Notes 3: Bottom-up Approach SABR for RFR

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Here are some necessary notations:

- $t \le T_s < T_e$: t is a generic valuation time, T_s is the accrual start date of a rate compounding period, and T_e is the accrual end date.
- τ: accrued, $\tau(T_s, T_e)$.
- $P(t,T)$: the value of a zero-coupon bond matures at time T.
- R: it stands for forward RFR rate.

1 A Bottom-Up Approach

In the previous notes, we have shown a top-down approach to model the dynamics of RFR rates. In a nutshell, we directly model $R(\cdot; T_s, T_e)$ via a single SABR process with volatility-decay feature. An alternative approach is to model each individual fixings in the accrual period $[T_s, T_e]$ with some parametric model, e.g., SABR, and then, we can derive the effective parameters by synthesizing parameters from the constituents.

Although the underlying of most liquid caplet/floorlet is compounding index R_c, we shall focus on the arithmetic average version R^a. Because 1) it is more tractable for the bottom-up approach; 2) the price of caplet between these two indices differ very little[1]. Denote the 1B forward rate as $R_i(t) := R_i(t; T_i, T_i + \tau_i)$, where t is the observation time[2]. It is the time t "estimation" of the RFR 1B rate from $[T_i, T_i + \tau_i]$. Under forward $T_i + \tau_i$ measure, $R_i(t)$ follows,

$$dR_i(t) = \sigma_{\theta_i}(t, R_i(t))dW_i(t) \tag{1}$$

where θ_i is the parameters of the model. For arithmetic average index R^a that consists of N fixings in its accrual period $[T_s, T_e]$,

$$\begin{aligned} R^a(t) &= \sum_{i=1}^{N} \omega_i R_i(t),\ \omega_i = \tau_i/\tau,\ \forall i \\ dW_i(t)dW_j(t) &= \gamma_{ij} \end{aligned} \tag{2}$$

The idea behind bottom-up approach is to derive the dynamics of $dR^a(t)$. Heuristically,

$$dR^a(t) = \sum_{i=1}^{N} \omega_i dR_i(t) \sim \sigma_\theta(t, R^a(t))dW(t) \tag{3}$$

[1] We showed in *Section 3* of *Notes 2*, it amounts to a tiny strike shift.

[2] Here τ_i stands for 1 business day, thus not necessarily 1/365.

and we would like to find the parameter θ for the dynamics of $dR^a(t)$ as sort of average across θ_i's,

$$\theta = G(\theta_1, ..., \theta_N, \gamma) \tag{4}$$

In addition, we also would like to have the dynamics of $dR^a(t)$ tractable so that pricing an European option has a (semi)-analytical formula.

Notice, in this approach, the volatility decay is being taken care of automatically. Indeed, suppose the valuation time is $T_0 \in [T_s, T_e]$, the fixings contribute to the volatility is $daycount(T_0, T_e)$. One day later, the valuation time becomes $T_0 + 1B$, then the number of fixings is decreased by 1, thus the aggregated volatility. Continue this way, the volatility decays as we march towards the end of the accrual period T_e.

2 Sketch with Bachelier Model

2.1 Effective Model

The simplest model for $R_i(t)$ is Bachelier model,

$$dR_i(t) = \sigma_i dW_i(t) \tag{5}$$

Obviously, in this case, σ_i is the parameter θ_i. (3) is not quite right, each $R_i(t)$ is a martingale under its own terminal forward measure. We need to unify all under T_e-forward measure, which will introduce a drift term for every $dR_i(t)$ except that last one. One can argue that the drift term is negligible given the short accrual period $[T_s, T_e]$ (see [1]). As a result, we have

$$dR^a(t) \approx \sum_i \omega_i \sigma_i dW_i(t) \tag{6}$$

To synthesize W_i;s to a single Brownian motion $W(t)$, notice,

$$Var\left(\sum_i \omega_i \sigma_i dW_i(t)\right) = \sum_{ij} \omega_i \omega_j \sigma_i \sigma_j \gamma_{ij} dt \equiv \sigma^2 dt \tag{7}$$

Therefore, we can re-write (6) with a single parameter σ,

$$dR^a(t) = \sigma \left(\frac{1}{\sigma} \sum_i \omega_i \sigma_i dW_i(t)\right) = \sigma dW(t) \tag{8}$$

(8) is exactly what we are after, R^a is again a normal model but with effective parameter σ as a function of σ_i's and γ. In this case, we also have a Bachelier formula to price caplet/floorlet.

2.2 Practical Considerations of Bottom-up Approach

Unlike the top-down approach, where trader only needs to mark a single set of parameters for expiry/tenor combination T_s and τ. The bottom-up approach requires marking parameters for every single fixings in the accrual period. This is not tractable as it is impossible for traders to manage a huge set of parameters.

To address that, we offer traders to mark a strip of parameters across different expiries (see Table 1). Then, we can build a 1D-Interpolator with marked volatilities (T_k, σ_k)'s, i.e., $\mathcal{I}(T_k's, \sigma_k's)$. Now, to compute the effective volatility with Bachelier model, we can use the interpolator to query the volatility for any fixings,

$$\sigma_i = \mathcal{I}(T_i; T_k's,\, \sigma_k's) \tag{9}$$

Expiry	1B
1M	σ_{1M}
2M	σ_{2M}
...	...
6M	σ_{6M}
9M	σ_{9M}
...	...

Table 1: Marking 1B Column

3 Synthesize SABR Model

3.1 Major Result

To have a more realistic dynamics for the $1B$ fixings, we equip $R^i(\cdot)$ with SABR dyanmics, $\forall i$

$$\begin{cases} dR^i(t) = \sigma^i(t) \left(R^i(t)\right)^{\beta_i} \mathrm{d}W_t^i \\ d\sigma^i(t) = \nu\sigma^i(t)\, \mathrm{d}B_t^i,\ \sigma^i(0) = \alpha^i \end{cases} \tag{10}$$

Notice, there are three types of correlations: 1) ρ_{ij} among rates process; 2) ξ_{ij} among volatility processes; 3)γ_{ij} between rates and volatility processes. Type (iii) is the part of SABR, as $\langle dW_t^i, dB_t^i \rangle = \rho_{ii}\mathrm{d}t$. We summarize them by a correlation matrix,

$$\Sigma \equiv \begin{bmatrix} \gamma & \rho \\ \rho & \xi \end{bmatrix} \tag{11}$$

The objective is:

- derive the dynamics for $\mathrm{d}R^a(t)$;
- project derived dynamics to a SABR-like system.

As you would expect, the derivations in this case is much more involved. Please refer to [1] for the details. Here, we directly give the SABR effective parameters for R^a,

$$\alpha^* = \sqrt{\bar{\gamma}} \sum_i \omega_i \alpha_i \sqrt{\frac{T_i}{\widetilde{T}}},\ \beta^* = \sum_i \omega_i \beta_i,\ \nu^* = \sqrt{\bar{\xi}} \sum_i \omega_i \nu_i \sqrt{\frac{T_i}{\widetilde{T}}},\ \rho^* = \frac{1}{\sqrt{\bar{\gamma}\bar{\xi}}} \sum_i \omega_i \rho_i \tag{12}$$

where $\widetilde{T} = \sum_i \omega_i T_i$, $\bar{\xi} = 1/N^2 \sum_{ij} \xi_{ij}$ and $\bar{\gamma} = 1/N^2 \sum_{ij} \gamma_{ij}$. To simplify the computation, we can assume $\xi = 1$, i.e., there is only one stochastic volatility driver. Eq (12) looks quite appealing, it states that all SABR parameters of R^a are just linear combination of SABR parameters of the constituent fixings.

3.2 Practical Considerations

Similar to the Bachelier case, it is impossible and unnecessary to mark SABR parameters for each fixing R_i. Rather, we can rely on the interpolators built from standard tenors $\mathcal{I}_\theta$, where $\theta = \alpha$, β, ν, ρ. It turns out we can even do better in this case. Notice, the sampling frequency for RFR is 1B, that means, for a 6M period caplet, it needs to interpolate 120 times. As we have linear combinations for all parameters, we can use integral instead of summation. Without loss of generality, let us take β as an example,

$$\beta^* = \sum_{i;T_i \in [T_s, T_e]} \omega_i \beta_i \;\rightleftharpoons\; \beta^* = \frac{1}{\tau} \int_{T_s}^{T_e} \mathcal{I}_\beta(s)\,\mathrm{d}s \tag{13}$$

The right hand side of the integral can be computed explicitly[3], which makes effective parameters calculation blazing fast.

4 Reference

[1] *Jianing Yao, "SABR Average Analytics", Technical Doc, 2021*

[3] In practice, we often use piecewise-constant interpolation or linear interpolation, or if one really wants, cubic interpolation. In any case, we can explicitly evalute the integral.