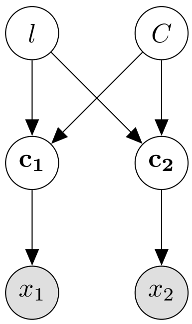
Divisive Normalization Notes



C is categorical variable for condition
l is level of background light
c_i is log contrast of grating i
x_i is noisy log measurement of c_i
$p(C = 1) = 0.25$
$p(C = 2) = 0.25(c_1 > 0)$
$p(C = 3) = 0.25(c_2 > 0)$
$p(C = 4) = 0.25(c_1 and c_2 > 0)$
$l = k$
$p(c_1|C = 1 \text{ or } 3, l) = \delta(c_1)$
$p(c_1|C = 2 \text{ or } 4, l) = \mathcal{N}(c_1; l, \sigma_c^2)$
$p(c_2|C = 1 \text{ or } 2, l) = \delta(c_2)$
$p(c_2|C = 3 \text{ or } 4, l) = \mathcal{N}(c_2; l, \sigma_c^2)$

$$p(x_i|c_i) = \mathcal{N}(x_i; c_i, \sigma^2)$$

$$\begin{aligned}
p(c_1, c_2|x_1, x_2) &\propto p(x_1, x_2, c_1, c_2) \\
&= \sum_C \int p(x_1, x_2, c_1, c_2, l, C) dl \\
&= \sum_C \int p(x_1|c_1) p(x_2|c_2) p(c_1, c_2, l, C) dl \\
&= \int p(x_1|c_1) p(x_2|c_2) p(c_1, c_2, l, C = 1) dl \\
&\quad + \int p(x_1|c_1) p(x_2|c_2) p(c_1, c_2, l, C = 2) dl \\
&\quad + \int p(x_1|c_1) p(x_2|c_2) p(c_1, c_2, l, C = 3 dl \\
&\quad + \int p(x_1|c_1) p(x_2|c_2) p(c_1, c_2, l, C = 4) dl \\
&= p(x_1|c_1) p(x_2|c_2) \delta(c_2) \delta(c_1) \\
&\quad + \int p(x_1|c_1) p(x_2|c_2) \delta(c_2) \mathcal{N}(c_1; l, \sigma_c^2) dl \\
&\quad + \int p(x_1|c_1) p(x_2|c_2) \delta(c_1) \mathcal{N}(c_2; l, \sigma_c^2) dl \\
&\quad + \int p(x_1|c_1) p(x_2|c_2) \mathcal{N}(c_1; l, \sigma_c^2) \mathcal{N}(c_2; l, \sigma_c^2) dl \\
&= p(x_1|c_1) p(x_2|c_2) \delta(c_2) \delta(c_1) + p(x_1|c_1) p(x_2|c_2) \delta(c_1) \\
&\quad + p(x_1|c_1) p(x_2|c_2) \delta(c_2) + \int p(x_1|c_1) p(x_2|c_2) \mathcal{N}(l; \frac{c_1 + c_2}{2}, 2\sigma_c^2) \mathcal{N}(c_1; c_2, 2\sigma_c^2) \\
&= \mathcal{N}(x_1; c_1, \sigma^2) \mathcal{N}(x_2; c_2, \sigma^2) \delta(c_1) \delta(c_2) + \mathcal{N}(x_1; c_1, \sigma^2) \mathcal{N}(x_2; c_2, \sigma^2) \delta(c_1) \\
&\quad + \mathcal{N}(x_1; c_1, \sigma^2) \mathcal{N}(x_2; c_2, \sigma^2) \delta(c_2) + \mathcal{N}(x_1; c_1, \sigma^2) \mathcal{N}(x_2; c_2, \sigma^2) \mathcal{N}(c_1; c_2, 2\sigma_c^2)
\end{aligned} \tag{1}$$

Normalization term(A):

$$\begin{aligned}
A = &\iint [p(x_1|c_1)p(x_2|c_2)\delta(c_2)\delta(c_1) + p(x_1|c_1)p(x_2|c_2)\delta(c_1) \\
&\quad + p(x_1|c_1)p(x_2|c_2)\delta(c_2) + p(x_1|c_1)p(x_2|c_2)\mathcal{N}(c_1; c_2, 2\sigma_c^2)]dc_1dc_2 \\
&= p(x_1|c_1=0)p(x_2|c_2=0) + p(x_1|c_1=0) + p(x_2|c_2=0) \\
&\quad + \int [\mathcal{N}(x_1; c_1, \sigma_c^2)\mathcal{N}(x_2; c_1, \sigma^2 + 2\sigma_c^2)]dc_1 \\
&= \mathcal{N}(x_1; 0, \sigma^2)\mathcal{N}(x_2; 0, \sigma^2) + \mathcal{N}(x_1; 0, \sigma^2) + \mathcal{N}(x_2; 0, \sigma^2) + \mathcal{N}(x_1; x_2, 2\sigma^2 + 2\sigma_c^2)
\end{aligned} \tag{2}$$

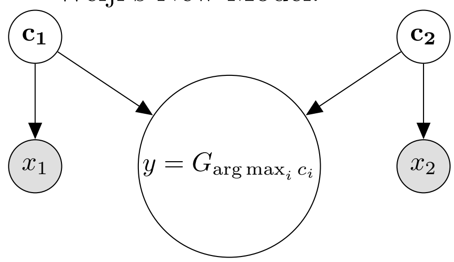
$$\begin{bmatrix} \hat{c_1} \\ \hat{c_2} \end{bmatrix} = \iint \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} p(c_1, c_2|x_1, x_2)dc_1dc_2 \tag{3}$$

$$\begin{aligned}
\hat{c_1} &= \frac{1}{A}\iint c_1p(x_1|c_1)p(x_2|c_2)[\delta(c_1)\delta(c_2) + \delta(c_1) + \delta(c_2) + \mathcal{N}(c_1; c_2, 2\sigma_c^2)]dc_1dc_2 \\
&\propto \int c_1p(x_1|c_1)[p(x_2|c_2=0)\delta(c_1) + \delta(c_1) + p(x_2|c_2=0) + \mathcal{N}(x_2; c_1, \sigma^2 + 2\sigma_c^2)]dc_1 \\
&= x_1\mathcal{N}(x_2; 0, \sigma^2) + \mathcal{N}(x_1; x_2, 2\sigma^2 + 2\sigma_c^2)\frac{\frac{x_1}{\sigma^2} + \frac{x_2}{\sigma^2+2\sigma_c^2}}{\frac{1}{\sigma^2} + \frac{1}{\sigma^2+2\sigma_c^2}} \\
&\frac{x_1\mathcal{N}(x_2; 0, \sigma^2) + \mathcal{N}(x_1; x_2, 2\sigma^2 + 2\sigma_c^2)\frac{\frac{x_1}{\sigma^2} + \frac{x_2}{\sigma^2+2\sigma_c^2}}{\frac{1}{\sigma^2} + \frac{1}{\sigma^2+2\sigma_c^2}}}{\mathcal{N}(x_1; 0, \sigma^2)\mathcal{N}(x_2; 0, \sigma^2) + \mathcal{N}(x_1; 0, \sigma^2) + \mathcal{N}(x_2; 0, \sigma^2) + \mathcal{N}(x_1; x_2, 2\sigma^2 + 2\sigma_c^2)}
\end{aligned} \tag{4}$$

$$\begin{aligned}
\hat{c_2} &= \frac{1}{A}\iint c_2p(x_1|c_1)p(x_2|c_2)[\delta(c_1)\delta(c_2) + \delta(c_1) + \delta(c_2) + \mathcal{N}(c_1; c_2, 2\sigma_c^2)]dc_1dc_2 \\
&\propto \int c_2p(x_2|c_2)[p(x_1|c_1=0)\delta(c_2) + \delta(c_2) + p(x_1|N=c_2) + \mathcal{N}(x_1; c_2, \sigma^2 + 2\sigma_c^2)]dc_2 \\
&= x_2\mathcal{N}(x_1; 0, \sigma_c^2) + \mathcal{N}(x_2; x_1, 2\sigma^2 + 2\sigma_c^2)\frac{\frac{x_2}{\sigma^2} + \frac{x_1}{\sigma^2+2\sigma_c^2}}{\frac{1}{\sigma^2} + \frac{1}{\sigma^2+2\sigma_c^2}} \\
&\frac{x_2\mathcal{N}(x_1; 0, \sigma_c^2) + \mathcal{N}(x_2; x_1, 2\sigma^2 + 2\sigma_c^2)\frac{\frac{x_2}{\sigma^2} + \frac{x_1}{\sigma^2+2\sigma_c^2}}{\frac{1}{\sigma^2} + \frac{1}{\sigma^2+2\sigma_c^2}}}{\mathcal{N}(x_1; 0, \sigma^2)\mathcal{N}(x_2; 0, \sigma^2) + \mathcal{N}(x_1; 0, \sigma^2) + \mathcal{N}(x_2; 0, \sigma^2) + \mathcal{N}(x_1; x_2, 2\sigma^2 + 2\sigma_c^2)}
\end{aligned} \tag{5}$$

	Our Model	Heeger Model
R	$\hat{c_1}G_1 + \hat{c_2}G_2$	$\frac{c_1^n G_1 + c_2^n G_2}{c_{50}^n + (c_1^2 + c_2^2)^{\frac{n}{2}}}$
w_1	$\frac{x_1 \mathcal{N}(x_2;0,\sigma^2) + \mathcal{N}(x_1;x_2,2\sigma^2+2\sigma_c^2)\frac{\frac{x_1}{\sigma^2}+\frac{x_2}{\sigma^2+2\sigma_c^2}}{\frac{1}{\sigma^2}+\frac{1}{\sigma^2+2\sigma_c^2}}}{\mathcal{N}(x_1;0,\sigma^2)\mathcal{N}(x_2;0,\sigma^2)+\mathcal{N}(x_1;0,\sigma^2)+\mathcal{N}(x_2;0,\sigma^2)+\mathcal{N}(x_1;x_2,2\sigma^2+2\sigma_c^2)}$	$\frac{c_1^n}{c_{50}^n + (c_1^2 + c_2^2)^{\frac{n}{2}}}$

Weiji's New Model:

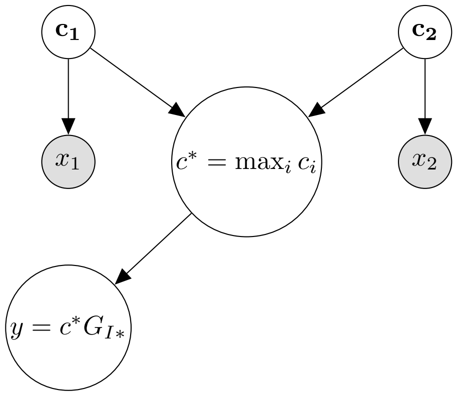


$$
\begin{aligned}
p(y|x_1,x_2) &\propto p(x_1,x_2,c_1,c_2|y) \\
&= \iint p(x_1|c_1)p(x_2|c_2)p(y|c_1,c_2)dc_1, dc_2 \\
&= \iint p(x_1|c_1)p(x_2|c_2)p(y|c_1,c_2)dc_1, dc_2 \\
&= \iint \mathcal{N}(x_1;c_1,\sigma^2)\mathcal{N}(x_2;c_2,\sigma^2)\delta(y-G_{\arg\max_i c_i})dc_1dc_2 \\
&= \iint_{c_1\geq c_2} \mathcal{N}(x_1;c_1,\sigma^2)\mathcal{N}(x_2;c_2,\sigma^2)\delta(y-G_1)dc_1dc_2 \\
&\quad + \iint_{c_1<c_2} \mathcal{N}(x_1;c_1,\sigma^2)\mathcal{N}(x_2;c_2,\sigma^2)\delta(y-G_2)dc_1dc_2 \\
&= \delta(y-G_1)\iint_{c_1\geq c_2} \mathcal{N}(x_1;c_1,\sigma^2)\mathcal{N}(x_2;c_2,\sigma^2)dc_1dc_2 \\
&\quad + \delta(y-G_2)\iint_{c_1<c_2} \mathcal{N}(x_1;c_1,\sigma^2)\mathcal{N}(x_2;c_2,\sigma^2)dc_1dc_2
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
w_1 &= \int_{-\infty}^{\infty}(\int_{-\infty}^{c_1}\mathcal{N}(x_1;c_1,\sigma^2)\mathcal{N}(x_2;c_2,\sigma^2)dc_2)dc_1 \\
&= \int \mathcal{N}(x_1;c_1,\sigma^2)(\frac{1}{2}+\frac{1}{2}\operatorname{erf}(\frac{c_1-x_2}{\sigma\sqrt{2}})dc_1 \\
&= \frac{1}{2}+\frac{1}{2}\int \mathcal{N}(x_1;c_1,\sigma^2)\operatorname{erf}(\frac{c_1-x_2}{\sigma\sqrt{2}}dc_1)
\end{aligned}
$$

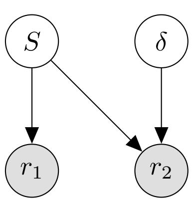
$$
\hat{y} = w_1G_1 + w_2G_2
$$

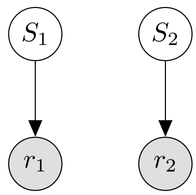
New New Model:



$$
\begin{aligned}
p(y|x_1, x_2) \propto\ & p(x_1, x_2, c_1, c_2|y) \\
&= \iint p(x_1|c_1)p(x_2|c_2)p(y|c_1, c_2)dc_1, dc_2 \\
&= \iint p(x_1|c_1)p(x_2|c_2)p(y|c_1, c_2)dc_1, dc_2 \\
&= \iint \mathcal{N}(x_1; c_1, \sigma^2)\mathcal{N}(x_2; c_2, \sigma^2)\delta(y - \max_i c_i G_i)dc_1 dc_2 \\
&= \iint_{c_1 \geq c_2} \mathcal{N}(x_1; c_1, \sigma^2)\mathcal{N}(x_2; c_2, \sigma^2)\delta(y - c_1 G_1)dc_1 dc_2 \\
&\quad + \iint_{c_1 < c_2} \mathcal{N}(x_1; c_1, \sigma^2)\mathcal{N}(x_2; c_2, \sigma^2)\delta(y - c_2 G_2)dc_1 dc_2
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
w_1 &= \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{c_1} \mathcal{N}(x_1; c_1, \sigma^2)\mathcal{N}(x_2; c_2, \sigma^2)dc_2 \Big) c_1 dc_1 ??? \\
&= \frac{1}{2} + \frac{1}{2}\int c_1 \mathcal{N}(x_1; c_1, \sigma^2)\,\mathrm{erf}\Big(\frac{c_1 - x_2}{\sigma\sqrt{2}} dc_1\Big)
\end{aligned}
$$





S_1
S_2
r_1
r_2