

Bits, Bytes, Ints

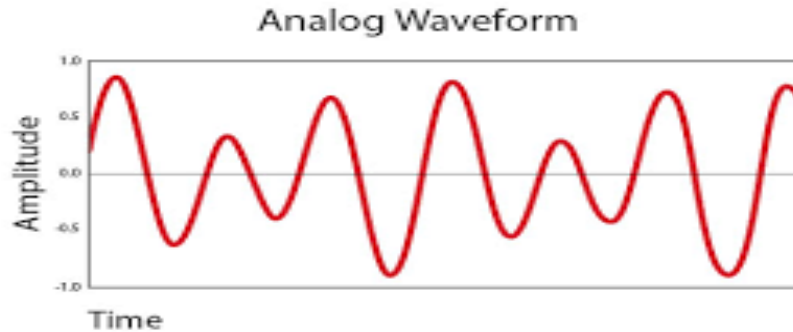
Jinyang Li

Some slides are due to Tiger Wang

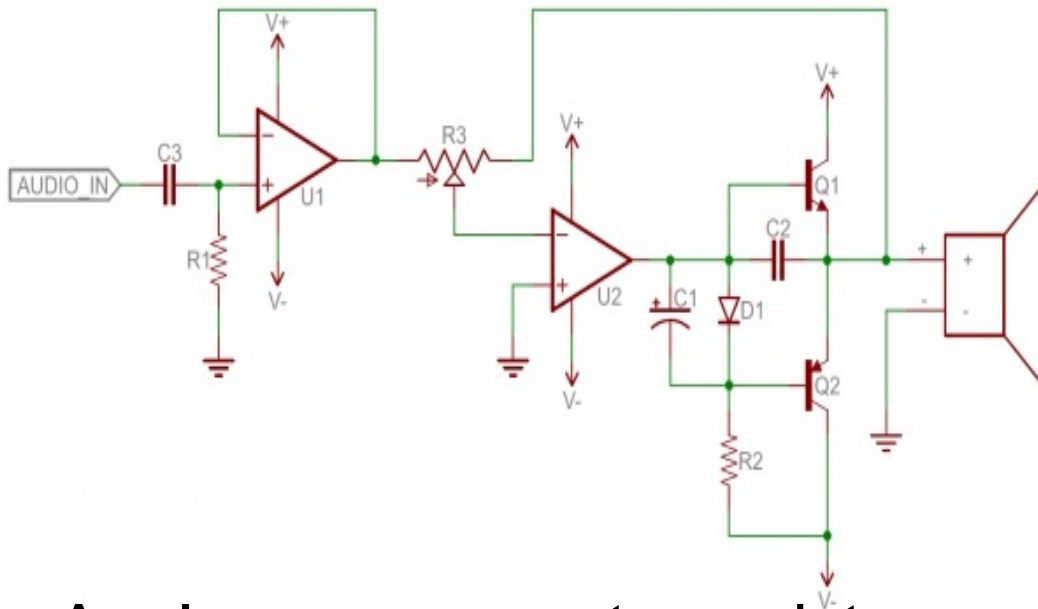
Lesson plan

- How computers represent integers in binary formats
 - Bit, Byte
- How to make binary formats readable to humans
 - Hex notation
- How computers add/subtract integers
- Unsigned vs. signed integer representation

The language of technology has evolved from analog signals...



Analog signals: smooth and continuous

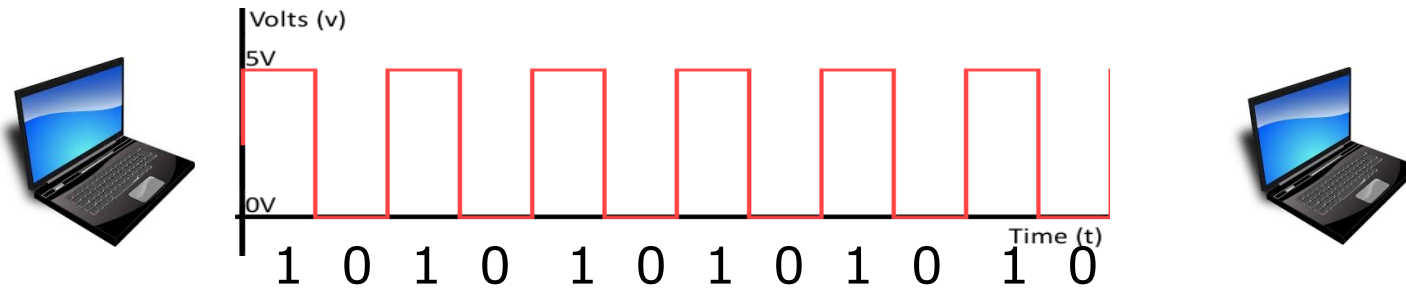


Hard

1. Difficult to design
2. Susceptible to noise

Analog components: resistors, capacitors, inductors, diodes, etc.

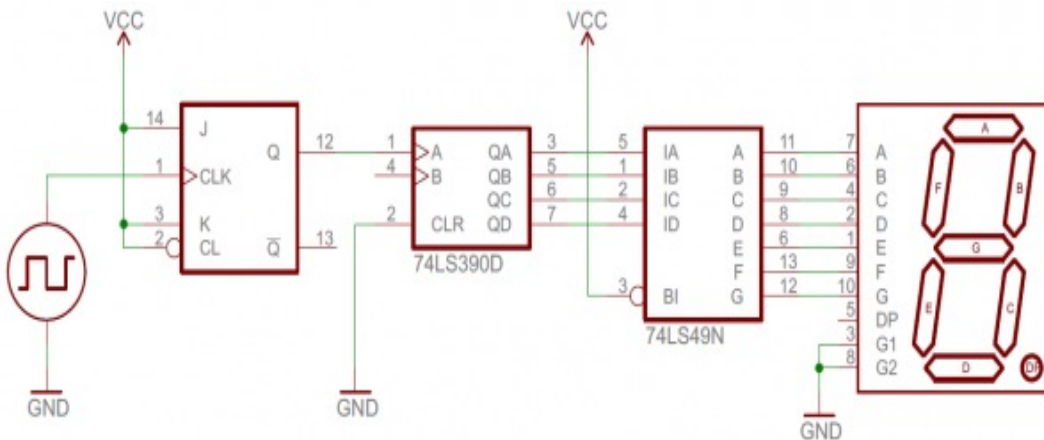
... to digital



Digital signals: discrete (0 or 1)

Easier

1. Easier to design
2. Robust to noise



Digital components: transistors, logic gates ...

Using bits to represent everything

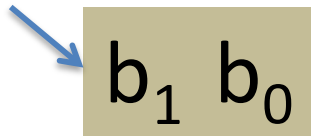
Bit = Binary digit, 0 or 1

- A bit is too small to be useful
 - A bit has 2 values; the English alphabet has 26 values (characters)
- Idea: use a group of bits
 - different bit patterns represent different “values”

Question

- How many bit patterns can a group of 2 bits have?

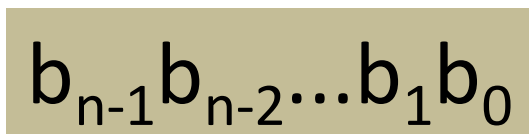
Can be
either 0 or 1



$b_1 b_0$

All patterns of 2-bits: 00, 01, 10, 11

- How many bit patterns does a group of n bits have?



$b_{n-1} b_{n-2} \dots b_1 b_0$

of patterns of n -bits: 2^n



n bits

Digression: All self-respecting CS students should memorize powers of 2

$$2^0 = 1$$

$$2^1 = 2$$

$$2^2 = 4$$

$$2^3 = 8$$

$$2^4 = 16$$

$$2^5 = 32$$

$$2^6 = 64$$

$$2^7 = 128$$

$$2^8 = 256$$

$$2^9 = 512$$

$$2^{10} = 1024$$



2^5



2^8

Approximations of powers of 2

$$2^{10} = 1024 \approx 10^3 \text{ (Kilo)}$$

$$2^{20} \approx 10^{3*2} = 10^6 \text{ (Mega)}$$

$$2^{30} \approx 10^{3*3} = 10^9 \text{ (Giga)}$$

$$2^{40} \approx 10^{3*4} = 10^{12} \text{ (Tera)}$$

$$2^{50} \approx 10^{3*5} = 10^{15} \text{ (Peta)}$$



verizon[✓]

200 Mbps
Speed

Stream and download movies, shows and photos.

\$39.99⁶

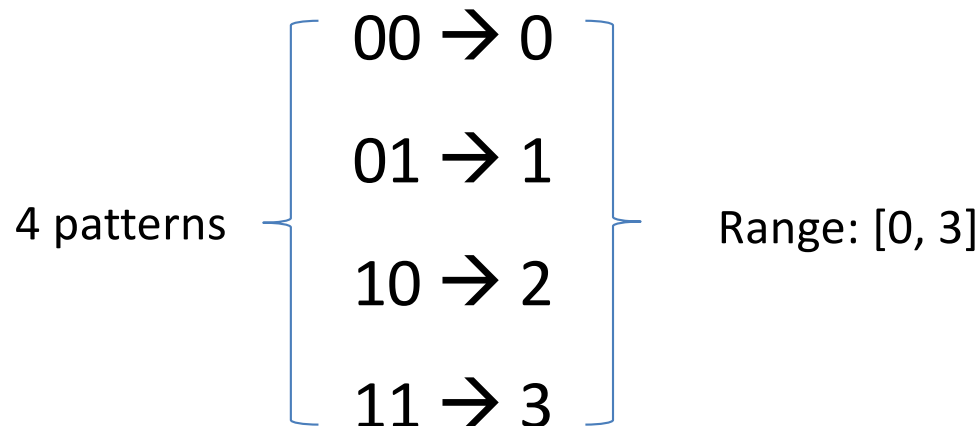
Per Month. With Auto Pay. Plus taxes and equipment charges.
200/200 Mbps

≈ 2^{??}

Represent ~~unsigned~~ non-negative integer

bits: b_1b_0

Goal: map each bit pattern to an integer



Represent unsigned integer

Bit pattern: $b_{n-1}b_{n-2}\dots b_2b_1b_0$

Range: $[0, 2^n - 1]$

Base-2 representation:

$$b_{n-1}b_{n-2}\dots b_2b_1b_0 = \sum_{i=0}^{n-1} b_i * 2^i$$

b_i is bit at i -th position (from right to left, starting at $i=0$)

Examples

Bit pattern: 00000110

Value: $0*2^7 + 0*2^6 + 0*2^5 + 0*2^4 + 0*2^3 + 1*2^2 + 1*2^1 + 0*2^0 = 6$

Bit pattern: 10000110

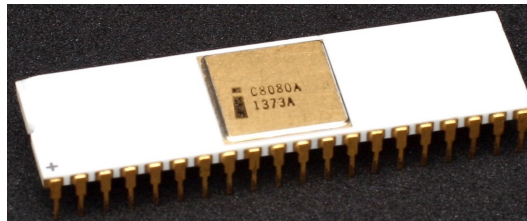
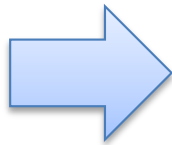
Value: ?



Byte



- Byte: a fixed size group of bits
 - The term is coined by Werner Buchholz (IBM).
 - Long long ago, different vendors use different byte sizes
- Now: Byte is 8-bit



IBM System/360, 1964

Introduced 8-bit byte

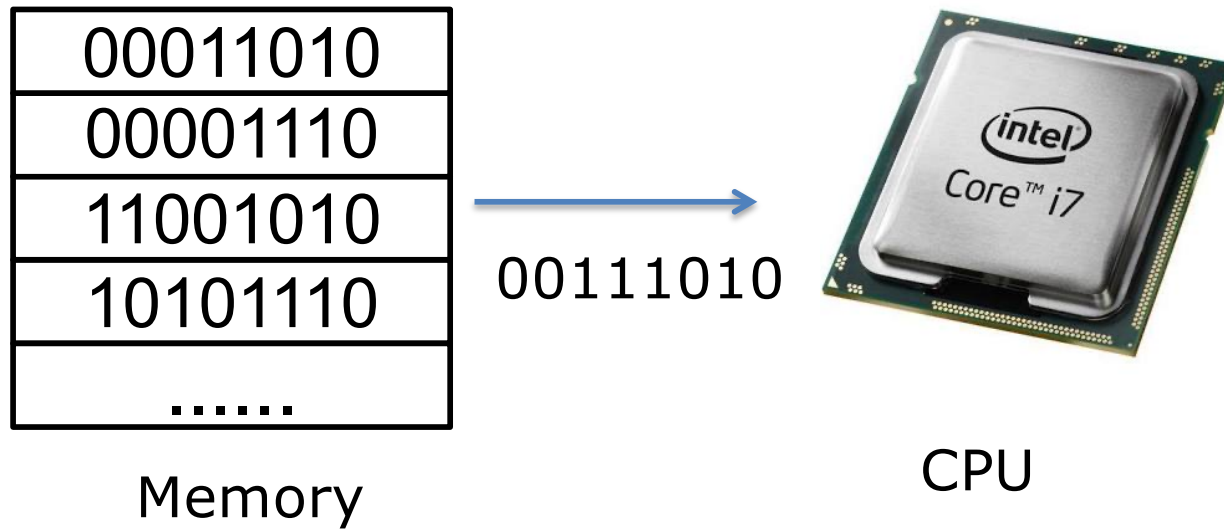
Intel 8080, 1974

Widely adopted

Modern processors

Standardized

Byte



Byte is the smallest unit of information storage, computation and transfer



Integers are represented by 1,2,4, or 8 bytes.



Range of 1-byte non-negative integers: $[0, ??]$

Bit-pattern of the largest integer?



Range of 4-byte non-negative integers: $[0, ??]$

Bit-pattern of the largest integer?

Most and least significant bit

MSB: bit position with the largest positional value

LSB: bit position with the smallest positional value

1-byte unsigned int:

10011010

4-byte unsigned int:

01110011 10001101 01010011 11011010

Most significant byte

Least significant byte

Describing bit patterns in a human-readable way

1-byte int: 10101110

C program:

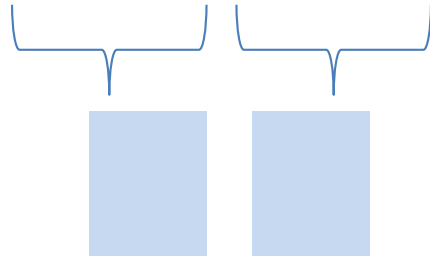
```
unsigned int a = 0b10101110;
```

If I ask you to type a 4-byte int, ...



Describing a bit pattern: hex notation

10101110



Use one (hex) symbol to represent a group of 4 bits



How many hex symbols are needed?

Binary	Hex
0000	0
0001	1
0010	2
0011	3
0100	4
0101	5
0110	6
0111	7

Binary	Hex
1000	8
1001	9
1010	a
1011	b
1100	c
1101	d
1110	e
1111	f

C program:

```
unsigned int a = 0xae;
```

What have we learnt

- How computers represent integers
 - Bit, Byte

Q: What is 10001111 in decimal? A: 143

Q: What's the least significant bit of any even number?

- Hex notation

Q: What is 10001111 in hex? A: 0x8F

Lesson plan

- How computers represent integers
 - Bit, Byte
- Hex notation
- How computers add/subtract integers
- Signed integer representation
 - 2's complement
- A short history of processors:
 - from 8-bit to 64-bit machines
- Byte ordering: big vs. small endian



Unsigned int addition

$$\begin{array}{r} 00001011 \\ + 00001010 \\ \hline \end{array}$$

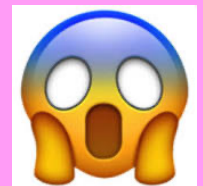
00010101



Grade school
method

$$\begin{array}{r} 10001011 \\ + 10001010 \\ \hline \end{array}$$

00010101



Overflow!



Unsigned int subtraction

$$\begin{array}{r} 00001110 \\ - 00001011 \\ \hline \end{array}$$

00000011



Grade school
method

$$\begin{array}{r} 00001010 \\ - 00001011 \\ \hline \end{array}$$

???



How to represent
negative numbers?

Represent negative numbers: a strawman

Most significant bit (MSB) represents the sign

Diagram illustrating the conversion of a 9-bit two's complement binary number to its decimal value:

- Top row: Binary number 00000001_2 . The leading 0 is labeled "sign". The remaining bits 00000001 are labeled "magnitude". An arrow points from the magnitude part to the decimal value 1 .
- Bottom row: Binary number 10000000_2 . The leading 1 is labeled "sign". The remaining bits 00000000 are labeled "magnitude". An arrow points from the magnitude part to the decimal value -1 .

$$\begin{array}{r} 00000001 \\ + 10000001 \\ \hline 10000010 \end{array}$$

-2 ??? 🤔

🙄 Need different h/w for signed vs. unsigned computation

Two's complement

Unsigned int

$$00010110 = 0*2^7 + 0*2^6 + 0*2^5 + 1*2^4 + 0*2^3 + 1*2^2 + 1*2^1 + 0*2^0$$

$$10010110 = 1*2^7 + 0*2^6 + 0*2^5 + 1*2^4 + 0*2^3 + 1*2^2 + 1*2^1 + 0*2^0$$

Signed int

$$00010110 = 0*(-2^7) + 0*2^6 + 0*2^5 + 1*2^4 + 0*2^3 + 1*2^2 + 1*2^1 + 0*2^0$$

$$10010110 = 1*(-2^7) + 0*2^6 + 0*2^5 + 1*2^4 + 0*2^3 + 1*2^2 + 1*2^1 + 0*2^0$$

Two's complement

- 1-byte bit pattern → signed int

Bit pattern	value
00000000	0
00000001	1
...	...
01111111	$2^7 - 1 = 127$
10000000	$-2^7 = -128$
10000001	$-2^7 + 1 = -127$
...	
11111111	$-2^7 + (2^7 - 1) = -1$

Two's complement

- ?-byte bit pattern → signed int



Source: xkcd.com

Basic facts of 2's complement

Signed int

Size (bytes)	Bit pattern of smallest	Bit pattern of largest	Range
1	0x80	0x7f	$[-2^7, 2^7-1]$
2	0x8000	0x7fff	$[-2^{15}, 2^{15}-1]$
4	0x80000000	0x7fffffff	$[-2^{31}, 2^{31}-1]$
8	0x8000000000000000	0x7fffffffffffffff	$[-2^{63}, 2^{63}-1]$



Home exercise: make a similar table for unsigned int

- Negative numbers \leftrightarrow MSBit=1
- A sequence of 1's (e.g. 0xff, 0xffffffff) \leftrightarrow -1

Two's complement: 8-bit signed integer

$$01011000 = 0*(-2^7) + 1*2^6 + 0*2^5 + 1*2^4 + 1*2^3 + 0*2^2 + 0*2^1 + 0*2^0 = 88$$

$$11011000 = 1*(-2^7) + 1*2^6 + 0*2^5 + 1*2^4 + 1*2^3 + 0*2^2 + 0*2^1 + 0*2^0 = -40$$

$$00000000 = 0$$

$$11111111 = -1$$

$$10000000 = -2^7 = -128$$

$$01111111 = 2^7 - 1 = 127$$

2's complement: find a number's negation

00101000 \longrightarrow ??
 $(40)_{10}$ $(-40)_{10}$

A useful trick to do negation:

Step-1: flip all bits

00101000 $(40)_{10}$

Step-1: flip bits

11010111

Step-2: +00000001

11011000 $(-40)_{10}$

Step-2: add 1

Why does the negation trick work

$$\vec{b} + (\sim \vec{b}) = 11\dots11_2 = -1$$

b with bits
flipped

$$-\vec{b} = (\sim \vec{b}) + 1$$

Using negation trick to find the bit-pattern of a negative number



The bit pattern of 8-bit signed integer -33?

Answer:

$$33 = (00100001)_2$$

$$\text{Apply negation trick: } (11011110)_2 + 1 = (11011111)_2$$

Negation trick helps computers do subtraction

Instead of doing this:

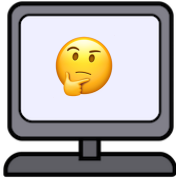
$$\begin{array}{r} 00000100 \text{ (4)}_{10} \\ - 00000011 \text{ (3)}_{10} \\ \hline 00000001 \text{ (1)}_{10} \end{array}$$

Do this instead:

$$\begin{array}{r} 00000100 \text{ (4)}_{10} \\ + \begin{array}{r} 11111100 \\ 00000001 \end{array} \left. \vphantom{\begin{array}{r} 11111100 \\ 00000001 \end{array}} \right\} \text{(-3)}_{10} \\ \hline 00000001 \text{ (1)}_{10} \end{array}$$

$00000011 \text{ (3)}_{10} \rightarrow$

Works for both unsigned and signed subtraction!



Unsigned addition

$$\begin{array}{r} 00000001 \text{ (1)}_{10} \\ + 00000011 \text{ (3)}_{10} \\ \hline \end{array}$$

$$00000100 \text{ (4)}_{10}$$

$$\begin{array}{r} 00000001 \text{ (1)}_{10} \\ + 10000001 \text{ (129)}_{10} \\ \hline \end{array}$$

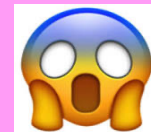
$$10000010 \text{ (130)}_{10}$$

$$\begin{array}{r} 01000001 \text{ (65)}_{10} \\ + 01000000 \text{ (64)}_{10} \\ \hline \end{array}$$

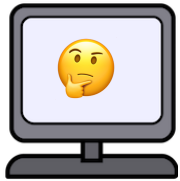
$$10000001 \text{ (129)}_{10}$$

$$\begin{array}{r} 10000001 \text{ (129)}_{10} \\ + 11111110 \text{ (254)}_{10} \\ \hline \end{array}$$

$$01111111 \text{ (127)}_{10}$$



Overflow!



Signed addition

$$\begin{array}{r} 00000001 \text{ (1)}_{10} \\ + 00000011 \text{ (3)}_{10} \\ \hline \end{array}$$

$$00000100 \text{ (4)}_{10}$$

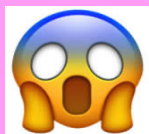
$$\begin{array}{r} 00000001 \text{ (1)}_{10} \\ + 10000001 \text{ (-127)}_{10} \\ \hline \end{array}$$

$$10000010 \text{ (-126)}_{10}$$

This is what 2's complement is designed to accomplish!

$$\begin{array}{r} 01000001 \text{ (65)}_{10} \\ + 01000000 \text{ (64)}_{10} \\ \hline \end{array}$$

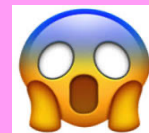
$$10000001 \text{ (-127)}_{10}$$



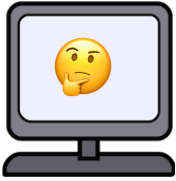
Overflow!

$$\begin{array}{r} 10000001 \text{ (-127)}_{10} \\ + 11111110 \text{ (-2)}_{10} \\ \hline \end{array}$$

$$01111111 \text{ (127)}_{10}$$



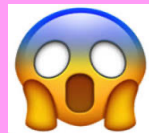
Overflow!



Unsigned subtraction

$$\begin{array}{r} 00000001 \text{ (1)}_{10} \\ - 00000011 \text{ (3)}_{10} \\ \hline \end{array}$$

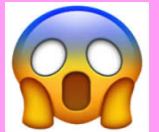
11111110 (254)₁₀



Overflow!

$$\begin{array}{r} 00000001 \text{ (1)}_{10} \\ - 11111111 \text{ (255)}_{10} \\ \hline \end{array}$$

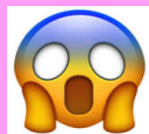
00000010 (2)₁₀



Overflow!

$$\begin{array}{r} 01111111 \text{ (127)}_{10} \\ - 11111110 \text{ (254)}_{10} \\ \hline \end{array}$$

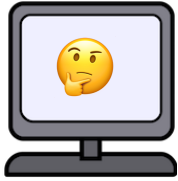
10000001 (129)₁₀



Overflow!

$$\begin{array}{r} 10000000 \text{ (128)}_{10} \\ - 00000001 \text{ (1)}_{10} \\ \hline \end{array}$$

01111111 (127)₁₀



Signed subtraction

$$\begin{array}{r} 00000001 \text{ (1)}_{10} \\ - 00000011 \text{ (3)}_{10} \\ \hline \end{array}$$

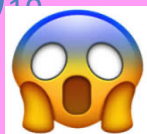
$$11111110 \text{ (-2)}_{10}$$

$$\begin{array}{r} 00000001 \text{ (1)}_{10} \\ - 11111111 \text{ (-1)}_{10} \\ \hline \end{array}$$

$$00000010 \text{ (2)}_{10}$$

$$\begin{array}{r} 01111111 \text{ (127)}_{10} \\ - 11111110 \text{ (-2)}_{10} \\ \hline \end{array}$$

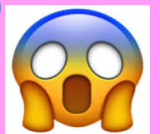
$$10000001 \text{ (-127)}_{10}$$



Overflow!

$$\begin{array}{r} 10000000 \text{ (-128)}_{10} \\ - 00000001 \text{ (1)}_{10} \\ \hline \end{array}$$

$$01111111 \text{ (127)}_{10}$$



Overflow!

Lesson plan

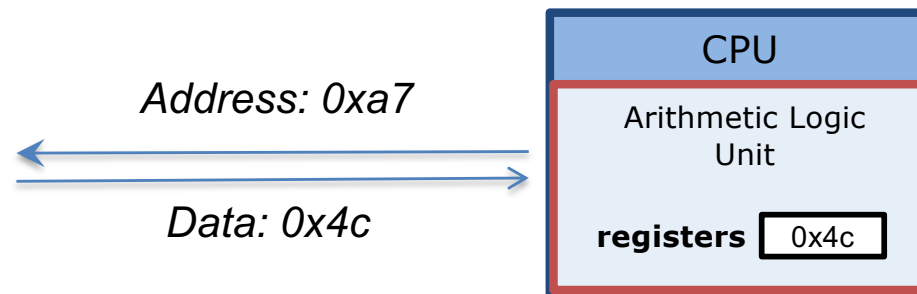
- How computers represent integers
 - Bit, Byte
- Hex notation
- How computers add/subtract integers
- Signed integer representation
 - 2's complement
- A short history of processors:
 - from 8-bit to 64-bit machines
- Byte ordering: big vs. small endian

THE EVOLUTION OF INTEGER SIZES IN PROCESSORS

8-bit processors: Intel 8080 (1974)

0xb0	0x0f
0xa9	0xaf
0xa8	0x1e
0xa7	0x4c
0xa6	0xea
0xa5	0xcb
0xa4	0xba
0xa3	0x00
0xa2	0xff
0xa1	0x8c
...

Memory



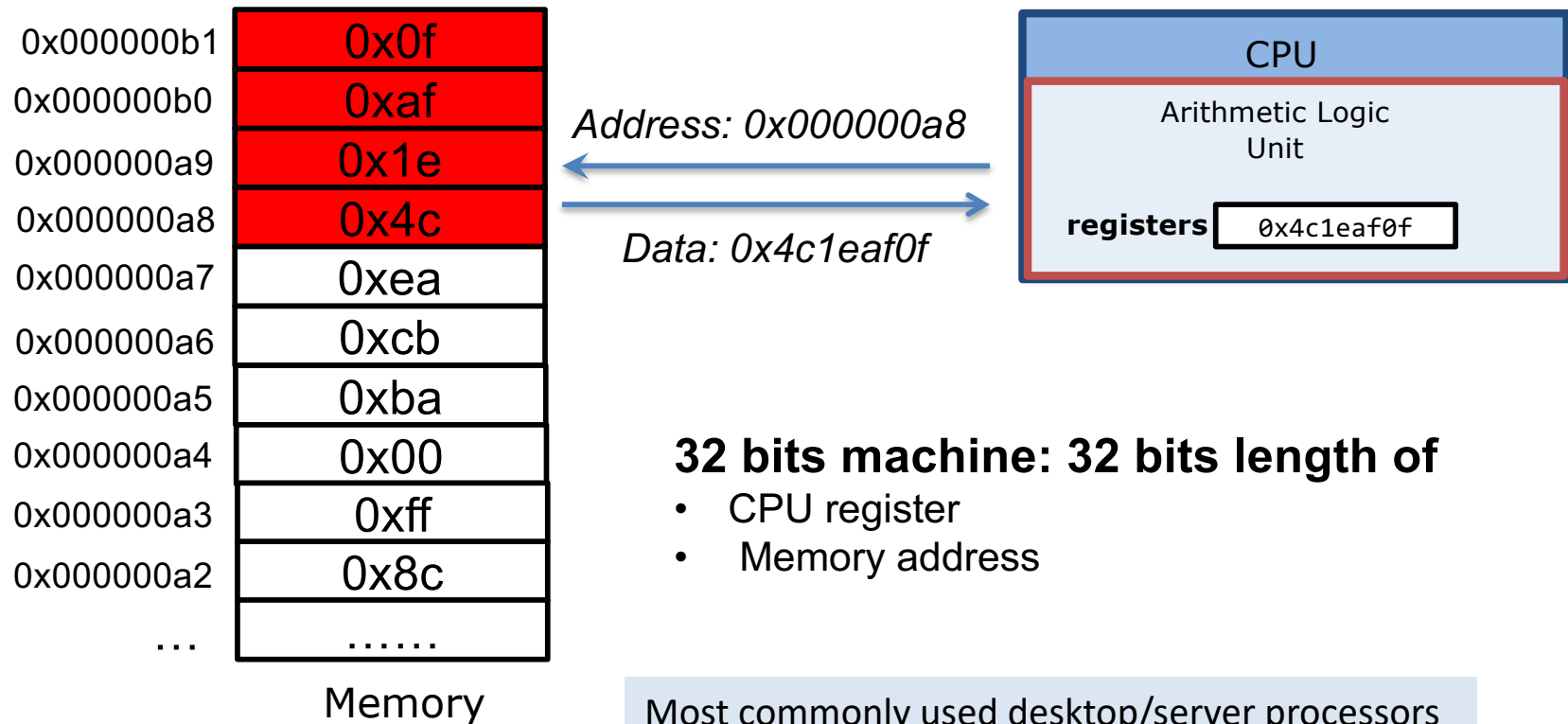
8 bits machine: 8 bits length of

- CPU register
- Memory address



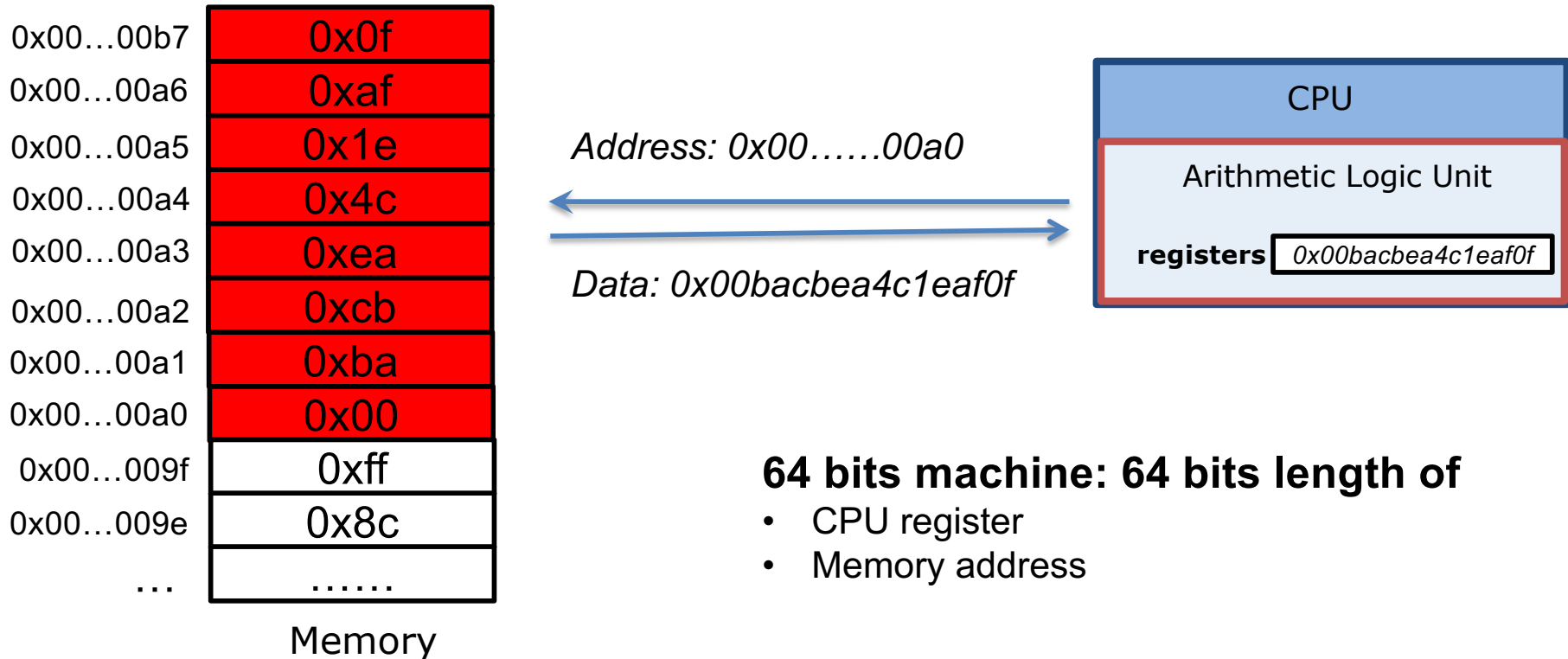
Nowadays: 8-bit processor (microcontroller)
Is used for embedded systems

32-bit processors: Intel 386 (1985)



Most commonly used desktop/server processors in the latest 80s to early 00s

64-bit processors: Intel Pentium 4 (2000)



64 bits machine: 64 bits length of

- CPU register
- Memory address

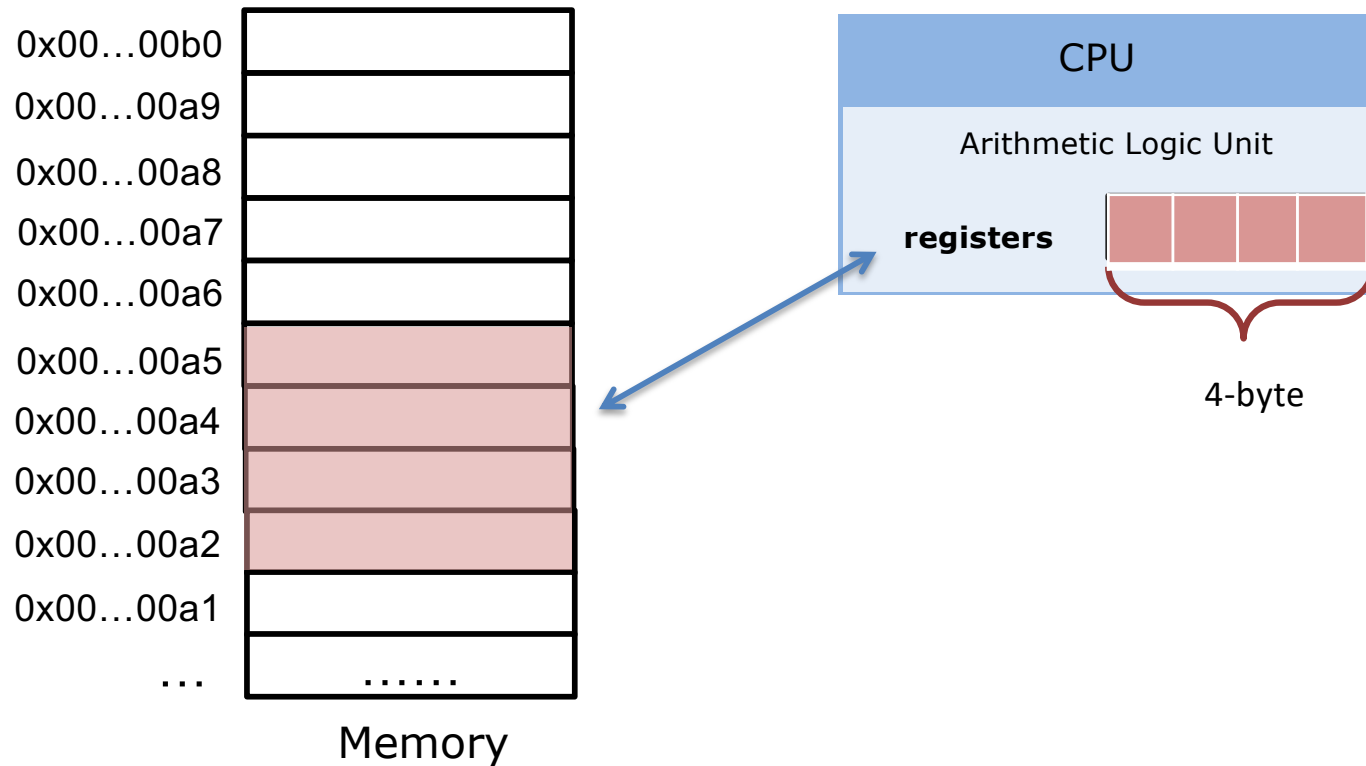
Nowadays:

- Servers/laptops: Intel/AMD 64-bit x86 processors
- Mobile phones/tablets: 64-bit ARM processors (made by Apple/Qualcomm/Samsung etc)

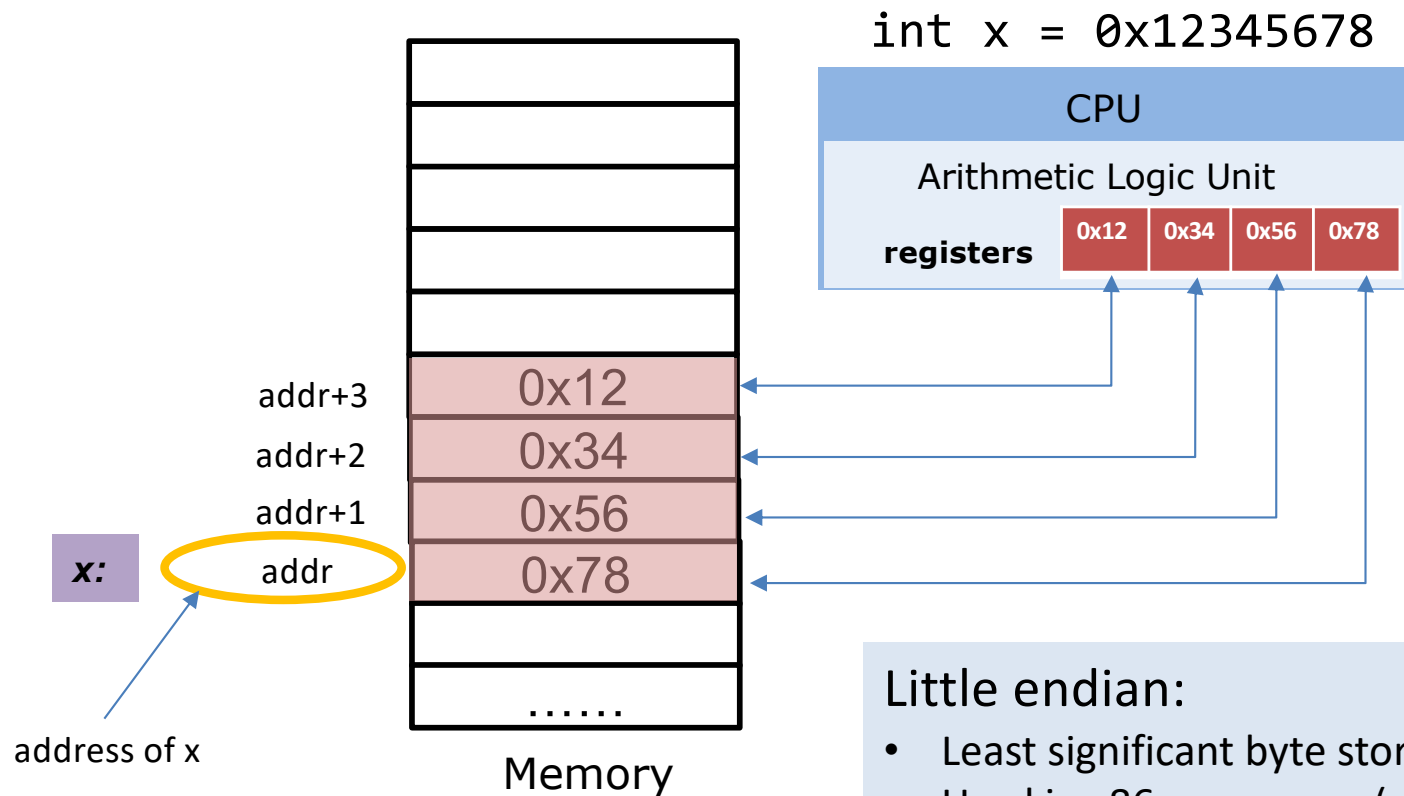
C's Integer data types on 64-bit machine

	Length	Min	Max
char	1 byte	-2^7	$2^7 - 1$
unsigned char	1 byte	0	$2^8 - 1$
short	2 bytes	-2^{15}	$2^{15} - 1$
unsigned short	2 bytes	0	$2^{16} - 1$
int	4 bytes	-2^{31}	$2^{31} - 1$
unsigned int	4 bytes	0	$2^{32} - 1$
long	8 bytes	-2^{63}	$2^{63} - 1$
unsigned long	8 bytes	0	$2^{64} - 1$

Memory layout for multi-byte integers




Memory layout: Little Endian



Little endian:

- Least significant byte stored at smallest address
- Used in x86 processors (servers/laptops) and most ARM (phones/tablets) implementation

Advantage of Little Endian

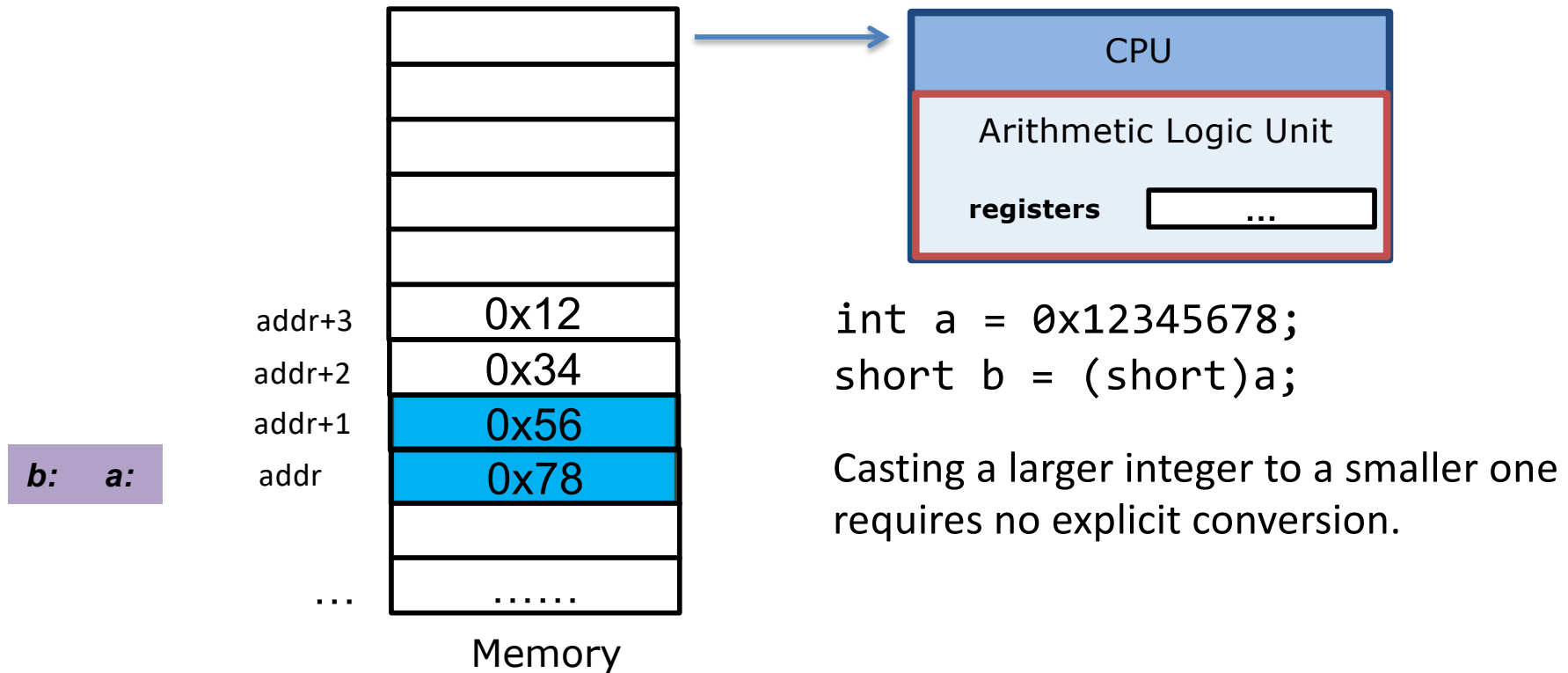
0x12345678
+ 0x12131415


Processor performs calculation
from the least significant bit

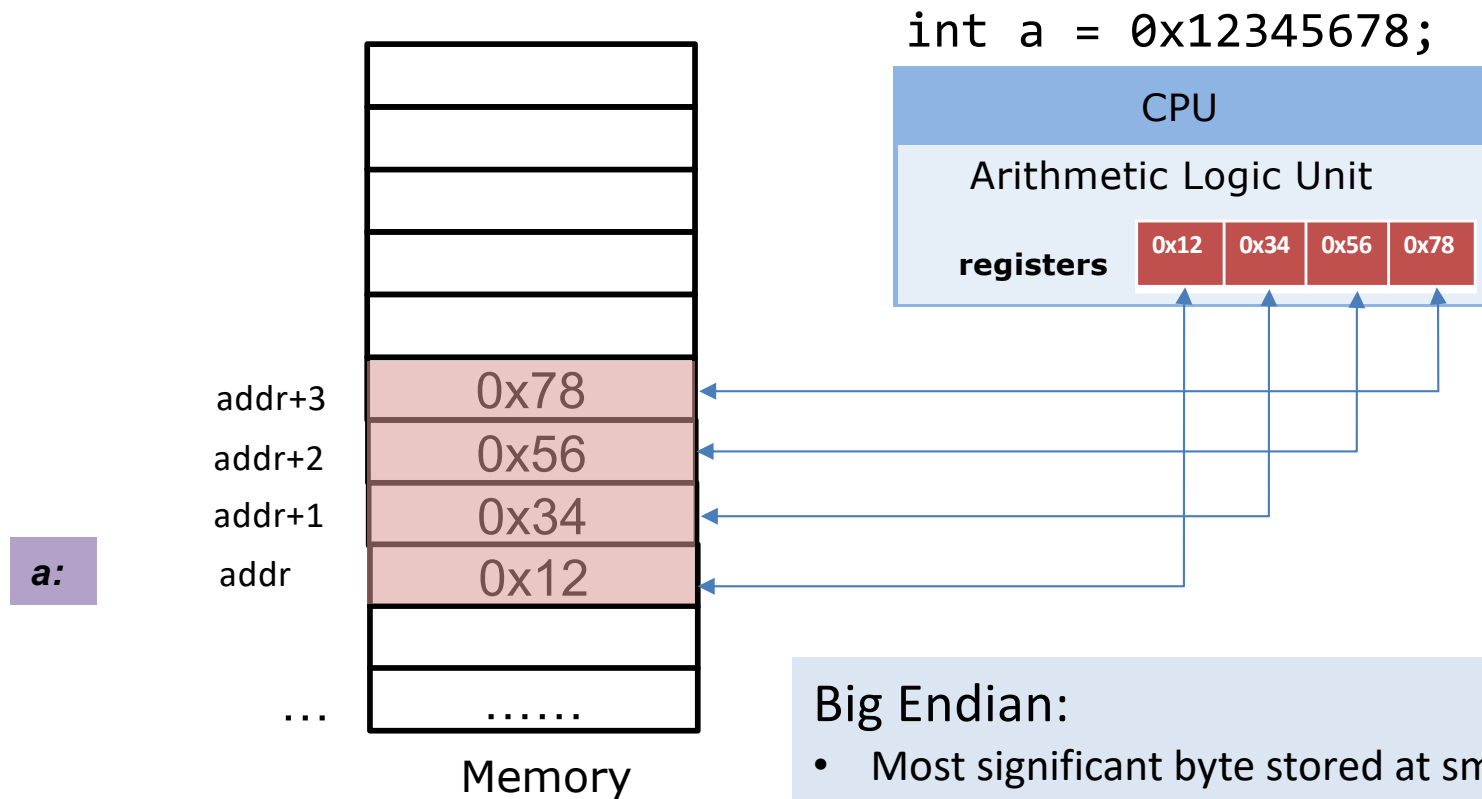


Processor can simultaneously
perform memory transfer and
calculation.

Another advantage of Little Endian



Memory layout: Big Endian



Big Endian:

- Most significant byte stored at smallest address
- ARM processors (phones/tablets) in principle support both endian

Advantages of Big Endian

Quick to test whether the number is positive or negative

- Examine byte stored at the address offset zero.

Big or Little Endian?

```
#include <stdio.h>

int main()
{
    int x = 0x05060708;
    int *p;
    p = &x; //p contains the address of x
    char y = *(char *)p; //y is the 1-byte value pointed to by p
    printf("y=%d\n", y);
}
```

```
$ gcc endian.c
```

```
$ ./a.out
```

Summary

- Integer representation
 - Unsigned (base-2)
 - Signed (2's complement)
- Hex notation
- Operations (e.g. add, subtract) on fixed-width integers can cause overflow
- Big vs. little endian