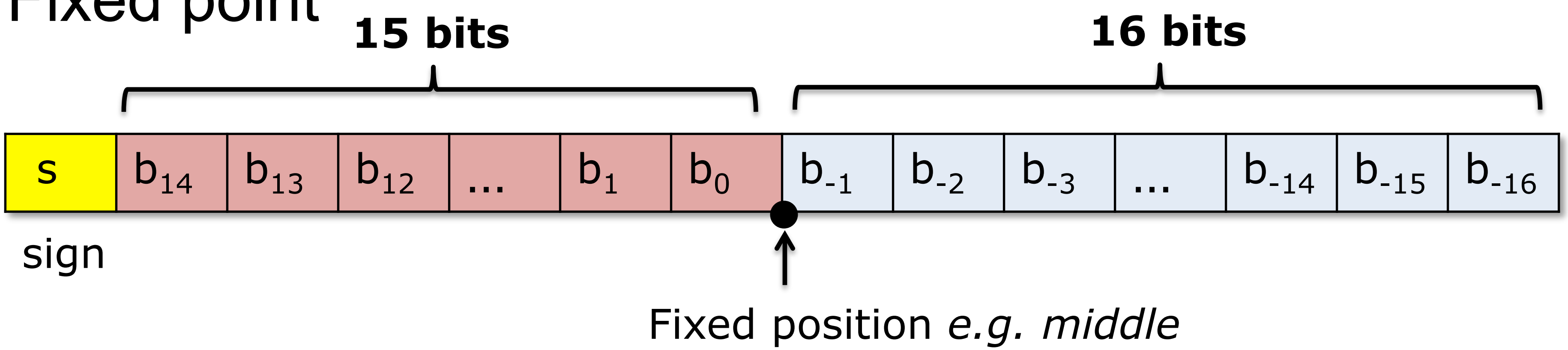
How to represent real numbers in fixed # of bits?

Fixed point



Naive idea: Fixed point

Fixed point



$(\ 10.011\ )_2$

0	0 0 0 0 0 0 0 0 0 0 0 0 0 1 0	0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0

Problems of Fixed Point

Limited range and precision: e.g., 32 bits

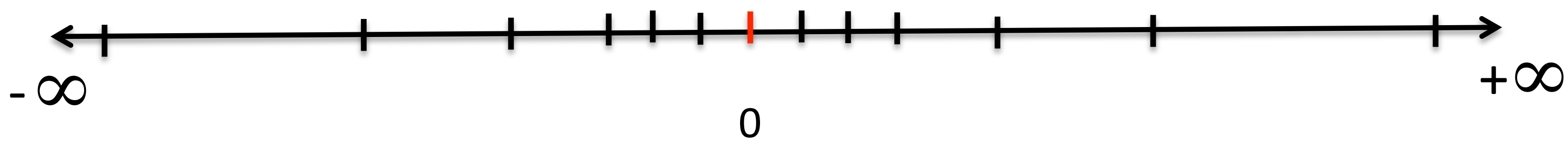
- Largest number: 2^{15} $(011...111)_2$
- Highest precision: 2^{-16}

→Rarely used (No built-in hardware support)

The idea

- Limitation of fixed point notation:
  – Represents evenly spaced fractional numbers
    - ➔ hard tradeoff between high precision and high magnitude

- How about un-even spacing between numbers?



Floating Point: decimal

Based on the normalized scientific notation

$r_{10} = \pm M * 10^E$, where $1 <= M < 10$

M: significant (mantissa), E: exponent

Floating Point: decimal

Example:

$365.25 = 3.6525 * 10^{2}$

$0.0123 = 1.23 * 10^{-2}$

Decimal point **floats** to the position immediately after the first nonzero digit.

Floating Point: binary

Binary (normalized) scientific notation:

$r_{10} = \pm M * 2^{E}$, where $1 <= M < 2$

$M = (1.b_1b_2b_3\ldots b_n)_2$

M: significant, E: exponent

$(5.5)_{10} = (101.1)_2 = (1.011)_2 * 2^2$

Exercises

The scientific notation of $(10.25)_{10}$ is ?

Exercises

The scientific notation of $(10.25)_{10}$ is ?

$$(10.25)_{10} = (1010.01)_2 = (1.01001)_2 * 2^3$$

How to represent a binary scientific notation in fixed # of bits?

$r_{10} = \underline{+}M * 2^E$, where $1 <= M < 2$

$M = (1.b_1b_2b_3...b_n)_2$

M: significant, E: exponent

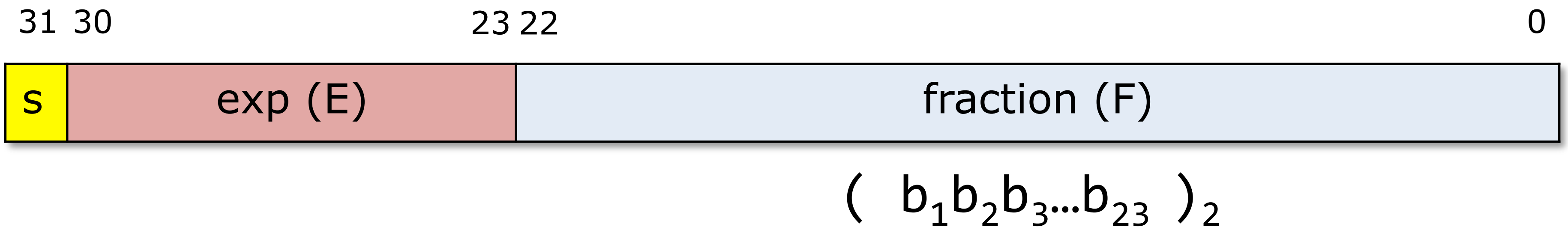
$(1.b_1b_2b_3...b_n)_2$

How to represent a binary scientific notation in fixed # of bits?

$r_{10} = \pm M * 2^{E}$, where $1 <= M < 2$

$M = (1.b_1b_2b_3 \ldots b_{23})_2$

M: significant, E: exponent



How to represent a binary scientific notation in fixed # of bits?

$r_{10} = \underline{+}M * 2^E$, where $1 <= M < 2$

$M = (\ 1.b_1b_2b_3...b_{23}\ )_2$

M: significant, E: exponent

31	30 … 23	22 … 0
0	0000 0010	0110 0000 0000 0000 0000 000
		$(\ b_1b_2b_3...b_{23}\ )_2$

$(5.5)_{10} = (101.1)_2 = (1.011)_2 * 2^2$

Exercise

What's the normalized representation of $(71)_{10}$ and $(10.25)_{10}$

Exercise

What's the normalized representation of $(71)_{10}$ and $(10.25)_{10}$

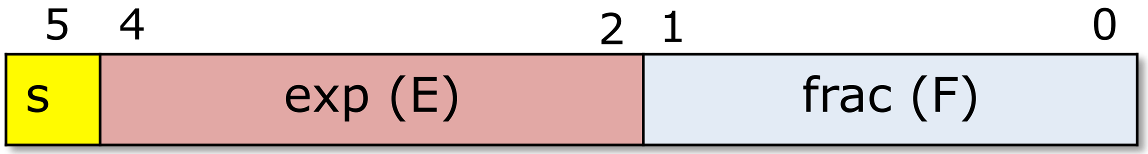
$(10.25)_{10} = (1010.01)_2 = (1.01001)_2 * 2^3$

31	30 … 23	22 … 0
0	0000 0011	0100 1000 0000 0000 0000 000

$(71)_{10} = (1000111)_2 = (1.000111)_2 * 2^6$

31	30 … 23	22 … 0
0	0000 0110	0001 1100 0000 0000 0000 000

Toy Number System



6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits

Largest positive number ?

Toy Number System

5	4 … 2	1 … 0
s	exp (E)	frac (F)

6-bit floating point representation

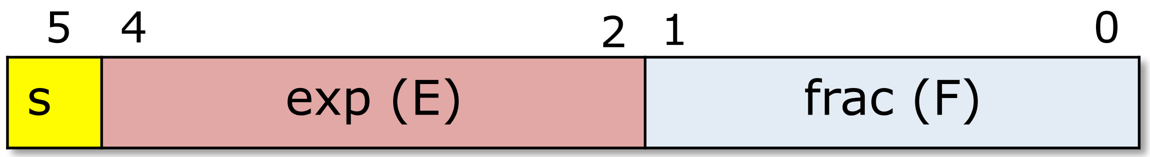
- exponent: 3 bits
- fraction: 2 bits

Largest positive number ?

0	1 1 1	1 1

$(1.11)_2 * 2^7 = 224$

Toy Number System



6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits

Largest positive number: 224

Smallest positive number ?

Toy Number System

5	4 … 2	1 … 0
s	exp (E)	frac (F)

6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits

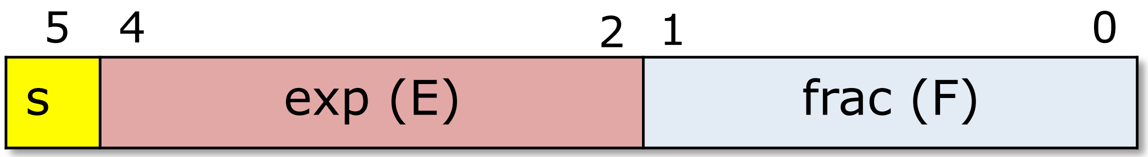
Largest positive number: 224

Smallest positive number: 1

0	0 0 0	0 0

$(1.00)_2 * 2^0 = 1$

Toy Number System



6-bit floating point representation
- exponent: 3 bits
- fraction: 2 bits

Positive number: 1 to 224

Negative number: -224 to -1

No more bit patterns left to represent numbers (-1, 1)

Questions

How to represent

1. numbers close or equal to 0?
2. special cases:
   - the result of dividing by 0, e.g. 1/0 ?
   - $$\infty * 0$$

Lots of different implementations around 1950s!

IEEE Floating Point Standard

IEEE p754
A standard for binary
floating point representation

Prof. William Kahan
University of California at Berkeley
Turing Award (1989)

National Semiconductor
The Sight & Sound of Information

intel®

hp

digital
EQUIPMENT
CORPORATION
MAYNARD, MASSACHUSETTS

The Only Book Focuses On IEEE Floating Point Standard

Numerical Computing with IEEE Floating Point Arithmetic

Including One Theorem, One Rule of Thumb, and One Hundred and One Exercises

Michael L. Overton
Courant Institute of Mathematical Sciences
New York University
New York, New York

hardware. This degree of altruism was so astonishing that MATLAB's creator Cleve Moler used to advise foreign visitors not to miss the country's two most awesome spectacles: the Grand Canyon, and meetings of IEEE p754."

https://cs.nyu.edu/overton/NumericalComputing/protected/NumericalComputingSIAM.pdf
With you nyu netid/password. You can also search the pdf with google.

Goals of IEEE Standard

- Consistent representation of floating point numbers
- Correctly rounded floating point operations, using several rounding modes.
- Consistent treatment of exceptional situations such as division by zero

Restrictions on Normalized Representation

$$r_{10} = \underline{+}M * 2^E \quad M = (\ 1.b_0b_1b_2b_3 \ldots b_n\ )_2$$

s	exp (E)	fraction (F)
31	30 … 23	22 … 0

$(\ b_0b_1b_2b_3 \ldots b_n\ )_2$

E can not be $(1111\ 1111)_2$ or $(0000\ 0000)_0$

E_{max} = ? 254, $(1111\ 1110)_2$

E_{min} = ? 1, $(0000\ 0001)_2$

Exponential Bias

$$r_{10} = \pm M * 2^E, \quad M = (\ 1.b_0 b_1 b_2 b_3 \ldots b_n\ )_2$$

To represent (-1,1), we must allow negative exponent.

- How to represent negative E?
  - ~~2's complement~~
  - use bias

31	30 … 23	22 … 0
s	exp (E) + 127	fraction (F)

Bias: 127

$(\ b_0 b_1 b_2 b_3 \ldots b_n\ )_2$

IEEE normalized representation

$$r_{10} = \pm M * 2^{E}, \quad M = (\ 1.b_0 b_1 b_2 b_3 \ldots b_n\ )_2$$

31	30 … 23	22 … 0
s	exp (E) + 127	fraction (F)

$$(\ b_0 b_1 b_2 b_3 \ldots b_n\ )_2$$

Bias: 127

$E_{max} = 254 - 127 = 127$

$E_{min} = 1 - 127 = -126$

Smallest positive number 2^{-126}

Negative number with smallest absolute value: -2^{-126}

Questions

Q1. Why using **bias**?

Q2. Why is **bias** 127?

Questions

Q1. Why using **bias** instead of 2’s complement?

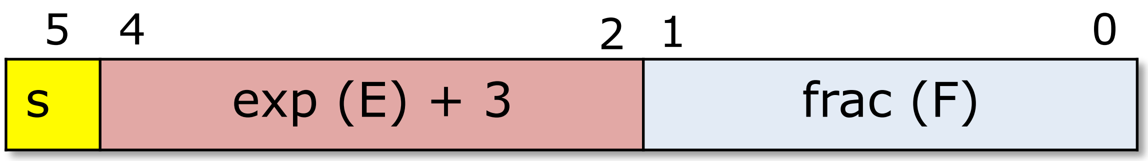
Answer: easier circuitry for comparison.

Questions

Q2. Why is bias 127?

A2. Balance positive exponents (magnitude) and negative exponents (precision)

Toy 6-bit Floating Point



6-bit floating point representation

– exponent: 3 bits
– fraction: 2 bits
– **bias: 3**

Smallest positive number ?

Toy 6-bit Floating Point

5	4 … 2	1 … 0
s	exp (E) + 3	frac (F)

6-bit floating point representation
- exponent: 3 bits
- fraction: 2 bits
- **bias: 3**

Smallest positive number: 0.25

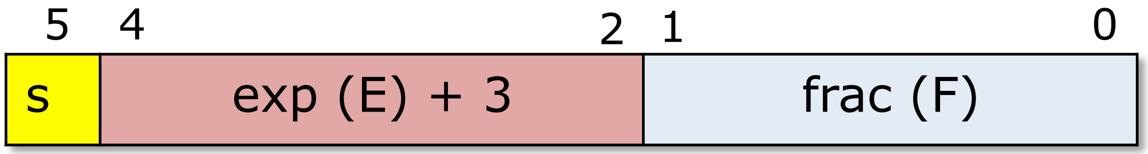
Smallest number >0.25?

0	0 0 1	0 0

$(1.00)_2 * 2^{-2} = 0.25$

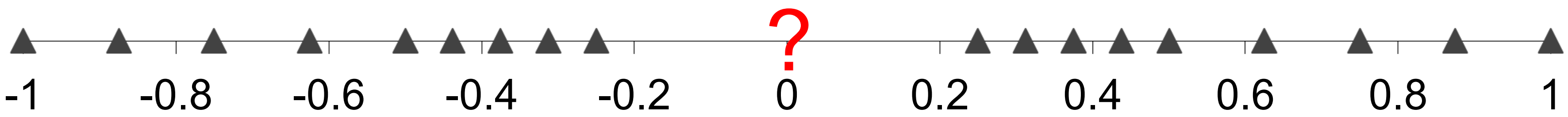
$(1.01)_2 * 2^{-2} = 0.25+0.0625$

Toy 6-bit Floating Point



6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits
- **bias: 3**



represent values which are close and equal to 0

IEEE denormalized representation

$$r_{10} = \pm M * 2^{E}$$

Normalized Encoding:

31	30 … 23	22 … 0
s	exp (E) + 127	fraction (F)

$1 <= M < 2$, $M = (1.F)_2$

Denormalized Encoding:

31	30 … 23	22 … 0
s	0000 0000	fraction (F)

$E = 1 – Bias = -126$

$0 <= M < 1$, $M = (0.F)_2$

Zeros

+0.0

0	0000 0000	0000 0000 0000 0000 0000 000

-0.0

1	0000 0000	0000 0000 0000 0000 0000 000

Examples

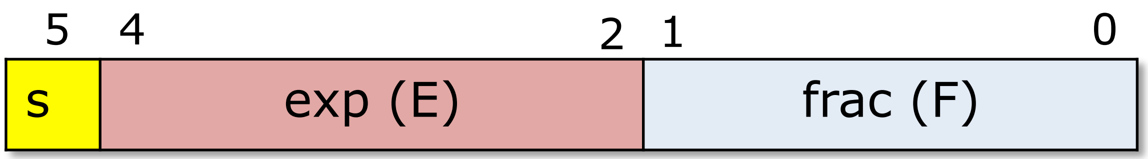
$(0.1)_2 * 2^{-126}$

0	0000 0000	1000 0000 0000 0000 0000 000

$-(0.010101)_2 * 2^{-126}$

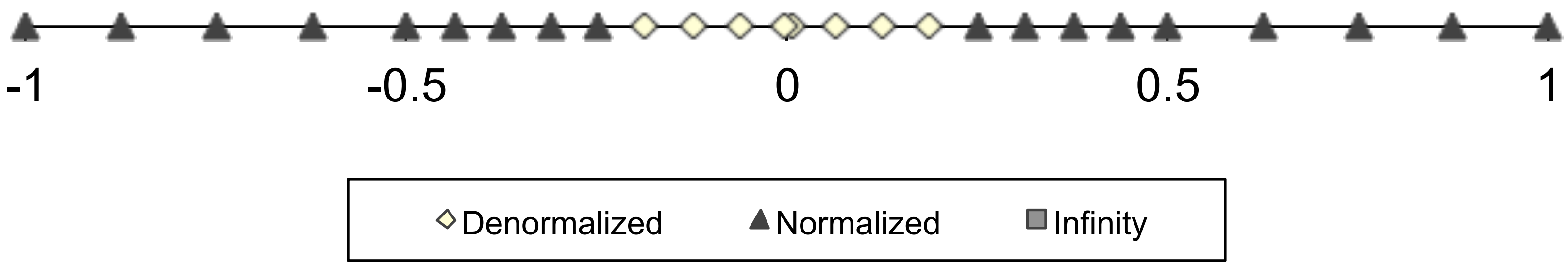
1	0000 0000	0101 0100 0000 0000 0000 000

Toy 6-bit Floating Point



6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits
- bias: 3
- **Denormalized encoding**



Special Values

Special Value's Encoding:

values	**sign**	**frac**
+∞	0	all zeros
- ∞	1	all zeros
NaN	any	non-zero

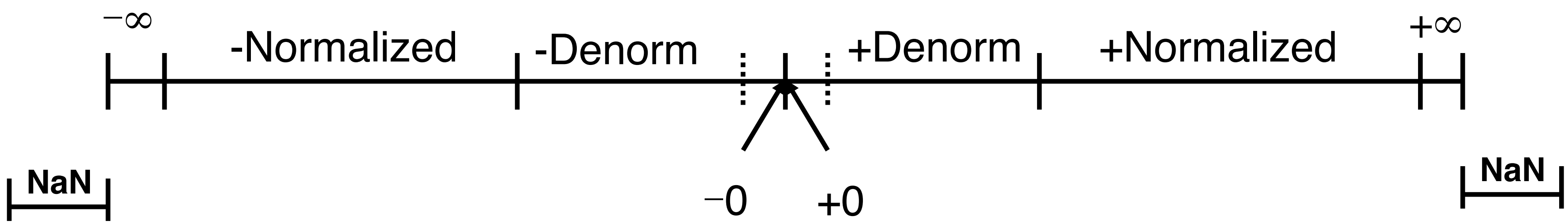
Exercises

representation	**E**	**M**	**V**
0100 1001 0101 0000 0000 0000 0000 0000			
			$2.5 * 2^{-127}$
			$-1.25 * 2^{-111}$
1111 1111 1111 1111 0000 0000 0000 0000			
1111 1111 1000 0000 0000 0000 0000 0000			
			$1.5 * 2^{-127}$

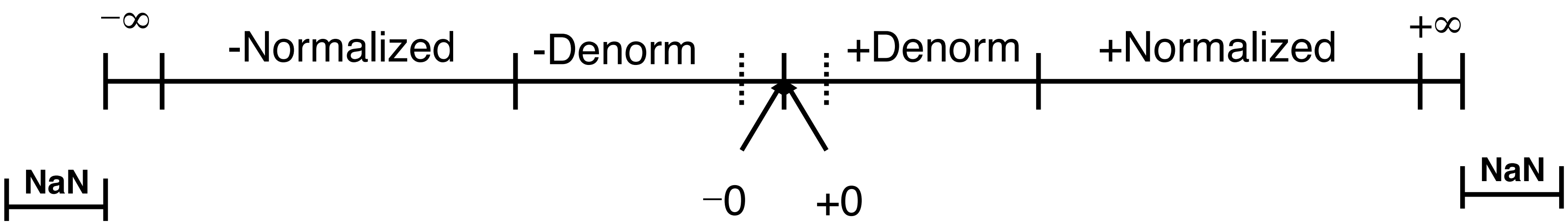
Exercises

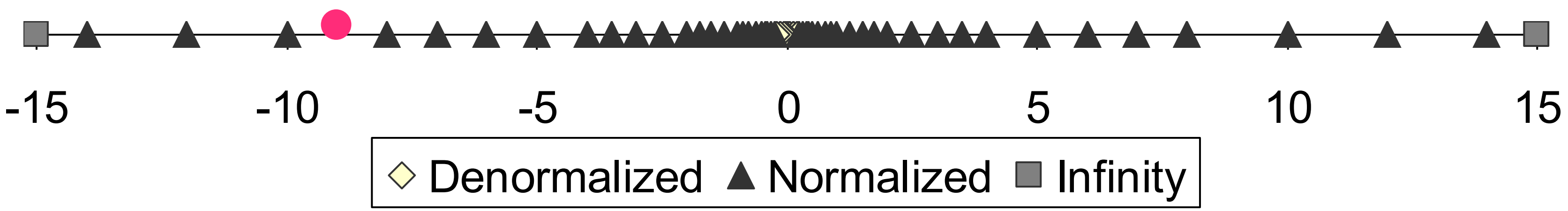
representation	E	M	V
0100 1001 0101 0000 0000 0000 0000 0000	146 – 127 = 19	$(1.101)_2$ = 1.625	$1.625 * 2^{19}$
0000 0000 1010 0000 0000 0000 0000 0000	1 – 127 = -126	$(1.01)_2$ = 1.25	$2.5 * 2^{-127}$ $= (1.01)_2 * 2^{-126}$
1000 1000 0010 0000 0000 0000 0000 0000	16 – 127 = -111	$(1.01)_2$ = 1.125	$-1.25 * 2^{-111}$
1111 1111 1111 1111 0000 0000 0000 0000	-	-	Nan
1111 1111 1000 0000 0000 0000 0000 0000	-	-	$-\infty$
0000 0000 0110 0000 0000 0000 0000 0000	-126	$(0.11)_2$	$(0.11)_2 * 2^{-126}$ $= 1.5 * 2^{-127}$

Distribution of Representable Values



Distribution of Representable Values





What if the result of computation is at ● ?

Rounding

Goal

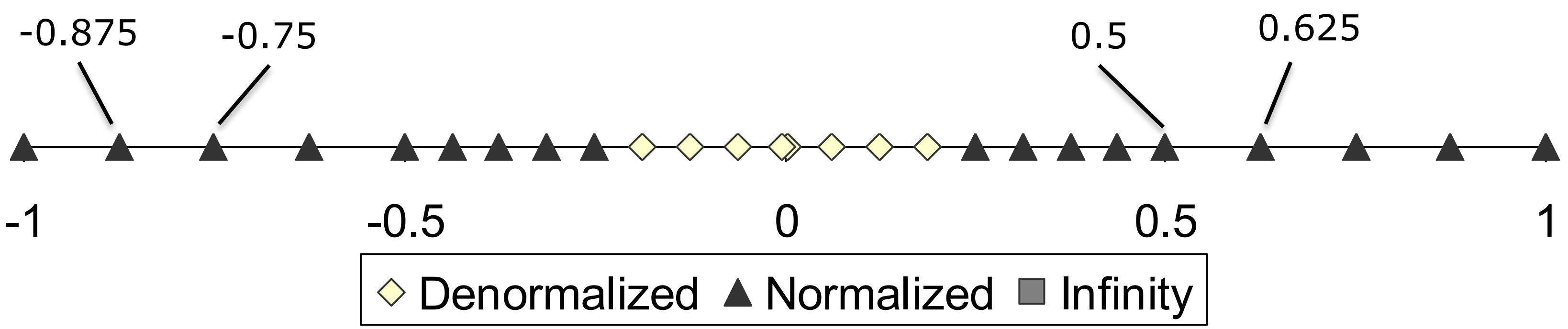
- Use the “closest” representable value *x′* to represent x.

Round modes

- Round-down
- Round-up
- Round-toward-zero
- Round-to-nearest (Round to even in text book)

Round down in toy 6-bit FP

$$Round(x) = x_{-} \ (x_{-} <= x)$$

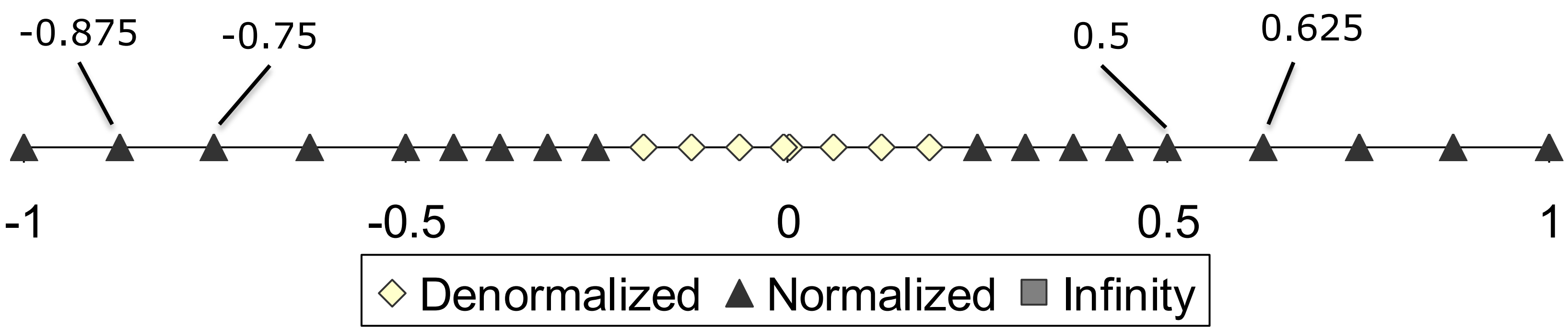


Round(-0.86) = ?

Round(0.55) = ?

Round down in toy 6-bit FP

Round(x) = x_- ($x_- <= x$)



Round(-0.86) = -0.875

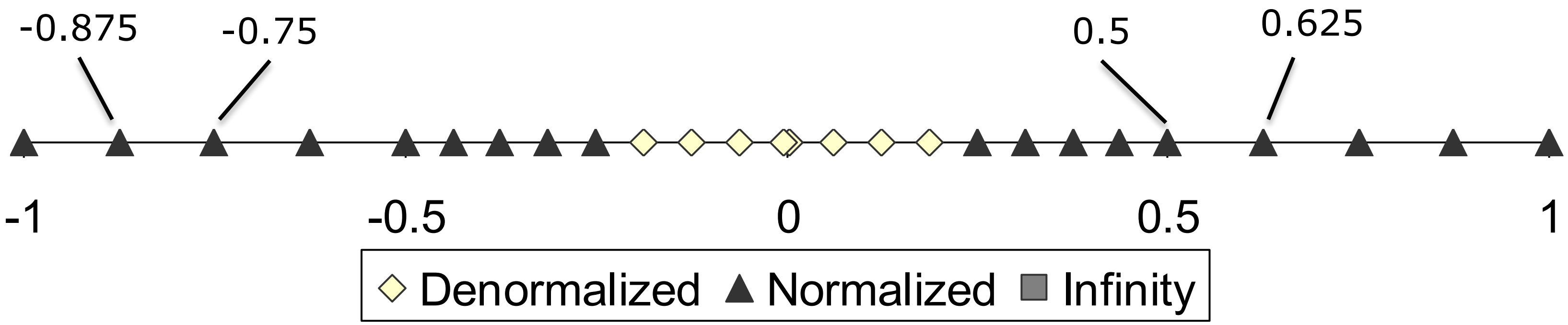
Round(0.55) = 0.5

Round up in toy 6-bit FP

$\mathrm{Round}(x) = x_+ \quad (x_+ >= x)$

Round up in toy 6-bit FP

$$\text{Round}(x) = x_+ \quad (x_+ >= x)$$

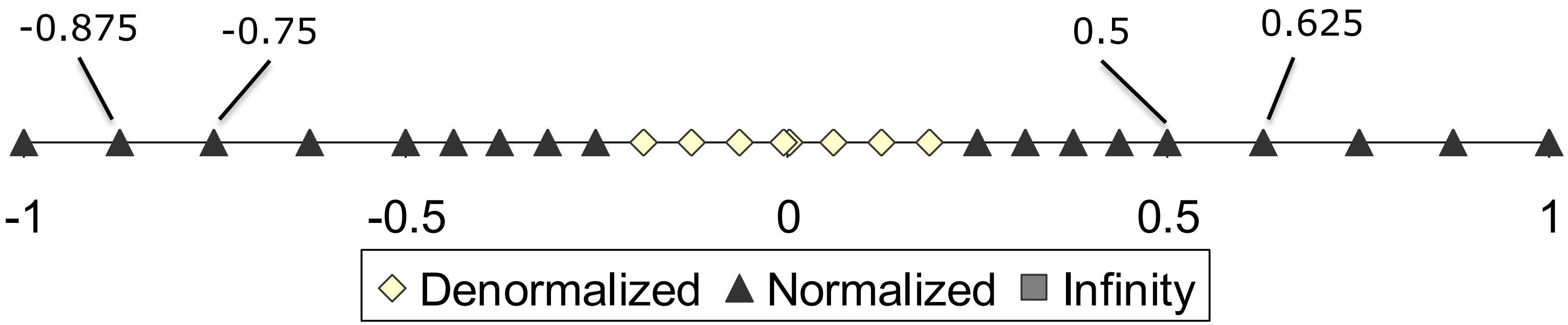


Round(-0.86) = ?

Round(0.55) = ?

Round up in toy 6-bit FP

$$\text{Round}(x) = x_+ \quad (x_+ >= x)$$



Round(-0.86) = -0.75

Round(0.55) = 0.625

Round towards zero in toy 6-bit FP

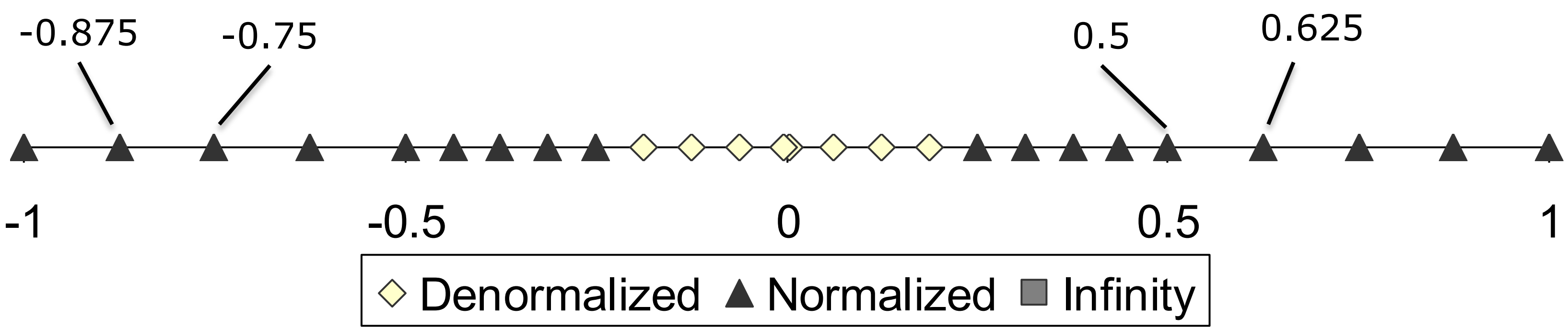
Round(x) = x_+ if x < 0

Round(x) = x_- if x > 0

Round towards zero in toy 6-bit FP

$$\text{Round}(x) = x_+ \text{ if } x < 0$$

$$\text{Round}(x) = x_- \text{ if } x > 0$$



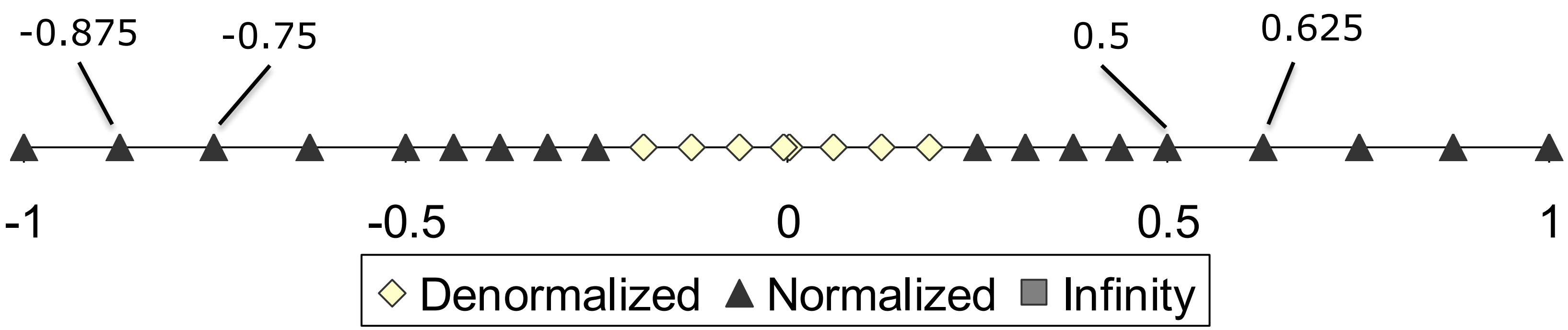
Round(-0.86) = ?

Round(0.55) = ?

Round towards zero in toy 6-bit FP

$\text{Round}(x) = x_+$ if $x < 0$

$\text{Round}(x) = x_-$ if $x > 0$



Round(-0.86) = -0.75

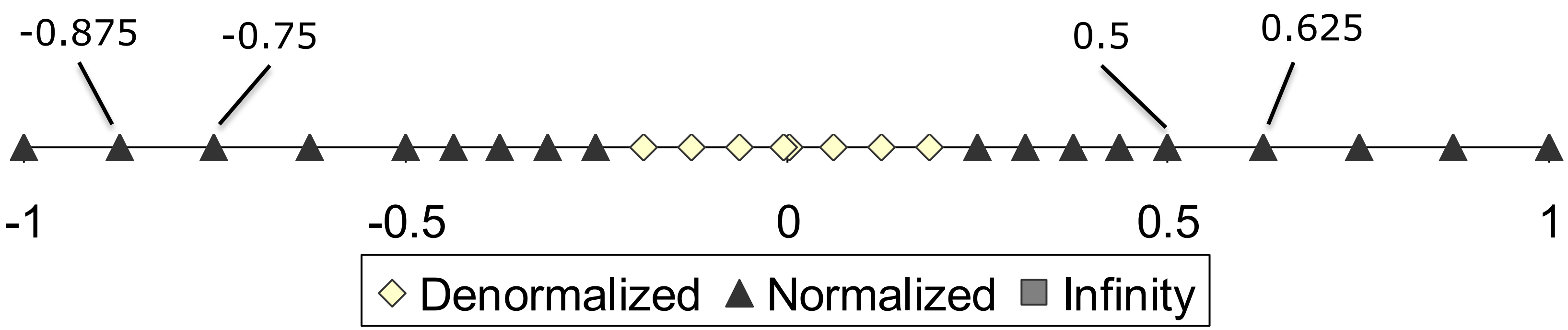
Round(0.55) = 0.5

Round to nearest in toy 6-bit FP

Round(x) either x_+ or x_- , whichever is nearer to x.

Round to nearest in toy 6-bit FP

Round(x) either x_+ or x_- , whichever is nearer to x.

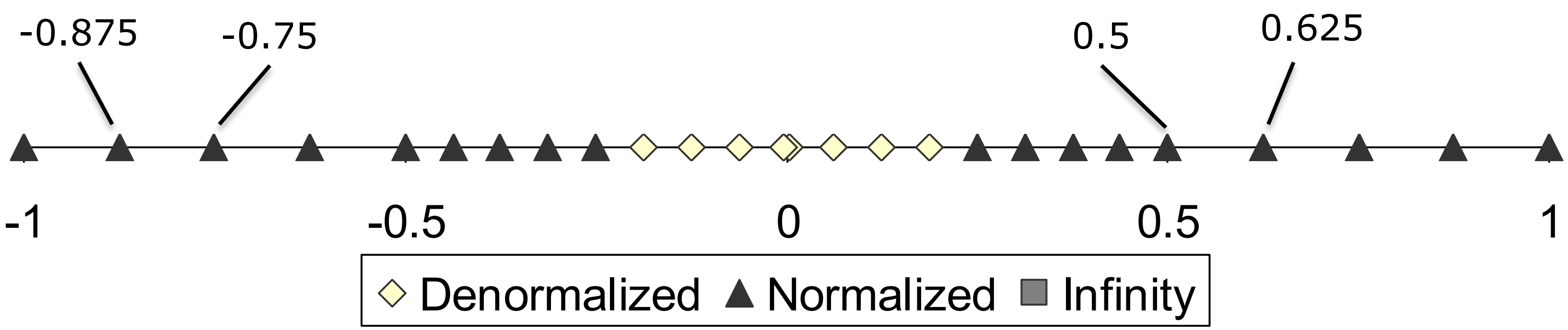


Round(-0.86) = ?

Round(0.55) = ?

Round to nearest in toy 6-bit FP

Round(x) either x_+ or x_- , whichever is nearer to x.

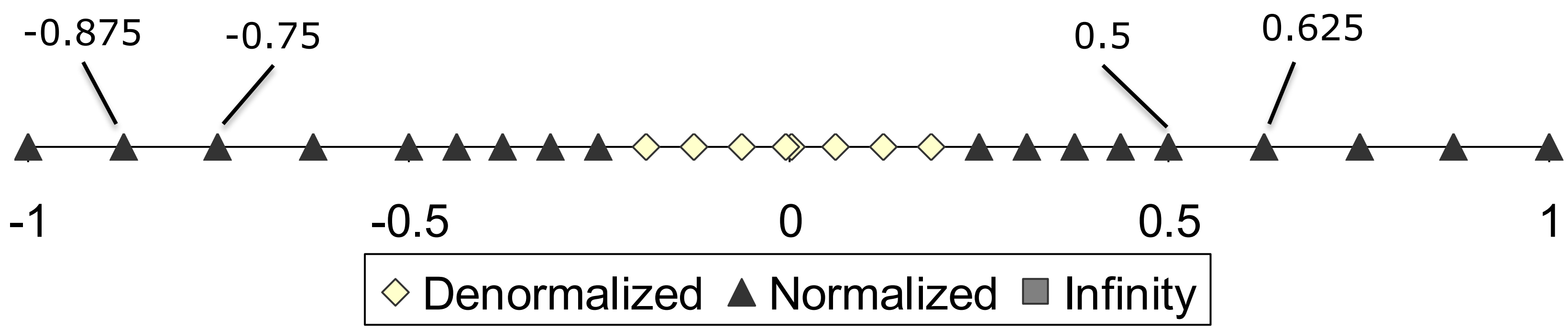


Round(-0.86) = -0.875

Round(0.55) = 0.5

Round to nearest; ties to even

Round(x) either x_+ or x_- , whichever is nearer to x.

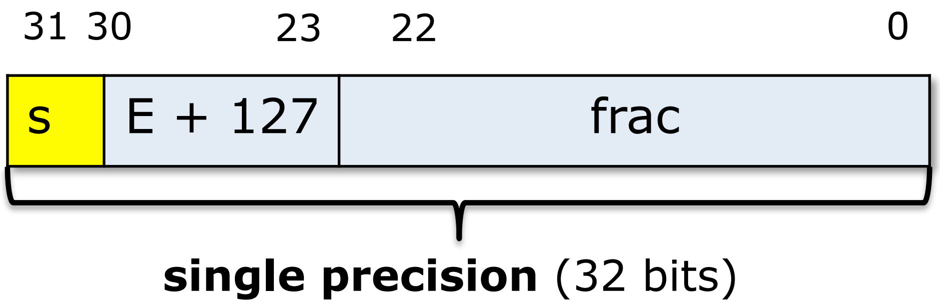


Round(-0.86) = -0.875

Round(0.55) = 0.5

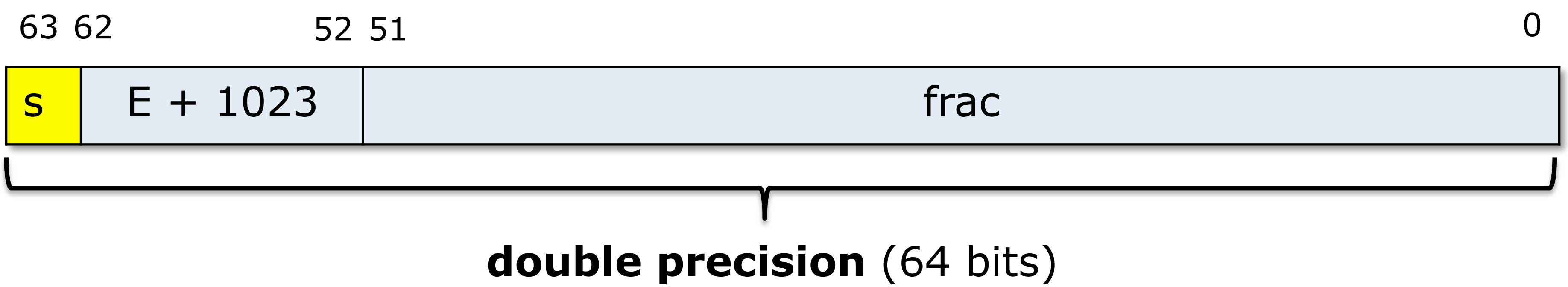
In case of a tie, the one with its least significant bit equal to zero is chosen.

single/ double precision



float f = 0.1

double d = 0.1

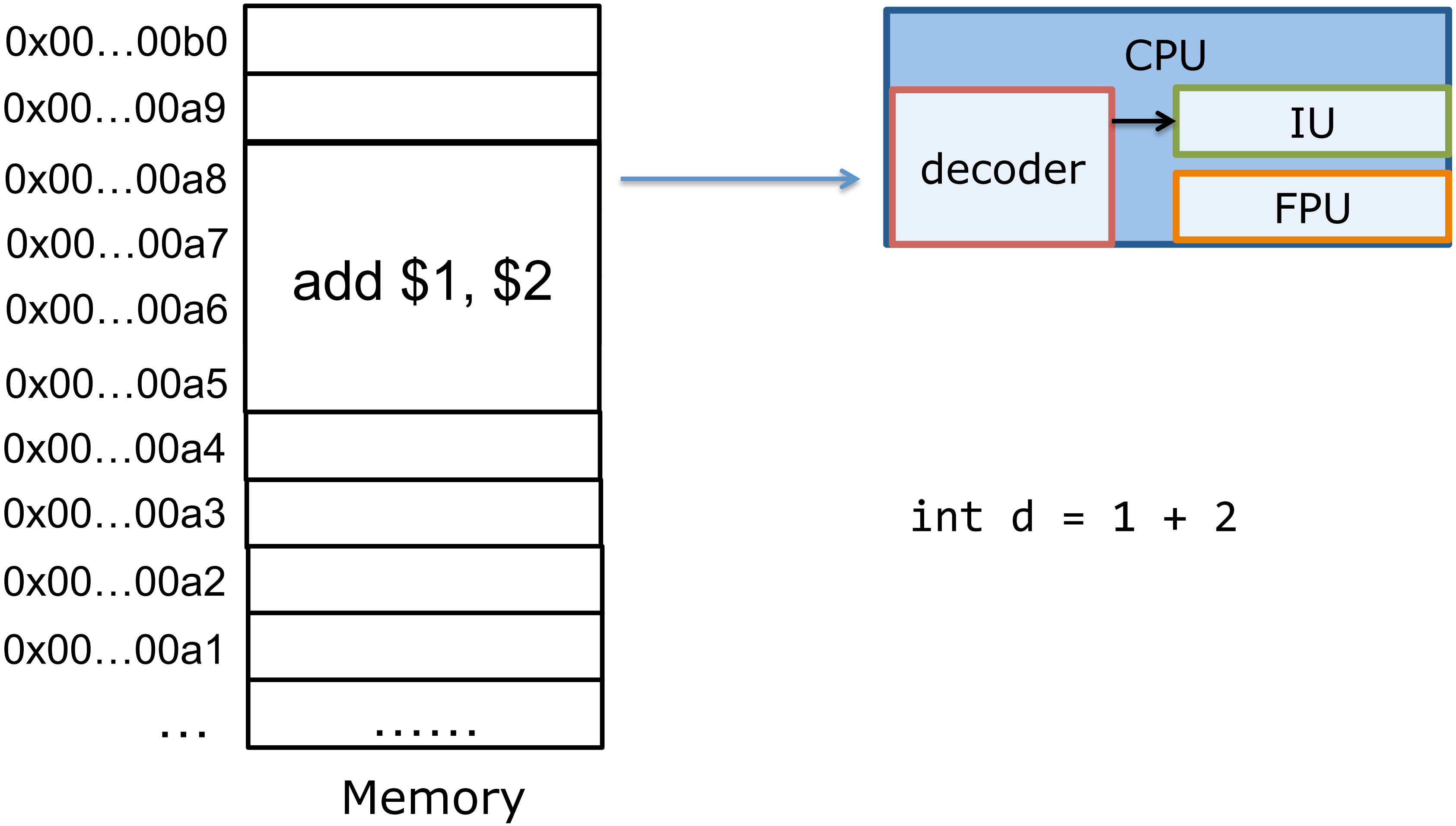


single/ double precision

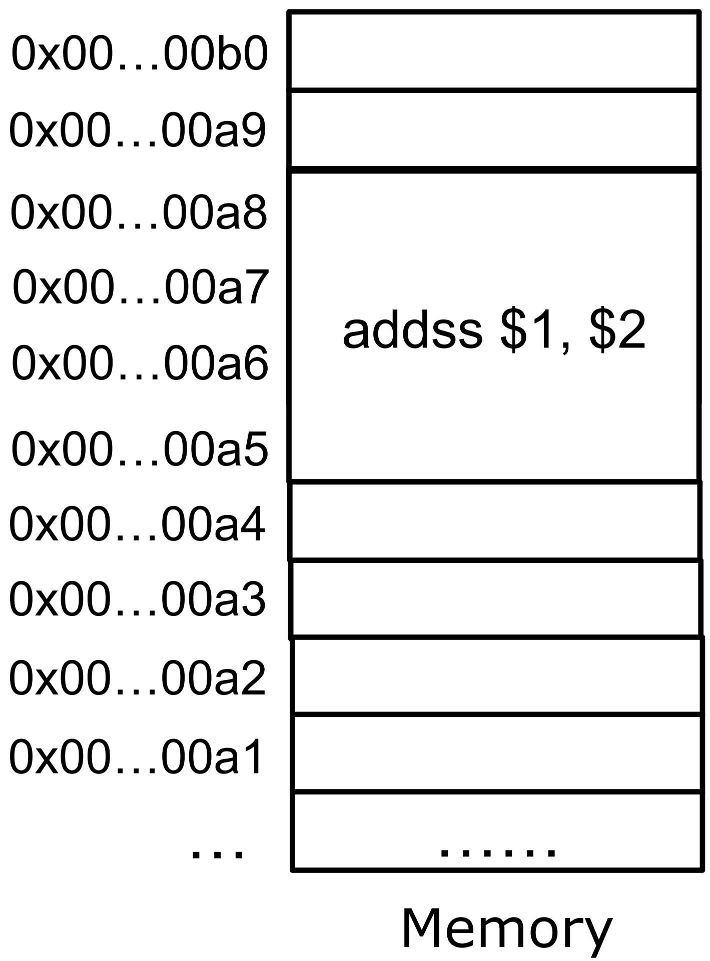
	E_{min}	E_{max}	N_{min}	N_{max}
Float	-126	127	$\approx 2^{-126}$	$\approx 2^{128}$
Double	-1022	1023	$\approx 2^{-1022}$	$\approx 2^{1024}$

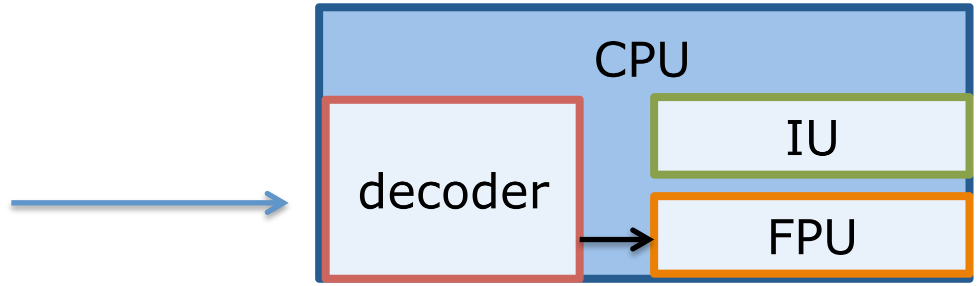
How does CPU know if it is floating point or integers ?

By having specific instruction for floating points operation.


0x00…00b0
0x00…00a9
0x00…00a8
0x00…00a7
0x00…00a6
0x00…00a5
0x00…00a4
0x00…00a3
0x00…00a2
0x00…00a1
…
add $1, $2
……
Memory
CPU
decoder
IU
FPU

```
int d = 1 + 2
```


0x00…00b0
0x00…00a9
0x00…00a8
0x00…00a7
0x00…00a6
0x00…00a5
0x00…00a4
0x00…00a3
0x00…00a2
0x00…00a1
…
addss $1, $2
……
Memory


CPU
decoder
IU
FPU

```
float f = 0.1 + 0.2
```