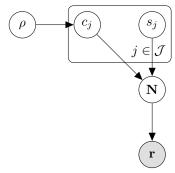
## **Divisive Normalization Notes**

We consider the problem of demixing stimuli that are mixed due to neurons having large receptive fields or wide tuning curves.

## 0.1 Generative model



 $s_j$  is the  $j^{\text{th}}$  stimulus,  $\mathbf{s} \sim \text{Unif}(0, \pi)$  $c_j$  is the contrast of the  $j^{\text{th}}$  stimulus,  $\mathbf{c} \sim \text{Dir}(\rho)$ 

 $f_i(s)$  is the tuning curve of the  $i^{\rm th}$  neuron in response to a single stimulus s

Tuning curve assumptions:

- Tuning curves cover the space so  $\sum_i f_i(s)$  is independent of s
- The tuning curves are Gaussian:  $f_i(s_j) \sim \mathcal{N}(s_i^{\text{pref}}, \sigma_{\text{tc}}^2)$

 $N_{ij}$  is the spike count of the  $i^{\rm th}$  neuron elicited by the  $j^{\rm th}$  stimulus  $N_{ij}|s_j\sim {\rm Poisson}(c_jf_i(s_j))$ 

 $r_i$  is the total spike count of the  $i^{\rm th}$  neuron in response to all stimuli  $r_i = \sum_j N_{ij}$ 

## 0.2 Inference

$$P(\mathbf{c}, \mathbf{s}|\mathbf{r}) = \sum_{\mathbf{N}} P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r})$$

$$P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) = P(\mathbf{r}|\mathbf{N})P(\mathbf{N}|\mathbf{c}, \mathbf{s})P(\mathbf{c})P(\mathbf{s})$$

$$\propto \prod_{i} \left[ \delta \left( r_{i} - \sum_{j} N_{ij} \right) \left( \prod_{j} \frac{(f_{i}(s_{j})c_{j})^{N_{ij}}}{N_{ij}!} \right) \right] \prod_{j} \frac{1}{\beta(\rho)} c_{j}^{\rho_{j}-1}$$
(1)

For the variational inference, we need

$$\log P(\mathbf{N}, \mathbf{c}, \mathbf{s} | \mathbf{r}) = \sum_{i} \left[ \log \delta \left( r_i - \sum_{j} N_{ij} \right) + \sum_{j} \left[ -\log N_{ij}! + N_{ij} \log(f_i(s_j)c_j) \right] \right] + \sum_{j} (\rho_j - 1) \log c_j$$
(2)

## 0.3 Variational approximation

Then we approximate this using a factorized distribution:

$$Q(\mathbf{N}, \mathbf{c}, \mathbf{s} | \mathbf{r}) = Q(\mathbf{N} | \mathbf{r}) Q(\mathbf{c} | \mathbf{r}) Q(\mathbf{s} | \mathbf{r})$$

$$\log Q(\mathbf{N} | \mathbf{r}) = \sum_{i} \log Q(\mathbf{N}_{i} | \mathbf{r}_{i})$$

$$\log Q(\mathbf{N}_{i} | \mathbf{r}_{i}) = \langle \log P(\mathbf{N}_{i}, \mathbf{c}, \mathbf{s} | \mathbf{r}_{i}) \rangle_{Q(\mathbf{c} | \mathbf{r}) Q(\mathbf{s} | \mathbf{r})}$$

$$= \log(\delta \left( r_{i} - \sum_{j} N_{ij} \right) + \sum_{j} N_{ij} \left( \langle \log c_{j} \rangle + \langle \log f_{i}(s_{j}) \rangle \right) - \sum_{j} \log N_{ij}!$$

$$\log Q(\mathbf{c} | \mathbf{r}) = \langle \log P(\mathbf{N}, \mathbf{c}, \mathbf{s} | \mathbf{r}) \rangle_{Q(\mathbf{N} | \mathbf{r}) Q(\mathbf{s} | \mathbf{r})}$$

$$= \sum_{j} [\log(c_{j} \rho_{j} - 1) + \sum_{i} \langle N_{ij} \rangle_{Q(\mathbf{N} | \mathbf{r})}]$$

$$\log Q(\mathbf{s} | \mathbf{r}) = \langle \log P(\mathbf{N}, \mathbf{c}, \mathbf{s} | \mathbf{r}) \rangle_{Q(\mathbf{N} | \mathbf{r}) Q(\mathbf{c} | \mathbf{r})}$$

$$= \sum_{ij} [\langle N_{ij} \rangle_{Q(\mathbf{N} | \mathbf{r})} \log(f_{i}(s_{j})) - f_{i}(s_{j}) \langle c_{j} \rangle_{Q(\mathbf{c} | \mathbf{r})}]$$

$$= \sum_{ij} \langle N_{ij} \rangle_{Q(\mathbf{N} | \mathbf{r})} \log(f_{i}(s_{j}))$$

So we now know

$$Q(\mathbf{c}|\mathbf{r}) \sim \text{Dir}(\boldsymbol{\alpha})$$

$$\alpha_j = \rho_j - 1 + \sum_i \langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})}$$
(5)

(4)

and

$$Q(s_{j}|r) \propto \frac{1}{2\sigma_{\text{tc}}^{2}} \sum_{i=1}^{I} \langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})} (s_{j} - s_{i}^{\text{pref}})^{2}$$

$$\sim \mathcal{N}(\mu_{j}, \tau_{j})$$

$$\tau_{j} = \sum_{i=1}^{I} \frac{\langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})}}{2\sigma_{\text{tc}}^{2}}$$

$$\mu_{j} = \frac{1}{\tau_{j}} \sum_{i=1}^{I} s_{i}^{\text{pref}} \frac{\langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})}}{2\sigma_{\text{tc}}^{2}}$$
(6)

$$Q(N_{ij}|r) \sim \text{Mult}(r_i, p_{ij})$$

$$p_{ij} = \frac{e^{\langle \log c_j \rangle + \langle \log f_i(s_j) \rangle}}{\sum_{j=1}^{J} e^{\langle \log c_j \rangle + \langle \log f_i(s_j) \rangle}}$$
(7)

From this we can compute:

$$\langle \log c_j \rangle_{Q(c_j)} = \Psi(\alpha_j) - \Psi(\sum_{j=1}^J \alpha_j)$$
 (8)

and

$$\langle \log f_i(s_j) \rangle_{Q(s_j)} = \frac{-\langle (s_j - s_i^{\text{pref}})^2 \rangle_{Q(s_j)}}{2\sigma_{\text{tc}}^2}$$

$$= \frac{-[(s_i^{\text{pref}} - \mu_j)^2 + \tau_j^{-1}]}{2\sigma_{\text{tc}}^2}$$
(9)

Therefore:

$$p_{ij} = \frac{e^{\Psi(\alpha_j) - \Psi(\sum_{j=1}^{J} \alpha_j) - \frac{(s_i^{\text{pref}} - \mu_j)^2 + \tau_j^{-1}}{2\sigma_{\text{tc}}^2}}}{\sum_{j=1}^{J} e^{\Psi(\alpha_j) - \Psi(\sum_{j=1}^{J} \alpha_j) - \frac{(s_i^{\text{pref}} - \mu_j)^2 + \tau_j^{-1}}{2\sigma_{\text{tc}}^2}}}$$
(10)

$$\alpha_j = \rho_j - 1 + \sum_i r_i p_{ij} \tag{11}$$

$$\tau_j = \sum_{i=1}^{I} \frac{r_i p_{ij}}{2\sigma_{\rm tc}^2} \tag{12}$$

$$\mu_j = \frac{1}{\tau_j} \sum_{i=1}^{I} s_i^{\text{pref}} \frac{r_i p_{ij}}{2\sigma_{\text{tc}}^2}$$
(13)

We actually want the natural parameters so

$$\eta_j = \mu_j \tau_j = \sum_{i=1}^{I} s_i^{\text{pref}} \frac{r_i p_{ij}}{2\sigma_{\text{tc}}^2}$$
(14)

Also we don't want complex division in the neural computations so:

$$F_{i}(\mu_{j}, \alpha_{j}, \tau_{j}) = e^{\Psi(\alpha_{j}) - \Psi(\sum_{j=1}^{J} \alpha_{j}) - \frac{(s_{i}^{\text{pref}} - \frac{\eta_{j}}{\tau_{j}})^{2} + \tau_{j}^{-1}}{2\sigma_{\text{tc}}^{2}}}$$

$$p_{ij} = F_{i}(\mu_{j}, \alpha_{j}, \tau_{j}) \pi_{i}$$

$$\pi_{i} = \frac{1}{\sum_{j=1}^{J} F_{i}(\mu_{j}, \alpha_{j}, \tau_{j})}$$
(15)

For simulation purposes:

$$f_{ij} = \Psi(\alpha_j) - \Psi(\sum_{j=1}^{J} \alpha_j) - \frac{(s_i^{\text{pref}} - \frac{\eta_j}{\tau_j})^2 + \tau_j^{-1}}{2\sigma_{\text{tc}}^2}$$

$$f_i^* = \max_{J} f_{ij}$$

$$\pi_i = \sum_{j=1}^{J} e^{f_{ij}} = e^{f_i^*} \sum_{j \neq \text{arg max}_J} e^{f_{ij} - f_i^*}$$

$$\log \pi_i = f_i^* + \log \sum_{j \neq \text{arg max}_J} e^{f_{ij} - f_i^*}$$

$$p_{ij} = \frac{e^{f_{ij}}}{\pi_i} = e^{f_{ij} - \log \pi_i}$$

$$\log p_{ij} = f_{ij} - \log \pi_i = f_{ij} - f_i^* + \log \sum_{j \neq \text{arg max}_J} e^{f_{ij} - f_i^*}$$

$$\log p_{ij} = f_{ij} - \log \pi_i = f_{ij} - f_i^* + \log \sum_{j \neq \text{arg max}_J} e^{f_{ij} - f_i^*}$$

And now we have update equations:

$$\frac{d\eta_{j}}{dt} = -\eta_{j} + \frac{1}{2\sigma_{\text{tc}}^{2}} \sum_{i=1}^{I} s_{i}^{\text{pref}} r_{i} p_{ij}$$

$$\frac{d\tau_{j}}{dt} = -\tau_{j} + \frac{1}{2\sigma_{\text{tc}}^{2}} \sum_{i=1}^{I} r_{i} p_{ij}$$

$$\frac{dp_{ij}}{dt} = -p_{ij} + F_{i}(\mu_{j}, \alpha_{j}, \tau_{j}) \pi_{i}$$

$$\frac{d\pi_{i}}{dt} = 1 - \pi_{i} \sum_{j=1}^{J} F_{i}(\mu_{j}, \alpha_{j}, \tau_{j})$$

$$\frac{d\alpha_{j}}{dt} = -\alpha_{j} + \rho_{j} - 1 + \sum_{i=1}^{I} r_{i} p_{ij}$$
(17)

Agan for simulation purposes  $(p_{ij} = \log p_{ij})$ :

$$\frac{d\eta_{j}}{dt} = -\eta_{j} + \frac{1}{2\sigma_{\text{tc}}^{2}} \sum_{i=1}^{I} s_{i}^{\text{pref}} r_{i} e^{p_{ij}}$$

$$\frac{d\tau_{j}}{dt} = -\tau_{j} + \frac{1}{2\sigma_{\text{tc}}^{2}} \sum_{i=1}^{I} r_{i} e^{p_{ij}}$$

$$\frac{dp_{ij}}{dt} = -p_{ij} + f_{ij} - \max_{J} f_{ij} + \log \sum_{j \neq \text{arg max}_{J}} e^{f_{ij} - \max_{J} f_{ij}}$$

$$\frac{d\alpha_{j}}{dt} = -\alpha_{j} + \rho_{j} - 1 + \sum_{i}^{I} r_{i} e^{p_{ij}}$$

$$\frac{df_{ij}}{dt} = -f_{ij} + \Psi(\alpha_{j}) - \Psi(\sum_{j=1}^{J} \alpha_{j}) - \frac{(s_{i}^{\text{pref}} - \frac{\eta_{j}}{\tau_{j}})^{2} + \tau_{j}^{-1}}{2\sigma_{\text{tc}}^{2}}$$
(18)