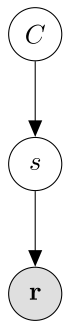
Variational Inference for Categorization Notes

We consider the problem of categorizing stimuli with overlapping distributions

0.1 Generative model



C is the category distribution ($\in \{0, 1\}$)
$P(C) = .5$
s is the presented stimulus, a draw from the selected category distribution
$P(s|C) = \mathcal{N}(s; 0, \sigma_{\mathrm{C}}^2) = \mathcal{N}(s; 0, \tau_{\mathrm{C}}^{-1})$
$\mathbf{r}$ is the vector of neural responses to s: $P(r_i|s) = \mathrm{Poisson}(r_i; f_i(s))$

$f_i(s)$ is the tuning curve of the i^{th} neuron in response to a stimulus s
Tuning curve assumptions:

- Tuning curves cover the space so $\sum_i f_i(s)$ is independent of s
- The tuning curves are Gaussian: $f_i(s) \sim \mathcal{N}(s_i^{\mathrm{pref}}, \sigma_{\mathrm{tc}}^2)$

0.2 Inference

$$\begin{aligned}
P(C, s|\mathbf{r}) &= \Big(\prod_i P(r_i|s)\Big) P(s|C) P(C) \\
&\propto \Big(\prod_i \mathrm{Poisson}(r_i; f_i(s))\Big) \mathcal{N}(s; 0, \tau_{\mathrm{C}}^{-1}) \\
&= \Big(\prod_i \frac{f_i(s)^{r_i}}{r_i!}\Big) \sqrt{\frac{\tau_C}{2\pi}} e^{\frac{-\tau_C s^2}{2}}
\end{aligned} \tag{1}$$

Let

$$\begin{aligned}
\tau_C &= \begin{cases} \tau_0, & \text{if } C = 0 \\ \tau_1, & \text{if } C = 1 \end{cases} \\
&= \tau_0(1 - C) + \tau_1 C \\
&= \tau_0 - (\tau_0 - \tau_1)C \\
&= \tau_0 - C\Delta\tau
\end{aligned} \tag{2}$$

For variational inference,

$$\log P(C, s|\mathbf{r}) = \sum_i \Big(-\log r_i! + r_i \log(f_i(s)) \Big) + \frac{-(\tau_0 - C\Delta\tau)s^2}{2} + \frac{1}{2}\log\left(\frac{\tau_0 - C\Delta\tau}{2\pi}\right) \tag{3}$$

0.3 Variational approximation

Then we approximate this using a factorized distribution:

$$Q(C, s|\mathbf{r}) = Q(C|\mathbf{r})Q(s|\mathbf{r}) \tag{4}$$

$$\begin{aligned}
Q(C|\mathbf{r}) &= \langle \log P(C, s|\mathbf{r}) \rangle_{Q(s|\mathbf{r})} \\
&= \frac{C\Delta\tau}{2}\langle s^2 \rangle_{Q(s|\mathbf{r})} + \frac{1}{2}\log\left(\frac{\tau_0 - C\Delta\tau}{2\pi}\right) \\
&\propto \frac{\langle s^2 \rangle_{Q(s|\mathbf{r})}\Delta\tau}{2} C \\
&\sim \text{Bernoulli}(C, p) \\
p &= \frac{1}{1 + \sqrt{\frac{\tau_0}{\tau_1}} e^{\frac{\langle s^2 \rangle_{Q(s|\mathbf{r})}\Delta\tau}{2}}}
\end{aligned} \tag{5}$$

$$
\begin{aligned}
Q(s|\mathbf{r}) &= \langle \log P(C,s|\mathbf{r})\rangle_{Q(C|\mathbf{r})} \\
&= \sum_i r_i \log(f_i(s)) + \frac{-(\tau_0 - \langle C\rangle_{Q(C|\mathbf{r})}\Delta\tau)}{2} s^2 \\
&\propto \frac{1}{2\sigma_{\text{tc}}^2}\sum_i r_i (s - s_i^{\text{pref}})^2 + \frac{-(\tau_0 - \langle C\rangle_{Q(C|\mathbf{r})}\Delta\tau)}{2} s^2 \\
&\sim \mathcal{N}(\mu,\tau) \\
\tau &= \sum_i \frac{r_i}{\sigma_{\text{tc}}^2} + (\tau_0 - \langle C\rangle_{Q(C|\mathbf{r})}\Delta\tau) \\
&= \sum_i \frac{r_i}{\sigma_{\text{tc}}^2} + (\tau_0 - p\Delta\tau) \\
\mu &= \frac{\sum_i \frac{r_i}{\sigma_{\text{tc}}^2} s_i^{\text{pref}}}{\sum_i \frac{r_i}{\sigma_{\text{tc}}^2} + (\tau_0 - \langle C\rangle_{Q(C|\mathbf{r})}\Delta\tau)} \\
&= \frac{\sum_i \frac{r_i}{\sigma_{\text{tc}}^2} s_i^{\text{pref}}}{\tau}
\end{aligned} \tag{6}
$$

Also

$$
\langle s^2\rangle_{Q(s|\mathbf{r})} = \mu^2 + \frac{1}{\tau} \tag{7}
$$

So we have update equations:

$$
\begin{aligned}
\frac{dp}{dt} &= 1 - p(1 + \sqrt{\frac{\tau_0}{\tau_1}} e^{\frac{\mu^2 + \frac{1}{\tau}\Delta\tau}{2}}) \\
\frac{d\tau}{dt} &= -\tau + \sum_i \frac{r_i}{\sigma_{\text{tc}}^2} + (\tau_0 - p\Delta\tau) \\
\frac{d\mu}{dt} &= \sum_i \frac{r_i}{\sigma_{\text{tc}}^2} s_i^{\text{pref}} - \mu\tau
\end{aligned} \tag{8}
$$