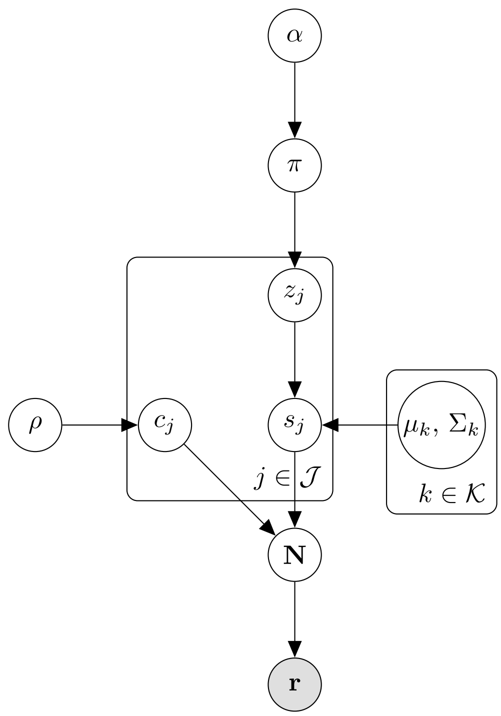
Variational Inference Notes



$s_j | z_j \sim \mathcal{N}(\mu_{z_j}, \Lambda_{z_j})$
$N_{ij} | \{s_j\} \sim \text{Poisson}(c_j f_i(s_j))$
$r_i | N_{ij} \sim \delta(r_i - \sum_j N_{ij})$
$\pi \sim \text{Dir}(\alpha)$
$c \sim \text{Dir}(\rho)$

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\begin{aligned}
\log Q(z) &= \langle \log P(z, s, c, \mu, \Lambda, N, r) \rangle_{Q(s)Q(N)Q(c)Q(\mu,\Sigma)} \\
&= \langle \log(\mathrm{Mult}(z;\pi)\mathrm{Dir}(\pi;\alpha) \prod_{j=1}^{J} \mathcal{N}(s_j; \mu_{z_j}, \Lambda_{z_j})\mathrm{Dir}(c,\rho)... \\
&\quad \prod_{j=1}^{J}((\delta(r_i - \sum_j N_{ij})\mathrm{Poisson}(c_j f_i(s_j)))... \\
&\quad \prod_{k=1}^{K} \mathcal{N}(\mu_k; m_k, (\beta_0 \Lambda_k))\mathcal{W}(\Lambda_k; \mathbf{W}_0, \nu_0)) \rangle_{Q(s)Q(N)Q(c)Q(\mu,\Lambda)} \\
\log Q(z_j = k) &= \langle \log(\mathrm{Mult}(z;\pi) \rangle_{Q(\pi)} + \langle \log(\prod_{j=1}^{J} \mathcal{N}(s_j; \mu_{z_j}, \Lambda_{z_j})\rangle_{Q(s)Q(\mu,\Lambda)}) \\
&= \frac{D}{2}\langle \log(|\Lambda_k|)\rangle_{Q(\Lambda)} - \frac{1}{2}\langle (s_j - \mu_k)^T \Lambda_k (s_k - \mu_k)\rangle_{Q(s)Q(\mu,\Lambda)} + \langle \log \pi_k \rangle_{Q(\pi)} \\
&= \frac{D}{2}(\sum_{d=1}^{D} \Psi(\frac{\nu_k^* + 1 - d}{2}) + \log(|\mathbf{W}_k^*|)) \\
&\quad - \frac{1}{2}(D{\beta_k^*}^{-1} + \nu_k^*((\mu_k^* - m_j^*)^T \mathbf{W}_k^* (\mu_k^* - m_j^*)) + \mathrm{Tr}(\mathbf{W}_k^* {\Lambda_j^*}^{-1})) \\
&\quad + \Psi(\alpha_k) - \Psi(\sum_{k=1}^{K} \alpha_k) \\
q_{jk}^* &= \frac{Q(z_j = k)}{\sum_{k=1}^{K} Q(z_j = k)}
\end{aligned} \tag{1}
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\begin{aligned}
\log Q(c) &= \langle \sum_j (\rho_j - 1) \log c_j \rangle + \sum_{ij} \langle N_{ij} \log c_j \rangle \\
&= \sum_j [\log c_j][\rho_j - 1 + \sum_i \langle N_{ij} \rangle] \\
Q(c) &\sim \mathrm{Dir}(\rho_j - 1 + \sum_i \langle N_{ij} \rangle) = \mathrm{Dir}(\gamma_j^*) \\
\gamma_j^* &= \rho_j - 1 + \sum_i r_i p_{ij}^* \\
\langle \log c_j \rangle &= \Psi(\rho_j - 1 + \sum_i r_i p_{ij}^*) - \Psi(\sum_j (\rho_j - 1 + \sum_i r_i p_{ij}^*))
\end{aligned}
\tag{2}
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\begin{aligned}
\log Q(s_j) &= \langle \log \mathcal{N}(\mu_{z_j}, \Lambda_{z_j}) \rangle_{Q(z)Q(\mu,\Lambda)} + \sum_{i=1}^{I} \langle N_{ij} \log f_i(s_j) \rangle_{Q(N)} \\
&= -\frac{1}{2}(\langle (s_j - \mu_j)^T \Lambda_{z_j}^{-1} (s_j - \mu_j) \rangle_{Q(z)Q(\mu,\Lambda)} + \sum_{i=1}^{I} \langle N_{ij} \rangle_{Q(N)} \log f_i(s_j)) \\
&= -\frac{1}{2}(\sum_{k=1}^{K} q_{jk}^* \nu_k^* (s_j - \mu_k)^T \mathbf{W}_k^* (s_j - \mu_k) - \frac{1}{2} \sum_{i=1}^{I} 2 \frac{||s_j - s_i^{\mathrm{pref}}||^2}{2\sigma^2}) \\
Q(s_j) &\sim \mathcal{N}(m_j^*, \Lambda_j^*) \\
\Lambda_j^* &= \sum_{l=1}^{K+I} \Lambda_l \\
m_j^* &= \Lambda_j^* \sum_{l=1}^{K+I} (\Lambda_l m_l) \\
m_l &\in \{m_k\}_{k=1}^{K}, \{s_i^{\mathrm{pref}}\}_{i=1}^{I} \\
\Lambda_l &\in \{q_{kj}^* \nu_k^* \mathbf{W}_k^*\}_{k=1}^{K}, \{\frac{2 r_i p_{ij}}{2\sigma^2} I\}_{i=1}^{I} \\
\langle \log f_i(s_j) \rangle_{Q(s_j)} &= \frac{-\langle ||s_j - s_i^{\mathrm{pref}}||^2 \rangle_{Q(s_j)}}{2\sigma^2} \\
&= \frac{-||s_i^{\mathrm{pref}} - m_j^*||^2 + \mathrm{Tr}(\Lambda_j^{*^{-1}})}{2\sigma^2}
\end{aligned}
\tag{3}
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\begin{aligned}
\log Q(N_i) &= \log(\delta(\sum_{j=1}^{J} N_{ij} - r_i)) + \sum_{j=1}^{J} N_{ij}[\langle \log c_j \rangle + \langle \log f_i(s_j) \rangle] - \sum_{j=1}^{J} \log N_{ij}! \\
Q(N_{ij}) &\sim \mathrm{Mult}(r_i, p_{ij}^*) \\
p_{ij}^* &= \frac{e^{\langle \log c_j \rangle + \langle \log f_i(s_j) \rangle}}{\sum_{j=1}^{J} e^{\langle \log c_j \rangle + \langle \log f_i(s_j) \rangle}} \\
&= \frac{e^{\Psi(\gamma_j^*) - \Psi(\sum_{j=1}^{J} \gamma_j^*) - \frac{||s_i^{\mathrm{pref}} - m_j^*||^2 + \mathrm{Tr}({\Lambda_j^*}^{-1})}{2\sigma^2}}}{\sum_{j=1}^{J} e^{\Psi(\gamma_j^*) - \Psi(\sum_{j=1}^{J} \gamma_j^*) - \frac{||s_i^{\mathrm{pref}} - m_j^*||^2 + \mathrm{Tr}({\Lambda_j^*}^{-1})}{2\sigma^2}}}
\end{aligned}
\tag{4}
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\begin{aligned}
\log Q(\mu_k, \Lambda_k) &= \langle \log(\prod_{j=1}^{J} \mathcal{N}(s_j; \mu_{z_j}, \Lambda_{z_j}))\rangle_{Q(s)Q(z)}) \\
&= \langle \sum_{j=1}^{J} [\frac{D}{2} \log |\Lambda_{z_j}| - \frac{1}{2}[(s_j - \mu_{z_j})^T \Lambda_{z_j} (s_j - \mu_{z_j}))\rangle_{Q(s)Q(z)} \\
&= \langle \sum_{j=1}^{J} [\frac{D}{2} \log |\Lambda_{z_j}| - \frac{1}{2}[(m_j^* - m_{z_j})^T \Lambda_{z_j} (m_j^* - m_{z_j}) + \mathrm{Tr}(\Lambda_{z_j} \Lambda_j^{*^{-1}})]]\rangle_{Q(z)} \\
&= \sum_{j=1}^{J} q_{jk}^* [\frac{D}{2} \log |\Lambda_{z_j}| - \frac{1}{2}[(m_j^* - m_{z_j})^T \Lambda_{z_j} (m_j^* - m_{z_j}) + \mathrm{Tr}(\Lambda_{z_j} \Lambda_j^{*^{-1}})] \\
Q(\mu_k, \Lambda_k) &\sim \mathcal{N}(\mu_k; \mu_k^*, (\beta_k^* \Lambda_k)) \mathcal{W}(\Lambda_k; \mathbf{W}_k^*, \nu_k^*) \\
\sum_{j=1}^{J} q_{jk}^* \mathrm{Tr}(\Lambda_k \Lambda_j^{*^{-1}}) &= \mathrm{Tr}(\Lambda_k \sum_{j=1}^{J} q_{jk}^* \Lambda_j^{*^{-1}}) \\
\mathbf{W}_k^* &= (\sum_{j=1}^{J} (q_{jk}^* \Lambda_j^{*^{-1}}))^{-1} \\
|\Lambda_{z_j}|^{\frac{\nu_k^* - 1}{2}} &= |\Lambda_{z_j}|^{\frac{D}{2} + \sum_{j=1}^{J} q_{jk}^*} \\
\nu_k^* &= D + 2 \sum_{j=1}^{J} q_{jk}^* + 1 \\
C &= \Lambda_k \sum_{j=1}^{J} q_{jk}^* \\
X &= \Lambda_k \sum_{j=1}^{J} q_{jk}^* m_j^* \\
\mu_k^* &= \frac{X}{C} = \frac{\sum_{j=1}^{J} q_{jk}^* m_j^*}{\sum_{j=1}^{J} q_{jk}^*} \\
(\beta_k^* \Lambda_k)^{-1} &= C \\
\beta_k^* &= \frac{1}{\sum_{j=1}^{J} q_{jk}^*}
\end{aligned}
\tag{5}
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\begin{aligned}
\frac{dq^*_{jk}}{dt} &= -q^*_{jk} + \frac{Q(z_j = k)}{\sum_{k=1}^{K} Q(z_j = k)} \\
\frac{dm^*_j}{dt} &= -m^*_j + \Lambda^*_j(\sum_{k=1}^{K}(q^*_{kj}\nu^*_k \mathbf{W}^*_k \mu^*_k) + \sum_{i=1}^{I}(\frac{2r_i p_{ij}}{2\sigma^2} I s_i^{\text{pref}})) \\
\frac{d\Lambda^*_j}{dt} &= -\Lambda^*_j + \sum_{k=1}^{K}(q^*_{kj}\nu^*_k \mathbf{W}^*_k) + \sum_{i=1}^{I}(\frac{2r_i p_{ij}}{2\sigma^2} I) \\
\frac{dp^*_{ij}}{dt} &= -p^*_{ij} + \frac{e^{\Psi(\gamma^*_j) - \Psi(\sum_{j=1}^{J}\gamma^*_j) - \frac{||s_i^{\text{pref}} - m^*_j||^2 + \text{Tr}({\Lambda^*_j}^{-1})}{2\sigma^2}}}{\sum_{j=1}^{J} e^{\Psi(\gamma^*_j) - \Psi(\sum_{j=1}^{J}\gamma^*_j) - \frac{||s_i^{\text{pref}} - m^*_j||^2 + \text{Tr}({\Lambda^*_j}^{-1})}{2\sigma^2}}} \\
\frac{d\gamma^*_j}{dt} &= -\gamma^*_j + \rho_j - 1 + \sum_i r_i p^*_{ij} \qquad (6) \\
\frac{d\mu^*_k}{dt} &= -\mu^*_k + \frac{\sum_{j=1}^{J} q^*_{jk} m^*_j}{\sum_{j=1}^{J} q^*_{jk}} \\
\frac{d\beta^*_k}{dt} &= -\beta^*_k + \frac{1}{\sum_{j=1}^{J} q^*_{jk}} \\
\frac{d\mathbf{W}^*_k}{dt} &= -\mathbf{W}^*_k + (\sum_{j=1}^{J}(q^*_{jk}{\Lambda^*_j}^{-1}))^{-1} \\
\frac{d\nu^*_k}{dt} &= -\nu^*_k + D + 2\sum_{j=1}^{J} q^*_{jk} + 1
\end{aligned}
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