Floating point

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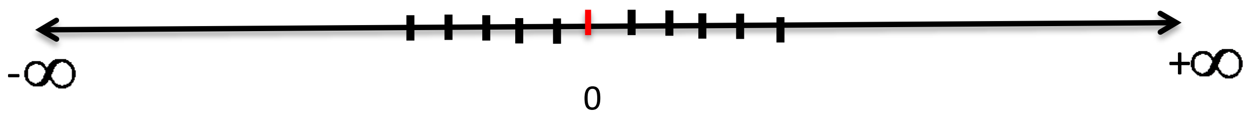
Floating Point (FP) lesson plan

- Normalized binary exponential notation
- Strawman 32-bit FP
- IEEE FP format
- Rounding
- FP operations and caveats

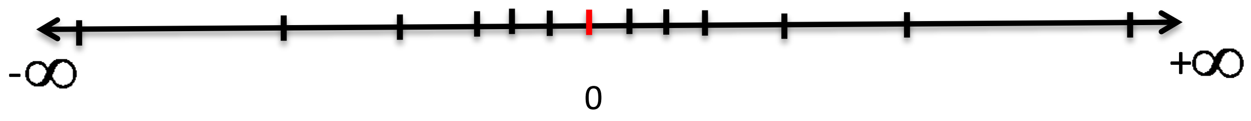
Floating point: key idea

- Limitation of fixed point:



  – Even spacing results in hard tradeoff between high precision and high magnitude

- How about un-even spacing between numbers?



Floating Point: decimal

Based on exponential notation (aka normalized scientific notation)

$r_{10} = \pm M * 10^{E}$, where $1 <= M < 10$

M: significant (mantissa), E: exponent

Floating Point: decimal

Example:

$365.25 = 3.6525 * 10^{2}$

$0.0123 = 1.23 * 10^{-2}$

Decimal point **floats** to the position immediately after the first nonzero digit.

Floating Point: binary

Binary exponential representation

$\underline{+}M * 2^E$, where $1 <= M < 2$

$M = (1.b_1b_2b_3\ldots b_n)_2$

M: significant, E: exponent

$(5.5)_{10} = (101.1)_2 = (1.011)_2 * 2^2$

Floating Point

Binary exponential representation

$\underline{+}M * 2^E$, where $1 <= M < 2$

$M = (1.b_1b_2b_3…b_n)_2$

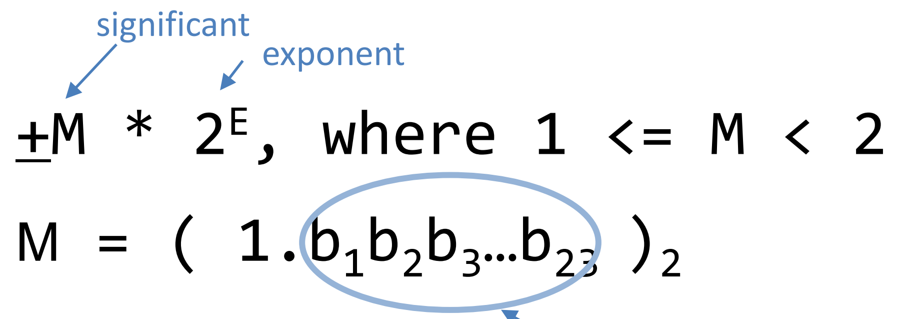
$\}$ Also called normalized representation

M: significant, E: exponent

$(5.5)_{10} = (101.1)_2 = (1.011)_2 * 2^2$

(Binary) normalized representation of $(10.25)_{10}$?

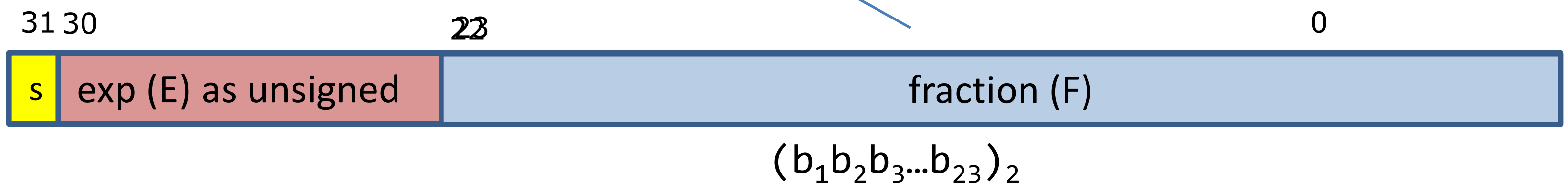
How to represent a normalized number in a fixed-length format?



$\underline{+}M * 2^E$, where $1 <= M < 2$

$M = (1.b_1b_2b_3…b_{23})_2$

Strawman 32-bit FP representation:



Normalized representation

significant

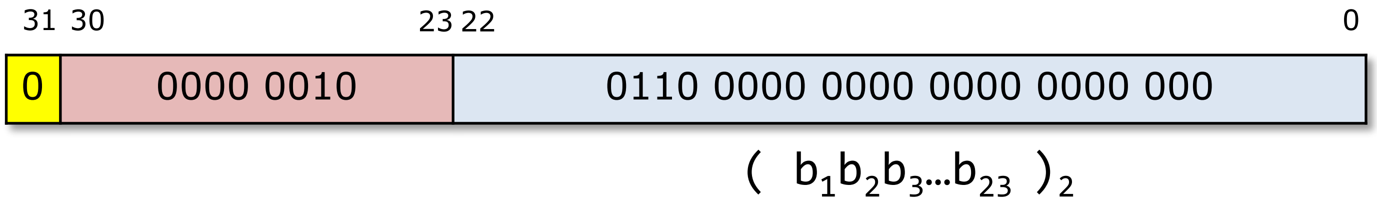
exponent

$$\underline{+}M * 2^E, \text{ where } 1 <= M < 2$$

$$M = (1.b_1b_2b_3 \ldots b_{23})_2$$

Example: $(5.5)_{10} = (101.1)_2 = (1.011)_2 * 2^2$

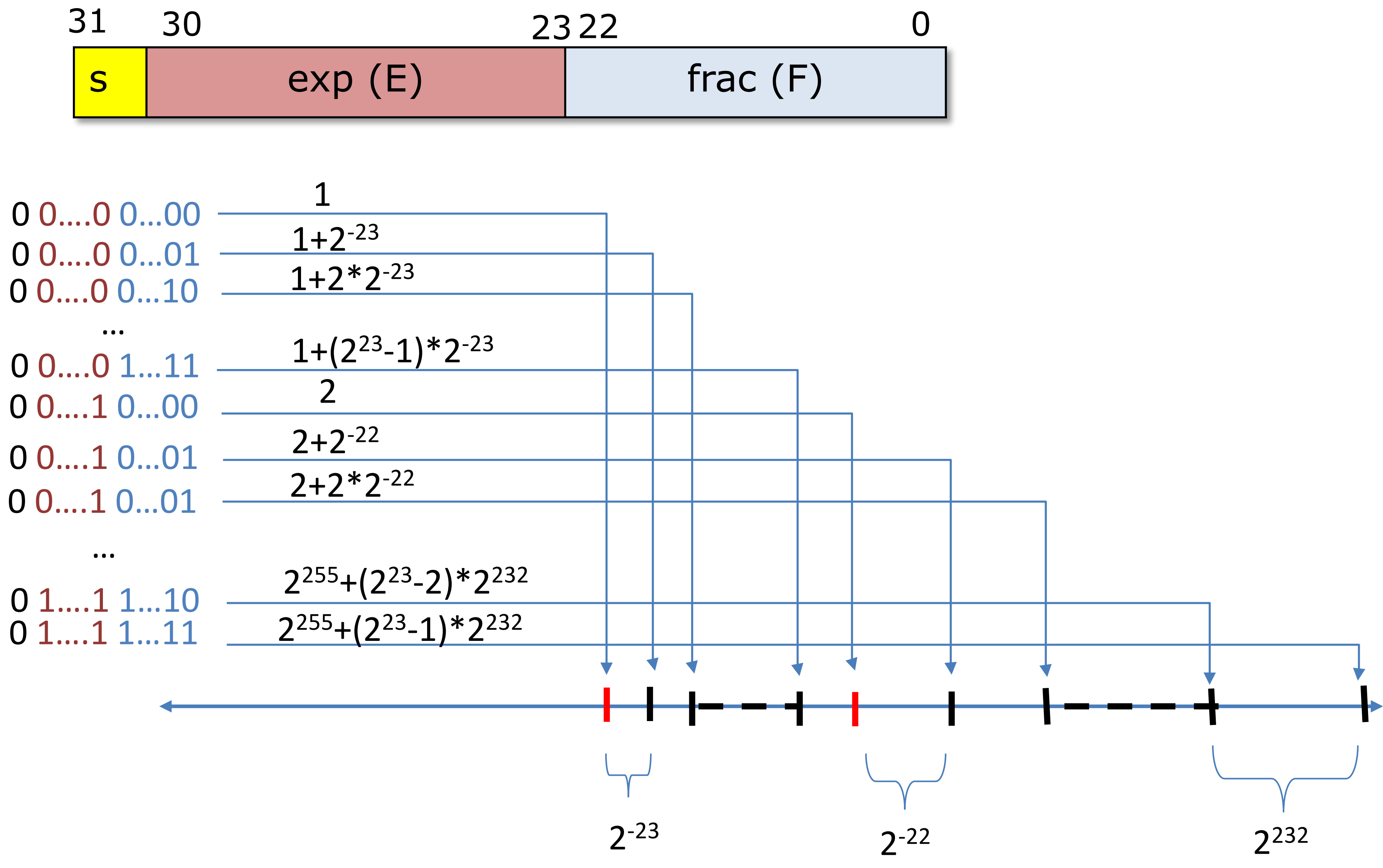
Strawman 32-bit FP representation:



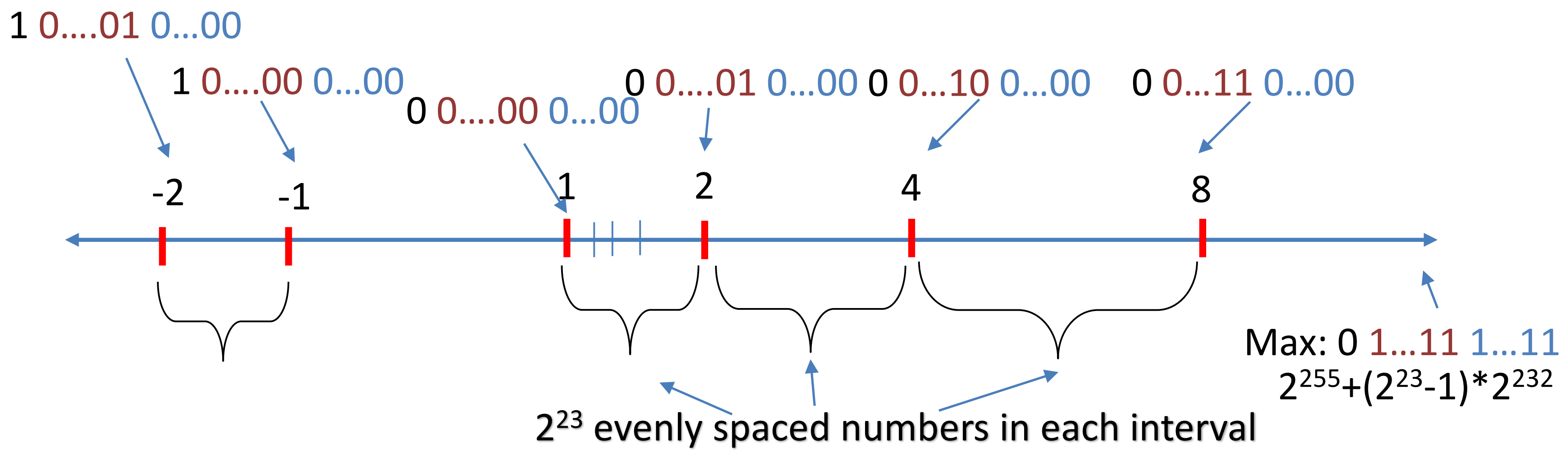
Exercise

The strawman 32-bit FP representation of $(65)_{10}$?

Strawman 32-bit FP



Strawman 32-bit FP



- The good 👍
  - Large range $[1, 2^{255}+(2^{23}-1)*2^{232}]$, $[-2^{255}-(2^{23}-1)*2^{232}, -1]$
  - Allows easy comparison: compare FPs by bit patterns
- The bad 👎
  - No 0!
  - No [-1, 1]
  - Max precision (2^{-23}) not high enough
  - No representation of special cases: ∞

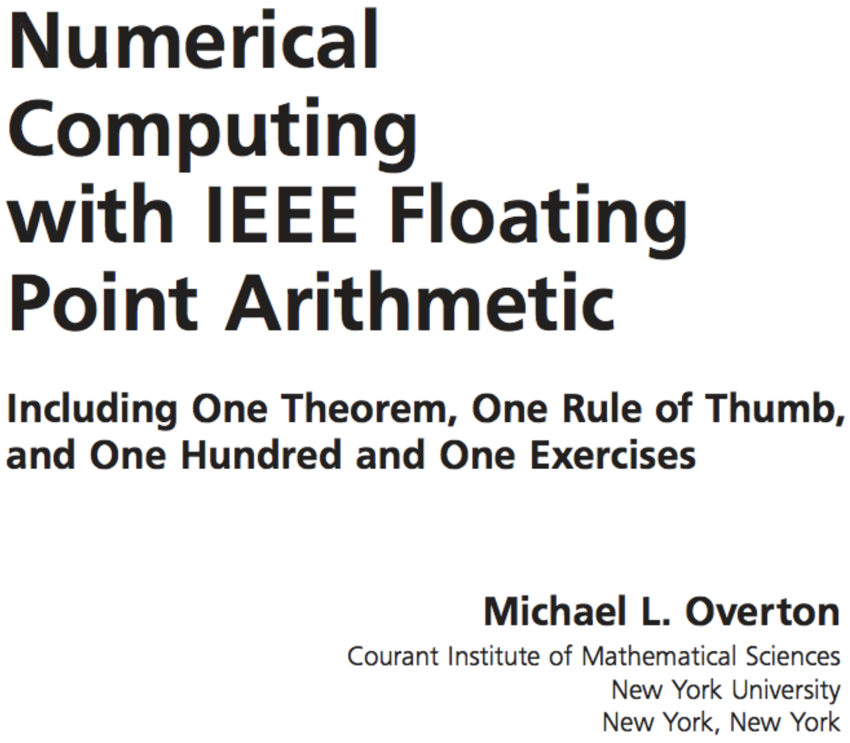
IEEE Floating Point Standard

- Lots of FP implementations in 60s/70s
  - Code not portable across processors
- IEEE formed a committee (IEEE.p754) to standardize FP format and specification.
  - IEEE FP standard published in 1985
  - Led by William Kahan

Prof. William Kahan
University of California at Berkeley
Turing Award (1989)

IEEE Floating Point Standard

- This class only covers basic FP materials
- A deep understanding of FP is crucial for numerical/scientific computing
  - Covered in undergrad/grad classes on numerical methods



Numerical Computing with IEEE Floating Point Arithmetic

Including One Theorem, One Rule of Thumb, and One Hundred and One Exercises

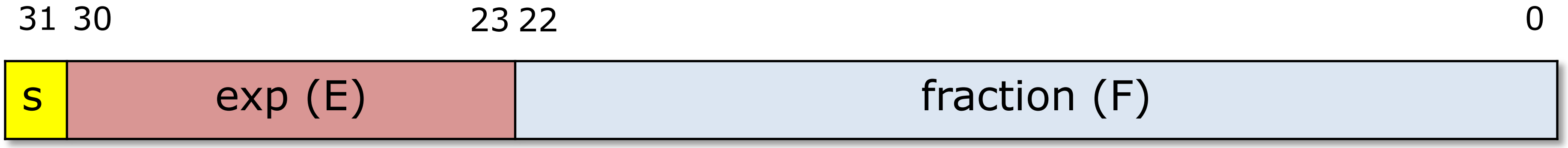
Michael L. Overton
Courant Institute of Mathematical Sciences
New York University
New York, New York

Goals of IEEE Standard

- Consistent representation of floating point numbers
  - Address the limitation of our FP strawman
- Correctly rounded floating point operations, using several rounding modes.
- Consistent treatment of exceptional situations such as division by zero

IEEE FP: Carve out subsets bit-patterns from normalized representation

$\underline{+}M * 2^E \quad M = (1.b_0b_1b_2b_3...b_n)_2$



$(b_0b_1b_2b_3...b_n)_2$

For normalization representation,
E can not be $(1111\ 1111)_2$ or $(0000\ 0000)_0$

E_{max} = ? 254, $(1111\ 1110)_2$

E_{min} = ? 1, $(0000\ 0001)_2$

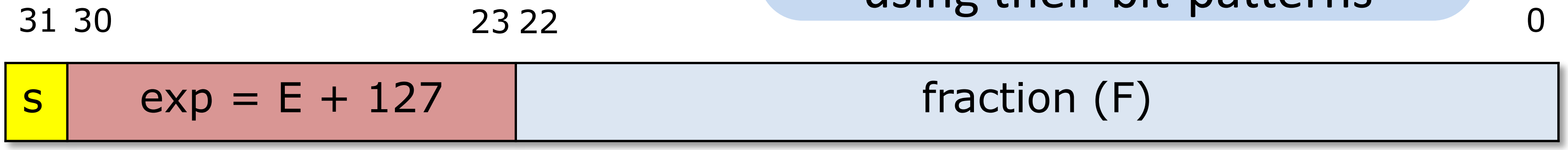
IEEE FP: Represent negative exponents using bias

$\underline{+}M * 2^E, \; M = (1.b_0b_1b_2b_3\ldots b_n)_2$

To represent FPs in (-1,1), we must allow negative exponent.

- How to represent negative E?
  - ~~2's complement~~
  - use bias

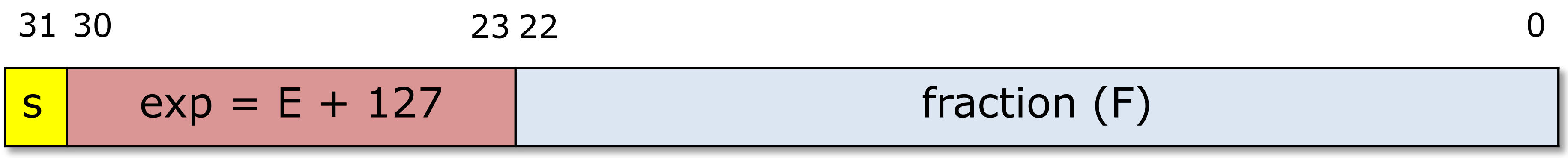
Why? Using bias instead of 2's complement allows simple comparison of FPs using their bit-patterns



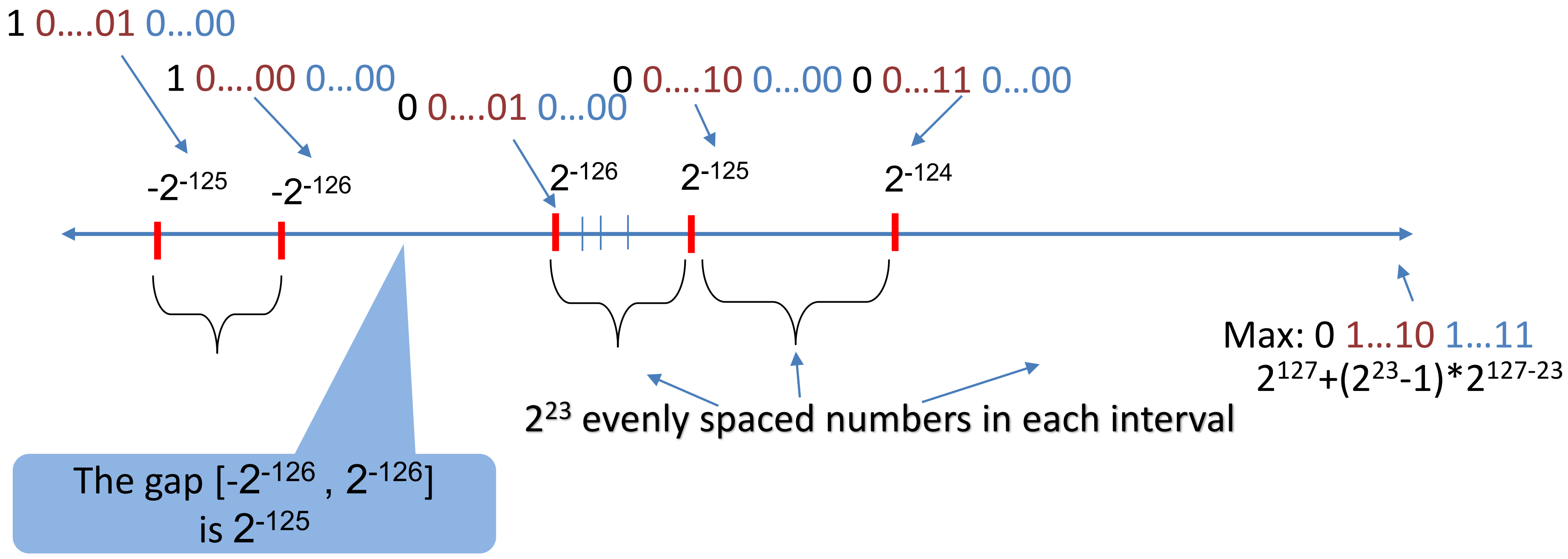
$(b_0b_1b_2b_3\ldots b_n)_2$

IEEE FP normalized representation

$\underline{+}M * 2^E, \; M = (1.b_0b_1b_2b_3\ldots b_n)_2$



$(b_0b_1b_2b_3\ldots b_n)_2$

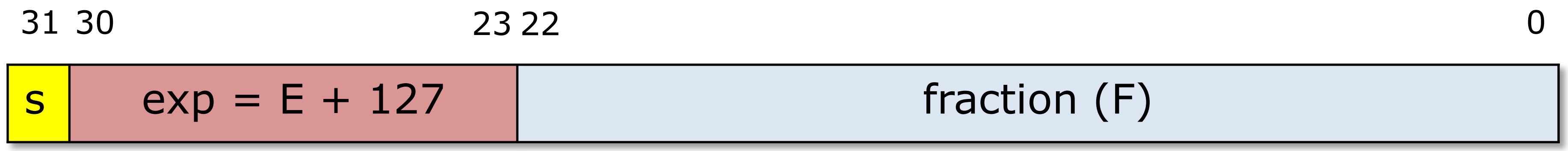


Represent values close and equal to 0

IEEE FP denormalized representation: represent values close and equal to 0

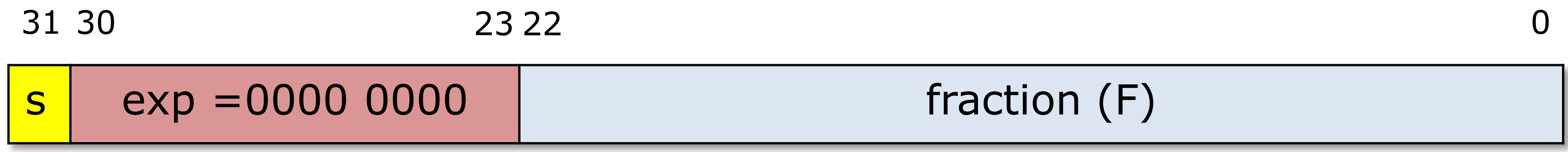
$\underline{+}M * 2^E$

Normalized Encoding:



$1 <= M < 2, M = (1.F)_2$

Denormalized Encoding:



$E = 1 – \text{Bias} = -126$

$0 <= M < 1, M = (0.F)_2$

Zeros

+0.0

0	0000 0000	0000 0000 0000 0000 0000 000

-0.0

1	0000 0000	0000 0000 0000 0000 0000 000

Denormalized FP example

Smaller than the smallest E (-126) of normalized encoding

What’s the IEEE FP format of $(1.0)_2 * 2^{-127}$?

$(1.0)_2 * 2^{-127} = (0.1)_2 * 2^{-126}$

0	0000 0000	1000 0000 0000 0000 0000 000

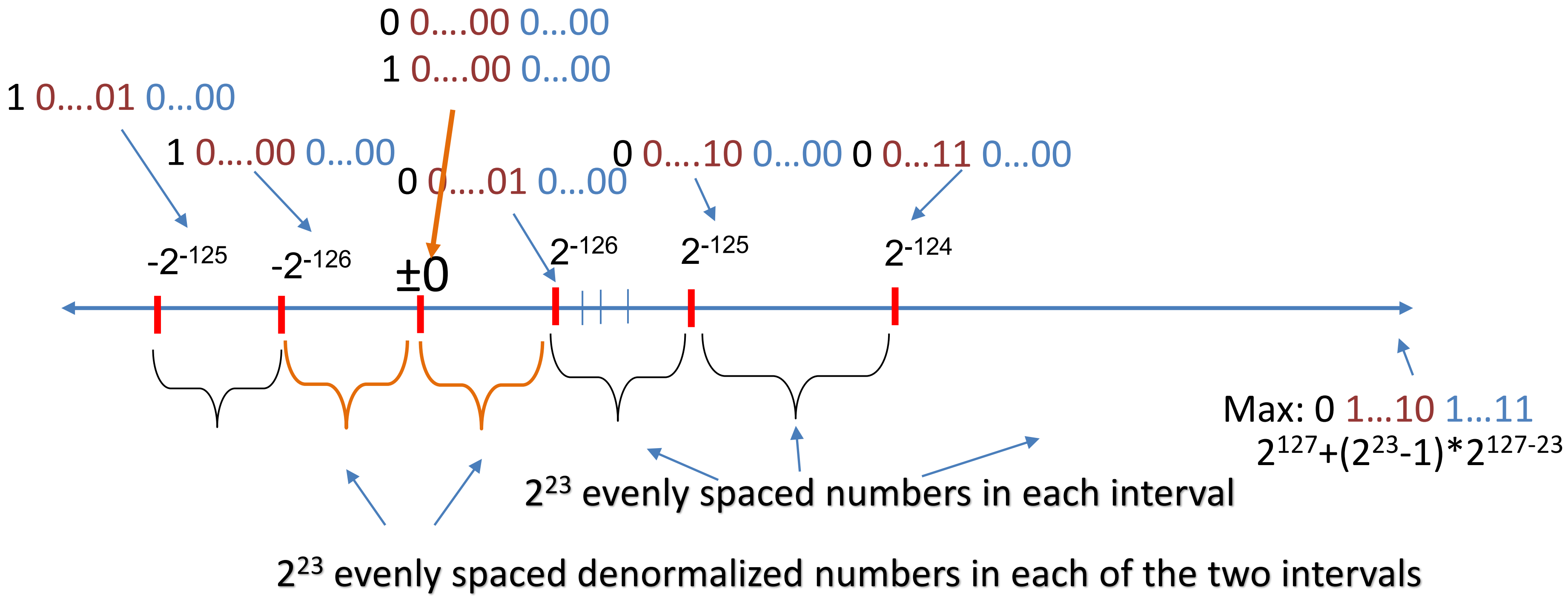
IEEE FP normalized + denormalized

s	exp = E + 127	fraction (F)

$1 <= M < 2$, $M = (1.F)_2$

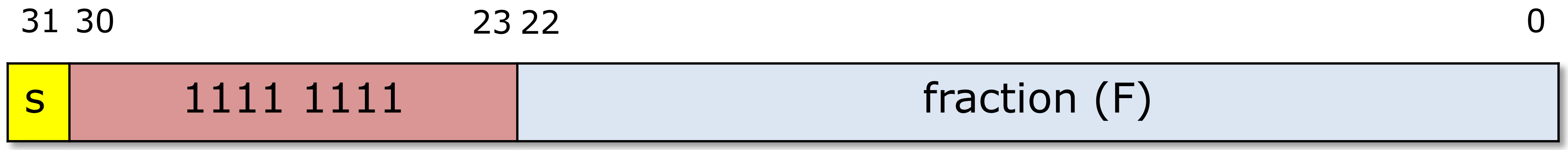
s	exp =0000 0000	fraction (F)

$0 <= M < 1$, $M = (0.F)_2$



IEEE FP: special values

Special Value's Encoding:



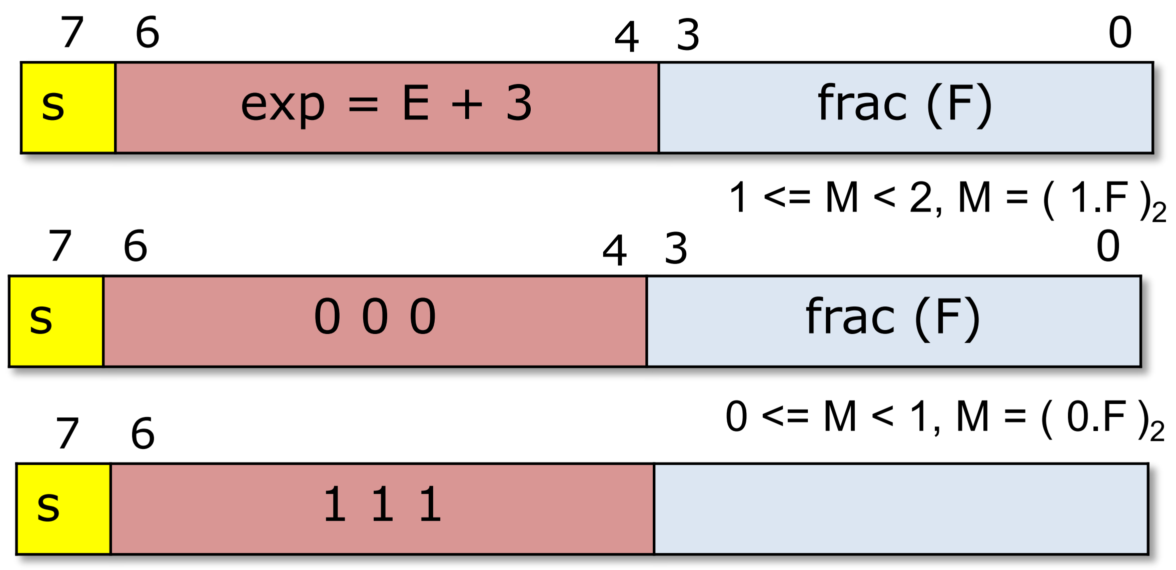
values	**sign**	**frac**
$+\infty$	0	all zeros
$-\infty$	1	all zeros
NaN	any	non-zero

Breakout time!

A toy 8-bit FP in the spirit of IEEE FP

$\pm M * 2^E$

- exponent: 3 bits
- fraction: 4 bits
- **bias: 3**



Normalized encoding
exp ≠ 000, 111

Denormalized encoding
exp = 000

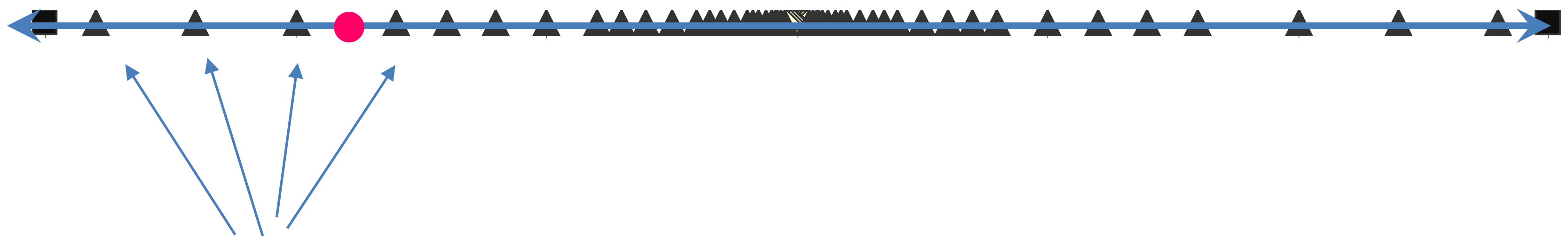
Special values encoding
exp = 111

- Smallest positive number?
- Range?
- How many distinct numbers?

Floating Point (FP) lesson plan

- Normalized binary exponential notation
- Strawman 32-bit FP
- IEEE FP format
- Rounding
- FP operations and caveats

FP: Rounding



Values that are represented precisely

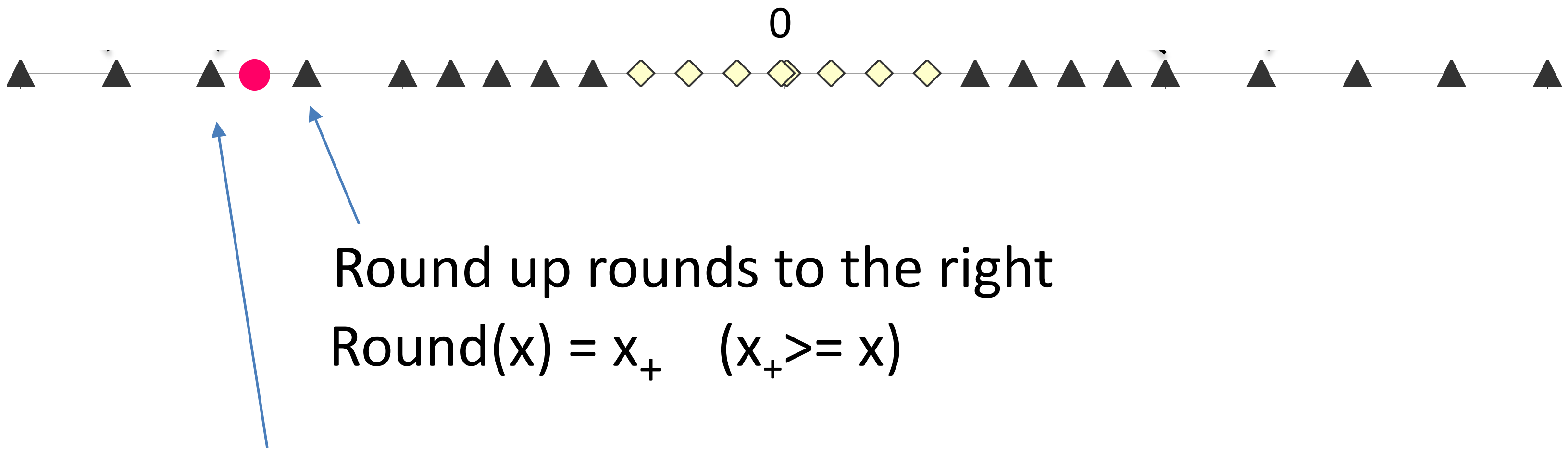
What if the result of computation is at ● ?

Rounding: Use the "closest" representable value *x'* for x.

4 modes:

- Round-down
- Round-up
- Round-toward-zero
- Round-to-nearest (Round-to-even in text book)

Round down; round up



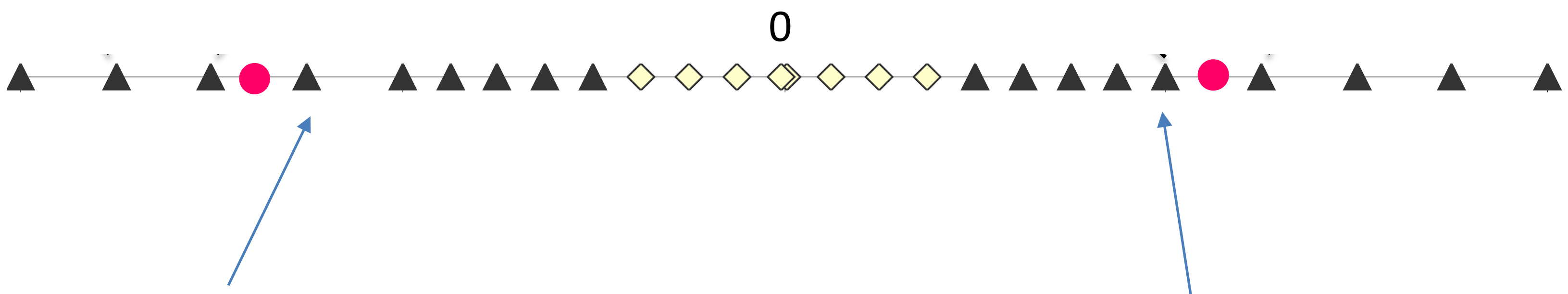
Round up rounds to the right

Round(x) = x_+ ($x_+ >= x$)

Round down rounds to the left

Round(x) = x_- ($x_- <= x$)

Round towards zero



Rounds to the right if $x < 0$

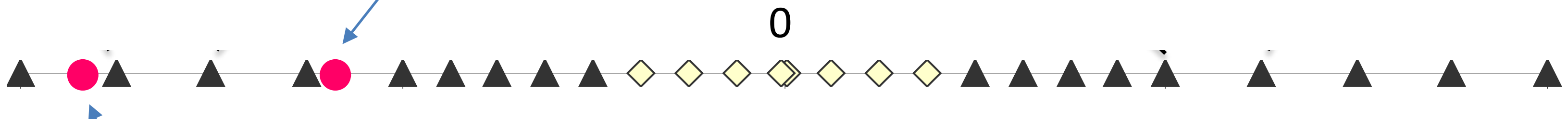
Round(x) = x_+ if $x < 0$

Rounds to the left if $x > 0$

Round(x) = x_- if $x < 0$

Round to nearest; ties to even

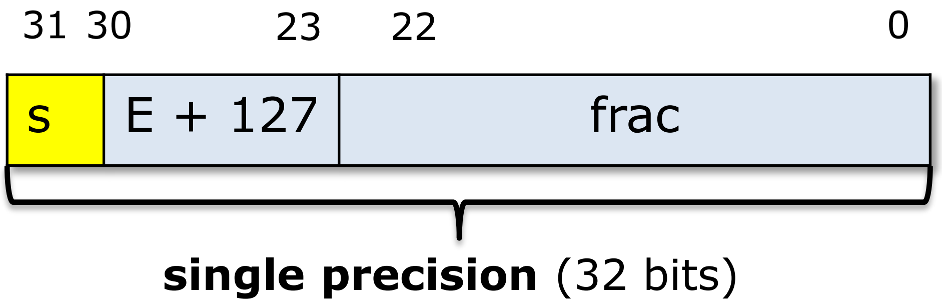
Round to the left if x_- is nearer to x than x_+



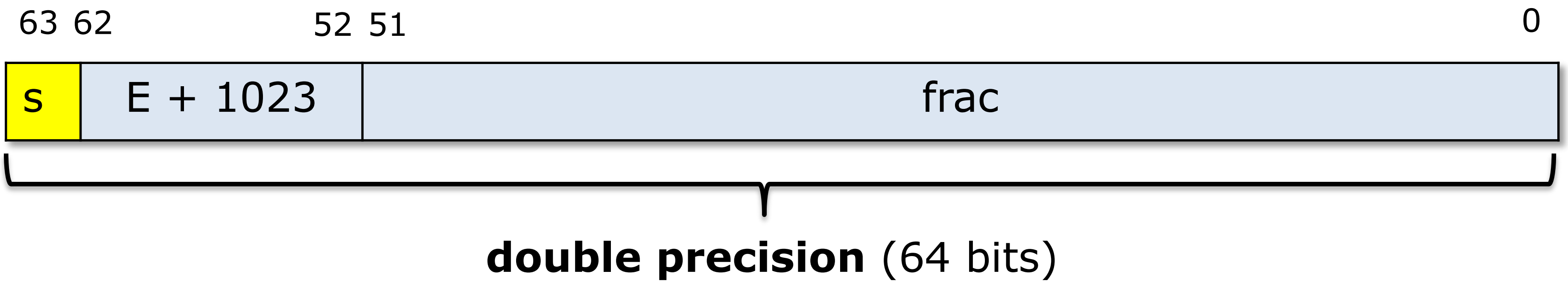
Round to the right if x_+ is nearer to x than x_-

In case of a tie, the one with its least significant bit equal to zero is chosen.

IEEE FP: single vs. double precision



float f = 0.1
double d = 0.1

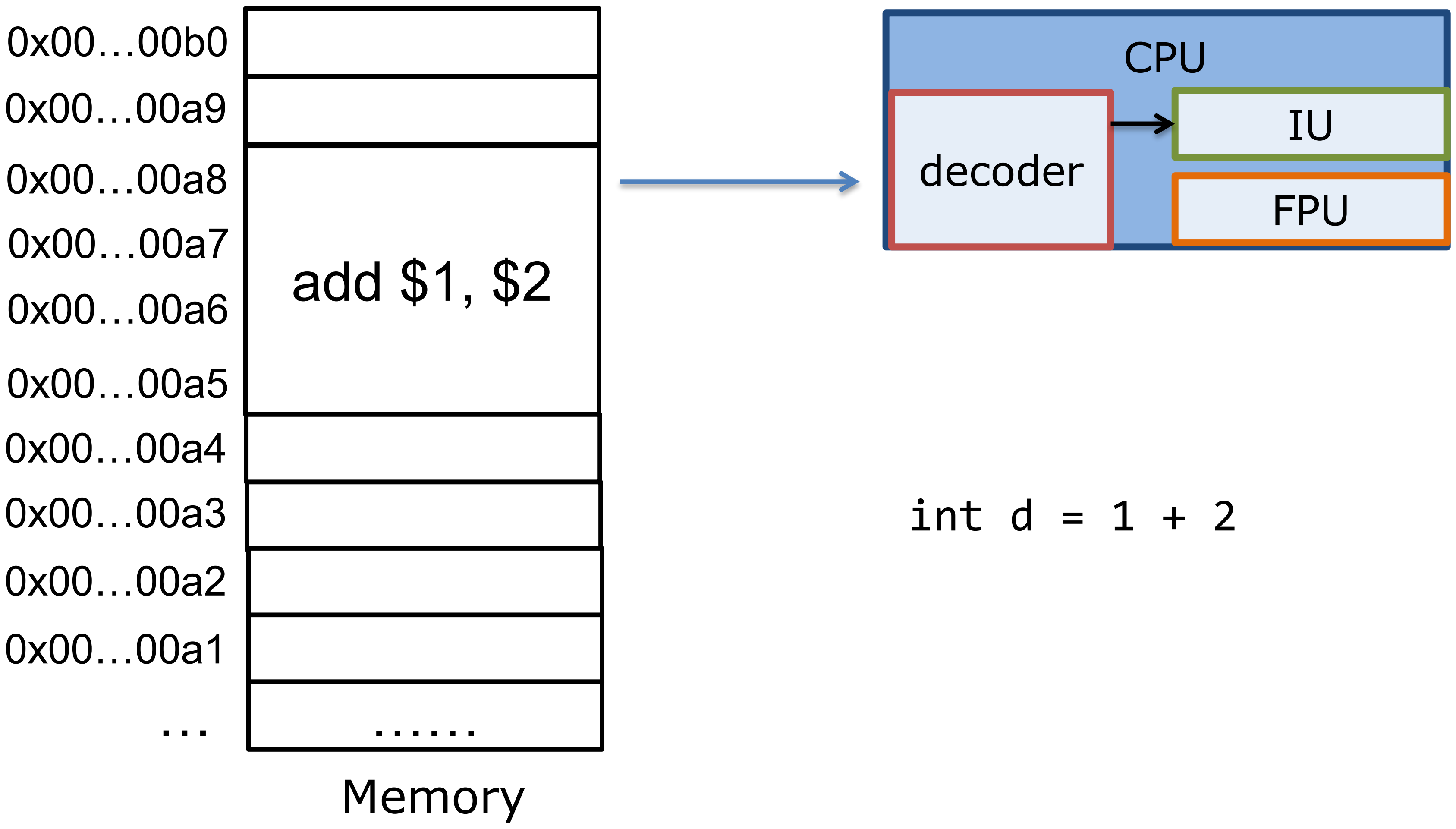


single/ double precision

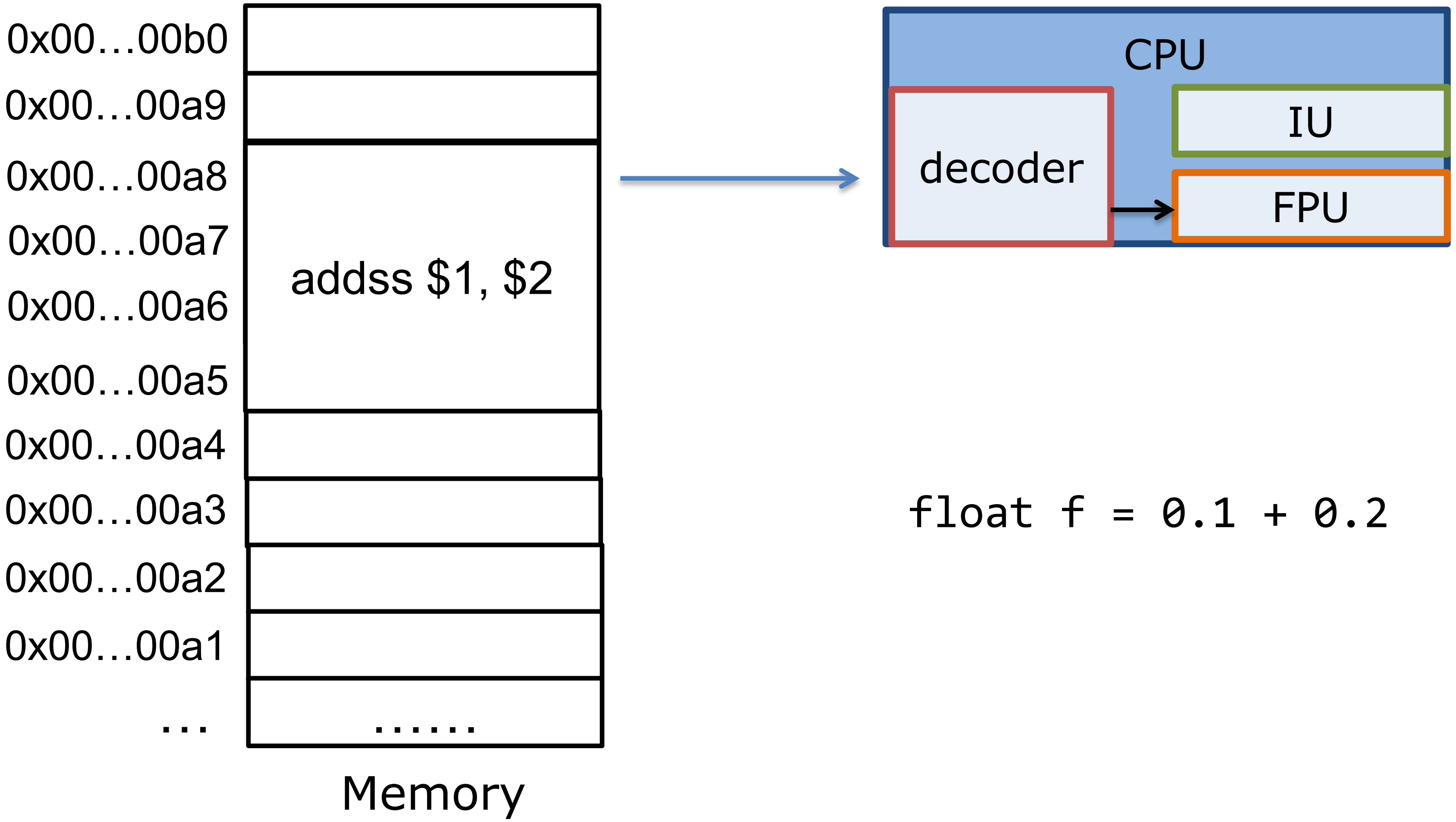
	E_{min}	E_{max}	N_{min}	N_{max}
Float	-126	127	$\approx 2^{-126}$	$\approx 2^{128}$
Double	-1022	1023	$\approx 2^{-1022}$	$\approx 2^{1024}$

How does CPU know if it is floating point or integers ?

By having specific instruction for floating points operation.


0x00…00b0
0x00…00a9
0x00…00a8
0x00…00a7
0x00…00a6
0x00…00a5
0x00…00a4
0x00…00a3
0x00…00a2
0x00…00a1
…
add $1, $2
……
Memory
CPU
decoder
IU
FPU

```
int d = 1 + 2
```


0x00…00b0
0x00…00a9
0x00…00a8
0x00…00a7
0x00…00a6
0x00…00a5
0x00…00a4
0x00…00a3
0x00…00a2
0x00…00a1
…
addss $1, $2
……
Memory
CPU
decoder
IU
FPU

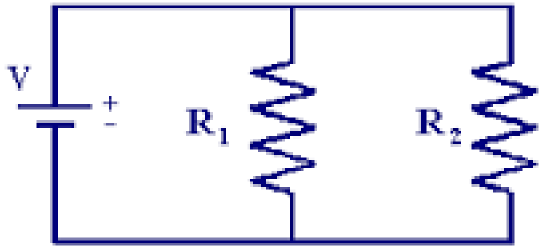
```
float f = 0.1 + 0.2
```

Floating point operations

- Addition, subtraction, multiplication, division etc.
- FP Caveats:
  - Invalid operation: 0/0, sqrt(-1), $\infty+\infty$
  - Divide by zero: $x/0 \rightarrow \infty$
  - Overflows: result too big to fit
  - Underflows: $0 <$ result $<$ smallest denormalized value
  - Inexact: round it!

Why divide by zero = ∞?

- Allow a calculation to continue and produce a valid result
- Example:



$$Parallel\ resistance = \frac{1}{\frac{1}{R1} + \frac{1}{R2}}$$

If R1 or R2 is 0, overall resistance should be 0

Floating point addition

- Commutative? x+y == y+x?
- Associative? (x+y)+z = x + (y+z)?
  - Rounding:

```
(3.14+1e10)-1e10 = 0
 3.14+(1e10-1e10) = 3.14
```

  - Overflow
- Every number has an additive inverse?
  - Yes except for ∞ and NaN

Floating point multiplication

- Commutative? x* y == y*x?
- Associative? (x*y)*z = x * (y*z)?
  - Overflow:

    `(1e20*1e20)*1e-20= inf, 1e20*(1e20*1e-20)= 1e20`
  - Rounding
- (x+y)*z = x*z + y*z?
  - `1e20*(1e20-1e20)=0.0, 1e20*1e20 – 1e20*1e20 =NaN`

Floating point in real world

- Storing time in computer games as a FP?
- Precision diminishes as time gets bigger

FP value	Time value	FP precision	Time precision
1	1 sec	1.19E-07	119 nanoseconds
100	~1.5 min	7.63E-06	7.63 microseconds
10 000	~3 hours	0.000977	.976 milliseconds
1000 000	~11 days	0.0625	62.5 milliseconds

Floating point in the real world

- Using floating point to measure distances

FP value	Length	FP precision	Precision size
1	1 meter	1.19E-07	Virus
100	100 meter	7.63E-06	red blood cell
10 000	10 km	0.000977	toenail thickness
1000 000	.16x earth radius	0.0625	credit card width

Table source: Random ASCII

Floating point trouble

- Comparing floats for equality is a bad idea!

```
float f = 0.1;
while (f != 1.0) {
        f += 0.1;
}
```

Floating point trouble

- Never count using floating points

```
count = 0;
for (float f = 0.0; f < 1.0; f += 0.1) {
    count++;
}
```

Floating point summary

- FP format is based on normalized exponential notation
- Floating points are tricky
  - Precision diminishes as magnitude grows
  - overflow, rounding error
- Many real world disasters due to FP trickiness
  - Patriot Missile failed to intercept due to rounding error (1991)
  - Ariane 5 explosion due to overflow in converting from double to int (1996)