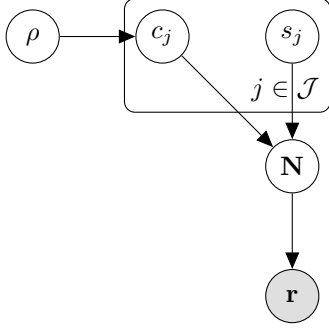


Divisive Normalization Notes

We consider the problem of demixing stimuli that are mixed due to neurons having large receptive fields or wide tuning curves.

0.1 Generative model



s_j is the j^{th} stimulus, $\mathbf{s} \sim \text{Unif}(0, \pi)$

c_j is the contrast of the j^{th} stimulus, $\mathbf{c} \sim \text{Dir}(\rho)$

$f_i(s)$ is the tuning curve of the i^{th} neuron in response to a single stimulus s

Tuning curve assumptions:

- Tuning curves cover the space so $\sum_i f_i(s)$ is independent of s
- The tuning curves are Gaussian: $f_i(s_j) \sim \mathcal{N}(s_i^{\text{pref}}, \sigma_{\text{tc}}^2)$

N_{ij} is the spike count of the i^{th} neuron elicited by the j^{th} stimulus

$N_{ij}|s_j \sim \text{Poisson}(c_j f_i(s_j))$

r_i is the total spike count of the i^{th} neuron in response to all stimuli

$r_i = \sum_j N_{ij}$

0.2 Inference

$$P(\mathbf{c}, \mathbf{s}|\mathbf{r}) = \sum_{\mathbf{N}} P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r})$$

$$P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) = P(\mathbf{r}|\mathbf{N})P(\mathbf{N}|\mathbf{c}, \mathbf{s})P(\mathbf{c})P(\mathbf{s}) \quad (1)$$

$$\propto \prod_i \left[\delta\left(r_i - \sum_j N_{ij}\right) \left(\prod_j \frac{(f_i(s_j)c_j)^{N_{ij}}}{N_{ij}!} \right) \right] \prod_j \frac{1}{\beta(\rho)} c_j^{\rho_j-1}$$

For the variational inference, we need

$$\begin{aligned} \log P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) = & \sum_i \left[\log \delta \left(r_i - \sum_j N_{ij} \right) + \sum_j \left[-\log N_{ij}! + N_{ij} \log(f_i(s_j)c_j) \right] \right] \\ & + \sum_j (\rho_j - 1) \log c_j \end{aligned} \quad (2)$$

0.3 Variational approximation

Then we approximate this using a factorized distribution:

$$Q(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) = Q(\mathbf{N}|\mathbf{r})Q(\mathbf{c}|\mathbf{r})Q(\mathbf{s}|\mathbf{r}) \quad (3)$$

$$\begin{aligned} \log Q(\mathbf{N}|\mathbf{r}) &= \sum_i \log Q(\mathbf{N}_i|\mathbf{r}_i) \\ \log Q(\mathbf{N}_i|\mathbf{r}_i) &= \langle \log P(\mathbf{N}_i, \mathbf{c}, \mathbf{s}|\mathbf{r}_i) \rangle_{Q(\mathbf{c}|\mathbf{r})Q(\mathbf{s}|\mathbf{r})} \\ &= \log \left(\delta \left(r_i - \sum_j N_{ij} \right) \right) + \sum_j N_{ij} \left(\langle \log c_j \rangle_{Q(\mathbf{c}|\mathbf{r})} + \langle \log f_i(s_j) \rangle_{Q(\mathbf{s}|\mathbf{r})} \right) - \sum_j \log N_{ij}! \\ \log Q(\mathbf{c}|\mathbf{r}) &= \langle \log P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) \rangle_{Q(\mathbf{N}|\mathbf{r})Q(\mathbf{s}|\mathbf{r})} \\ &= \sum_j [\log(c_j \rho_j - 1) + \sum_i \langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})}] \\ \log Q(\mathbf{s}|\mathbf{r}) &= \langle \log P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) \rangle_{Q(\mathbf{N}|\mathbf{r})Q(\mathbf{c}|\mathbf{r})} \\ &= \sum_{ij} [\langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})} \log(f_i(s_j)) - f_i(s_j) \langle c_j \rangle_{Q(\mathbf{c}|\mathbf{r})}] \\ &= \sum_{ij} \langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})} \log(f_i(s_j)) \end{aligned} \quad (4)$$

So we now know

$$\begin{aligned} Q(\mathbf{c}|\mathbf{r}) &\sim \text{Dir}(\boldsymbol{\alpha}) \\ \alpha_j &= \rho_j - 1 + \sum_i \langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})} \end{aligned} \quad (5)$$

and

$$\begin{aligned}
Q(s_j|r) &\propto \frac{1}{2\sigma_{\text{tc}}^2} \sum_{i=1}^I \langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})} (s_j - s_i^{\text{pref}})^2 \\
&\sim \mathcal{N}(\mu_j, \tau_j) \\
\tau_j &= \sum_{i=1}^I \frac{\langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})}}{2\sigma_{\text{tc}}^2}
\end{aligned} \tag{6}$$

$$\begin{aligned}
\mu_j &= \frac{1}{\tau_j} \sum_{i=1}^I s_i^{\text{pref}} \frac{\langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})}}{2\sigma_{\text{tc}}^2} \\
Q(N_{ij}|r) &\sim \text{Mult}(r_i, p_{ij}) \\
p_{ij} &= \frac{e^{\langle \log c_j \rangle + \langle \log f_i(s_j) \rangle}}{\sum_{j=1}^J e^{\langle \log c_j \rangle + \langle \log f_i(s_j) \rangle}}
\end{aligned} \tag{7}$$

From this we can compute:

$$\langle \log c_j \rangle_{Q(c_j)} = \Psi(\alpha_j) - \Psi\left(\sum_{j=1}^J \alpha_j\right) \tag{8}$$

and

$$\begin{aligned}
\langle \log f_i(s_j) \rangle_{Q(s_j)} &= \frac{-\langle (s_j - s_i^{\text{pref}})^2 \rangle_{Q(s_j)}}{2\sigma_{\text{tc}}^2} \\
&= \frac{-[(s_i^{\text{pref}} - \mu_j)^2 + \tau_j^{-1}]}{2\sigma_{\text{tc}}^2}
\end{aligned} \tag{9}$$

Therefore:

$$p_{ij} = \frac{e^{\Psi(\alpha_j) - \Psi(\sum_{j=1}^J \alpha_j) - \frac{(s_i^{\text{pref}} - \mu_j)^2 + \tau_j^{-1}}{2\sigma_{\text{tc}}^2}}}{\sum_{j=1}^J e^{\Psi(\alpha_j) - \Psi(\sum_{j=1}^J \alpha_j) - \frac{(s_i^{\text{pref}} - \mu_j)^2 + \tau_j^{-1}}{2\sigma_{\text{tc}}^2}}} \tag{10}$$

$$\alpha_j = \rho_j - 1 + \sum_i r_i p_{ij} \tag{11}$$

$$\tau_j = \sum_{i=1}^I \frac{r_i p_{ij}}{2\sigma_{\text{tc}}^2} \tag{12}$$

$$\mu_j = \frac{1}{\tau_j} \sum_{i=1}^I s_i^{\text{pref}} \frac{r_i p_{ij}}{2\sigma_{\text{tc}}^2} \tag{13}$$

We actually want the natural parameters so

$$\eta_j = \mu_j \tau_j = \sum_{i=1}^I s_i^{\text{pref}} \frac{r_i p_{ij}}{2\sigma_{\text{tc}}^2} \quad (14)$$

Also we don't want complex division in the neural computations so:

$$\begin{aligned} F_i(\mu_j, \alpha_j, \tau_j) &= e^{\Psi(\alpha_j) - \Psi(\sum_{j=1}^J \alpha_j) - \frac{(s_i^{\text{pref}} - \frac{\eta_j}{\tau_j})^2 + \tau_j^{-1}}{2\sigma_{\text{tc}}^2}} \\ p_{ij} &= F_i(\mu_j, \alpha_j, \tau_j) \pi_i \\ \pi_i &= \frac{1}{\sum_{j=1}^J F_i(\mu_j, \alpha_j, \tau_j)} \end{aligned} \quad (15)$$

For simulation purposes:

$$\begin{aligned} f_{ij} &= \Psi(\alpha_j) - \Psi(\sum_{j=1}^J \alpha_j) - \frac{(s_i^{\text{pref}} - \frac{\eta_j}{\tau_j})^2 + \tau_j^{-1}}{2\sigma_{\text{tc}}^2} \\ f_i^* &= \max_J f_{ij} \\ \pi_i &= \sum_{j=1}^J e^{f_{ij}} = e^{f_i^*} \sum_{j \neq \arg \max_J} e^{f_{ij} - f_i^*} \\ \log \pi_i &= f_i^* + \log \sum_{j \neq \arg \max_J} e^{f_{ij} - f_i^*} \\ p_{ij} &= \frac{e^{f_{ij}}}{\pi_i} = e^{f_{ij} - \log \pi_i} \\ \log p_{ij} &= f_{ij} - \log \pi_i = f_{ij} - f_i^* + \log \sum_{j \neq \arg \max_J} e^{f_{ij} - f_i^*} \end{aligned} \quad (16)$$

And now we have update equations:

$$\begin{aligned}
\frac{d\eta_j}{dt} &= -\eta_j + \frac{1}{2\sigma_{\text{tc}}^2} \sum_{i=1}^I s_i^{\text{pref}} r_i p_{ij} \\
\frac{d\tau_j}{dt} &= -\tau_j + \frac{1}{2\sigma_{\text{tc}}^2} \sum_{i=1}^I r_i p_{ij} \\
\frac{dp_{ij}}{dt} &= -p_{ij} + F_i(\mu_j, \alpha_j, \tau_j) \pi_i \\
\frac{d\pi_i}{dt} &= 1 - \pi_i \sum_{j=1}^J F_i(\mu_j, \alpha_j, \tau_j) \\
\frac{d\alpha_j}{dt} &= -\alpha_j + \rho_j - 1 + \sum_i r_i p_{ij}
\end{aligned} \tag{17}$$

Agan for simulation purposes ($p_{ij} = \log p_{ij}$):

$$\begin{aligned}
\frac{d\eta_j}{dt} &= -\eta_j + \frac{1}{2\sigma_{\text{tc}}^2} \sum_{i=1}^I s_i^{\text{pref}} r_i e^{p_{ij}} \\
\frac{d\tau_j}{dt} &= -\tau_j + \frac{1}{2\sigma_{\text{tc}}^2} \sum_{i=1}^I r_i e^{p_{ij}} \\
\frac{dp_{ij}}{dt} &= -p_{ij} + f_{ij} - \max_J f_{ij} + \log \sum_{j \neq \arg \max_J} e^{f_{ij} - \max_J f_{ij}} \\
\frac{d\alpha_j}{dt} &= -\alpha_j + \rho_j - 1 + \sum_i r_i e^{p_{ij}} \\
\frac{df_{ij}}{dt} &= -f_{ij} + \Psi(\alpha_j) - \Psi\left(\sum_{j=1}^J \alpha_j\right) - \frac{(s_i^{\text{pref}} - \frac{\eta_j}{\tau_j})^2 + \tau_j^{-1}}{2\sigma_{\text{tc}}^2}
\end{aligned} \tag{18}$$