Asset pricing

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Key Equation

(*) $E(m R^i) = 1$ for $i = 1, \dots, I$

m = stochastic discount factor

R^i = gross return

$E \sim$ mathematical expectation

(*) $\Rightarrow$

$1 = E m E R^i + cov(m, R^i)$

$R^f = (1/E(m))$ = risk free gross return

$\Rightarrow R^f = E R^i + cov(m, R^i) R^f$

$E R^i = R^f - cov(m, R^i) R^f$

or

$E R^i - R^f = -cov(m, R^i) R^f$ (= excess return)

$E m R = 1$

$R = R^f$

$\Rightarrow R^f E m = 1$

$\Rightarrow E m = (R^f)^{-1}$

Parallel presentation - RMT 5

$q_t = E_t(m_{t+1} P_{t+1})$ payoff

$q_t = E_t m_{t+1} E_t P_{t+1} + cov_t(m_{t+1}, P_{t+1})$

apply Cauchy-Schwartz

$\frac{q_t}{E_t m_{t+1}} \geq E_t P_{t+1} - \frac{\sigma_t(m_{t+1})}{E_t m_{t+1}} \sigma_t(P_{t+1})$

for a return $q_t = 1$, $P_{t+1} = R_{t+1}$, $E_t m_{t+1} = R^f_{t+1}$

$R^f_{t+1} \geq R_{t+1} - \frac{\sigma_t(m_{t+1})}{E_t(m_{t+1})} \sigma_t(R_{t+1})$

or

$R_{t+1} \leq R^f_{t+1} + \frac{\sigma_t(m_{t+1})}{E_t(m_{t+1})} \sigma_t(R_{t+1})$ || mean variance frontier → upper bound.

market price of risk

amount of risk

$1 = E_t(m_{t+1} R_{t+1})$

$R_{t+1} = \frac{P_{t+1}}{P_t}$ return

expected return - beta representation

$$E R^i = R^f + \underbrace{\left(\frac{cov(R^i, m)}{var(m)}\right)}_{\beta_{i,m}} \underbrace{\left(-\frac{var(m)}{E(m)}\right)}_{\lambda_m}$$

$\beta_{i,m}$ = regression coefficient

λ_m = "price of risk"

$E R^i = R^f + \beta_{i,m} \lambda_m$

← returned to & repeated below

Mean-variance frontier

Cauchy-Schwartz inequality

~ "R^2 are less than or equal to 1"

$1 = E(m R^i) = E(m) E(R^i) + \rho_{m,R^i} \sigma(m) \sigma(R^i)$

where $\rho_{m,R^i} \equiv \frac{cov(m, R^i)}{\sigma(m)\sigma(R^i)}$ ← standard deviation

$1 = E(m) E(R_i) + cov(m, R^i)$

$\rho_{m,R^i} = \frac{cov(m, R^i)}{\sigma(m)\sigma(R^i)}$

$\Rightarrow cov(m, R^i) = \rho_{m,R^i} \sigma(m) \sigma(R^i)$

$\Rightarrow E R^i = R^f - \rho_{m,R^i} \frac{\sigma(m)}{E(m)} \sigma(R^i)$ ← From $E R^i = R^f - cov(m, R^i) R^f$ above

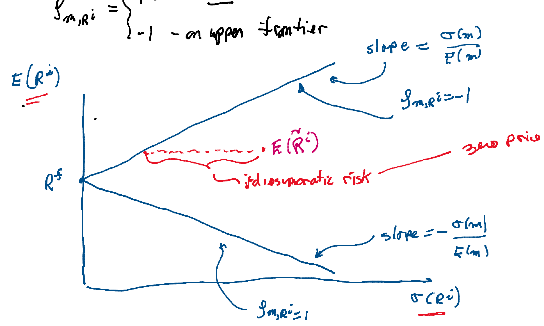
$\Rightarrow |E R^i - R^f| \leq \frac{\sigma(m)}{E(m)} \sigma(R^i)$ because $\rho_{m,R^i} \in [-1, 1]$, $|\rho_{m,R^i}| \leq 1$

$\boxed{\frac{\sigma(m)}{E(m)} \geq \frac{|E R^i - R^f|}{\sigma(R^i)}}$ HJ bound

return inside frontier:

$R^f - \frac{\sigma(m)}{E(m)} \sigma(R^i) \leq E R^i \leq R^f + \frac{\sigma(m)}{E(m)} \sigma(R^i)$

$\rho_{m,R^i} = \begin{cases} +1 & \text{- on lower frontier} \\ -1 & \text{- on upper frontier} \end{cases}$



Upper frontier:

$E R^i = R^f + \frac{\sigma(m)}{E(m)} \sigma(R^i)$

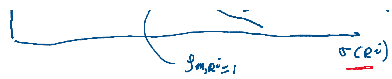
"price of risk" ↳ (Hansen)

lower frontier:

$E R^i = R^f - \frac{\sigma(m)}{E(m)} \sigma(R^i)$

for a return inside frontier

$E(\tilde{R}^i) < R^f + \frac{\sigma(m)}{E(m)} \sigma(\tilde{R}^i)$



$E(\tilde{R}^i) < R^f + \frac{\sigma(m)}{E(m)}\,\sigma(\tilde{R}^i)$

$\tilde{R}^i = E(\tilde{R}^i) + \tilde{\epsilon}_i + \tilde{u}_i$ ‖ $\sigma(\tilde{R}_i) = \sigma(\tilde{\epsilon}_i) + \sigma(\tilde{u}_i)$

perfectly correlated with m — $\perp m$ — systematic risk — $\perp m$ — idiosyncratic risk

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Implications

- all returns on frontier are perfectly correlated

  Let R^m, R^{mv} be two returns on frontier

  $R^{mv} = R^f + a(R^m - R^f)$ for some scalar a

- each return on frontier is perfectly correlated with m (negatively)

  $(\rho_{m,R^i} = -1) \Rightarrow$

  $m = a + b R^{mv}$

  $R^{mv} = e + d m$ for some scalars a, b, e, d

  "any return on frontier is a stochastic discount factor"

Single β representation derivation:

$1 = E\, m R^i = E(m)E(R^i) + cov(m, R^i)$

$\Rightarrow E(R^i) = \frac{1}{E(m)} - \frac{cov(m,R^i)}{E(m)}$

$= \gamma + \frac{cov(m,R^i)}{var(m)} \cdot \left(\frac{-var(m)}{E(m)}\right)$

$= \gamma + \beta_{i,m}\lambda_m$, $\gamma = \frac{1}{E(m)}$

- for any mv-efficient return (not R^f), single-beta representation is

  $E R^i = R^f + \beta_{i,R^{mv}}\left[E(R^{mv}) - R^f\right] \quad \forall i = 1, \dots, I$ ← here $R^{mv} = m$, so $\beta_{i,m} = \beta_{i,R^{mv}}$

  and

  $E R^{mv} = R^f + 1 \cdot \left[E(R^{mv}) - R^f\right]$, for $R^i = R^{mv}$ ← see graph $R^i = R^f + \beta_{i,R^{mv}}\left[E(R^{mv}) - R^f\right] + \epsilon_i$

Expected return – β representation

implications & "testing"

- $E R^i = \gamma + \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \dots$ $\quad i = 1, 2, \dots, I$ ~ returns

  λ_j, $j = a, b, \dots$ price of exposure to factor $a, b, \dots$

(*) $R^i_t = a_i + \beta_{i,a} f^a_t + \beta_{i,b} f^b_t + \dots + \epsilon^i_t$, $t = 1, 2, \dots, T$

$\epsilon^i_t \perp f^j_t$, $j = a, b, \dots$ $\quad i = 1, \dots, I$

e.g.: f_t = return on market portfolio

~ sdf. $\beta \frac{u'(c_{t+1})}{u'(c_t)}$ ...

$\beta_{i,a}$ – exposure to factor f_a risk

λ_a ~ price of exposure to factor f_a risk

R_t ← from t to $t+1$ ✓

Check timing – ↘ rent this notebook

$E R^i = R^f + \frac{cov(m,R^i)}{var(m)}\left(\frac{-var(m)}{E(m)}\right)$

$= R^f + \beta_{i,m} \cdot \lambda_m$

To test the model:

(i) estimate $a_i, \beta_{i,a}, \beta_{i,b} \dots$ by running time series regression

R^i_t on $f^a_t, f^b_t, \dots$

(ii) run cross-section regression

$E(R^i) = \gamma + \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \dots + \alpha_i$, $i = 1, \dots, I$

average return over time series — regressor — regression coefficient — regressor — regressor coefficient — pricing error

$\alpha_i \perp \beta_{i,j}$, $j = a, b, \dots$ ← least squares orthogonality condition

estimate $\gamma, \lambda_a, \lambda_b, \dots$ by regression

- for risk-free rate $R^f = \gamma$

- for excess returns –

$ER^{ei} = \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + \cdots + \alpha_i \quad , i=1,\ldots,n$

Hansen-Jagannathan bound : purpose – quick check for candidate s.d.

m ~ "unknown" s.d.f.

p ~ $n\times 1$ vector of asset prices

x ~ $n\times 1$ vector of pay offs

$p = E(mx)$ $\quad m$ ~ scalar ; $p=1$ for returns

regress anything on anything

$m = a + x'b + e \quad , \quad e \perp x$

b ~ $n\times 1$ vector of regression coefficients

$b = [cov(x,x)]^{-1} cov(x,m)$ $\quad$ population l.s. formula

$a = Em - Ex'b$, constant term (scalar)

Note: $Emx = p \Rightarrow cov(x,m) = p - E(m)E(x)$

$$\Rightarrow \quad b = \underbrace{[cov(x,x)^{-1}]}_{\text{know}}\,[\underbrace{p}_{\text{know}} - \underbrace{E(m)}_{\text{don't know}}\,\underbrace{E(x)}_{\text{know}}]$$

strategy: for "made up" $E(m)$ compute $b[E(m)] = \tilde{b}$

from l.s. regression

$var(m) = var(x'b) + var(e) \quad , \quad b \perp$

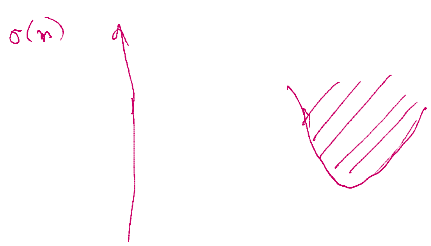
note: $var(x'b) = b'\,cov(x,x)\,b$

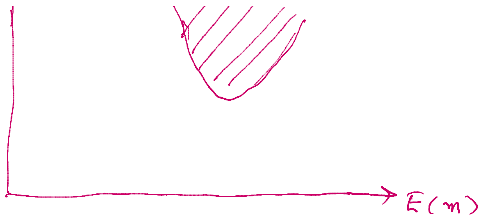
$$\frac{\sigma(m)}{E(m)} \geq \frac{|ER^i - ER^0|}{\sigma(R^i)} \geq \frac{|E\,i - f|}{\sigma(i)} \quad \forall i$$

HJ bound

$$\underbrace{\sqrt{b'\,cov(x,x)\,b}}_{b(E(m))\ \ b(E(m))} \leq \sigma(m) \quad , \quad b(E(m)) = [cov(x,x)^{-1}][p - E(m)\overset{n\times 1}{E(x)}]$$

lower bound on $\sigma(m)$ as fn of $E(m)$ to





Note: for returns – $p=1$, x are ex post returns

$$b = [\mathrm{cov}(x,x)]^{-1} [1 \;\; -E(m)\, E(x)]$$