# Floating point

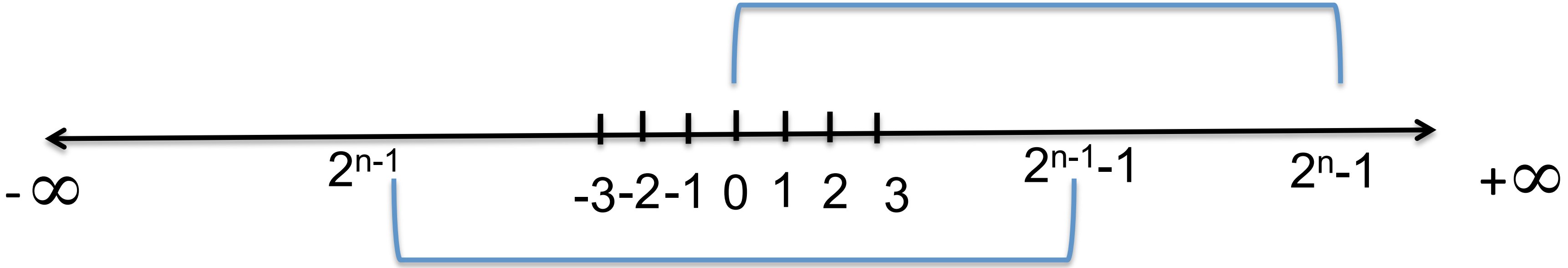
Jinyang Li

based on Tiger Wang's slides

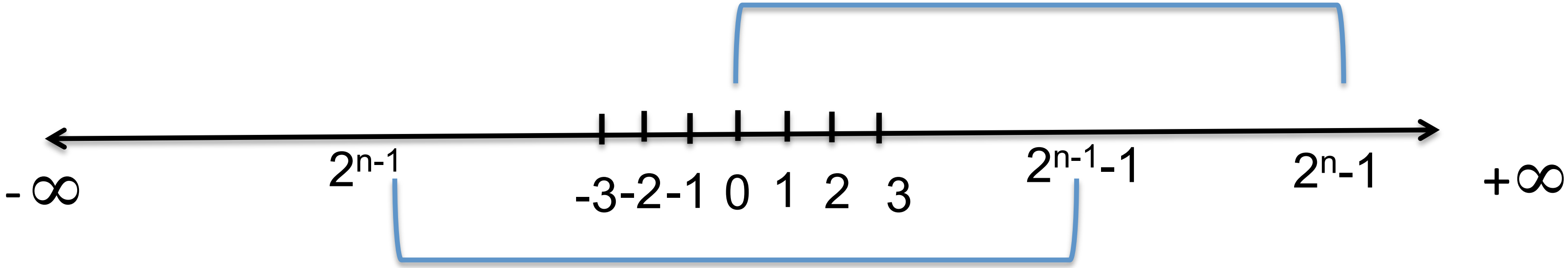
# Representing Real Numbers using bits

# Representing Real Numbers using bits

## What we have studied



# Representing Real Numbers using bits



Today: How to represent fractional numbers?

# Decimal Representation

| Real Numbers | Decimal Representation (Expansion) |
|---|---|
| 11 / 2 | $(5.5)_{10}$ |
| 1 / 3 | $(0.3333333...)_{10}$ |
| $\sqrt{2}$ | $(1.4128...)_{10}$ |

# Decimal Representation

| Real Numbers | Decimal Representation (Expansion) |
|---|---|
| $11 / 2$ | $(5.5)_{10}$ |
| $1 / 3$ | $(0.3333333...)_{10}$ |
| $\sqrt{2}$ | $(1.4128...)_{10}$ |

$$(5.5)_{10} = 5 * 10^{0} + 5 * 10^{-1}$$

$$(0.333333...)_{10} = 3 * 10^{-1} + 3 * 10^{-2} + 3 * 10^{-3} + ...$$

$$(1.4128...)_{10} = 1 * 10^{0} + 4 * 10^{-1} + 1 * 10^{-2} + 2 * 10^{-3} + ...$$

# Decimal Representation

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$$(1.4128...)_{10} = 1 * 10^{0} + 4 * 10^{-1} + 1 * 10^{-2} + 2 * 10^{-3} + ...$$

$$r_{10} = (d_m d_{m-1} \ldots d_1 d_0 \,.\, d_{-1} d_{-2} \ldots d_{-n})_{10}$$

$$= \sum_{i=-n}^{m} 10^{i} \times d_i$$

# Binary Representation

$(5.5)_{10} = 4 + 1 + 1 / 2$

$= 1 * 2^{2} + 1 * 2^{0} + 1 * 2^{-1}$

# Binary Representation

$$
\begin{aligned}
(5.5)_{10} &= 4 + 1 + 1 / 2 \\
&= 1 * 2^{2} + 0 * 2^{1} + 1 * 2^{0} + 1 * 2^{-1}
\end{aligned}
$$

# Binary Representation

$(5.5)_{10} = 4 + 1 + 1 / 2$

$= 1 * 2^{2} + 0 * 2^{1} + 1 * 2^{0} + 1 * 2^{-1}$

$= (101.1)_{2}$

# Binary Representation

$(5.5)_{10} = 4 + 1 + 1 / 2$

$= 1 * 2^2 + 0 * 2^1 + 1 * 2^0 + 1 * 2^{-1}$

$= (101.1)_2$

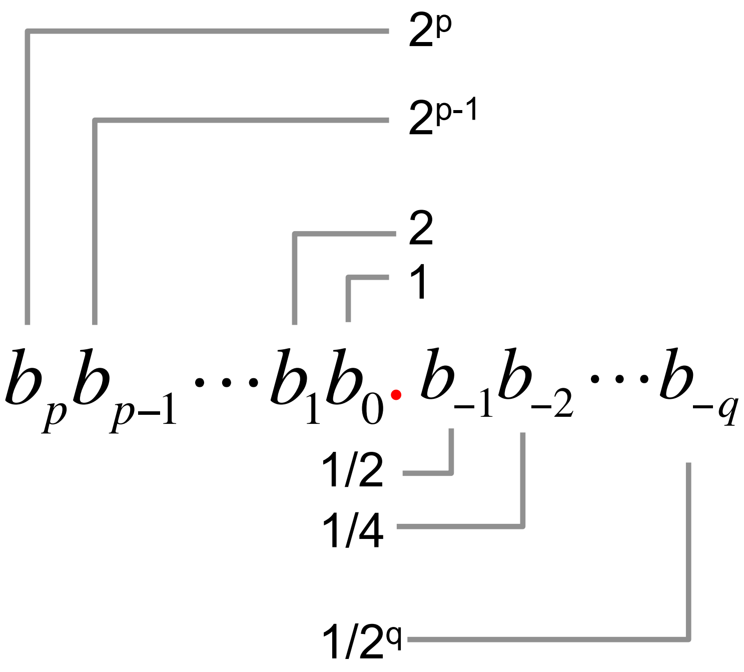
$(0.333333...)_{10} = 1 / 4 + 1 / 16 + 1 / 64 + ...$

$= (0.01010101...)_2$

# Binary Representation

$$r_{10} = (d_m d_{m-1} d_1 d_0 . d_{-1} d_{-2} \ldots d_{-n})_{10}$$

$$= (b_p b_{p-1} b_1 b_0 . b_{-1} b_{-2} \ldots b_{-q})_2$$



$$b_p b_{p-1} \cdots b_1 b_0 . b_{-1} b_{-2} \cdots b_{-q} = \sum_{i=-q}^{p} 2^i \times b_i$$

# Exercise

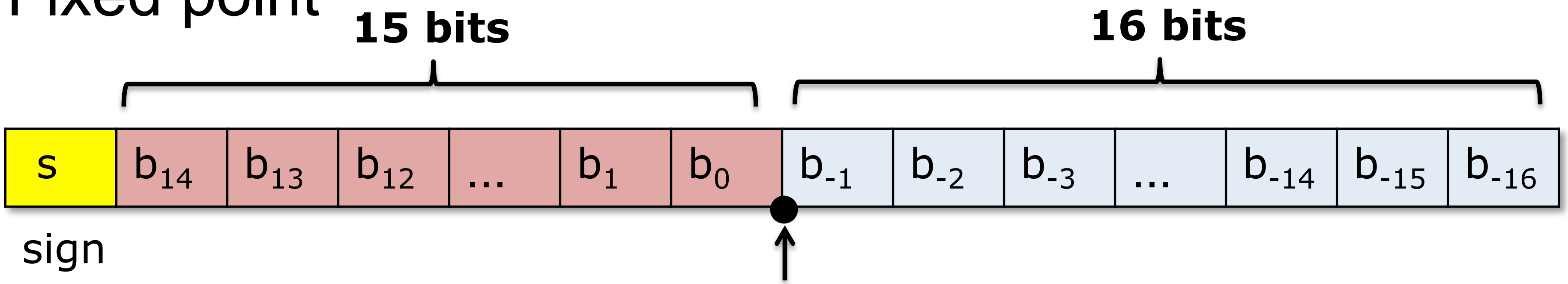
| Binary Expansion | Formula | Decimal |
|---|---|---|
| $10.011_2$ | | |
| | $2^{-3} + 2^{-4} + 2^{-6}$ | |
| | $2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}$ | |

# Exercise

| Binary Expansion | Formula | Decimal |
|---|---|---|
| $10.011_2$ | $2^1 + 2^{-2} + 2^{-3}$ | $2.375_{10}$ |
| $0.001101_2$ | $2^{-3} + 2^{-4} + 2^{-6}$ | $0.203125_{10}$ |
| $0.1111_2$ | $2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}$ | $0.9375_{10}$ |

# Intuitive Idea

Fixed point



15 bits

16 bits

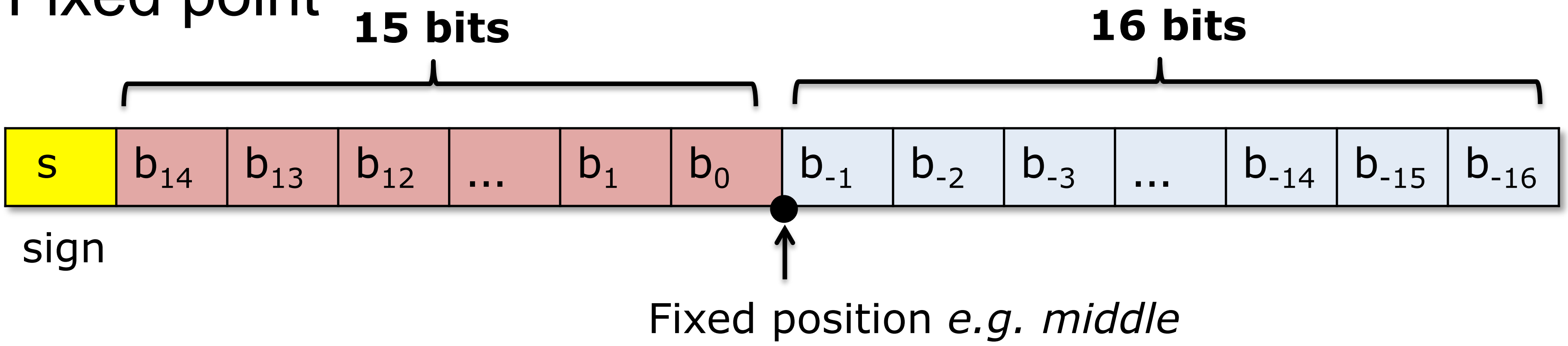
| s | $b_{14}$ | $b_{13}$ | $b_{12}$ | … | $b_1$ | $b_0$ | $b_{-1}$ | $b_{-2}$ | $b_{-3}$ | … | $b_{-14}$ | $b_{-15}$ | $b_{-16}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

sign

Fixed position *e.g. middle*

# Intuitive Idea

Fixed point



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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

sign

Fixed position *e.g. middle*

$( 10.011 )_2$

| 0 | 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 | 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 |
|---|---|---|

# Problems of Fixed Point

Limited range and precision: e.g., 32 bits

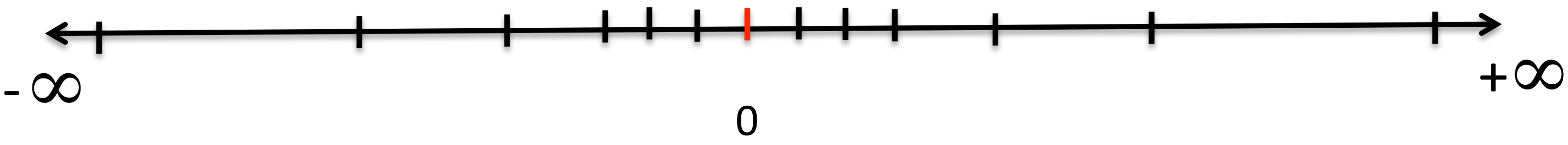
- Largest number: $2^{15}$ $(011...111)_2$
- Highest precision: $2^{-16}$

→Rarely used (No built-in hardware support)

# The idea

- Limitation of fixed point notation:
  - Represents evenly spaced fractional numbers
    - ➔ hard tradeoff between high precision and high magnitude

- How about un-even spacing between numbers?



# Floating Point: decimal

Based on exponential notation (aka normalized scientific notation)

$r_{10} = \pm M * 10^{E}$, where $1 <= M < 10$

M: significant (mantissa), E: exponent

# Floating Point: decimal

Example:

$365.25 = 3.6525 * 10^{2}$

$0.0123 = 1.23 * 10^{-2}$

Decimal point **floats** to the position immediately after the first nonzero digit.

# Floating Point: binary

Binary exponential representation

$r_{10} = \underline{+}M * 2^{E}$, where $1 <= M < 2$

$M = ( 1.b_1b_2b_3…b_n )_2$

M: significant, E: exponent

$(5.5)_{10} = (101.1)_2 = (1.011)_2 * 2^2$

# Floating Point

Binary exponential representation

$$\left.\begin{array}{l} r_{10} = \pm M * 2^{E}, \text{ where } 1 <= M < 2 \\ M = ( 1.b_1b_2b_3 \ldots b_n )_2 \end{array}\right\} \text{Normalized representation of r}$$

M: significant, E: exponent

$(5.5)_{10} = (101.1)_2 = (1.011)_2 * 2^2$

Normalization: give a number r, obtain its normalized representation

# Exercises

The normalized representation of $(10.25)_{10}$ is ?

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The normalized representation of $(10.25)_{10}$ is ?

$$(10.25)_{10} = (1010.01)_2 = (1.01001)_2 * 2^3$$

# Floating Point

Binary exponential representation

$$\left.\begin{array}{l} r_{10} = \underline{+}M * 2^{E}, \text{ where } 1 <= M < 2 \\ M = (\ 1.b_1b_2b_3\ldots b_n\ )_2 \end{array}\right\} \text{Normalized representation of r}$$

M: significant, E: exponent

$(5.5)_{10} = (101.1)_2 = (1.011)_2 * 2^2$

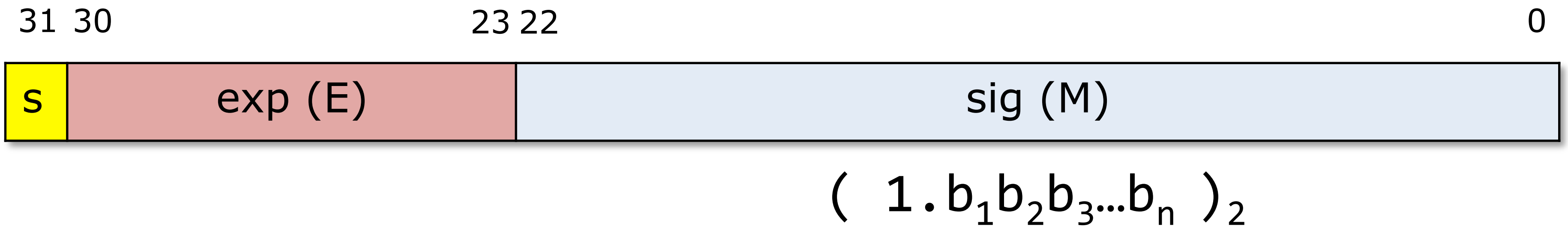
**How to represent a normalized number?**

# Normalized representation

$r_{10} = \underline{+}M * 2^{E}$, where $1 <= M < 2$

$M = ( 1.b_1b_2b_3 \ldots b_n )_2$

M: significant, E: exponent

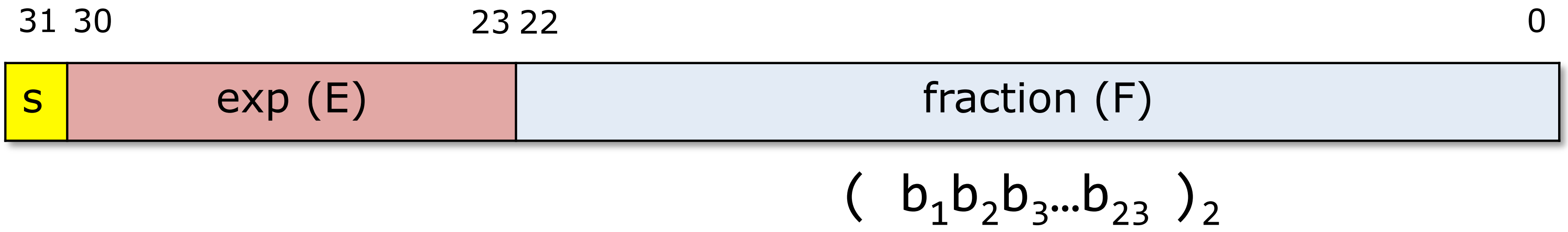


# Normalized representation in computer

$r_{10} = \underline{+}M * 2^{E}$, where $1 <= M < 2$

$M = ( 1.b_1b_2b_3 \ldots b_{23} )_2$

M: significant, E: exponent

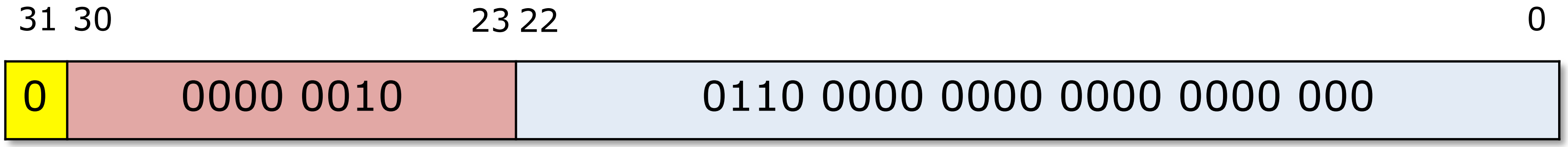


# Normalized representation

$r_{10} = \pm M * 2^E$, where $1 <= M < 2$

$M = ( 1.b_1b_2b_3 \ldots b_{23} )_2$

M: significant, E: exponent



$( b_1b_2b_3 \ldots b_{23} )_2$

$(5.5)_{10} = (101.1)_2 = (1.011)_2 * 2^2$

# Exercise

Given the normalized representation of $(71)_{10}$ and $(10.25)_{10}$

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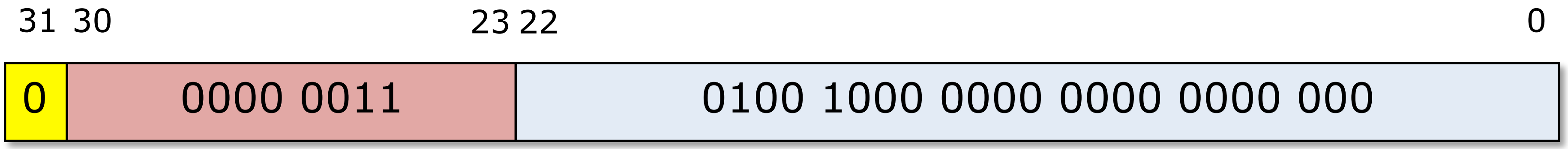
Given the normalized representation of $(71)_{10}$ and $(10.25)_{10}$

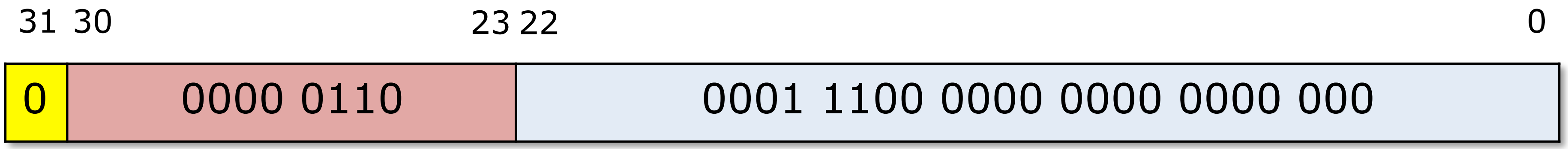
$(10.25)_{10} = (1010.01)_2 = (1.01001)_2 * 2^3$



31 30 23 22 0

0 | 0000 0011 | 0100 1000 0000 0000 0000 000

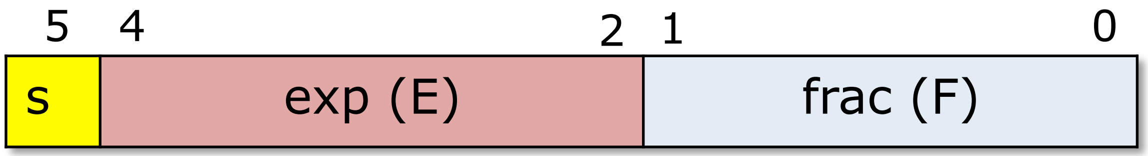
$(71)_{10} = (1000111)_2 = (1.000111)_2 * 2^6$



31 30 23 22 0

0 | 0000 0110 | 0001 1100 0000 0000 0000 000

# Toy Number System

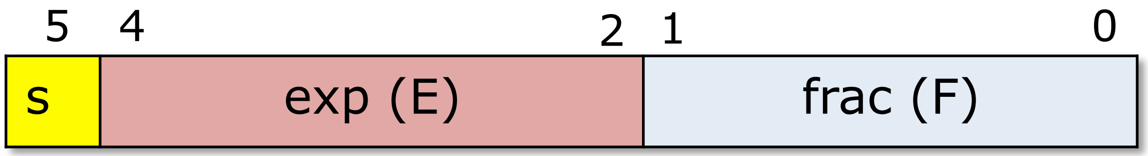


6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits

**Largest positive number ?**

# Toy Number System



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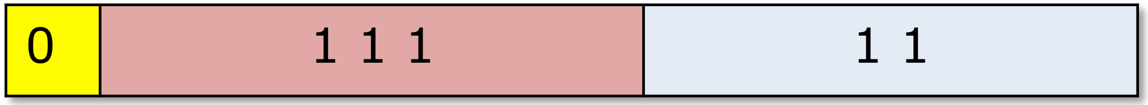
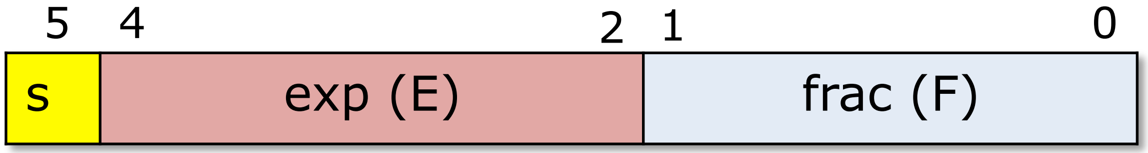


$( 1.11 )_2 * 2^7 = 224$

# Toy Number System



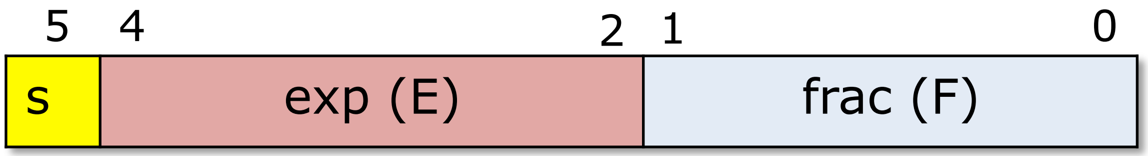
6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits

**Largest positive number: 224**

**Smallest positive number ?**

# Toy Number System

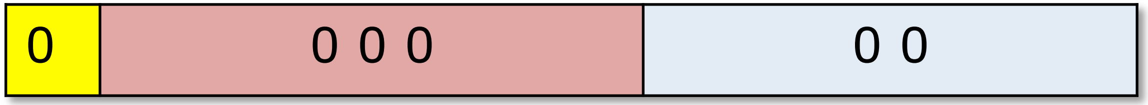


6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits

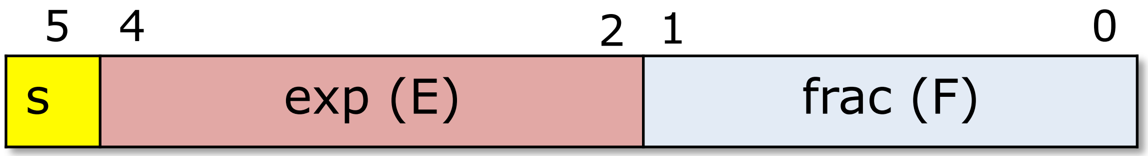
**Largest positive number: 224**

**Smallest positive number: 1**



$( 1.00 )_2 * 2^0 = 1$

# Toy Number System



6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits

**Positive number: 1 to 224**

**Negative number: -224 to -1**

No more bit patterns left to represent numbers
(-1, 1)

# Questions

How to represent

1. numbers close or equal to 0?
2. special cases:
   - the result of dividing by 0, e.g. 1/0 ?
   - $\infty * 0$

**Lots of different implementations around 1950s!**

# IEEE Floating Point Standard

IEEE p754

A standard for binary floating point representation

Prof. William Kahan
University of California at Berkeley
Turing Award (1989)

National Semiconductor

The Sight & Sound of Information

intel®

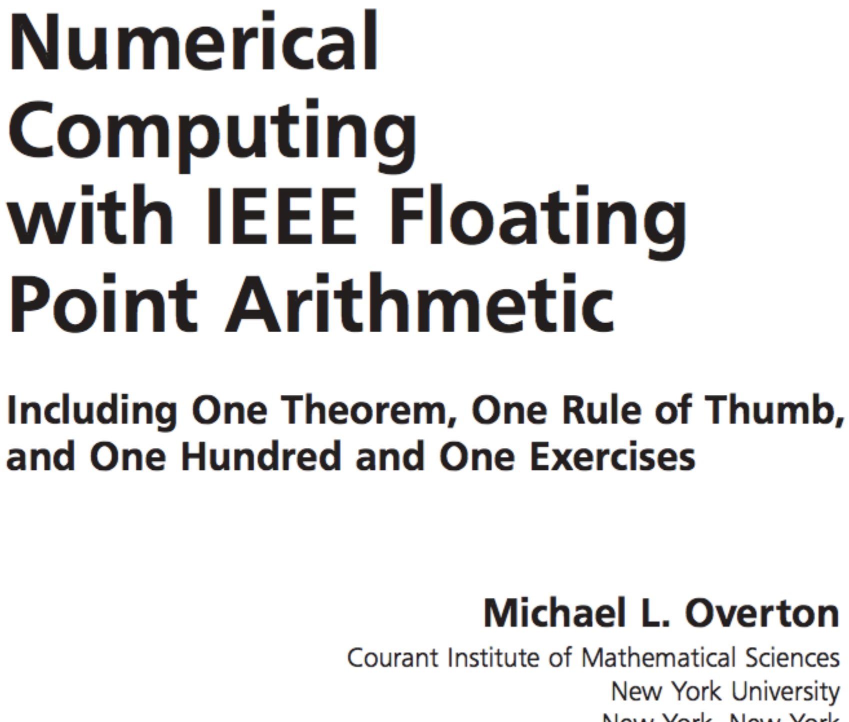
hp

digital

EQUIPMENT CORPORATION

MAYNARD, MASSACHUSETTS

# The Only Book Focuses On IEEE Floating Point Standard

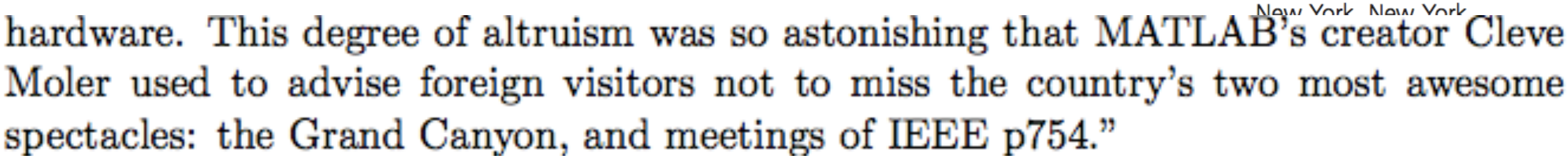


Numerical Computing with IEEE Floating Point Arithmetic

Including One Theorem, One Rule of Thumb, and One Hundred and One Exercises

Michael L. Overton

Courant Institute of Mathematical Sciences
New York University
New York, New York



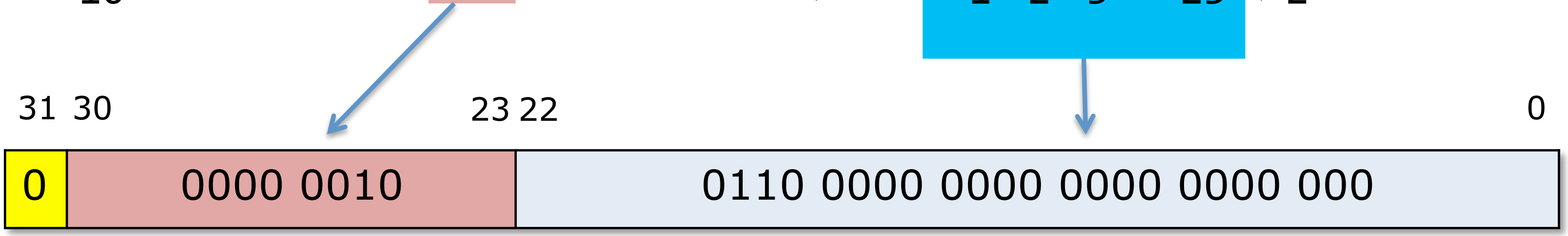
hardware. This degree of altruism was so astonishing that MATLAB's creator Cleve Moler used to advise foreign visitors not to miss the country's two most awesome spectacles: the Grand Canyon, and meetings of IEEE p754."

https://cs.nyu.edu/overton/NumericalComputing/protected/NumericalComputingSIAM.pdf
With you nyu netid/password. You can also search the pdf with google.

# What we have learnt so far

- normalized representation of floating point

$r_{10} = \underline{+}M * 2^{E}$     $M = ( 1.b_1b_2b_3\ldots b_{23} )_2$



| 31 | 30 23 | 22 0 |
|---|---|---|
| 0 | 0000 0010 | 0110 0000 0000 0000 0000 000 |

$(5.5)_{10} = (101.1)_2 = (1.011)_2 * 2^2$

- how to represent numbers in range (-1,1)
- how to represent special cases? e.g. $\infty$

# Goals of IEEE Standard

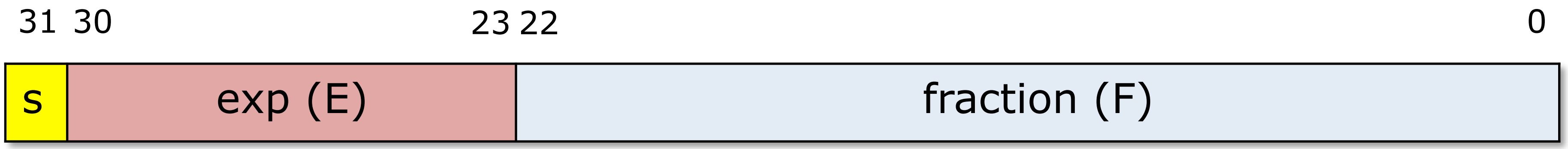
- Consistent representation of floating point numbers

- Correctly rounded floating point operations, using several rounding modes.

- Consistent treatment of exceptional situations such as division by zero

# Restrictions on Normalized Representation

$r_{10} = \underline{+}M * 2^{E}$  $M = (\ 1.b_0b_1b_2b_3...b_n\ )_2$



$(\ b_0b_1b_2b_3...b_n\ )_2$

E can not be $(1111\ 1111)_2$ or $(0000\ 0000)_0$

$E_{max}$ = ? 254, $(1111\ 1110)_2$

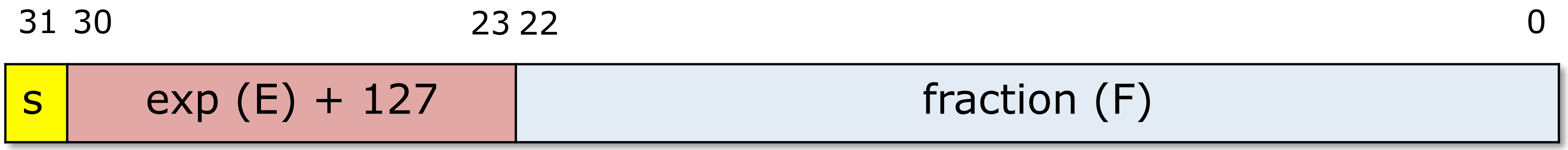
$E_{min}$ = ? 1, $(0000\ 0001)_2$

# Exponential Bias

$$r_{10} = \pm M * 2^{E},\ M = (\ 1.b_0b_1b_2b_3\ldots b_n\ )_2$$

To represent (-1,1), we must allow negative exponent.

- How to represent negative E?
  - ~~2's complement~~
  - use bias

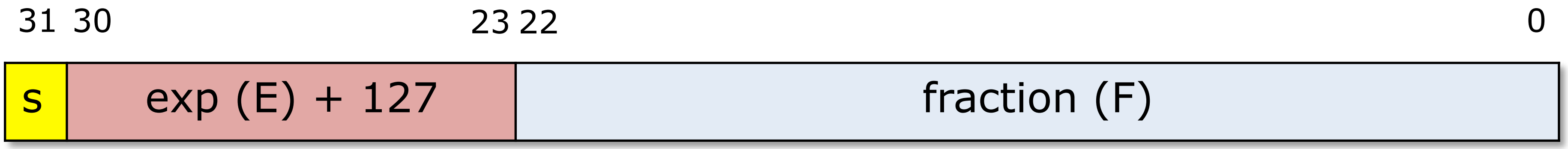


Bias: 127

$(\ b_0b_1b_2b_3\ldots b_n\ )_2$

# IEEE normalized representation

$$r_{10} = \underline{+}M * 2^{E},\quad M = (\ 1.b_0b_1b_2b_3\ldots b_n\ )_2$$



31 30 23 22 0

s

exp (E) + 127

fraction (F)

$(\ b_0b_1b_2b_3\ldots b_n\ )_2$

Bias: 127

$E_{max} = 254 - 127 = 127$

$E_{min} = 1 - 127 = -126$

Smallest positive number $2^{-126}$

Negative number with smallest absolute value: $-2^{-126}$

# Questions

Q1. Why using **bias**?

Q2. Why is **bias** 127?

# Questions

Q1. Why using **bias** instead of 2’s complement?

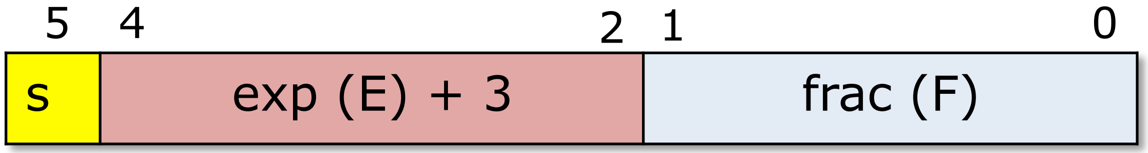
Answer: easier circuitry for comparison.

# Questions

Q2. Why is bias 127?

A2. Balance positive numbers (magnitude) and negative numbers (precision)

# Example Toy Number System

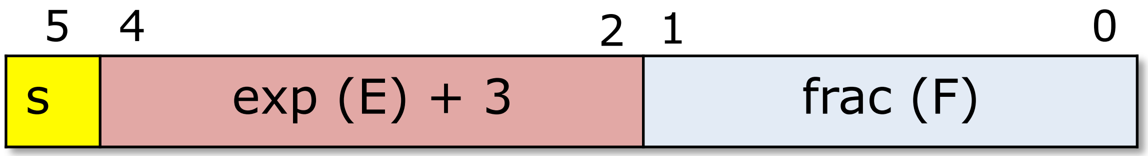


6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits
- **bias: 3**

Smallest positive number ?

# Toy Number System



5 4 2 1 0

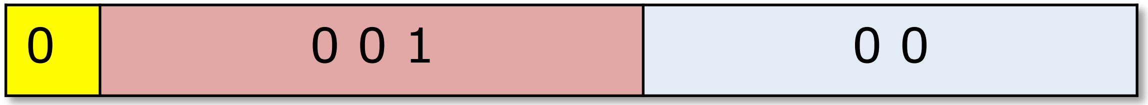
s | exp (E) + 3 | frac (F)

6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits
- **bias: 3**

Smallest positive number: 0.25

Smallest number >0.25?

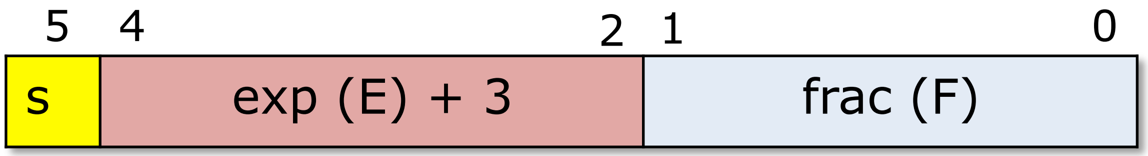


0 | 0 0 1 | 0 0

$( 1.00 )_2 * 2^{-2} = 0.25$

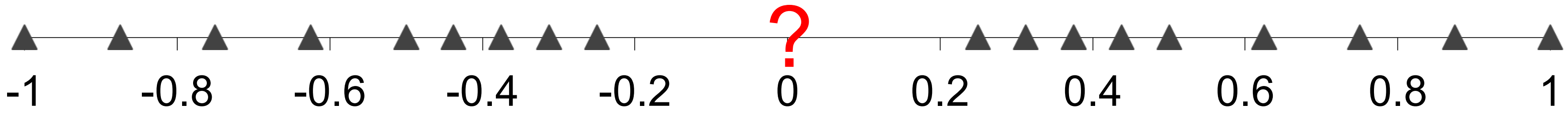
$( 1.01 )_2 * 2^{-2} = 0.25+0.0625$

# Toy Number System



6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits
- **bias: 3**

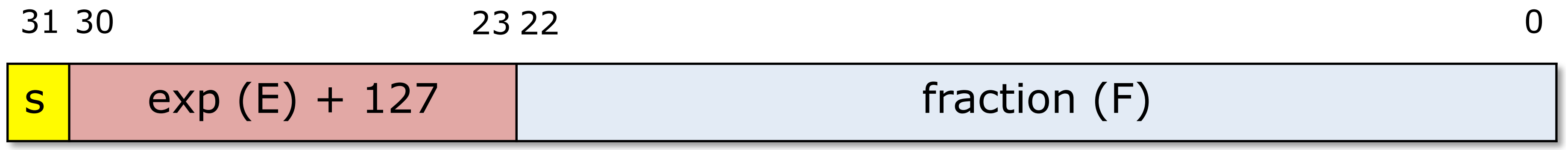


**represent values which are close and equal to 0**

# IEEE denormalized representation

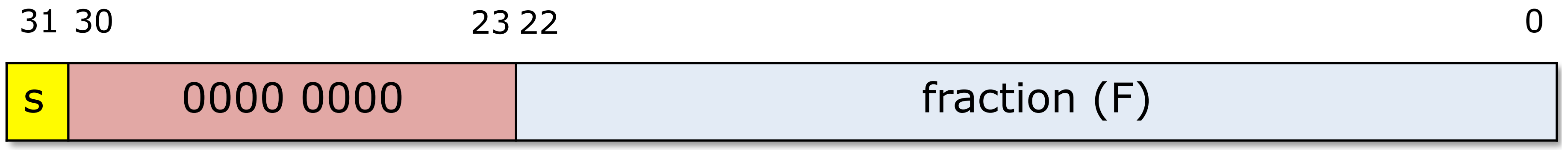
$r_{10} = \underline{+}M * 2^E$

**Normalized Encoding:**



$1 <= M < 2, M = ( 1.F )_2$

**Denormalized Encoding:**



E = 1 – Bias = -126

$0 <= M < 1, M = ( 0.F )_2$

# Zeros

+0.0

| | | |
|---|---|---|
| 0 | 0000 0000 | 0000 0000 0000 0000 0000 000 |

-0.0

| | | |
|---|---|---|
| 1 | 0000 0000 | 0000 0000 0000 0000 0000 000 |

# Examples

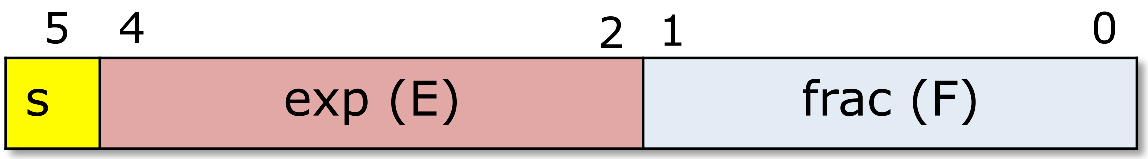
$(0.1)_2 * 2^{-126}$

| | | |
|---|---|---|
| 0 | 0000 0000 | 1000 0000 0000 0000 0000 000 |

$-(0.010101)_2 * 2^{-126}$

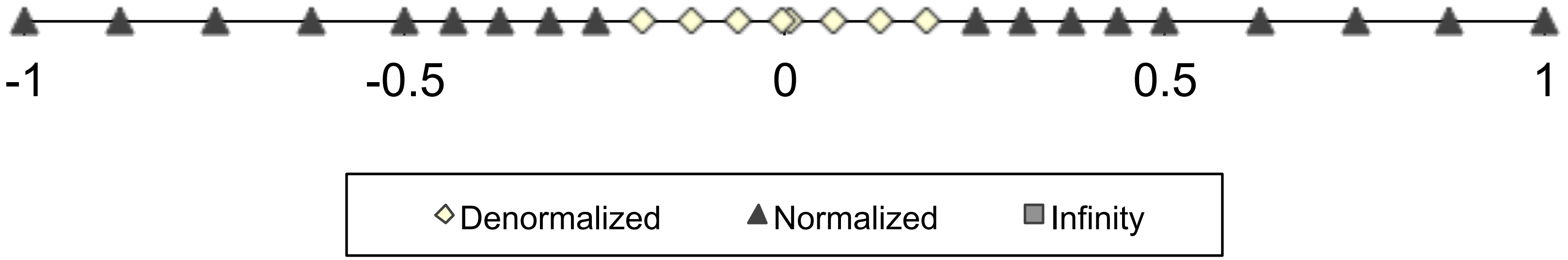
| | | |
|---|---|---|
| 1 | 0000 0000 | 0101 0100 0000 0000 0000 000 |

# Toy Number System



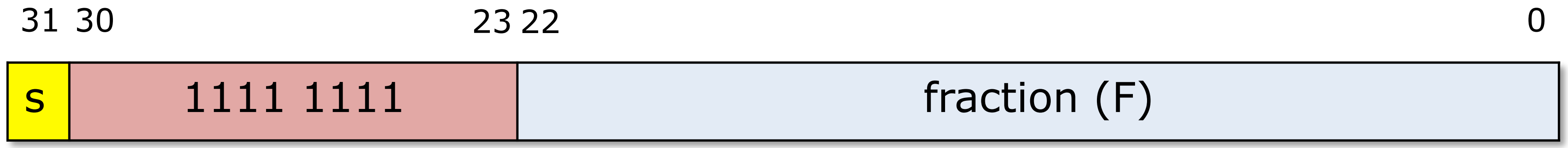
6-bit floating point representation

- exponent: 3 bits
- fraction: 2 bits
- bias: 3
- **Denormalized encoding**



# Special Values

**Special Value's Encoding:**



| **values** | **sign** | **frac** |
|---|---|---|
| +∞ | 0 | all zeros |
| - ∞ | 1 | all zeros |
| NaN | any | non-zero |

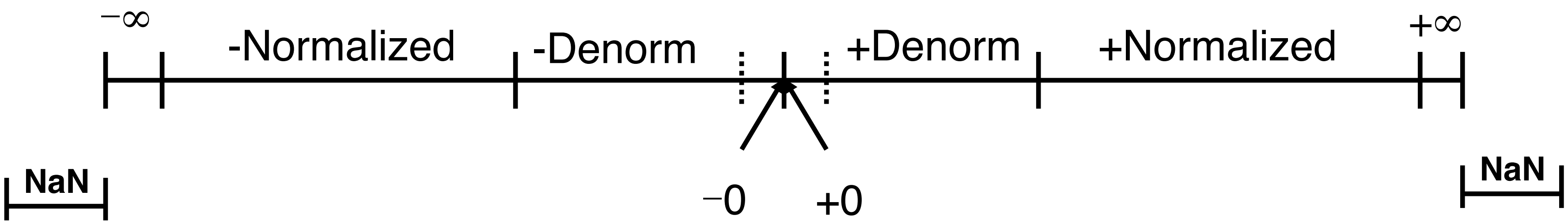
# Exercises

| representation | E | M | V |
|---|---|---|---|
| 0100 1001 0101 0000<br>0000 0000 0000 0000 | | | |
| | | | $2.5 * 2^{-127}$ |
| | | | $-1.25 * 2^{-111}$ |
| 1111 1111 1111 1111<br>0000 0000 0000 0000 | | | |
| 1111 1111 1000 0000<br>0000 0000 0000 0000 | | | |
| | | | $1.5 * 2^{-127}$ |

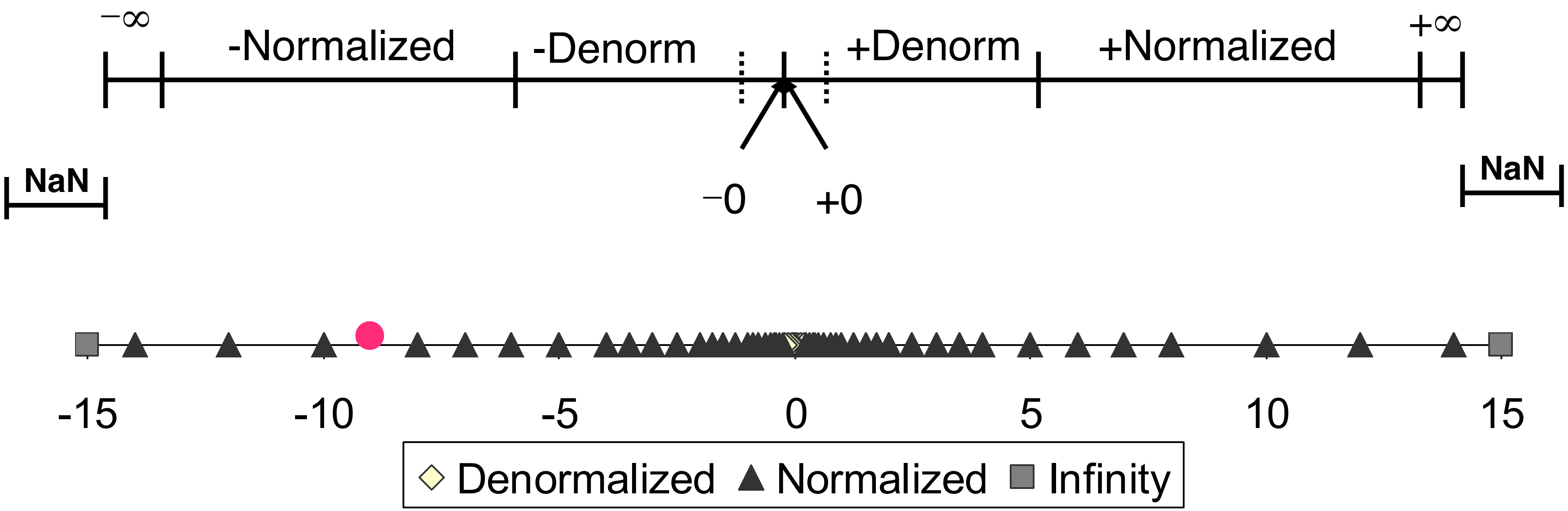
# Exercises

| **representation** | **E** | **M** | **V** |
|---|---|---|---|
| 0100 1001 0101 0000 0000 0000 0000 0000 | 146 – 127 = 19 | $(1.101)_2$ = 1.625 | $1.625 * 2^{19}$ |
| 0000 0000 1010 0000 0000 0000 0000 0000 | 1 – 127 = -126 | $(1.01)_2$ = 1.25 | $2.5 * 2^{-127} = (1.01)_2 * 2^{-126}$ |
| 1000 1000 0010 0000 0000 0000 0000 0000 | 16 – 127 = -111 | $(1.01)_2$ = 1.125 | $-1.25 * 2^{-111}$ |
| 1111 1111 1111 1111 0000 0000 0000 0000 | - | - | Nan |
| 1111 1111 1000 0000 0000 0000 0000 0000 | - | - | $- \infty$ |
| 0000 0000 0110 0000 0000 0000 0000 0000 | -126 | $(0.11)_2$ | $(0.11)_2 * 2^{-126} = 1.5 * 2^{-127}$ |

# Distribution of Representable Values



# Distribution of Representable Values



What if the result of computation is at ● ?

# Rounding

Goal

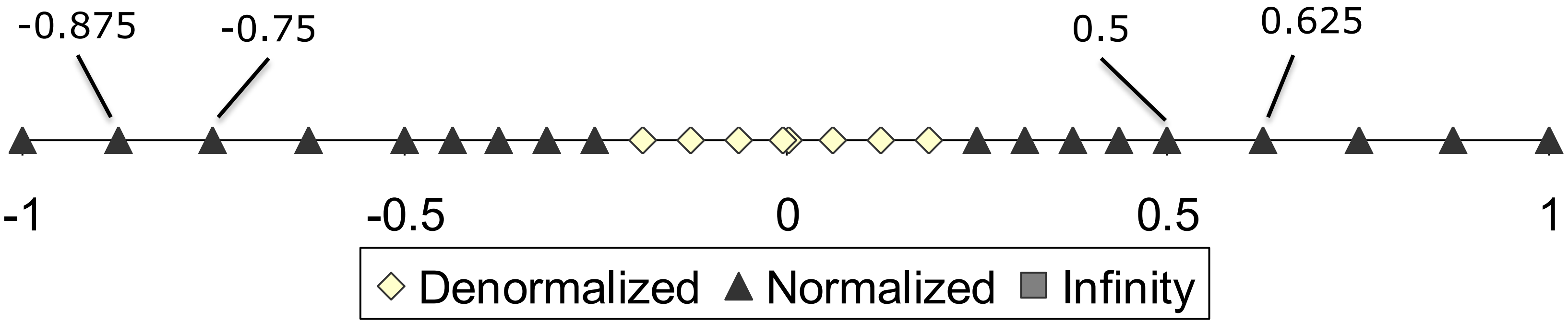
- Use the “closest” representable value *x′* to represent x.

Round modes

- Round-down
- Round-up
- Round-toward-zero
- Round-to-nearest (Round to even in text book)

# Round down

Round(x) = $x_-$ ($x_- <= x$)

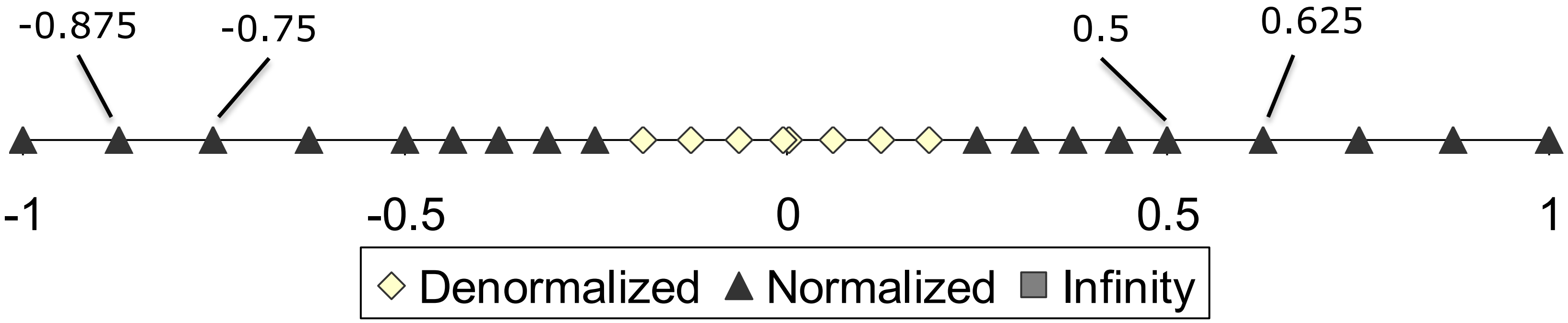


Round(-0.86) = ?

Round(0.55) = ?

# Round down

Round(x) = $x_-$ ($x_- <= x$)



Round(-0.86) = -0.875

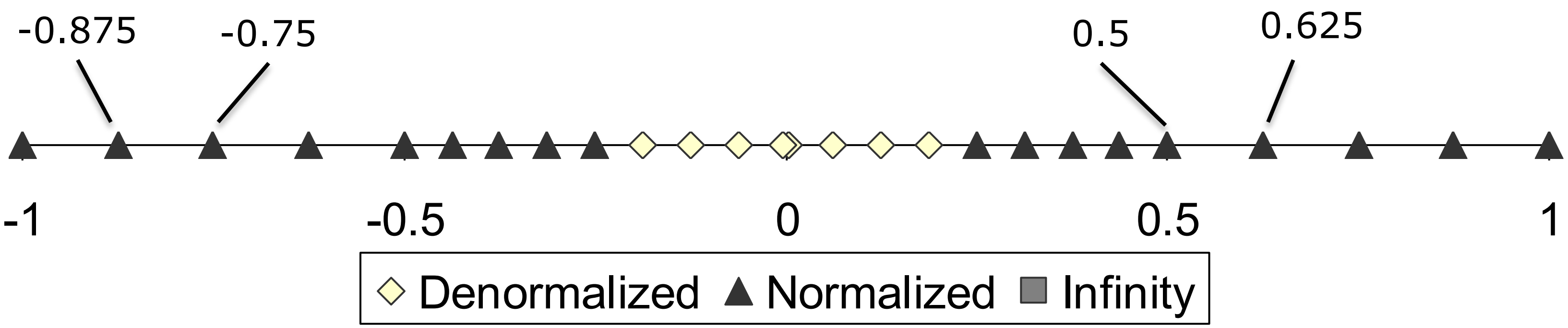
Round(0.55) = 0.5

# Round up

Round(x) = $x_+$ ($x_+ >= x$)

# Round up

$\mathrm{Round}(x) = x_+ \quad (x_+ >= x)$

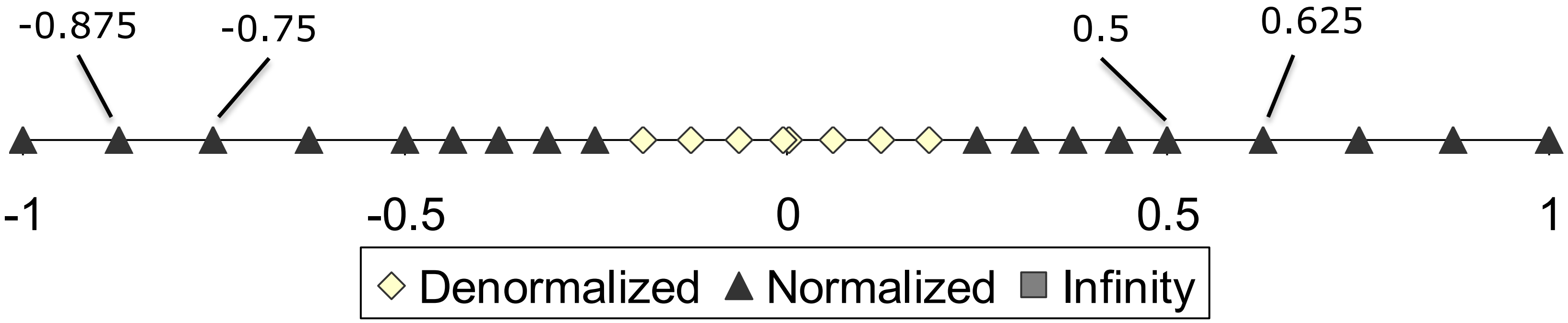


Round(-0.86) = ?

Round(0.55) = ?

# Round up

$\text{Round}(x) = x_{+} \quad (x_{+} >= x)$



Round(-0.86) = -0.75

Round(0.55) = 0.625

# Round towards zero

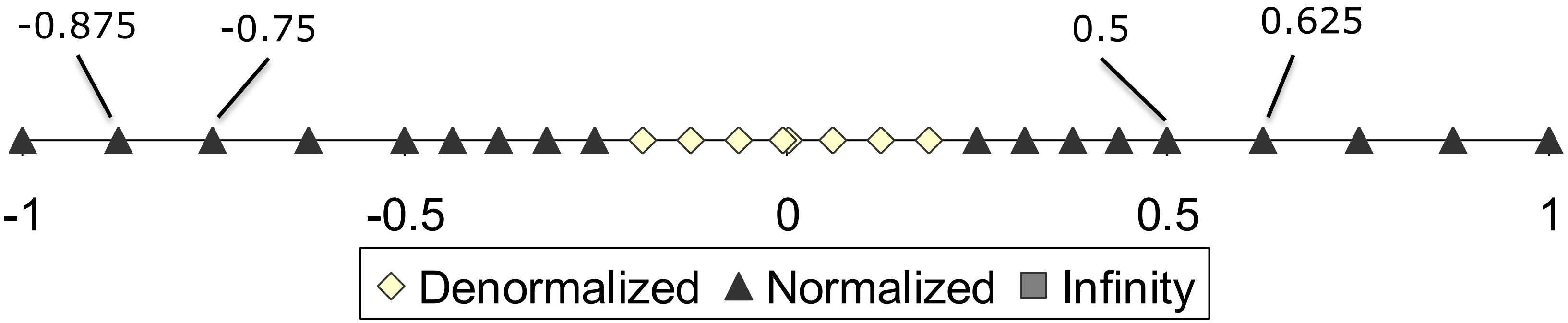
Round(x) = $x_+$ if $x < 0$

Round(x) = $x_-$ if $x > 0$

# Round towards zero

Round(x) = $x_+$ if x < 0

Round(x) = $x_-$ if x > 0



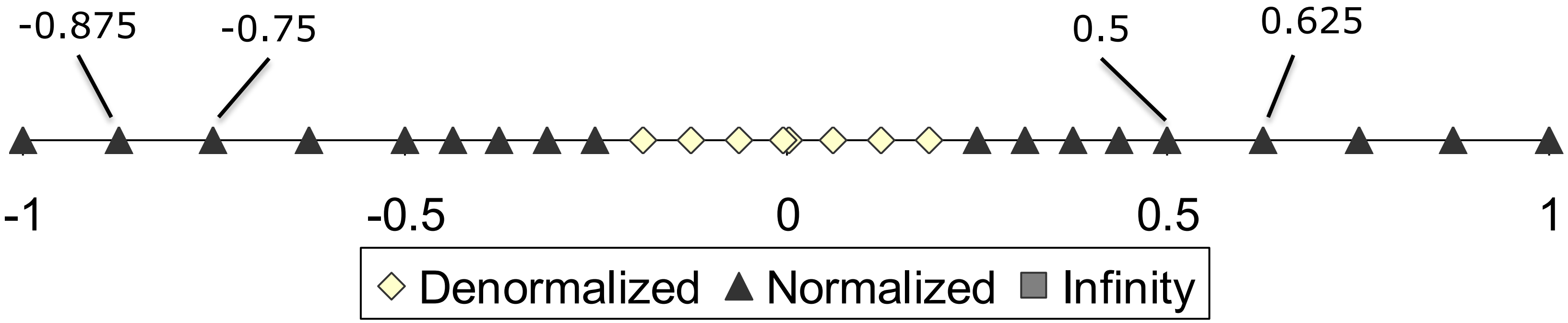
Round(-0.86) = ?

Round(0.55) = ?

# Round towards zero

Round(x) = $x_+$ if x < 0

Round(x) = $x_-$ if x > 0



Round(-0.86) = -0.75

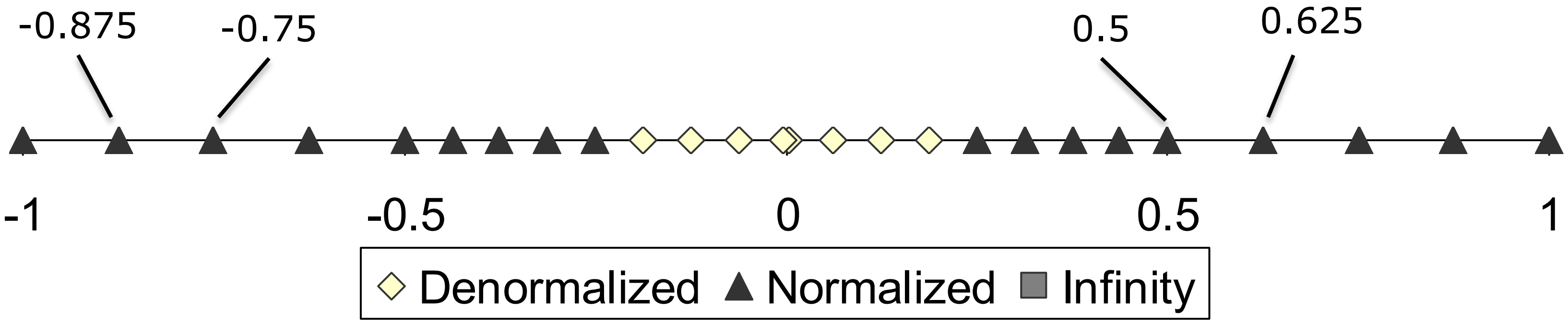
Round(0.55) = 0.5

# Round to nearest

Round(x) either $x_+$ or $x_-$ , whichever is nearer to x.

# Round to nearest

Round(x) either $x_+$ or $x_-$ , whichever is nearer to x.

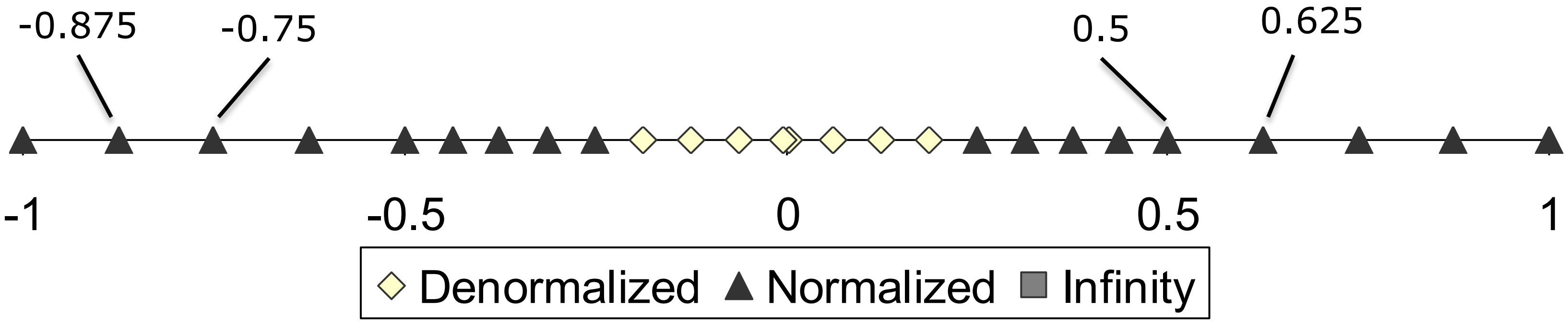


Round(-0.86) = ?

Round(0.55) = ?

# Round to nearest

Round(x) either $x_+$ or $x_-$ , whichever is nearer to x.

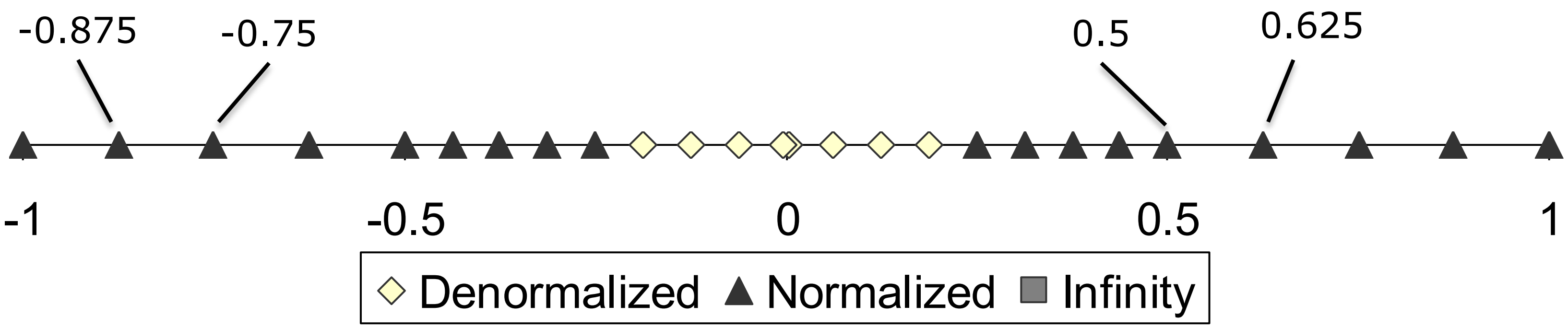


Round(-0.86) = -0.875

Round(0.55) = 0.5

# Round to nearest; ties to even

Round(x) either $x_+$ or $x_-$ , whichever is nearer to x.

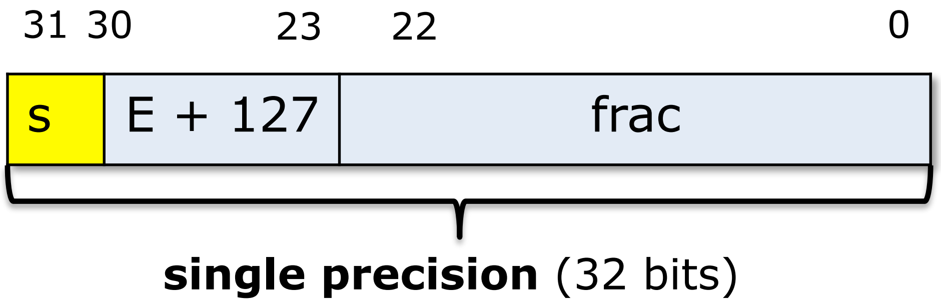


Round(-0.86) = -0.875

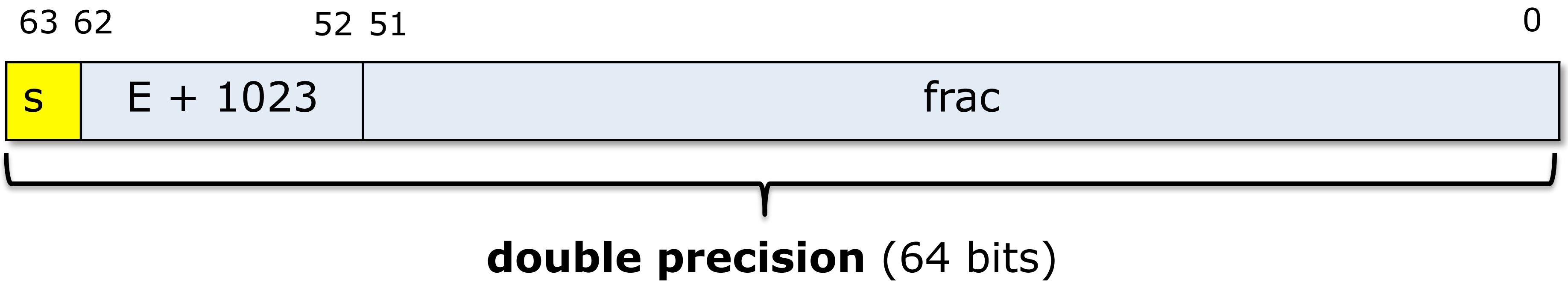
Round(0.55) = 0.5

In case of a tie, the one with its least significant bit equal to zero is chosen.

# single/ double precision



float f = 0.1
double d = 0.1

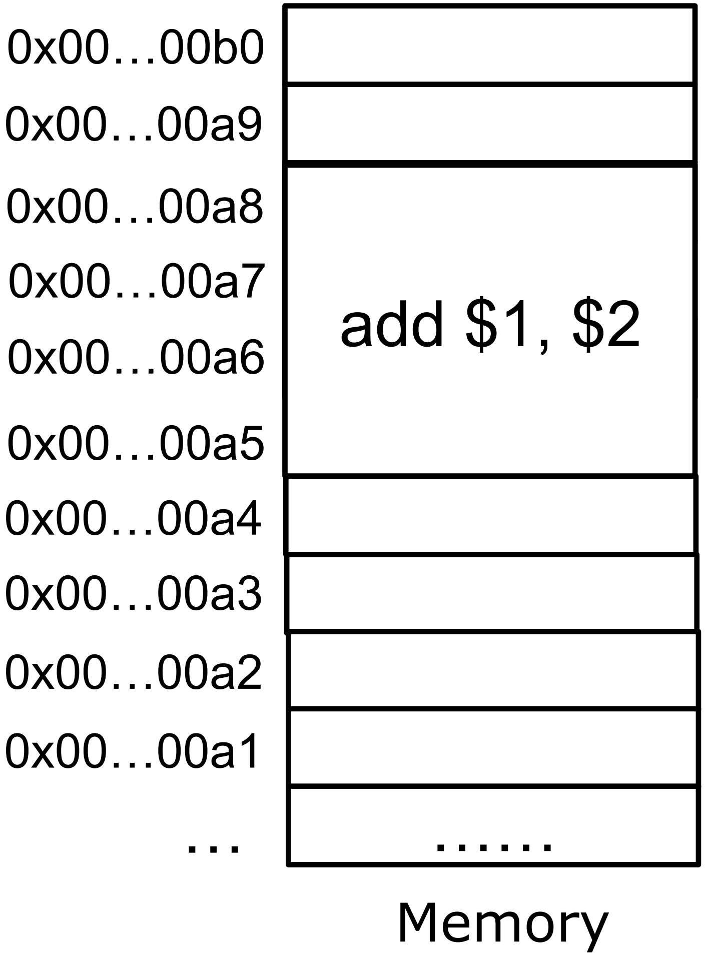


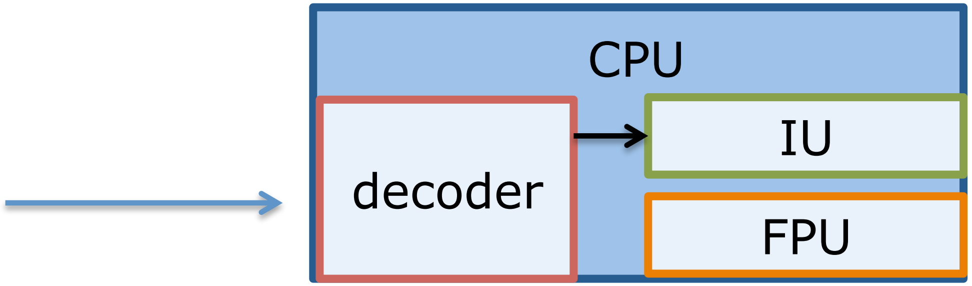
# single/ double precision

| | $E_{min}$ | $E_{max}$ | $N_{min}$ | $N_{max}$ |
|---|---|---|---|---|
| Float | -126 | 127 | $\approx 2^{-126}$ | $\approx 2^{128}$ |
| Double | -1022 | 1023 | $\approx 2^{-1022}$ | $\approx 2^{1024}$ |

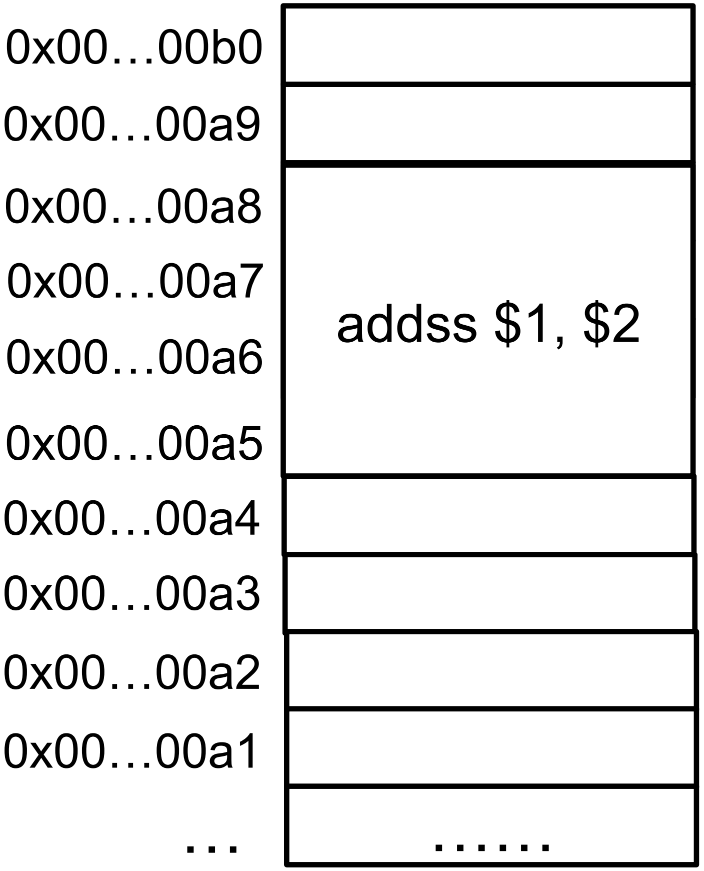
# How does CPU know if it is floating point or integers ?

By having specific instruction for floating points operation.

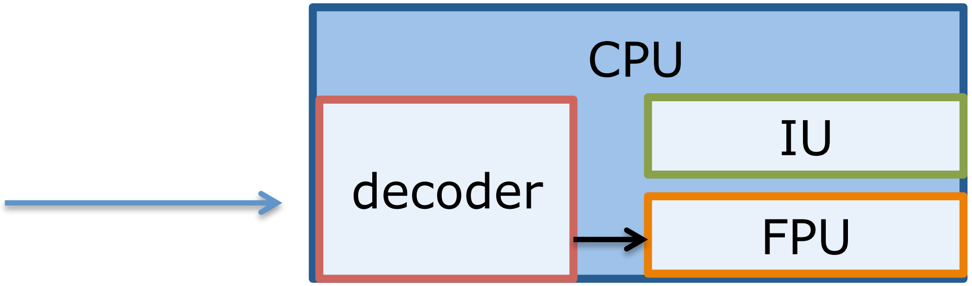

0x00…00b0
0x00…00a9
0x00…00a8
0x00…00a7
0x00…00a6
0x00…00a5
0x00…00a4
0x00…00a3
0x00…00a2
0x00…00a1
…
add $1, $2
……
Memory


CPU
decoder
IU
FPU

```
int d = 1 + 2
```



Memory



```
float f = 0.1 + 0.2
```