Bits, Bytes, Ints

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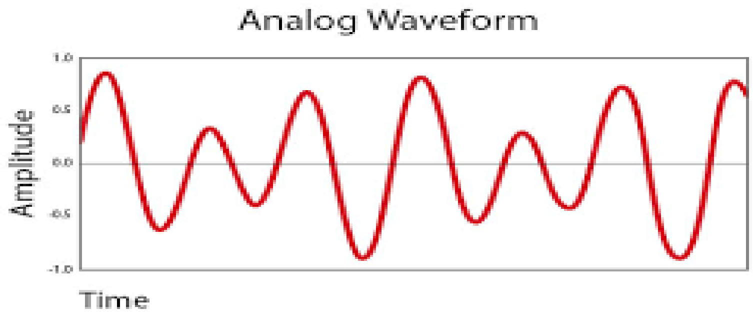
Some slides are due to Tiger Wang

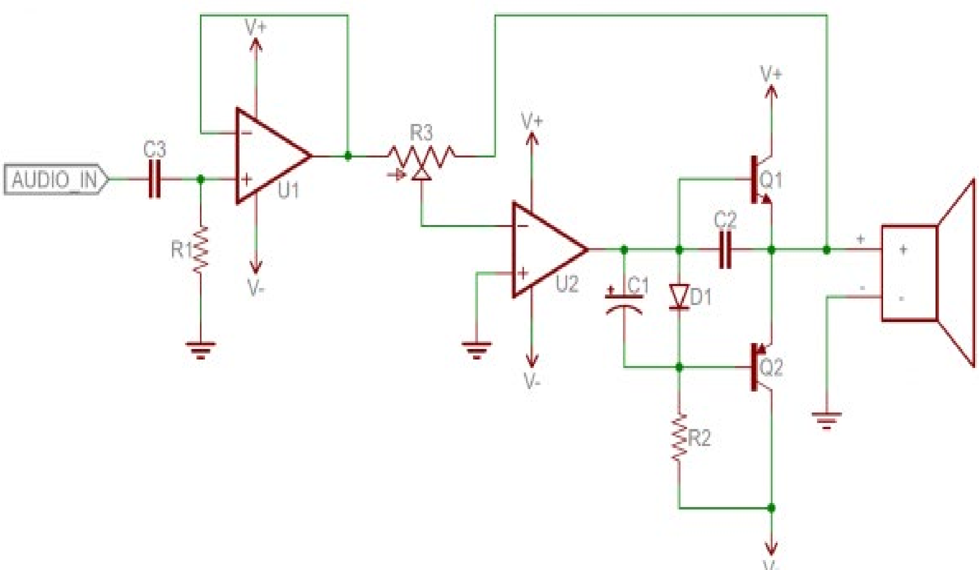
Lesson plan

- How computers represent integers in binary formats
  - Bit, Byte
- How humans represent binary formats
  - Hex notation
- How computers add/subtract integers
- Unsigned vs. signed integer representation

The language of technology has evolved from analog signals…



Analog signals: smooth and continuous

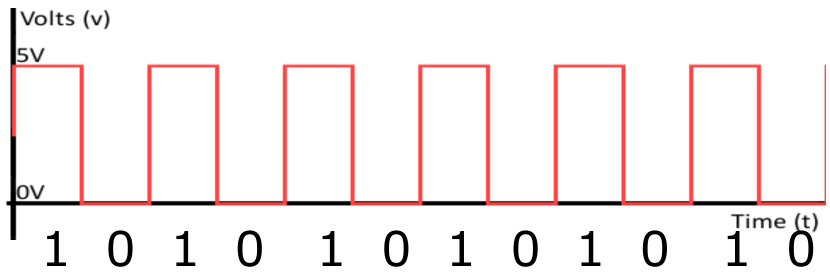


Hard

1. **Difficult to design**
2. **Susceptible to noise**

Analog components: resistors, capacitors, inductors, diodes, etc.

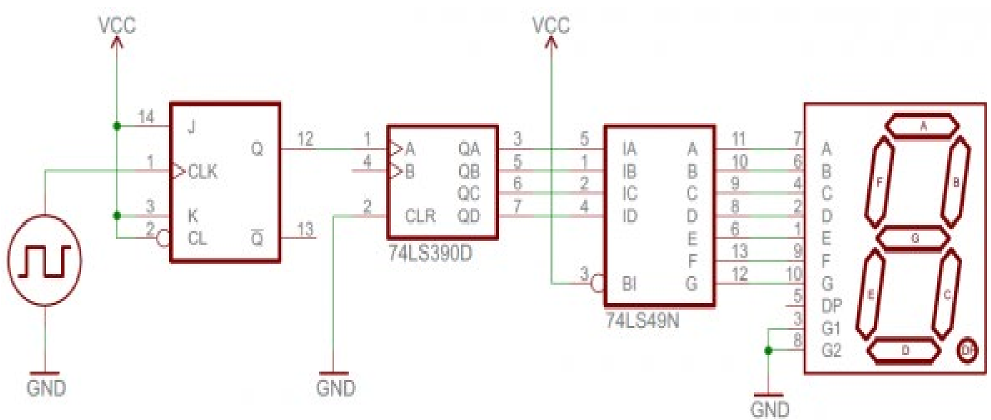
… to digital



Digital signals: discrete (0 or 1)

Easier

1. **Easier to design**
2. **Robust to noise**



Digital components: transistors, logic gates …

Using bits to represent everything

Bit = Binary digit, 0 or 1

- A bit is too small to be useful
  - A bit has 2 values; the English alphabet has 26 values (characters)

- Idea: use a group of bits
  - different bit patterns represent different “values”

Question

- How many bit patterns can a group of 2 bits have?

Can be
either 0 or 1

$b_1 \; b_0$

All patterns of 2-bits: 00, 01, 10, 11

- How many bit patterns does a group of n bits have?

$\underbrace{b_{n-1}b_{n-2}\ldots b_1b_0}_{\text{n bits}}$

# of patterns of n-bits: 2^n

Digression: Any self-respecting CS person must memorize powers of 2

$2^0 = 1$

$2^1 = 2$

$2^2 = 4$

$2^3 = 8$

$2^4 = 16$

$2^5 = 32$

$2^6 = 64$

$2^7 = 128$

$2^8 = 256$

$2^9 = 512$

$2^{10} = 1024$

2^5

2^8

Approximations of powers of 2

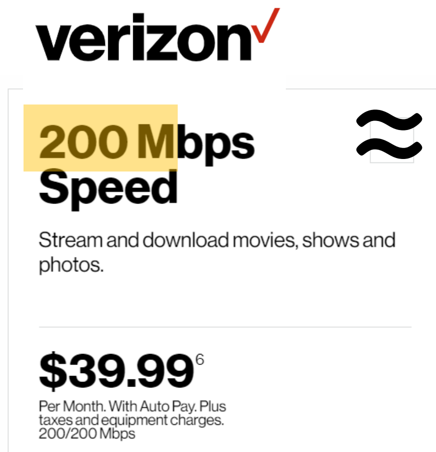
$2^{10} = 1024 \approx 10^3$ (Kilo)

$2^{20} \approx 10^{3*2} = 10^6$ (Mega)

$2^{30} \approx 10^{3*3} = 10^9$ (Giga)

$2^{40} \approx 10^{3*4} = 10^{12}$ (Tera)

$2^{50} \approx 10^{3*5} = 10^{15}$ (Peta)


verizon
200 Mbps
Speed
Stream and download movies, shows and photos.
$39.99[6]
Per Month. With Auto Pay. Plus taxes and equipment charges.
200/200 Mbps

$\approx 2^{??}$

Represent ~~non-negative~~ unsigned integer

bits: $b_1 b_0$

Goal: map each bit pattern to an integer

4 patterns

00 → 0

01 → 1

10 → 2

11 → 3

Range: [0, 3]

Represent unsigned integer

Bit pattern: $b_{n-1}b_{n-2}...b_2b_1b_0$

Range: $[0, 2^n - 1]$

Base-2 representation:

$$b_{n-1}b_{n-2}...b_2b_1b_0 = \sum_{i=0}^{n-1} b_i * 2^i$$

b_i is bit at i-th position (from right to left, starting at i=0)

Examples

Bit pattern: 00000110

Value: $0*2^7+0*2^6+0*2^5+0*2^4+0*2^3+1*2^2+1*2^1+0*2^0 = 6$

Bit pattern: 10000110

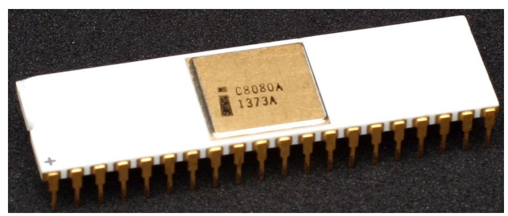
Value:  ?

Byte

- Byte: a fixed size group of bits
  - The term is coined by Werner Buchholz (IBM).
  - Long long ago, different vendors use different byte sizes
- Now: Byte is 8-bit

IBM System/360, 1964

Introduced 8-bit byte



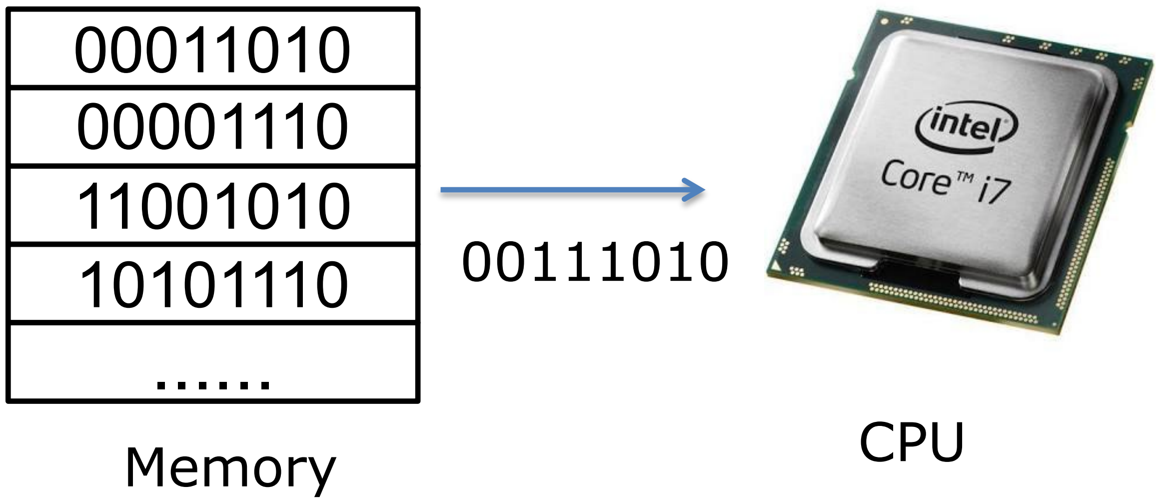
Intel 8080, 1974

Widely adopted

Modern processors

Standardized

Byte



Byte is the smallest unit of information storage, computation and transfer

Integers are represented by 1,2,4, or 8 bytes.

Range of 1-byte non-negative integers: [0, ??]

Bit-pattern of the largest integer?

Range of 4-byte non-negative integers: [0, ??]

Bit-pattern of the largest integer?

Most and least significant bit

MSB: bit position with the largest positional value
LSB: bit position with the smallest positional value

1-byte unsigned int:

10011010

4-byte unsigned int:

01110011 10001101 01010011 11011010

Most significant byte

Least significant byte

Describing bit patterns in a human-readable way

`1-byte int: 10101110`

C program:

```
unsigned int a = 0b10101110;
```

If I ask you to type a 4-byte int, …

Describing a bit pattern: hex notation

10101110

Use one (hex) symbol to represent a group of 4 bits

How many hex symbols are needed?

Binary	Hex
0000	0
0001	1
0010	2
0011	3
0100	4
0101	5
0110	6
0111	7

Binary	Hex
1000	8
1001	9
1010	a
1011	b
1100	c
1101	d
1110	e
1111	f

C program:

```
unsigned int a = 0xae;
```

Breakout time!

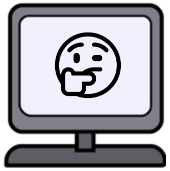
Breakout exercises

- What's the least significant bit of any even number? `0`
- Write the hex notation of this 2-byte bit pattern: `1111000011000011` `0xf0c2`
- What's the decimal value of unsigned int: `0xff`

`255`

Lesson plan

- How computers represent integers
  - Bit, Byte
- Hex notation
- How computers add/subtract integers
- Unsigned vs. signed integer representation



Unsigned int addition

```
    0 0 0 0 1 0 1 1
+   0 0 0 0 1 0 1 0
-------------------
    0 0 0 1 0 1 0 1
```

```
    1 0 0 0 1 0 1 1
+   1 0 0 0 1 0 1 0
-------------------
    0 0 0 1 0 1 0 1
```

Overflow!

Unsigned int subtraction

```
    0 0 0 0 1 1 1 0
-   0 0 0 0 1 0 1 1
-------------------
    0 0 0 0 0 0 1 1
```

```
    0 0 0 0 1 0 1 0
-   0 0 0 0 1 0 1 1
-------------------
    1 1 1 1 1 1 1 1
```

Overflow!

How to represent negative numbers?

Strawman

Most significant bit (MSB) represents the sign

$\underbrace{0}_{\text{sign}}\ \underbrace{0\ 0\ 0\ 0\ 0\ 0\ 1}_{\text{magnitude}}{}_2 \longrightarrow 1$

$1\ 0\ 0\ 0\ 0\ 0\ 0\ 1_2 \longrightarrow -1$

$$
\begin{array}{rl}
 & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \\
+ & 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \\
\hline
 & 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0
\end{array}
$$

-2 ???

☹ Need different h/w for signed vs. unsigned computation

Two's complement

Unsigned int

`00010110` $= 0*2^7 + 0*2^6 + 0*2^5 + 1*2^4 + 0*2^3 + 1*2^2 + 1*2^1 + 0*2^0$

`10010110` $= 1*2^7 + 0*2^6 + 0*2^5 + 1*2^4 + 0*2^3 + 1*2^2 + 1*2^1 + 0*2^0$

Signed int

`00010110` $= 0*(-2^7) + 0*2^6 + 0*2^5 + 1*2^4 + 0*2^3 + 1*2^2 + 1*2^1 + 0*2^0$

`10010110` $= 1*(-2^7) + 0*2^6 + 0*2^5 + 1*2^4 + 0*2^3 + 1*2^2 + 1*2^1 + 0*2^0$

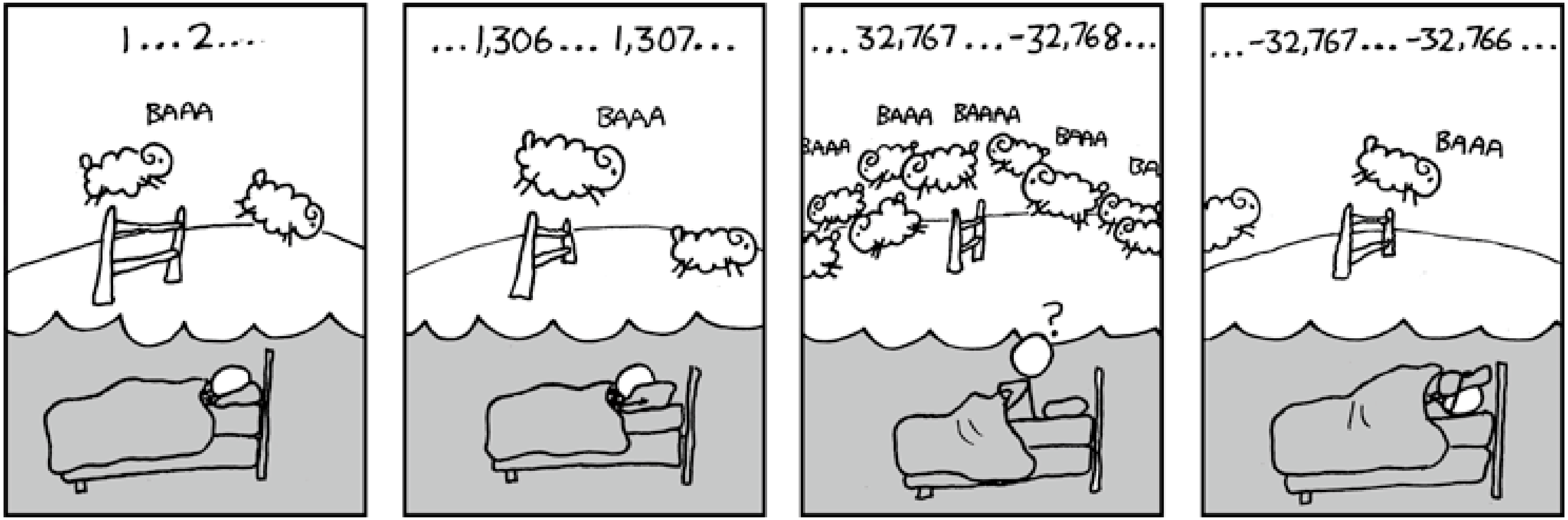
Two's complement

- 1-byte bit pattern → signed int

Bit pattern	value
00000000	0
00000001	1
…	…
01111111	$2^7-1 = 127$
10000000	$-2^7 = -128$
10000001	$-2^7 +1= -127$
…	
11111111	$-2^7+(2^7-1) = -1$

Two's complement

- 1-byte bit pattern → signed int



Source: xkcd.com

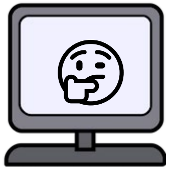
Basic facts of 2's complement

Signed int

Size (bytes)	Bit pattern of smallest	Bit pattern of largest	Range
1	`0x80`	`0x7f`	$[-2^{-7}, 2^{7}-1]$
2	`0x8000`	`0x7fff`	$[-2^{-15}, 2^{15}-1]$
4	`0x80000000`	`0x7fffffff`	$[-2^{-31}, 2^{31}-1]$
8	`0x8000000000000000`	`0x7fffffffffffffff`	$[-2^{-63}, 2^{63}-1]$

🤔 **Home exercise: make a similar table for unsigned int**

- Negative numbers ←→ MSB=1
- A sequence of 1's (e.g. `0xff`, `0xffffffff`) ←→ -1



Signed addition on hardware

```
  0 0 0 0 0 0 0 1   (1)10
+ 0 0 0 0 0 0 1 1   (3)10
-----------------
  0 0 0 0 0 1 0 0   (4)10
```

$(1)_{10} + (3)_{10} = (4)_{10}$

```
  0 0 0 0 0 0 0 1   (1)10
+ 1 0 0 0 0 0 0 1   (-127)10
-----------------
  1 0 0 0 0 0 1 0   (-126)10
```

$(1)_{10} + (-127)_{10} = (-126)_{10}$

This is what 2's complement is designed to accomplish!

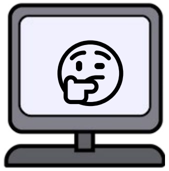
```
  0 1 0 0 0 0 0 1   (65)10
+ 0 1 0 0 0 0 0 0   (64)10
-----------------
  1 0 0 0 0 0 0 1   (-127)10
```

$(65)_{10} + (64)_{10} \rightarrow (-127)_{10}$

```
  1 0 0 0 0 0 0 1   (-127)10
+ 1 1 1 1 1 1 1 0   (-2)10
-----------------
  0 1 1 1 1 1 1 1   (127)10
```

$(-127)_{10} + (-2)_{10} \rightarrow (127)_{10}$

Overflow!



Signed subtraction on hardware

$$
\begin{array}{rl}
& 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ (1)_{10} \\
- & 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ (3)_{10} \\
\hline
& 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ (-2)_{10}
\end{array}
\qquad
\begin{array}{rl}
& 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ (1)_{10} \\
- & 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ (-1)_{10} \\
\hline
& 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ (2)_{10}
\end{array}
$$

This is what 2's complement is designed to accomplish!

$$
\begin{array}{rl}
& 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ (127)_{10} \\
- & 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ (-2)_{10} \\
\hline
& 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ (-127)_{10}
\end{array}
\qquad
\begin{array}{rl}
& 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ (-128)_{10} \\
- & 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ (1)_{10} \\
\hline
& 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ (127)_{10}
\end{array}
$$

Overflow!

Summary

- Integer representation
  - Unsigned (base-2)
  - Signed (2's complement)
- Hex notation
- Operations (e.g. add,subtract) on fixed-width integers can cause overflow