Asset pricing lecture

Friday, April 2, 2021 10:17 AM

$f(z_{t+1}, z_t, z_{t-1}, \ldots, z_1, z_0)$ joint probability density over $\{z_\tau\}_{\tau=0}^{t+1}$ $t \sim$ indexing time

$$\overbrace{f(z_{t+1}, z_t, z_{t-1}, \ldots, z_1, z_0)}^{\text{joint}} =$$

$$\underbrace{f(z_{t+1} \mid z_t, z_{t-1}, \ldots, z_0)}_{\text{conditional}} \underbrace{f(z_t \mid z_{t-1}, \ldots, z_0)} \cdots f(z_1 \mid z_0)\, f(z_0)$$

factored a joint density into a product of conditional densities.

i.i.d. $f(z_{t+1} \mid z_t, z_{t-1}, \ldots, z_0) \equiv f(z_{t+1})$

marginal distribution of z_{t+1}

independence

i.i.d. $f(z_{t+1}) = f(z_t) \cdots f(z_0)$

no t

identically

if i.i.d. $E\, z_{t+1} = \int f(z_{t+1})\, dz_{t+1}$

$$E\, z_{t+1} \mid z_t, z_{t-1}, \ldots, z_0 = E\, z_{t+1}$$

Law of iterated expectations:

$f(z_{t+1}, z_t, \ldots, z_0)$ joint distrn

not i.i.d.

Law of iterated expectation:

w.r.t. marginal distribution z_{t+1}

$$E\Big(\underbrace{E(z_{t+1} \mid z_t, \dots, z_0)}_{\text{is a r.v.}}\Big) = E\, z_{t+1}$$

expectation w.r.t. to the marginal distribution of whatever is in ( )

$\mid z_t, z_{t-1}, \dots, z_0)$

mean

$$E\big(\underbrace{E_t(z_{t+1})}_{\text{best forecast of } z_{t+1} \mid z_t, \dots, z_0}\big) = E\, z_{t+1}$$

$$E_t\big(E_{t+1}(z_{t+2})\big) = E_t\, z_{t+2}$$

Proof:

$E(E\, z_{t+1} \mid z_t, \dots, z_0)$

$$= \int \Big[\overbrace{\int z_{t+1} f(z_{t+1} \mid z_t, \dots, z_0)\, dz_{t+1}}^{E_t z_{t+1}}\Big] f(z_t, z_{t-1}, \dots, z_0)\, dz_t\, dz_{t-1} \dots dz_0$$

Change order of integration –

$$\int z_{t+1} \Big(\underbrace{\int f(z_{t+1} \mid z_t, z_{t-1}, \dots, z_0)\, f(z_t, z_{t-1}, \dots, z_0)\, dz_t\, dz_{t-1} \dots dz_0}_{f(z_{t+1}) \text{ marginal distribution}} \Big)\, dz_{t+1}$$

$$\int z_{t+1} f(z_{t+1})\, dz_{t+1} = E\, z_{t+1}$$

law of iterated expectations:

Law of iterated expectations:

R. Shiller, Lars Hansen

Lucas asset pricing model:

Lucas's theory

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

any information

(*) $\boxed{E_t\left(m_{t+1} R^i_{t+1}\right) = 1}$ m $E\left(m_{t+1} R^i_{t+1} \mid I_t\right)$

l. of iterated expectation

(**) $\Rightarrow$ $\boxed{E\, m_{t+1} R^i_{t+1} = 1}$

(*) & (**) are much more general than just the Lucas asset pricing theory

"Often m_{t+1}'s work"

Very general

$$\boxed{\begin{aligned} &E_t\, m_{t+1} R^i_{t+1} = 1 \\ &\Downarrow \\ &E\, m_{t+1} R^i_{t+1} = 1 \end{aligned}}$$

R^i_{t+1} = gross rate of return between dates t & $t+1$

$E_t(\cdot) = E(\cdot \mid I_t)$

$i = 1, \ldots, I$ any gross return.

$E(\cdot)$ ~ w.r.t. unconditional prob of (a)

m_{t+1} scalar random variable

the stochastic discount factor.

• Lucas asset pricing

Special

examples of m_{t+1} $\left(\beta \frac{u'(c_{t+1})}{u'(c_t)}\right)$

$\exists$ many many other examples of

$\exists$ many many other examples of m_{t+1} $\left(\sim \beta \frac{u'(c_{t+1})}{u'(c_t)}\right)$

m_{t+1}'s. . . .

Ross, Kreps, Harrison

$\exists$ a scalar s.d.f. that satisfies

$$\boxed{E_t\, m_{t+1} R^i_{t+1} = 1} \quad \forall i$$

trial cases

under the assumption that "the law of one price prevails"

Principle that there exist no arbitrages

two different assets are really claim to the same bundle of payoffs, then they have the same price.

An arbitrage is a transaction or trade you can make that gives positive profits with zero risk.

Proof just involves some linear algebra - Riesz representation theorem.

$E\, m R^i = 1 \quad \forall i$, gross return

$i = 1, \ldots, n$

m - universal stochastic discount

one r.v.

$E\, m R^i = 1$

Asset pricing "theory" or empirics is all about covariances"

$E\, m R^i$ — not a covariance

"close to a covariance"

(sol) $E(mR^i)=1$

m ~ stochastic discount factor

R^i ~ gross return asset i

E - mathematical

$$R_t^i = \frac{P_{t+1}^i - P_t^i + d_{t+1}^i}{P_t^i}$$

gross $\simeq$ 1.05, 1.10

(*) $\Rightarrow$ $\boxed{1 = E(mR^i)}$

$= E m \cdot E(R^i) + cov(m, R^i)$ ← definition of covariance

$cov(m,R^i) \equiv E(m - E(m))(R^i - E(R^i))$

$= E(mR^i - mER^i - R^iEm + EmER^i)$

$= EmR^i - EmER^i - ER^iEm + EmER^i$

$cov(m,R^i) = E(m \cdot R^i) - EmER^i$

$E(mR^i) = EmER^i + cov(m,R^i)$

R risk R^f ← not random

$1 = EmR^f$ ← because R^f is constant.

$1 = R^f Em$

$\boxed{(R^f)^{-1} = Em}$ ← ↑ not risky

$\boxed{R^f = (Em)^{-1}}$

$1 = EmER^i + cov(m,R^i)$, $\forall i = 1, \dots, I$

$1 = (R^f)^{-1} ER^i + cov(m,R^i)$ ←

$R^f = ER^i + cov(m,R^i) R^f$ $(\times R^f)$

$ER^i = R^f - cov(m,R^i) R^f$

$\boxed{ER^i - R^f = -cov(m,R^i) R^f}$ ← $E(mR^i) = 1$

$R^i - R^f = R^{ei}$

$\boxed{ER^{ei} = -cov(m,R^i) R^f}$

Representations - rewriting

Representations — rewriting

expected return - beta representation β_i

$$E R^i = R^f - \mathrm{cov}(m, R^i) R^f \quad , \quad R^f = (E m)^{-1}$$

$$= R^f + \frac{\mathrm{cov}(m, R^i)}{\mathrm{var}(m)} \left(- \frac{\mathrm{var}(m)}{E(m)} \right)$$

$$\boxed{E R^i = R^f + \beta_{i,m} \lambda_m} \quad , \quad i = 1, \dots, n$$

$$\beta_{i,m} = \frac{\mathrm{cov}(m, R^i)}{\mathrm{var}(m)}$$ = regression coefficient ~ "quantity of risk" ~ "beta of asset i"

$$\lambda_m = \frac{-\mathrm{var}(m)}{E(m)}$$ ~ "price of risk"

if we know "m" $\Rightarrow \lambda_m$,

$(m, R^i) \quad \forall i$ — risk - vs. return

Say more about m — what is m

Hansen - Richard — $E(m R^i) = 1 \Rightarrow$

mean variance frontier $\Rightarrow$ information about admissible, possible s.d.f.'s m.

mean frontier: $\Leftarrow$ • $E(m R^i) = 1 \Leftarrow$, $E_t m_{t+1} R^i_{t+1} = 1$

• C.S. inequality "$(R^2 < 1)$"

$$1 = E(m R^i) = E(m) E(R^i) + \rho_{m, R^i} \sigma(m) \sigma(R^i)$$

$$1 = E(mR^i) = E(m)E(R^i) + \rho_{m,R^i}\,\sigma(m)\,\sigma(R^i)$$

correlation coefficient

$$\rho_{m,R^i} \equiv \frac{\mathrm{cov}(m,R^i)}{\sigma(m)\,\sigma(R^i)}$$

$-1, +1$

algebra

$$\rho_{m,R^i} = \begin{cases} -1 \Rightarrow \text{upper} \\ +1 = \text{bottom of frontier} \end{cases}$$

$$R^f = (Em)^{-1}$$

$\Rightarrow$

(**) $$\boxed{ER^i = R^f - \rho_{m,R^i}\frac{\sigma(m)}{E(m)}\,\sigma(R^i)}$$

apply CS

$$ER^i = R^f - \mathrm{cov}(m,R^i)\,R^f$$

(***) $$|ER^i - R^f| \le \frac{\sigma(m)}{E(m)}\,\sigma(R^i)$$

because $|\rho_{m,R^i}| < 1$ in absolute value

(+) $$\boxed{\frac{|ER^i - R^f|}{\sigma(R^i)} \le \frac{\sigma(m)}{E(m)}}$$

upper bound on Sharpe-ratio

"price of risk" LPH

"maximum Sharpe ratio"

$$= \frac{ER^i - R^f}{\sigma(R^i)}$$

(+)

everything that the theory implies about risk-return tradeoffs

conditional version of (+)

$$E(mR^i) = 1 \quad \forall\, i = 1, \dots, I$$

$$|ER^i - R^f| \le \frac{\sigma(m)}{E(m)}\,\sigma(R^i) \Rightarrow \boxed{\text{mean variance frontier}}$$

the frontier: $\rho_{m,R^i} = -1$ one piece of frontier

$\rho_{m,R^i} = 1$ another piece of frontier

Upper frontier:

$$ER^i = R^f + \frac{\sigma(m)}{E(m)} \cdot \sigma(R^i) \qquad \rho_{m,R^i} = -1$$

slope

lower frontier

slope

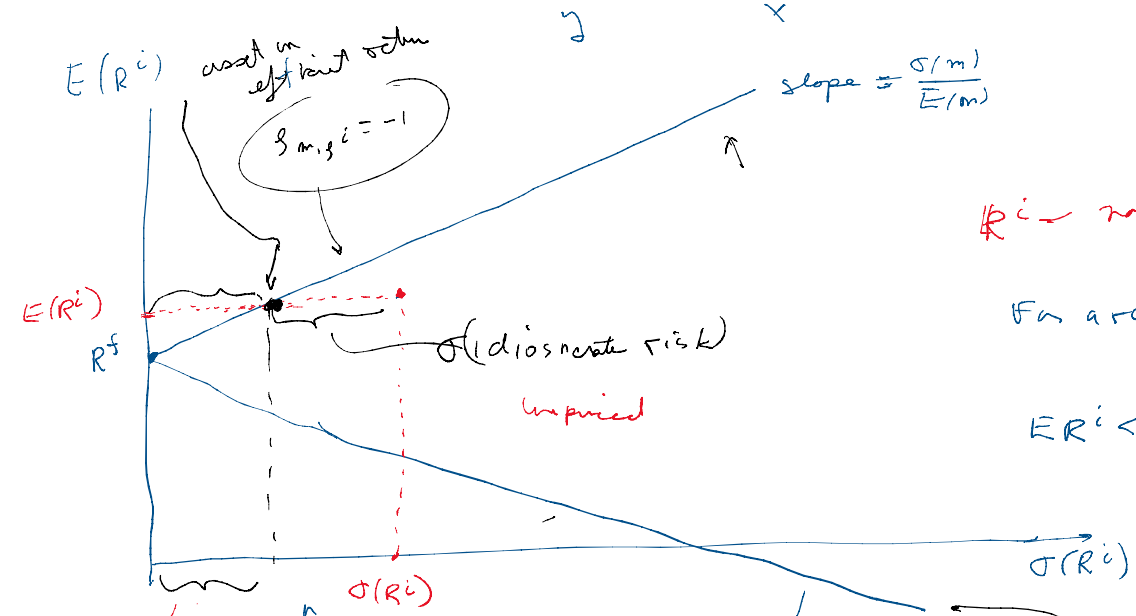
$$ER^i = R^f - \frac{\sigma(m)}{E(m)} \sigma(R^i) \qquad \rho_{m,R^i} = +1$$

intercept

plot two lines $E(R^i)$, $\sigma(R^i)$

y x

$$\frac{\sigma(m)}{E(m)} = \text{price of risk}$$



R^i — not on the frontier

For a return inside frontier —

$$ER^i < R^f + \boxed{\frac{\sigma(m)}{E(m)}} \sigma(R^i)$$

$E\, m\, R^i = 1$

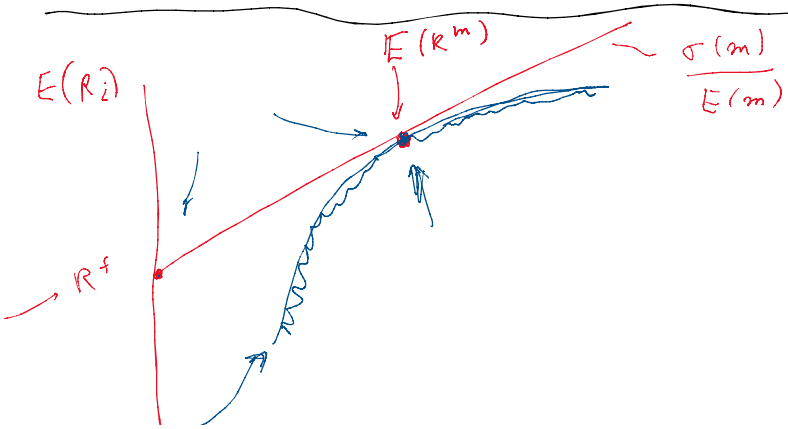
covariance

$\rho^2_{m,R^i} \le 1$

$\rho_{m,R^i} = +1$

priced

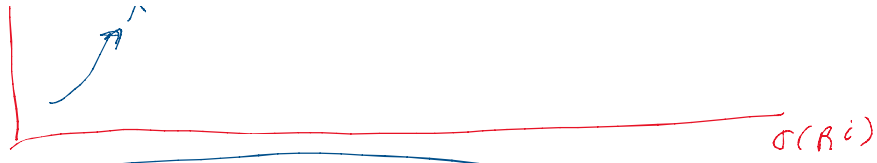
$$R^i = E(R^i) + \epsilon_i + u_i$$



Roll's critique:

there might be no single security on the frontier.

mutual fund? → market portfolio.



mutual fund? $\longrightarrow$ market portfolio

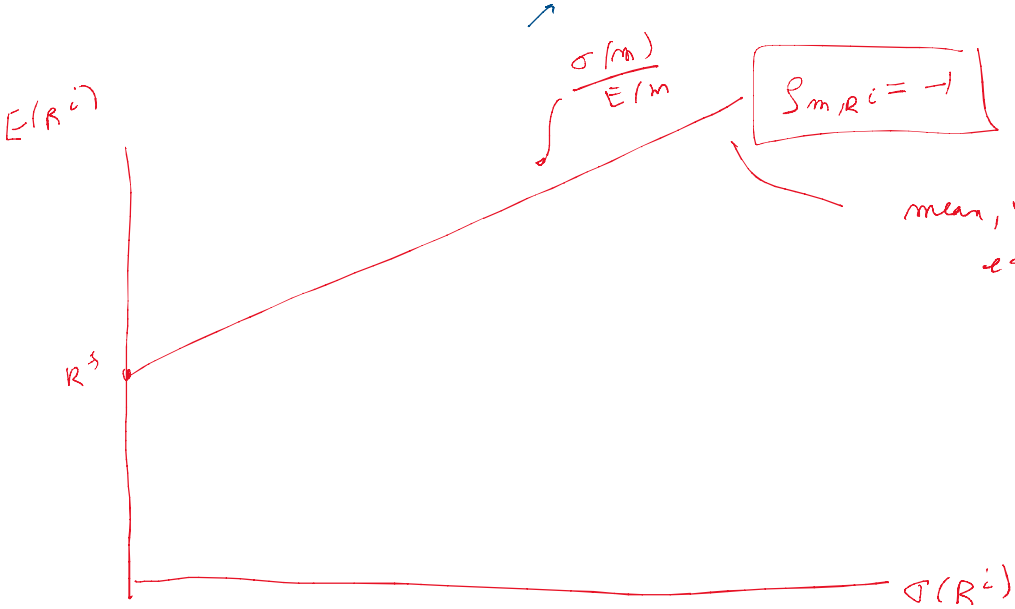
Sharpe, Lintner

R^m is on frontier

$\boxed{\max E R^p \text{ given } \sigma(R^p)}$ R^i, optimal

Markowitz

$R \sim (\mu, \Sigma)$



mean, variance efficient frontier

Implication

- all returns on frontier, are perfectly correlated with each other & (negatively) correlated with m

Let R^m, R^{mv} be two returns on frontier

random $\longrightarrow$ $R^{mv} = R^f + \tilde{a}(R^m - R^f)$, no error.

- each return on frontier is perfectly (negatively) correlated with m $(\rho_{m,R^i} = -1)$

$\Rightarrow$ we can <u>represent</u>

$m = a + b R^{mv}$ a, b constants

$R^{mv} = c + d\, m$ $\qquad$ c, d constants

"any return on the frontier can serve as a stochastic discount factor m"

- for any mv-efficient return (not R^f), single-β representation is

$$E R^i = R^f + \underset{1}{\beta_{i,R^{mv}}} \underbrace{\left[E(R^{mv}) - R^f \right]}_{\text{single factor}}, \quad i = 1, \dots, n$$

$$E R^{mv} = R^f + 1 \cdot \left[E(R^{mv}) - R^f \right], \qquad R^i = R^{mv}$$

there can be multiple s.d.f.'s m that "work"

<u>Empirical work</u>: • implement or use theory ← time series regression
• "test the theory" ← cross section regression

assume that R_t^i — return on asset i between date t & $t+1$.

$$f_t \sim R_t^m$$
$$R_t^f \qquad R^f \qquad , \underline{\text{i.i.d.}} \quad \forall i$$

- returns are not forecastable
- $R_t^f = R$

R_t^i, $t = 0, \dots, T-1$; $i = 1, \dots, n$.

- Time series regression: "use ε (first step of "test")

$$R_t^i = \alpha_i + \beta_i R_t^m + \varepsilon_t^i, \quad \leftarrow \text{OLS}$$

$$R_t^i = \alpha_i + \beta_i R_t^m + \epsilon_t^i, \quad \leftarrow \text{OLS}$$

constant — regression coefficient — regressor

$$t = 0, \dots, T-1$$

$$(R_t^i - R^f) = \beta_i (R_t^m - R^f) + \epsilon_t^i$$

excess return on asset i

excess return on market portfolio

one regression $\Rightarrow \beta_i$

$\hat{\beta}_i, \; i = 1, \dots, I$.

"beta's" on asset i. — measure undiversifiable risk you assume by buying asset i.

Cross-section regressions — to test theory.

$$R_t^i, \quad t = 0, \dots, T-1 \implies \frac{1}{T} \sum_{t=0}^{T-1} R_t^i = E_T(R^i)$$

to each stock $i = 1, \dots, I$

run this cross-section regression across $i = 1, \dots, I$

constant for cross section

$$(**) \quad E_T(R^i) = \hat{\beta}_i \lambda + \xi + \epsilon_i \quad \text{l.s. regression}$$

varies across i — regressor — scalar regression coefficient from the cross section

theory: $$E R^i = R^f + \beta_{m,i} \cdot \lambda_m$$

return on market portfolio

$$\lambda_m = -\frac{\text{var}(m)}{E(m)}$$

return on market portfolio

price of risk.

quantity of risk

CAPM – single factor model

↑ return on market portfolio.

$$\boxed{E\, m R^i = 1}$$ — CAPM,

Many extensions follow easily from ↑

$E(m R^i) = 1$ ← Hansen & Jagannathan

diagnostic tool for constructing structural macroeconomic or public finance models of m – they link m to real variables.

$$m_t = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$ Lucas 1978

$$E(m_{t+1} R^i_{t+1}) = 1$$ real macro model.

Class of s.d.f.s – m_{t+1} legitimate s.d.f.

"Regress what you don't know on what know"

m – unknown s.d.f. ← don't have a candidate

r.v.

p – $n \times 1$ vector of prices

x – $n \times 1$ vector of payoffs on those asset prices.

$$p = E(m x)$$

returns

$$1 = E\left(m, \frac{x}{p}\right)$$

know

$$p_t = E_t(m_{t+1} x_{t+1})$$

return

$$1 = E_t\left(m_{t+1}\left(\frac{x_{t+1}}{p_t}\right)\right)$$

don't know $\underline{m}$, || data on p, x

$$m = a + \underset{1\times n}{x'}\underset{n\times 1}{b} + e, \quad e \perp x$$

regression

$$b = [\mathrm{cov}(x,x)]^{-1}\,\mathrm{cov}(x,m)$$

$$a = Em - Ex'b \quad \leftarrow \text{constant}$$

$$Emx = p \Rightarrow \mathrm{cov}(x,m) = p - E(m)E(x)$$

$$b = \underbrace{[\mathrm{cov}(x,x)]^{-1}}_{\text{know}}\,[\underset{\text{know}}{p} - \underset{\text{don't know}}{E(m)}\,\underset{\text{know}}{E(x)}]$$

HJ strategy:

for "made up" $E(m)$, compute $b[E(m)] = \tilde{b}$

HJ strategy:

for "made up" $E(m)$, compute $b[E(m)] = \tilde{b}$

from l.s. regression theory — $\boxed{var(m) \geq var(x'b)}$

$$var(m) = var(x'b) + var(e), \quad e \perp x$$

$$var(x'b) = b' \, cov(x,x) \, b$$

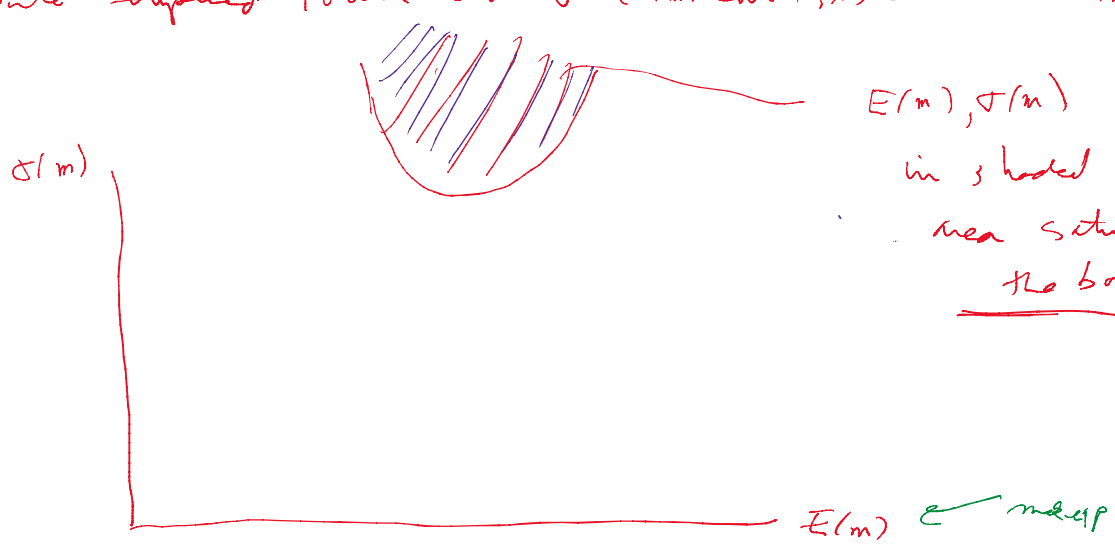
HJ bound $\Rightarrow$

$$\sqrt{\underbrace{b'}_{b'(E(m))} cov(x,x) \underbrace{b}_{E(m)}} \leq \sigma(m)$$

m, x

- make up a $E(m)$ ← put on x axis —
- compute ~~tripled~~ lower bnd $\sqrt{b'(E(m)) \, cov(x,x) \, b}$ on $\sigma(m)$

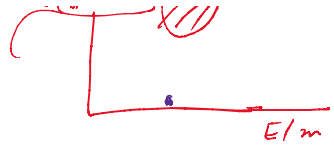
plot



$E(m), \sigma(m)$ in shaded area satisfy the bound.

HJ // universal - any s.d.f. m that satisfies $E(mR^i) = 1$ must be in the $\sigma(m)$

must be in the



$$m = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

$E(m), \sigma(m)$

this looks as an asset price model.

must have high enough variance for a given mean.

m_{t+1}