Introduction to Robot Intelligence (CSCI-UA 480-073) Homework 3

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October 8, 2024

Submission Instructions

This problem set is due on **October 22, 2024, 11:59 PM**.
You must submit solutions to both the theory and coding portions of this homework to be eligible for full credit on this assignment.

Please see the **Assignments page** of the course website for the coding portion of the assignment.

You are strongly encouraged to typeset your answers to the theory questions below using LaTeX, with the provided template (also on the Assignments page). You must submit your answers to the coding problems by filling out the provided IPython notebook. We encourage you to use Google Colab to write and test your code.

When you have completed both portions of the homework, submit them **as two separate files, with the coding portion in `.ipynb` format**. **No other forms of submissions will be accepted. Late submissions will also not be accepted.**

You may not discuss the questions in this problem set with other students.

Theory Questions

Question 1: Vector Identities

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ be 3-vectors and let $\cdot$ and $\times$ denote the dot and cross products in $\mathbf{R}^3$, respectively. Verify the identities below.

(a) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

(b) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Question 2: Rigid Bodies

(a) Describe the configurations of each of the everyday objects below as a combination of links and joints.

(1) A cabinet.

(2) A bicycle.

(3) A skateboard.

(b) Let $T(\vec{v}) = \boldsymbol{R}\vec{v} + \vec{t}$ be a 3-D rigid body transformation, where $\boldsymbol{R}$ is a rotation matrix. Consider a pair of arbitrary 3-D vectors, $\vec{u}, \vec{w} \in \mathbb{R}^3$.

Prove that $T(\cdot)$ preserves Euclidean distance, i.e. $||T(\vec{u}) - T(\vec{w})||^2 = ||\vec{u} - \vec{w}||^2$.

(c) A rigid body moving in $\mathbb{R}^2$ has three degrees of freedom (two from translation and one from rotation), while a rigid body in $\mathbb{R}^3$ has six degrees of freedom (three from translation and three from rotation).

Prove that for general $n \in \mathbb{N}_{\geq 2}$, an n-dimensional rigid body has $\frac{n}{2}(n+1)$ degrees of freedom. How many of these are due to rotation? How many are due to translation?

Question 3: Rotations

1. Prove that the set of all 3-D rotation matrices form a algebraic group under matrix multiplication. Namely, given two arbitrary rotation matrices $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$, demonstrate that each of the following properties hold.

   (a) **Closure**: $\boldsymbol{R}' = \boldsymbol{R}_1\boldsymbol{R}_2$ is a valid rotation matrix.

   (b) **Identity**: there exists a valid rotation matrix e such that $\boldsymbol{R}_1 e = e\boldsymbol{R}_1 = \boldsymbol{R}_1$ and $\boldsymbol{R}_2 e = e\boldsymbol{R}_2 = \boldsymbol{R}_2$.

   (c) **Inverse**: there exist rotation matrices $\boldsymbol{R}_1^{-1}, \boldsymbol{R}_2^{-1}$ such that $\boldsymbol{R}_1\boldsymbol{R}_1^{-1} = e = \boldsymbol{R}_2\boldsymbol{R}_2^{-1}$.

   Recall that a 3×3 matrix $\boldsymbol{R}$ is a valid rotation matrix if and only if $\boldsymbol{R}^T = \boldsymbol{R}^{-1}$ and $\det(\boldsymbol{R}) = 1$.

   An algebraic group is called *Abelian* if the group operation is *commutative*. Is the group formed by 3-D rotation matrices under matrix multiplication *Abelian*? Why, or why not?

2. Consider the robot shown on the following page. Express the position and orientation of the green point $p = (x, y)$ relative to the base of the arm in terms of joint angles and link parameters. Note that joint j_2 is a *prismatic*

joint, able to extend the link by sliding up and down, affecting the height of l_2.