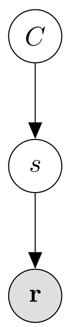
# Variational Inference for Categorization Notes

We consider the problem of categorizing stimuli with overlapping distributions

## 0.1 Generative model




$C$ is the category distribution ($\in \{0, 1\}$)
$P(C) = .5$
$s$ is the presented stimulus, a draw from the selected category distribution
$P(s|C) = \mathcal{N}(s; 0, \sigma_{\mathrm{C}}^2) = \mathcal{N}(s; 0, \tau_{\mathrm{C}}^{-1})$
$\mathbf{r}$ is the vector of neural responses to $s$: $P(r_i|s) = \mathrm{Poisson}(r_i; f_i(s))$

$f_i(s)$ is the tuning curve of the $i^{\mathrm{th}}$ neuron in response to a stimulus $s$
Tuning curve assumptions:

- Tuning curves cover the space so $\sum_i f_i(s)$ is independent of $s$
- The tuning curves are Gaussian: $f_i(s) \sim \mathcal{N}(s_i^{\mathrm{pref}}, \sigma_{\mathrm{tc}}^2)$

## 0.2 Inference

$$\begin{aligned} P(C, s|\mathbf{r}) &= \Big(\prod_i P(r_i|s)\Big) P(s|C) P(C) \\ &\propto \Big(\prod_i \mathrm{Poisson}(r_i; f_i(s))\Big) \mathcal{N}(s; 0, \tau_{\mathrm{C}}^{-1}) \\ &= \Big(\prod_i \frac{f_i(s)^{r_i}}{r_i!}\Big) \sqrt{\frac{\tau_C}{2\pi}} e^{\frac{-\tau_C s^2}{2}} \end{aligned} \tag{1}$$

Let

$$\begin{aligned}\tau_C &= \begin{cases}\tau_0, & \text{if } C = 0\\ \tau_1, & \text{if } C = 1\end{cases}\\ &= \tau_0(1-C) + \tau_1 C\\ &= \tau_0 - (\tau_0 - \tau_1)C\\ &= \tau_0 - C\Delta\tau\end{aligned} \tag{2}$$

For variational inference,

$$\log P(C, s|\mathbf{r}) = \sum_i \Big( -\log r_i! + r_i \log(f_i(s))\Big) + \frac{-(\tau_0 - C\Delta\tau)s^2}{2} + \frac{1}{2}\log\left(\frac{\tau_0 - C\Delta\tau}{2\pi}\right) \tag{3}$$

## 0.3 Variational approximation

Then we approximate this using a factorized distribution:

$$Q(C, s|\mathbf{r}) = Q(C|\mathbf{r})Q(s|\mathbf{r}) \tag{4}$$

$$\begin{aligned}\log Q(C|\mathbf{r}) &= \langle \log P(C, s|\mathbf{r})\rangle_{Q(s|\mathbf{r})}\\ &= \frac{C\Delta\tau}{2}\langle s^2\rangle_{Q(s|\mathbf{r})} + \frac{1}{2}\log\left(\frac{\tau_0 - C\Delta\tau}{2\pi}\right)\\ &\propto \frac{C\Delta\tau}{2}\langle s^2\rangle_{Q(s|\mathbf{r})} + \frac{1}{2}\log(\tau_0 - C\Delta\tau)\\ \log Q(C=1|\mathbf{r}) &\propto \frac{\Delta\tau}{2}\langle s^2\rangle_{Q(s|\mathbf{r})} + \frac{1}{2}\log(\tau_0 - \Delta\tau)\\ \log Q(C=0|\mathbf{r}) &\propto \frac{1}{2}\log(\tau_0)\\ \log\frac{Q(C=1|\mathbf{r})}{Q(C=0|\mathbf{r})} &= \frac{\Delta\tau}{2}\langle s^2\rangle_{Q(s|\mathbf{r})} + \frac{1}{2}\log\left(\frac{\tau_1}{\tau_0}\right)\\ Q(C=1|\mathbf{r}) &= \frac{1}{1 + e^{-\log\frac{Q(C=1|\mathbf{r})}{Q(C=0|\mathbf{r})}}}\\ Q(C|\mathbf{r}) &\sim \text{Bernoulli}(p)\\ p &= \frac{1}{1 + \sqrt{\frac{\tau_0}{\tau_1}}e^{-\frac{\Delta\tau}{2}\langle s^2\rangle_{Q(s|\mathbf{r})}}}\end{aligned} \tag{5}$$

$$
\begin{aligned}
Q(s|\mathbf{r}) &= \langle \log P(C, s|\mathbf{r}) \rangle_{Q(C|\mathbf{r})} \\
&= \sum_i r_i \log(f_i(s)) + \frac{-(\tau_0 - \langle C \rangle_{Q(C|\mathbf{r})} \Delta\tau)}{2} s^2 \\
&\propto \frac{1}{2\sigma_{\text{tc}}^2} \sum_i r_i (s - s_i^{\text{pref}})^2 + \frac{-(\tau_0 - \langle C \rangle_{Q(C|\mathbf{r})} \Delta\tau)}{2} s^2 \\
&\sim \mathcal{N}(\mu, \tau) \\
\tau &= \sum_i \frac{r_i}{\sigma_{\text{tc}}^2} + (\tau_0 - \langle C \rangle_{Q(C|\mathbf{r})} \Delta\tau) \\
&= \sum_i \frac{r_i}{\sigma_{\text{tc}}^2} + (\tau_0 - p\Delta\tau) \\
\mu &= \frac{\sum_i \frac{r_i}{\sigma_{\text{tc}}^2} s_i^{\text{pref}}}{\sum_i \frac{r_i}{\sigma_{\text{tc}}^2} + (\tau_0 - \langle C \rangle_{Q(C|\mathbf{r})} \Delta\tau)} \\
&= \frac{\sum_i \frac{r_i}{\sigma_{\text{tc}}^2} s_i^{\text{pref}}}{\tau} \\
\eta &= \mu\tau \\
&= \sum_i \frac{r_i}{\sigma_{\text{tc}}^2} s_i^{\text{pref}}
\end{aligned}
\tag{6}
$$

($\eta$ and $\tau$ are natural parameters)
Also

$$
\langle s^2 \rangle_{Q(s|\mathbf{r})} = (\frac{\eta}{\tau})^2 + \frac{1}{\tau} \tag{7}
$$

So we have update equations ($\eta$ is a constant):

$$
\begin{aligned}
\frac{dp}{dt} &= 1 - p(1 + \sqrt{\frac{\tau_0}{\tau_1}} e^{-\frac{((\frac{\eta}{\tau})^2 + \frac{1}{\tau})\Delta\tau}{2}}) \\
\frac{d\tau}{dt} &= -\tau + \sum_i \frac{r_i}{\sigma_{\text{tc}}^2} + (\tau_0 - p\Delta\tau)
\end{aligned}
\tag{8}
$$

Analytic posterior:

From Qamar et al. (2013) we have

$$\sigma_0^2 = \frac{1}{\tau_0}$$
$$\sigma_1^2 = \frac{1}{\tau_1}$$
$$\log\frac{P(\mathbf{r}|C=0)}{P(\mathbf{r}|C=1)} = \frac{1}{2}\log\frac{1+\sigma_1^2\mathbf{a}\cdot\mathbf{r}}{1+\sigma_0^2\mathbf{a}\cdot\mathbf{r}} - \frac{(\sigma_1^2-\sigma_0^2)(\mathbf{b}\cdot\mathbf{r})^2}{2(1+\sigma_0^2\mathbf{a}\cdot\mathbf{r})(1+\sigma_1^2\mathbf{a}\cdot\mathbf{r})} \tag{9}$$
$$P(\mathbf{r}|C=0) = \frac{1}{1+e^{-(\frac{1}{2}\log\frac{1+\sigma_1^2\mathbf{a}\cdot\mathbf{r}}{1+\sigma_0^2\mathbf{a}\cdot\mathbf{r}} - \frac{(\sigma_1^2-\sigma_0^2)(\mathbf{b}\cdot\mathbf{r})^2}{2(1+\sigma_0^2\mathbf{a}\cdot\mathbf{r})(1+\sigma_1^2\mathbf{a}\cdot\mathbf{r})})}}$$

$$\frac{1}{|\mathcal{D}|}\mathcal{L}(\theta=\{W,b\},\mathcal{D}) = \frac{1}{|\mathcal{D}|}\sum_{i=0}^{|\mathcal{D}|}\log(P(Y=y^{(i)}|x^{(i)},W,b))\ell(\theta=\{W,b\},\mathcal{D}) \tag{10}$$