Notes 1: Introduction to Risk-Free-Rate (RFR)

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Here are some necessary notations:

- $t \leq T_s < T_e$: t is a generic valuation time, T_s is the accrual start date of a rate compounding period, and T_e is the accrual end date.
- $P(t, T)$: the value of a zero-coupon bond matures at time T.
- F and R: F stands for forward LIBOR rate, R stands for forward RFR rate.

1 Recall: Forward LIBOR Rate

Recall LIBOR is fixed *in advance*, which means, at time T_s, the LIBOR fixing is available. For the whole accrual period $[T_s, T_e]$, a "fixed" simple rate is applied. The forward LIBOR rate is the par rate of an FRA that pays at T_e, namely[1],

$$P^L(t, T_e)E_t^{T_e}[L(T_s, T_e) - K] = 0 \Rightarrow F(t; T_s, T_e) := K = \mathbb{E}_t^{T_e}[L(T_s, T_e)] \tag{1}$$

where $L(T_s, T_e)$ is the LIBOR rate at time T_s. It can be shown, the conditional expectation can be expressed explicitly via ZCBs,

$$F(t; T_s, T_e) = \frac{1}{\tau}\left(\frac{P^L(t, T_s)}{P^L(t, T_e)} - 1\right) \tag{2}$$

We call *LIBOR* a term rate, as the tenor $\tau = \tau(T_s, T_e)$ are usually of length $1M$, $3M$, $6M$.

2 RFR Forward Rate

2.1 Definitions

The Risk-Free-Rate(RFR), on the contrary, is fixed *in arrear*. In other words, the fixing for the period $[T_s, T_e]$ is only known at time T_e. The most prevalent compounding style for RFR are,

$$\begin{aligned} R^c(T_e; T_s, T_e) &= \frac{1}{\tau}\left(\prod_{i=0}^{n-1}(1 + \tau_i R_i) - 1\right) \\ R^a(T_e; T_s, T_e) &= \frac{1}{\tau}\sum_{i=0}^{n-1}\tau_i R_i \end{aligned} \tag{3}$$

[1]the subscript t stands for conditioning on filtration up to time t, i.e., $\mathcal{F}(t)$.

Here, we assume there are in total n days in the accrual period $[T_s, T_e]$, R_i is the risk-free rate of i-th day, and τ_i is the 1-day accrual[2]. For instance, in USD, R_i is the *SOFR (Secured Overnight Financing Rate)* – a benchmark interest rate based on actual overnight repo transactions collateralized by U.S. Treasury securities.

Staring at (3), we can easily convince ourselves that RFR is fixed in arrear, because we will need to know all daily rates R_i's in the accrual period to determine its value. Mathematically, $R^x(T_e; T_s, T_e)$ is $\mathcal{F}(T_e)$-measurable, as oppose to $F(T_s; T_s, T_e)$ that is $\mathcal{F}(t_s)$-measurable. In addition, as RFR is daily compounding style, the accrual period τ is very flexible. It doesn't have to be $1M$, $3M$, etc, although the standard tenors are most liquid traded, such as, SOFR-3M.

2.2 Conventional Approximations

"$\prod$" is always nasty to deal with, one standard approximation is to use *continuously compounded* RFR index,

$$R(T_e; T_s, T_e) = \frac{1}{\tau}\left(e^{\int_{T_s}^{T_e} r(u)du} - 1\right) \tag{4}$$

where $r(u)$ is the instantaneous RFR rate (imagine $\tau_i \to 0$). To show R is a good approximation of R^c, we use Taylor expansion

$$\begin{aligned}
R(T_e; T_s, T_e) &= \frac{1}{\tau}\left(e^{\int_{T_s}^{T_e} r(u)du} - 1\right) \\
&\approx \frac{1}{\tau}\left(e^{\sum_{i=0}^{n-1} R_i\tau_i} - 1\right) \\
&= \frac{1}{\tau}\left(\prod_{i=0}^{n-1} e^{R_i\tau_i} - 1\right) \\
&= \frac{1}{\tau}\left(\prod_{i=0}^{n-1}\left(1 + R_i\tau_i + \mathcal{O}^2(R_i\tau_i)\right) - 1\right) \\
&\approx \frac{1}{\tau}\left(\prod_{i=0}^{n-1}(1 + R_i\tau_i) - 1\right) \\
&= R^c(T_e; T_s, T_e)
\end{aligned}$$

This allows us to focus on $R(\cdot; T_s, T_e)$, and all results apply to R holds for R^c.

Next, we examine the relationship between R^c and R^a. Let us recall the well-known exponential limit,

$$\lim_{x\to 0}(1 + ax)^{\frac{1}{x}} = e^a \tag{5}$$

Then,

$$\begin{aligned}
R^c(T_e; T_s, T_e) &\approx \frac{1}{\tau}\left(e^{\sum_i \tau_i R_i} - 1\right) \\
&= \frac{1}{\tau}\left(e^{\tau R^a(T_e; T_s, T_e)} - 1\right) \\
&= \frac{1}{\tau}\left(1 + \tau R^a(T_e; T_s, T_e) + \frac{1}{2}\tau^2(R^a(T_e; T_s, T_e))^2 + \mathcal{O}((\tau R^a(T_e; T_s, T_e))^3) - 1\right)
\end{aligned}$$

[2]Here, 1-day refers to business day, so it is not always 1d, it will be 3-day for a weekend, for instance.

Ignore the higher order terms,

$$R^a(T_e;T_s,T_e) \approx R^c(T_e;T_s,T_e) - \frac{\tau(R^c(T_e;T_s,T_e))^2}{2} \tag{6}$$

For any payoff with underlying R^a, we can re-express in terms of R^c via (6), i.e., $H(R^a) = H\left(R^c - \frac{\tau(R^c)^2}{2}\right)$.

2.3 Forward Rate

The RFR forward rate is defined as the par rate of RFR FRA, where one exchange a fixed rate K with RFR rate $R(T_e;T_s,T_e)$ for the accrual period $[T_s,T_e]$, and paid at time T_e. Similar to (1),

$$R(t;T_s,T_e) = E_t^{T_e}[R(T_e;T_s,T_e)] \tag{7}$$

We can derive the same formula as in the LIBOR case via change of numerarire,

$$\begin{aligned} 1+\tau R(t;T_s,T_e) &= E_t^{T_e}\left[e^{\int_{T_s}^{T_e} r(u)du}\right] \\ &= \frac{1}{P(t,T_e)} E_t^Q\left[e^{-\int_t^{T_s} r(u)du}\right] = \frac{P(t,T_s)}{P(t,T_e)} \end{aligned}$$

Equivalently,

$$R(t;T_s,T_e) = \frac{1}{\tau}\left(\frac{P^R(t,T_s)}{P^R(t,T_e)} - 1\right) \tag{8}$$

We recover the same formula as in the LIBOR case, but $P^R(t,\cdot)$ corresponds to instantaneous RFR rate $r(\cdot)$. As our focus is RFR, for notational convenience, we will skip the superscript R in $P^R(t,\cdot)$.

2.4 Understanding the Randomness

When $t \le T_s$, R refers to a collection of fixings in $[T_s,T_e]$, none of which are known. As soon as we kick into the accrual period, i.e., $t \in [T_s,T_e]$, R becomes increasingly deterministic, as increasing number of fixings are getting realized. Let us breakdown R into realized portion (R_h) and forward portion (R_f). Set an intermediate point $T_0 \in [T_s,T_e]$, and $\tau_h := \tau(T_0,T_s]$, $\tau_f := \tau(T_0,T_e)$, we have,

$$\begin{aligned} R_h &:= \frac{1}{\tau_h}\left(e^{\int_{T_s}^{T_0} r(u)du} - 1\right) \\ R_f &:= \frac{1}{\tau_f}\left(e^{\int_{T_0}^{T_e} r(u)du} - 1\right) \end{aligned}$$

We can re-express R as

$$1+\tau R = (1+\tau_h R_h)(1+\tau_f R_f) \tag{9}$$

For pricing derivatives on R, we have to know how to model the dynamics of R_F, whose uncertainty decays in the accrual period $[T_s,T_e]$. Notice when $\tau_h = 0$, $R_f = R$.