Notes 2: Time Decay SABR for RFR

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Here are some necessary notations:

- $t \leq T_s < T_e$: t is a generic valuation time, T_s is the accrual start date of a rate compounding period, and T_e is the accrual end date.
- τ: accrued, $\tau(T_s, T_e)$.
- $P(t, T)$: the value of a zero-coupon bond matures at time T.
- F and R: F stands for forward LIBOR rate, R stands for forward RFR rate.

1 Vanilla RFR Caplet/Floorlet

In RFR world, caplet/floorlet has two different flavors. One is *backward-looking* caplet, pays at T_e,

$$V^b(T_e) = \tau \left(R(T_e; T_s, T_e) - K\right)^+ , \tag{1}$$

As discussed in the previous notes, this payoff is only known at time T_e. The other one is *forward-looking* caplet, pays at time T_e,

$$V^f(T_e) = \tau \left(R(T_s; T_s, T_e) - K\right)^+ , \tag{2}$$

where $R(T_s; T_s, T_e)$ is the time T_s-observed compounded RR rate of the period $[T_s, T_e]$. Obviously, the payoff is known at time T_s, but paid at time T_e, which has no difference from a standard LIBOR caplet. Therefore, we will focus on the backward-looking caplet. Under T_e-forward measure, the pricing formula reads,

$$V^b(t) = \tau P(t, T_e) \cdot E_t^{T_e} \left[\left(R(T_e; T_s, T_e) - K\right)^+ \right] \tag{3}$$

We had some discussions regarding the randomness of $R(T_e; T_s, T_e)$. Basically, decomposing R into R_h and R_f, we shall only focus on R_f. When $t < T_s$, the underlying is R ($R_h = 0$, thus $R_f = R$), whose volatility decreases after kicking into $[T_s, T_e]$; when $t > T_s$, if we look at a particular forward portion $[T_0, T_e]$, it's volatility again is going to decrease as t approaches T_e. However, the industry standard model to price vanilla fixed income derivatives is SABR, which does not account for such volatility decay feature of RFR rate. Ideally, we would like to customize SABR to incorporate the volatility-decay effect, in the meanwhile, preserve all nice properties of SABR model.

1.1 Deterministic Volatility

Before delving into customized SABR model, let us consider a simplified version – CEV model – in which the volatility is deterministic, i.e., vol-of-vol $\nu = 0$. Let us introduce the decay function:

$$\varphi(t) = \min\left(1, \frac{T_e - t}{T_e - T_s}\right)^q \tag{4}$$

and the corresponding CEV SDE for $R(\cdot)$ is,

$$dR(t) = \alpha\varphi(t)R^\beta(t)dW(t) \tag{5}$$

For $t \leq T_s$, R follows a standard CEV process, for $T_s < t \leq T_e$, the volatility is scaled down so that R becomes increasingly certain for as $t \to T_e$. The parameter q controls how fast the volatility is reduced. Large values of q lead to a faster reduction, as shown in *Figure 1*.

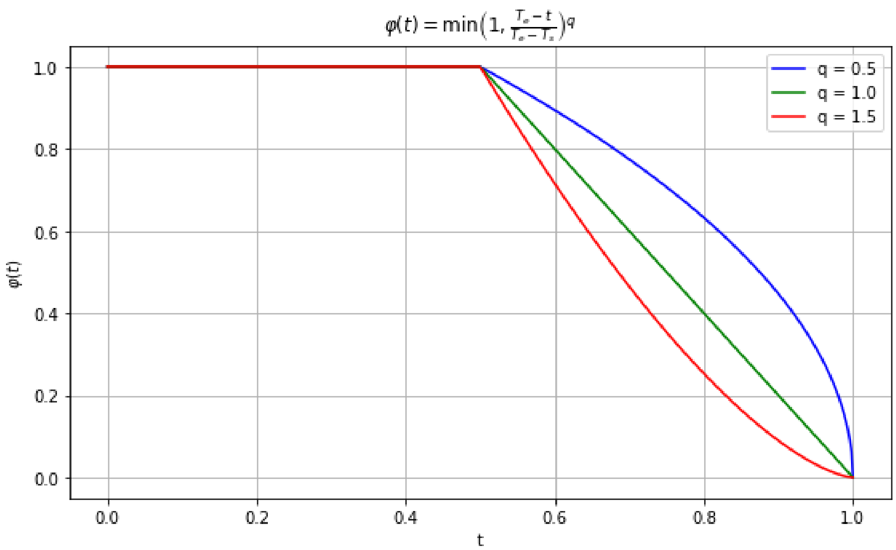

Figure 1: Decay function for different values of q with $T_s = 0.5$, $T_e = 1$

Let us compare (5) to

$$dR(t) = \widehat{\alpha}R(t)^\beta dW(t) \tag{6}$$

We want to find $\widehat{\alpha}$ such that the two models have the same marginal distribution of $R(T_e)$. Let's define a new process $X(t)$ such that:

$$X(t) = \frac{R(t)^{1-\beta}}{1-\beta} \tag{7}$$

By Itô's lemma:

$$dX(t) = \alpha\phi(t)dW(t) \tag{8}$$

This transformation linearizes the SDE — all dependence on $R(t)^\beta$ disappears in $X(t)$. So in the transformed world:

$$dX(t) = \alpha\phi(t)dW(t) \quad \Rightarrow \quad X(T_e) \sim \mathcal{N}\left(X(0),\, \alpha^2\theta(T_e)\right) \tag{9}$$

Under the effective model:

$$dX(t) = \widehat{\alpha} dW(t) \quad \Rightarrow \quad X(T_e) \sim \mathcal{N}\left(X(0),\, \widehat{\alpha}^2 T_e\right) \tag{10}$$

Now we equate distributions of $X(T_e)$ to ensure the same marginal for $R(T_e)$:

$$\alpha^2 \theta(T_e) = \widehat{\alpha}^2 T_e \quad \Rightarrow \quad \widehat{\alpha} = \sqrt{\frac{\theta(T_e)}{T_e}}\alpha \tag{11}$$

This shows in deterministic volatility case, we can derive the effective parameter for a standard CEV process.

1.2 Stochastic Volatility

In the stochastic case, the idea is the same, we want to derive the effective SABR parameters for $R(t)$ that follows

$$\begin{cases} dR(t) = \phi(t)\sigma(t)R^{\beta}(t)dW_1(t), \\ d\sigma(t) = \nu\sigma(t)dW_2(t),\ \ \sigma(0) = \alpha, \\ dW_1(t)dW_2(t) = \rho dt \end{cases} \tag{12}$$

Then we can use the standard Hagan's formula for pricing. In [1], the author derives the effective SABR parameters for the system (12),

$$\widehat{\alpha}^2 = \frac{\alpha^2}{2q+1}\frac{T}{T_e}e^{\frac{1}{2}HT_e},\ \ \widehat{\beta} = \beta,\ \ \widehat{\rho} = \rho\frac{3T^2 + 2qT_s^2 + T_e^2}{\sqrt{\gamma}(6q+4)},\ \ \widehat{\nu}^2 = \nu^2\gamma\frac{2q+1}{T^3 T_e} \tag{13}$$

where,

$$\begin{aligned} T &:= 2qT_s + T_e, \\ H &:= \nu^2\frac{T^2 + 2qT_s^2 + T_e^2}{2T_e T(q+1)} - \widehat{\nu}^2, \\ \gamma &:= \frac{2T^3 + T_e^3 + (4q^2 - 2q)T_s^3 + 6qT_s^2 T_e}{(4q+3)(2q+1)} + 3q\rho^2(T_e - T_s)^2\frac{3T^2 - T_e^2 + 5qT_s^2 + 4T_s T_e}{(4q+3)(3q+2)^2} \end{aligned}$$

2 Pricing Details

The caplet/floorlet pricing in the case $R = R_f$ (none of the fixings are realized) is straightforward,

- Given marked parameters $(\alpha, \beta, \nu, \rho)$, determine the effective parameters of s-shifted process[1] $R(t) + s$ via (13).
- Proceed with standard SABR implied vol approximation, i.e., *Hagan's* formula.

When fixings start to be realized, it is a little more involved. Suppose the valuation time $T_0 \in (T_s, T_e)$, the caplet pricing formula is

$$V^b(T_0) = \tau P(T_0, T_e) \cdot E_{T_0}^{T_e}\left[(R(T_e; T_s, T_e) - K)^+\right] \tag{14}$$

[1] As usual, when dealing with interest rate, we have to account for the case where rates dives into negative territory. Thus a positive shift, e.g., $s = 4\%$, is applied on log-normal/SABR model, and more.

Recall,

$$1 + \tau R = (1 + \tau_h R_h)(1 + \tau_f R_f) \tag{15}$$

Thus,

$$\begin{aligned} V^b(T_0) &= \tau P(T_0, T_e) \cdot E_{T_0}^{T_e}\left[\left(\frac{1}{\tau}\left((1+\tau_h R_h)\left(1+\tau_f R_f(T_e; T_0, T_e)\right) - 1\right) - K\right)^+\right] \\ &= \tau P(T_0, T_e) \cdot E_{T_0}^{T_e}\left[\left(\frac{\tau_f}{\tau}\left(1+\tau_h R_h\right) R_f(T_e; T_0, T_e) - \left(K - \frac{\tau_h}{\tau} R_h\right)\right)^+\right] \\ &= \tau P(T_0, T_e) \cdot E_{T_0}^{T_e}\left[\left(R_f^*(T_e; T_0, T_e) - K^*\right)^+\right] \end{aligned} \tag{16}$$

where

$$R_f^*(t; T_0, T_e) := \omega(R_f(t; T_0, T_e) + s),\ \omega := \frac{\tau_f}{\tau}\left(1 + \tau_h R_h\right),\ K^* := K + \omega s - \frac{\tau_h}{\tau} R_h \tag{17}$$

Given R_f follows a shifted version of (12), apply Itô's rule on $R_f^*(t)$, we have

$$\begin{aligned} dR_f^*(t) &= \omega dR_f(t) \\ &= \omega \phi(t)\sigma(t)\left(R_f(t) + s\right)^\beta dW_1(t) \\ &= \omega^{1-\beta}\phi(t)\sigma(t)\left(R_f(t) + s\right)^\beta dW_1(t) \\ &= \omega^{1-\beta}\phi(t)\sigma(t)\left(R_f^*(t)\right)^\beta dW_1(t) \end{aligned}$$

To re-write in the time-decay SABR format, we have,

$$\begin{cases} dR_f^*(t) = \phi(t)\sigma(t)\left(R_f^*(t)\right)^\beta dW_1(t),\ R_f^*(0) = \omega(R_f(0) + s) \\ d\sigma(t) = \nu\sigma(t) dW_2(t),\ \sigma(0) = \alpha^* = \alpha\omega^{1-\beta}, \\ dW_1(t) dW_2(t) = \rho dt \end{cases} \tag{18}$$

To evaluate (16), we proceed as follows:

- Given trader marked parameters $(\alpha, \beta, \nu, \rho)$ and caplet strike K, we compute transformed α^* and $R_f^*(0)$ as in (18), strike $K^* = K + \omega s - \frac{\tau_h}{\tau} R_h$.
- Derive effective parameters via (13) based on $(\alpha^*, \beta, \nu, \rho)$.
- Proceed with standard SABR implied vol approximation, i.e., *Hagan's* formula.

3 Comments on Pricing Caplet with Underlying R^a

Recall the arithmetic average index R^a, which can be approximated by

$$R^a(t; T_s, T_e) \approx R^c(t; T_s, T_e) - \frac{\tau(R^c(t; T_s, T_e))^2}{2} \approx R(t; T_s, T_e) - \frac{\tau(R(t; T_s, T_e))^2}{2} \tag{19}$$

Then, to price a caplet on $R^a(T_e;T_e,T_s)$,

$$\begin{aligned} V_a^b(t) &= \tau P(t,\,T_e)\cdot E_t^{T_e}\left[(R_a(T_e;T_s,T_e)-K)^+\right] \\ &\approx \tau P(t,\,T_e)\cdot E_t^{T_e}\left[\left(R(T_e;T_s,T_e)-\frac{\tau(R(T_e;T_s,T_e))^2}{2}-K\right)^+\right] \\ &\approx \tau P(t,\,T_e)\cdot E_t^{T_e}\left[\left(R(T_e;T_s,T_e)-\frac{\tau(R(t;T_s,T_e))^2}{2}-K\right)^+\right] \\ &= \tau P(t,\,T_e)\cdot E_t^{T_e}\left[\left(R(T_e;T_s,T_e)-K'\right)^+\right] \end{aligned} \tag{20}$$

with $K'=K+\frac{\tau R^2(t)}{2}$. Here, we made an approximation to freeze $R(\cdot;T_s,T_e)$ at time t. It makes the stochastic correction deterministic so that it can be absorbed by the strike. This is a sensible assumption because the square of R multiplied by τ is a smaller number. In other words, the pricing of a caplet on an arithmetic average index can be approximated by the standard compounding index with a strike shift. Thus, to continue on (20), we can follow the same procedure as in *Section 2*.

4 Reference

[1] *Sander Willems, "SABR Smiles For RFR Caplets", Risk Magazine, 2021*