SABR Average Analytics

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List of Notations:

- $\mathbb{E}_t[\cdot]$: conditional expectation given partial information $\mathcal{F}_t$;
- $R^i(\cdot)$: forward rate starting at some future time instance indexed by i, i.e., $R^i(t) = R(t; T_i, T_i + \delta_0)$, where δ_0 is the tenor;
- $R(\cdot)$: δ-tenor forward rate compounded through R^i, $R(t) = R(t; T, T + \delta)^1$;

1 Objective

For a future period $[T, T + \delta]$, we use the following discretizations: $0 < T = T_0 < T_1, ..., T_N = T + \delta$ with $T_{i+1} - T_i = \delta_0$(assume no short stubs in the begining or end). Equip $R^i(\cdot)$ with SABR dyanmics, $\forall i \in \{0, ..., N-1\}$

$$\begin{cases} dR_t^i = z_t^i g^i(R_t^i)\,\mathrm{d}W_t^i \\ dz_t^i = z_t^i \nu^i\,\mathrm{d}B_t^i \\ z_0^i = 1 \end{cases} \tag{1}$$

Notice, there are three types of correlations: 1) ρ_{ij} among rates process; 2) ξ_{ij} among volatility processes; 3)γ_{ij} between rates and volatility processes. Type (iii) is the part of SABR, as $\langle dW_t^i, dB_t^i \rangle = \rho_{ii}\mathrm{d}t$. We summarize them by a correlation matrix,

$$\Sigma \equiv \begin{bmatrix} \gamma & \rho \\ \rho & \xi \end{bmatrix} \tag{2}$$

In general, $R = \phi(R^1, ..., R^N)$, where ϕ is subject to compounding method. The objective is:

- derive the dynamics for $\mathrm{d}R(t)$;
- project derived dynamics to a SABR-like system.

The first one is Itô, the second one requires loads of assumptions and approximation (as always).

[1]Obviously, $\delta_0 < \delta$, for instance, δ could be 3M while δ_0 is just 1B, which is the case for RFR.

2 Itô's Step

Eq.1 is not precise. We know each R^i is martingale under corresponding terminal measure, i.e., T_i-forwad measure, if we unify to T_N-forward measure, each constituents would have a drift term. Apply Itô on $\phi(R^1, ..., R^N)$, under T_N-forward measure,

$$dR_t = \sum_i [\partial_i \phi] dR_t^i + (\cdots)dt \tag{3}$$

The drift term comes from two sources: 1) the drift of each constituents; 2) the second order/cross term from Itô's formula. We want to justify that both of them are "small" thus can be ignored. The former one is small because the drift term has scaling factor δ_0 and it is bounded above by δ, which in most cases is just 3M[2], the Itô higher order term is scaled by the sensitvity of $\phi(\cdot)$, one can verify easily that for both arithmetic and geometric method, it's negligible.

Here $\partial\phi_i$ is the interesting quantity. If we have arithmetic average compounding, $\partial_i\phi$ is a deterministic quantity($1/acc(T_i, T_{i+1})$), therefore it's exact. We end up with a deterministic weights vector $\omega = [\partial\phi_1, ..., \partial\phi_N]$. Rerwite (3) as:

$$dR_t = \sum_i \omega^i z_t^i g^i(R_t^i)\,\mathrm{d}W_t^i \equiv \sum_i z_t^i f_t^i, \mathrm{d}W_t^i \tag{4}$$

Here, we glue ω_i with $g^i(R_t^i)$ as f_t^i. On the other hand, under terminal measure $R(\cdot)$ should follow a driftless SDE, and we would like project the state process on,

$$\mathrm{d}R_t = z_t g(R_t)\mathrm{d}W_t \tag{5}$$

and the target volatility process,

$$dz_t = \nu z_t dB_t \tag{6}$$

where $\langle dW_t, dB_t\rangle = \rho dt$. In general R_t could have a more general diffusion coefficient, we put the above structure, $z_t g(R_t)$, makes it the closest to SABR(z_t and R_t are separated). Comparing (4) to (5)-(6), we derive the following:

- To synthesize a basket of W_t^i to W_t, we notice

$$Var\left(\sum_i w^i f^i(R_t^i) dW_t^i\right) = \sum_{ij} f_t^i f_t^j \gamma_{ij} z_t^i z_t^j dt \equiv \sigma_t^2 dt \tag{7}$$

  To map to a standard Brownian motion W_t, we just need to normalize, $dW_t = \frac{1}{\sigma_t}\sum_i f_t^i z_t^i\,\mathrm{d}W_t^i$.

[2]One can refer to the change of measure in LMM model.

- Forz_t part, if there were no $g(R_t)$ term in state equation, z_t should just be σ_t to match the variance. In the meanwhile, to have z_t eventually being SABR-Like, we need to have $z_0 = 1$, which can be easily achieved by dividing σ_0. Also, SABR's volatility process does not interact directly with forward process, while σ_t has explicit dependency on R_t^i, R_t^j, to address that, we freeze the forward in σ_t at time 0, i.e.,

$$z_t \equiv \frac{1}{\sigma_0^2} \sum_{ij} f_0^i f_0^j \gamma_{ij} z_t^i z_t^j \tag{8}$$

  Eq.(8) is our proposal for z_t, if we plug it back into (5), it's kinda reasonable as the freezing part will be pick up by $g(R_t)$. Notice, we haven't really written down what $\mathrm{d}z_t$ looks like yet;

- to determine $g(R_t)$, we leave to the next section.

3 Gyöngy's Step

To motivate ourself, let us think in a very general setting, where we have $\{X_t\}$ driven by:

$$\mathrm{d}X_t = \beta_t \,\mathrm{d}W_t \tag{9}$$

where β_t is adapted stochastic process. This is a very general SDE, and in practice, people usually found Markovian structure very appealing, that is β_t can be characterized by a deterministic function of time and state itself. Namely, we would like to find another process Y_t given by:

$$\mathrm{d}Y_t = b(t, Y_t) \,\mathrm{d}W_t \tag{10}$$

such that for any $t \geq 0$, X_t and Y_t has same terminal distribution, i.e.,

$$\mathbb{P}(X_t \in B) = \mathbb{P}(Y_t \in B), \;\; B \in \mathcal{B}(\mathbb{R}) \tag{11}$$

Gyöngy's Lemma says such process always exist (under mild conditions) as long as we have the following condition satisfied:

$$\mathbb{E}[\beta_t^2 \,|\, X_t] = \mathbb{E}[b^2(t, X_t) \,|\, X_t]. \tag{12}$$

We can apply this theorem directly (4) and (5) to determine $g(R_t)$. That is,

$$\mathbb{E}[\sigma_t^2 \,|\, R_t] = \mathbb{E}[z_t^2 g^2(R_t) \,|\, R_t] = g(R_t)\mathbb{E}[z_t^2 \,|\, R_t] \tag{13}$$

which implies:

$$g^2(R_t) = \frac{\mathbb{E}[\sigma_t^2 \,|\, R_t]}{\mathbb{E}[z_t^2 \,|\, R_t]} \tag{14}$$

For the rest, we will need to explore conditional expectation appear above to make it more explicit by reasonable approximation.

4 Approximations Step

We outline the essential workflow here, the main technique is Taylor expansion and short-time expansion, which is widely used for such considerations in math finance. Notice, the algebraic manipulation is not of particular importance, thus they are omitted when we can.

4.1 Treat $\mathbb{E}[\sigma_t^2 \mid R_t]$

Recall, σ_t^2 has the following expression:

$$\sigma_t^2 = \sum_{ij} f_t^i f_t^j \gamma_{ij} z_t^i z_t^j \tag{15}$$

Let us apply Taylor expansion on f_t^i, z_t^i,

$$\begin{aligned} f_t^i &= f_0^i + f_0^{i'}(R_t^i - R_0^i) + o(R_t^i), \\ z_t^i &= z_0 + z_t^i - z_0 = 1 + (z_t^i - 1) \end{aligned} \tag{16}$$

Substitute (16) into (15), we shall end up with:

$$\sigma_t^2 \approx \sigma_0^2 \left(z_t + \frac{2}{\sigma_0^2} \sum_{ij} f_0^{i'} f_0^j \gamma_{ij} \left(R_t^i - R_0^i \right) \right) \tag{17}$$

Taking conditional expectation,

$$\mathbb{E}[\sigma_t^2 \mid R_t] = \sigma_0^2 \left(z_t + \frac{2}{\sigma_0^2} \sum_{ij} f_0^{i'} f_0^j \gamma_{ij} \left(\mathbb{E}[R_t^i - R_0^i \mid R_t] \right) \right) \tag{18}$$

We made a bit progress to have R_t^i presented above, the ultimate goal is to write it as function of R_t. Let's carry on. We consider the short-time expansion of R_t^i, linearize diffusion coefficients at $t = 0$, i.e.,

$$dR_t^i = \frac{z_t^i}{\omega_i} f_t^i \,\mathrm{d}W_t^i \approx \frac{z_t^i}{\omega_i} f_t^i \big|_{t=0} \,\mathrm{d}W_t^i = \frac{1}{\omega_i} f_0^i \,\mathrm{d}W_t^i \tag{19}$$

Notice, z_t is freezed at 0 therefore disappeared as $z_0^i = 1$. This allows us to approximate R_t by,

$$dR_t \approx \sigma_0 dW_t \tag{20}$$

Since R_t is a linear combination of R_t^i, then $[R_t, R_t^i]$ is jointly normal with

$$\begin{aligned} Var(R_t) &= \sigma_0^2 \cdot t, \;\; Var(R_t^i) = \left(\frac{R_0^i}{\omega^i} \right)^2 \cdot t \\ Cov(R_t^i, R_t) &= \frac{f_0^i \cdot t}{\omega^i} \sum_j f_0^j \gamma_{ij} \end{aligned} \tag{21}$$

Let us introduce a formula that facilitate the following calculation(and will be used again later). For random variables $[X, Y]$ that are jointly normal distributed, we have

$$\mathbb{E}[X|Y=y] = \mathbb{E}[X] + \frac{Cov(X,Y)}{Var(Y)}(y - \mathbb{E}[Y]) \tag{22}$$

Apply to our case, we have,

$$\begin{aligned}
&\mathbb{E}[R_t^i - R_0^i \,|\, R_t] \\
=&\mathbb{E}[R_t^i - R_0^i] + \frac{Cov(R_t^i, R_t)}{Var(R_t)}\big(R_t - \mathbb{E}[R_t]\big) \\
=&\frac{f_0^i}{\omega^i \sigma_0^2} \sum_j f_0^j \gamma_{ij} (R_t - R_0)
\end{aligned} \tag{23}$$

4.2 Treat $\mathbb{E}[z_t^2 \,|\, R_t]$

Substitute (16) into (8), we can simplify:

$$\begin{aligned}
z_t &= \sqrt{\frac{1}{\sigma_0^2}\bigg(\sum_{ij} f_0^i f_0^j \gamma_{ij} + 2\sum_{ij} f_0^i f_0^j \gamma ij (z_t^i - 1)\bigg)} \\
&= \sqrt{1 + \frac{2}{\sigma_0^2}\sum_{ij} f_0^i f_0^j \gamma_{ij} (z_t^i - 1)}
\end{aligned} \tag{24}$$

Taking conditional expectation renders,

$$\mathbb{E}[z_t \,|\, R_t] = \sqrt{1 + \frac{2}{\sigma_0^2}\sum_{ij} f_0^i f_0^j \gamma_{ij} (\mathbb{E}[z_t^i - 1 \,|\, R_t])} \tag{25}$$

The rest follows the same procedures as the above section: using short maturity expansion, we have

$$\mathrm{d}z_t^i \approx \nu^i \mathrm{d}B_t^i \tag{26}$$

Obviously, z_t^i are normal and have the following moments of z_t^i,

$$Var(z_t^i) = (\nu^i)^2 \cdot t, \;\; Cov(z_t^i, R_t) = \nu^i t \sum_j f_0^j \rho_{ij} \tag{27}$$

Using the same identity as in (22),

$$\begin{aligned}
\mathbb{E}[z_t^i - 1 \,|\, R_t] &= \mathbb{E}[z_t^i - 1] + \frac{Cov(z_t^i, R_t)}{Var(R_t)}\bigg(R_t - \mathbb{E}[R_t]\bigg) \\
&= \frac{\nu^i}{\sigma_0^2}\sum_j f_0^j \rho_{ij}\big(R_t - R_0\big).
\end{aligned} \tag{28}$$

4.3 Characterize $g(R_t)$

In the previous two steps, we have successfully represented denominator and numerator in (14) as function of R_t. Let us summarize here, set

$$c^i(x) \equiv \frac{f_0^i}{\omega^i \sigma_0^2} \sum_j f_0^j \gamma_{ij} x, \;\; d^i(x) \equiv \frac{\nu^i}{\sigma_0^2} \sum_j f_0^j \rho_{ij} x, \tag{29}$$

we have

$$g^2(R_t) \approx \sigma_0^2 \left(1 + \frac{2 \sum_{ij} f_0^{i'} f_0^j \gamma_{ij} c^i (R_t - R_0)}{\sigma_0^2 + 2 \sum_{ij} f_0^i f_0^j \gamma_{ij} d^i (R_t - R_0)} \right) \tag{30}$$

To simplify (30) further(no one likes quotient and square root), we apply again Taylor approximation, which brings us to:

$$g(R_t) = \sigma_0 + \left(\frac{1}{\sigma_0^3 \omega^i} \sum_{ij} f_0^{i'} f_0^i \gamma_{ij} f_0^j \sum_k f_0^k \gamma_{ik} \right) (R_t - R_0) \tag{31}$$

4.4 Characterize ν for z_t

We managed to characterize state equation by expressing $g(\cdot)$ as function of R_t. Now, we need to take care of volatility process z_t, in particular, the vol-of-vol ν. Recall the form of z_t^2 (Eq.(8)). To get $\mathrm{d}z_t$, we want to get rid off square root by Taylor approximation and then,

$$z_t \approx 1 + \frac{1}{\sigma_0^2} \sum_{ij} f_0^i f_0^j \gamma_{ij} (z_t^i - 1) \tag{32}$$

or, equivalently,

$$dz_t = \frac{1}{\sigma_0^2} \sum_{ij} f_0^i f_0^j \gamma_{ij} \nu^i z_t^i dB_t^i \tag{33}$$

The target form of z_t is $\mathrm{d}z_t = z_t \nu dB_t$, this reminds us Gyöngy again, it is sufficient if we have,

$$\frac{1}{\sigma_0^4} \sum_{i,j,p,q} f_0^i f_0^j \gamma \nu^i \xi_{ip} f_0^p f_0^q \gamma_{pq} \nu^p z_t^i z_t^p = \mathbb{E}[\nu^2 z_t^2 \mid z_t] \tag{34}$$

By freezing z^i and z^p at time $t = 0$,

$$L.H.S \approx \frac{1}{\sigma_0^4} \sum_{i,j,p,q} f_0^i f_0^j \gamma_{ij} \nu^i \xi_{ip} f_0^p f_0^q \gamma_{pq} \nu^p \tag{35}$$

Freezing z_t at time $t = 0$ gives,

$$\nu^2 \approx \frac{1}{\sigma_0^4} \sum_{i,j,p,q} f_0^i f_0^j \gamma_{ij} \nu^i \xi_{ip} f_0^p f_0^q \gamma_{pq} \nu^p \tag{36}$$

4.5 Finalize Correlation Between B_t and W_t

Lastly, the correlation between B_t and W_t needs to be estimated. The way we go about it is as follows, take volatility processes as example, we know the two equivalent expressions are(up to approximation)

$$\begin{aligned}
\mathrm{d}z_t &= \frac{1}{\sigma_0^2}\sum_{ij} f_0^i f_0^j \gamma_{ij}\nu^i z_t^i \mathrm{d}B_t^i, \\
\mathrm{d}z_t &= \nu z_t \nu \mathrm{d}B_t
\end{aligned} \tag{37}$$

Hence, we can approximate B_t by

$$\begin{aligned}
\mathrm{d}Z_t &= \sum_{ij} \frac{f_0^i f_0^j \gamma_{ij}\nu^i z_t^i}{\nu z_t \sigma_0^2}\,\mathrm{d}B_t^i \\
&\approx \sum_{ij} \frac{f_0^i f_0^j \gamma_{ij}\nu^i z_0^i}{\nu z_0 \sigma_0^2}\,\mathrm{d}B_t^i \\
&= \sum_{ij} \frac{f_0^i f_0^j \gamma_{ij}\nu_{ij}\nu^i}{\nu \sigma_0^2}\mathrm{d}B_t^i
\end{aligned}$$

Similarly, same story for R_t,

$$\mathrm{d}W_t \approx \sum_i \frac{f_0^i}{\sigma_0}\mathrm{d}W_t^i \tag{38}$$

The rest is straightforward calculation of quadratic variation, i.e., $\mathrm{d}[W_t, B_t]$, which renders

$$\rho = \frac{1}{\nu\sigma_0^3}\sum_{i,j,u} f_0^i f_0^j f_0^u \nu^i \gamma_{ij}\rho_{iu} \tag{39}$$

5 Simplification Under Homogeneity

So far, we arrived at general SABR[3] dynamics for R_t and z_t,

$$\begin{cases}
\mathrm{d}R_t = z_t g(R_t)\,\mathrm{d}W_t \\
\mathrm{d}z_t = \nu z_t\,\mathrm{d}B_t, \\
\langle \mathrm{d}W_t, \mathrm{d}B_t\rangle = \mathrm{d}t
\end{cases} \tag{40}$$

where $g(\cdot)$, ν and ρ are specified in (31)-(36)-(39). In this section, we make further assumptions, i.e., all parameters are homogeneous, to significantly simplify calculations and the resulting dynamics will be of classic SABR

[3]The classic SABR has explicit form of $g(x) = x^\beta$.

fashion.

Homogeneity Assumption: the correlations satisfy, for $i = j$, γ_{ii} and ξ_{ii} are both equal to 1, otherwise,

$$\gamma_{ij} = \gamma,\ \xi_{ij} = \xi \tag{41}$$

for correlation between volatilities and forward, $\rho_{ij} \equiv \sum_i \omega^i \rho_{ii}$.For other SABR parameters,

$$\nu_i \equiv \sum_i \omega^i \nu^i,\ g_i(x) = \bar{\alpha}(x+s)^{\bar{\beta}} \tag{42}$$

where $\bar{\alpha} \equiv \sum_i \omega^i \alpha^i$ and $\bar{\beta} \equiv \sum_i \omega^i \beta^i$($s$ is the shift).

Under this assumption,

$$f_0^i = \frac{1}{n}\bar{g}(R_0) \equiv \bar{f}_0,\ f_0^{i'} = \frac{1}{n}\bar{g}'(R_0) \equiv \bar{f}_0^{i'}, \tag{43}$$

we can compute σ_0 as follows

$$\sigma_0 = \sqrt{\sum_{ij} f_0^i f_0^j \gamma_{ij} z_0^i z_0} = n\bar{f}_0\sqrt{\bar{\gamma}} \tag{44}$$

where $\bar{\gamma} = \frac{1}{n^2}\sum_{ij}\gamma_{ij} = \frac{1+(n-1)\gamma}{n}$. Therefore, we can compute ν as

$$\begin{aligned}
\nu &= \frac{1}{\sigma_0^2}\sqrt{\sum_{i,j,p,q} f_0^i f_0^j \gamma_{ij}\nu^i \xi_{ip} f_0^p f_0^q \gamma_{pq}\nu^p} \\
&= \frac{\bar{\nu}\bar{f}_0^2}{\sigma_0^2}\sqrt{\sum_{ip}\bar{\xi}_{ip}\sum_{jq}\gamma_{ij}\gamma_{pq}} \\
&= \frac{\bar{\nu}}{n^2\bar{\gamma}}\sqrt{n^2\bar{\xi}(1+(n-1)\gamma)^2} \\
&= \nu\sqrt{\bar{\xi}}
\end{aligned}$$

where $\bar{\xi} \equiv \frac{1+(n-1)\xi}{n}$. For ρ, we have

$$\begin{aligned}
\rho &= \frac{1}{\nu\sigma_0^3}\sum_{iju} f_0^i f_0^j f_0^u \nu_i \gamma_{ij}\rho_{iu} \\
&= \frac{\bar{f}_0^3}{\sigma_0^3}\frac{\bar{\nu}\bar{\rho}}{\bar{\nu}\sqrt{\xi}}\sum_{ju}\sum_{i}\gamma_{ij} \\
&= \frac{1}{(n\sqrt{\bar{\gamma}})^3}\frac{\bar{\rho}}{\sqrt{\xi}}\sum jun\bar{\gamma} \\
&= \frac{\bar{\rho}}{\sqrt{\bar{\gamma}\bar{\xi}}}
\end{aligned}$$

Finally, for $g(\cdot)$, we have

$$
\begin{aligned}
g(R_t) &= \sigma_0 + \left(\frac{1}{\sigma_0^3\omega^i}\sum_{ij} f_0^{i'} f_0^i \gamma_{ij} f_0^j \sum_k f_0^k \gamma_{ik}\right)(R_t - R_0) \\
&= \bar{g}(R_0)\sqrt{\bar{\gamma}} + \left(\frac{n\bar{f}_0'}{(n\sqrt{\bar{\gamma}})^3}\sum_j\sum_i \gamma_{ij}\sum_k \gamma_{ik}\right)(R_t - R_0) \\
&= \bar{g}(R_0)\sqrt{\bar{\gamma}} + \left(\frac{n\bar{f}_0'}{(n\sqrt{\bar{\gamma}})^3}\sum_j (1+(n-1)\gamma)^2\right)(R_t - R_0) \\
&= \bar{g}(R_0)\sqrt{\bar{\gamma}} + \left(\frac{n^2\bar{f}_0'}{(n\sqrt{\bar{\gamma}})^3} n^2\bar{\gamma}^2\right)(R_t - R_0) \\
&= \bar{g}(R_0)\sqrt{\bar{\gamma}} + (n\bar{f}_0'\sqrt{\bar{\gamma}})(R_t - R_0) \\
&= \bar{g}(R_0)\sqrt{\bar{\gamma}} + (\bar{g}'(R_0)\sqrt{\bar{\gamma}})(R_t - R_0)
\end{aligned}
$$

Since $g(x) = \alpha(x+s)^\beta$, we have

$$
g(R_t) = \sqrt{\bar{\gamma}}\bar{\alpha}(R_0+s)^\beta + \bar{\beta}\sqrt{\bar{\gamma}}\bar{\alpha}(R_0+s)^{\bar{\beta}-1}(R_t - R_0), \tag{45}
$$

which is just the first order Taylor expansion around R_0 of $\bar{g}(R_t)$, i.e.,

$$
\bar{g}(R_t) = \sqrt{\bar{\gamma}}\bar{\alpha}(R_t+s)^{\bar{\beta}}. \tag{46}
$$

We will use $\bar{g}(R_t)$ to replace the true $g(R_t)$, therefore, we have the following SABR dynamics under homogeneous parameter assumption:

$$
\begin{cases}
\mathrm{d}R_t = \alpha z_t (R_t+s)^\beta \,\mathrm{d}W_t \\
\mathrm{d}z_t = \nu z_t \,\mathrm{d}B_t, \\
\langle \mathrm{d}W_t, \mathrm{d}B_t\rangle = \mathrm{d}t
\end{cases} \tag{47}
$$

where

$$
\begin{aligned}
\alpha &= \sqrt{\bar{\gamma}}\sum_i \omega^i\alpha^i, \\
\beta &= \sum_i \omega^i\beta^i, \\
\nu &= \sqrt{\bar{\xi}}\sum_i \omega^i\nu^i, \\
\rho &= \frac{1}{\sqrt{\bar{\gamma}\bar{\xi}}}\sum_i \omega^i\rho^i.
\end{aligned}
$$

6 Final Adjustments

In this final section, we try to address two issues: 1) the time inconsistency in the constituents SABR; 2) the estimation of γ.

6.1 Time-Correction

The synthetic index $R = \phi(R^1, ..., R^N)$ has an imaginary fixing time $\widetilde{T} = \sum_i \omega^i T^i$. However, for each constituents SABR, say i-th one, it is suppose to produce the right terminal distribution at time T^i. To make sure we retrieve $\widetilde{T}$ the same density we had at T^i, we can scale the time by $\theta^i = \frac{T^i}{\widetilde{T}^i}$ and work with $\widetilde{R}^i_t = R^i_{\theta^i t}$ and $\widetilde{z}^i_t = z^i_{\theta^i t}$. They following the following SDE:

$$\begin{cases} \mathrm{d}\widetilde{R}^i_t = \widetilde{z}^i_t g^i(\widetilde{R}^i_t)\,\mathrm{d}W^i_{\theta^i t} = \widetilde{z}^i_t g^i(\widetilde{R}^i_t)\sqrt{\theta^i}\,\mathrm{d}\widetilde{W}^i_t \\ \mathrm{d}\widetilde{z}^i_t = \widetilde{z}^i_t \nu^i \sqrt{\theta^i}\,\mathrm{d}\widetilde{B}^i_t \end{cases} \tag{48}$$

where $\widetilde{W}^i_t$ and $\widetilde{Z}^i_t$ are the time-changed processes. At this point, we see that in order to obtain at $\widetilde{T}$ the density we had at T^i, we need to scale α, ν as follows:

$$\alpha^i \to \alpha^i \sqrt{T^i/\widetilde{T}},\ \nu^i \to \nu^i \sqrt{\frac{T^i}{\widetilde{T}}}. \tag{49}$$

6.2 Estimation of Correlation γ

Recall our definition of $\bar{\gamma} = \frac{1}{n^2}\sum_{ij}\gamma_{ij}$. We assume that the correlation γ_{ij} can be approximated as linear with distance between the fixing time of R^i and R^j:

$$\gamma_{ij} \approx 1 - \mu|T^i - T^j| \tag{50}$$

The damping parameter μ controls the decay of correlation. Since T^i are almost equally spaced, we can simplify it further as:

$$\gamma_{ij} = 1 - \mu|i - j|| \tag{51}$$

Under this assumption,

$$\bar{\gamma} \approx \frac{1}{n^2}\sum_{ij}\big(1 - \mu|i-j|\big) = 1 - \frac{\mu}{n^2}\sum_{ij}|i-j| = 1 - \mu\frac{n(n+1)(n-1)}{3} \tag{52}$$

Notice, the last step requires little algebra. To calibrate μ, we can use the correlation between the first and last fixing forward rate, γ_{1N}, i.e.,

$$\gamma_{1N} \approx 1 - \mu|n-1| \Rightarrow \mu = \frac{1 - \mu_{1N}}{n-1}. \tag{53}$$

Consequently, the average correlation is:

$$\bar{\gamma} \approx 1 - (1 - \gamma_{1N})\frac{N+1}{3N} \tag{54}$$

The problem of using (54) is that when doing risk, it will be projected out only to γ_{1N}, which is not ideal. To compromise, we consider 4-point correlation structure, where there are four equal-spaced points in $[T^1, T^N]$, denoted by t_1, t_2, t_3, t_4. Let $\hat{\gamma}_{ij}$ be the correlation between t_i and t_j, then by (51), we have

$$\begin{cases} \hat{\gamma}_{ii} = 1, & i = 1, 2, 3, 4, \\ \hat{\gamma}_{i(i+1)} = \hat{\gamma}_{(i+1)i} = \hat{\gamma}_{12}, & i = 1, 2, 3, \\ \hat{\gamma}_{i(i+2)} = \hat{\gamma}_{(i+2)i} = \hat{\gamma}_{13}, & i = 1, 2, \\ \hat{\gamma}_{i(i+3)} = \hat{\gamma}_{(i+3)i} = \hat{\gamma}_{14}, & i = 1 \end{cases} \tag{55}$$

Hence,

$$\bar{\gamma}^{(4)} = \frac{1}{16} \sum_{i,j=1}^{4} \gamma_{ij} \approx \frac{4 + 6 \cdot \gamma_{12} + 4 \cdot \gamma_{13} + 2 \cdot \gamma_{14}}{16}. \tag{56}$$

Equating (55) and (54), we obtain a representation of γ_{1N} in terms of four points,

$$\gamma_{1N} \approx 0.9\hat{\gamma}_{12} + 0.6\hat{\gamma}_{13} + 0.3\hat{\gamma}_{14} - 0.8 \equiv \gamma^* \tag{57}$$

Using γ^* instead of γ_{1N} gives

$$\bar{\gamma} \approx 1 - (1 - \gamma^*)\frac{n+1}{3n} \tag{58}$$

which makes correlation risk spread out on four points in the correlation structure.