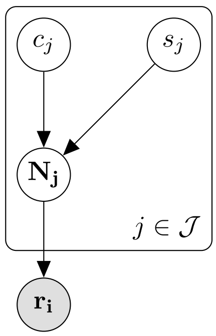
Divisive Normalization Notes



s_j is stimulus j
c_j is contrast of grating j
r_i is response of neuron i
$f_i(s)$ is tuning curve of neuron i
Tuning curve assumptions:
Tuning curves cover the space so $\sum_i f_i(s_j)$ is independent of s_j
The tuning curves are von Mises: $f_i(s_j) = e^{\kappa(\cos(s_j - s_i^{\text{pref}}) - 1)}$
N_{ij} is the response of neuron i to grating j
$P(N|c,s) = \prod_i \text{Poisson}(N_j; f_i(s_j)c_j)$
$P(c) = \text{Gamma}(c_j; \alpha, \beta)$
$P(s) = \text{Unif}(0, 2\pi)$

$$\begin{aligned} P(\mathbf{c},\mathbf{s}|\mathbf{r}) &= \sum_N P(\mathbf{N},\mathbf{c},\mathbf{s}|\mathbf{r}) = \sum_N P(\mathbf{r}|\mathbf{N})P(\mathbf{N}|\mathbf{c},\mathbf{s})P(\mathbf{c})P(\mathbf{s}) \\ &\propto \prod_i (\delta(r_i - \sum_j N_{ij}) \prod_j \frac{(f_i(s_j)c_j)^{N_{ij}} e^{-f_i(s_j)c_j}}{N_{ij}!})(\prod_j \frac{\beta^\alpha}{\Gamma(\alpha)} c_j^{\alpha-1} e^{\beta c_j}) \end{aligned} \tag{1}$$

For the variational inference, we need

$$\begin{aligned} \log P(\mathbf{N},\mathbf{c},\mathbf{s}|\mathbf{r}) = \sum_i [\log(\delta(r_i - \sum_j N_{ij})) - \sum_j [-\log N_{ij}! + N_{ij}\log(f_i(s_j)c_j) - f_i(s_j)c_j]] \\ + \sum_j [(\alpha - 1)\log c_j - \beta c_j - \log(\beta^{-\alpha}\Gamma(\alpha))] \end{aligned} \tag{2}$$

Then we approximate this using a factorized distribution:

$$Q(\mathbf{N},\mathbf{c},\mathbf{s}|\mathbf{r}) = Q(\mathbf{N}|\mathbf{r})Q(\mathbf{c}|\mathbf{r})Q(\mathbf{s}|\mathbf{r}) \tag{3}$$

Then

$$\log Q(\mathbf{N}|\mathbf{r}) = \langle \log P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) \rangle_{Q(\mathbf{c}|\mathbf{r})Q(\mathbf{s}|\mathbf{r})}$$
$$= \sum_i [\log(\delta(r_i - \sum_j N_{ij})) - \sum_j [N_{ij} \langle \log(f_i(s_j)) c_j \rangle_{Q(\mathbf{c}|\mathbf{r})Q(\mathbf{s}|\mathbf{r})} - \log N_{ij}!]] \tag{4}$$

$$\begin{aligned}
\log Q(\mathbf{c}|\mathbf{r}) &= \langle \log P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) \rangle_{Q(\mathbf{N}|\mathbf{r})Q(\mathbf{s}|\mathbf{r})} \\
&= \sum_{ij} [\langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})} \log c_j - \langle f_i(s_j) \rangle_{Q(\mathbf{s}|\mathbf{r}} c_j] \\
&\quad + \sum_j [(\alpha - 1) \log c_j - \beta c_j - \log(\beta^{-\alpha} \Gamma(\alpha))] \\
&= \sum_{ij} [(\alpha + \langle N_{ij} \rangle_{Q(\mathbf{c}|\mathbf{r})} - 1) \log c_j - (\beta + \langle f_i(s_j) \rangle_{Q(\mathbf{s}|\mathbf{r})}) c_j] \\
&\quad - \sum_j \log(\beta^{-\alpha} \Gamma(\alpha))
\end{aligned} \tag{5}$$

$$\log Q(\mathbf{s}|\mathbf{r}) = \langle \log P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) \rangle_{Q(\mathbf{N}|\mathbf{r})Q(\mathbf{c}|\mathbf{r})}$$
$$= \sum_{ij} [\langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})} \log(f_i(s_j)) - f_i(s_j) \langle c_j \rangle_{Q(\mathbf{c}|\mathbf{r})}] = \sum_{ij} [\langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})} \log(f_i(s_j))] \tag{6}$$

Last equality is because of space covering assumption

So $Q(\mathbf{N}|\mathbf{r})$ is a multinomial, $Q(\mathbf{c}|\mathbf{r})$ is a Gamma and $Q(\mathbf{s}|\mathbf{r})$ is von Mises

$$Q(\mathbf{N}|\mathbf{r}) = \prod_i \delta(r_i - \sum_j N_{ij}) r_i! \prod_j \frac{\langle (f_i(s_j) c_j) \rangle_{Q(\mathbf{c}|\mathbf{r})Q(\mathbf{s}|\mathbf{r})}^{N_{ij}}}{N_{ij}!} \tag{7}$$

$$Q(\mathbf{c}|\mathbf{r}) = \prod_j \text{Gamma}(c_j | \alpha + \langle N_{ij} \rangle_{Q(\mathbf{c}|\mathbf{r})}, \beta + \langle f_i(s_j) \rangle_{Q(\mathbf{s}|\mathbf{r})}) \tag{8}$$

$$
\begin{aligned}
Q(\mathbf{s}|\mathbf{r}) &\propto \prod_j e^{\kappa(\sum_{ij}[\langle N_{ij}\rangle_{Q(\mathbf{N}|\mathbf{r})}\cos(s_j - s_i^{\text{pref}})} \\
&= \prod_j e^{\kappa(\sum_j \langle N_{ij}\rangle_{Q(\mathbf{N}|\mathbf{r})}(\cos s_j \cos s_i^{\text{pref}} + \sin s_j \sin s_i^{\text{pref}}))} \\
&= \prod_j e^{\kappa(\sum_j (\cos s_j \sum_i \langle N_{ij}\rangle_{Q(\mathbf{N}|\mathbf{r})}\cos s_i^{\text{pref}} + \sin s_j \sum_i \langle N_{ij}\rangle_{Q(\mathbf{N}|\mathbf{r})}\sin s_i^{\text{pref}})})} \\
&= \prod_j e^{\kappa(\cos s_j \cos(\sum_i \langle N_{ij}\rangle_{Q(\mathbf{N}|\mathbf{r})} s_i^{\text{pref}}) + \sin s_j \sin(\sum_i \langle N_{ij}\rangle_{Q(\mathbf{N}|\mathbf{r})} s_i^{\text{pref}})}} \\
&= \prod_j e^{\tilde{\kappa}\cos(s_j - \hat{s})} \\
\text{where } \hat{s} &= \sum_i \langle N_{ij}\rangle_{Q(\mathbf{N}|\mathbf{r})} s_i^{\text{pref}}
\end{aligned} \tag{9}
$$

Calculations with: $P(r|c,s) = \prod_i \text{Poisson}(r_i; \phi(\sum_j f_i(s_j)c_j))$

$$
\begin{aligned}
P(\mathbf{c},\mathbf{s}|\mathbf{r}) &= P(\mathbf{r}|\mathbf{c},\mathbf{s})P(\mathbf{c})P(\mathbf{s}) \\
&\propto \prod_i \frac{(\phi(\sum_j (f_i(s_j)c_j)))^{r_i} e^{-\phi(\sum_j (f_i(s_j)c_j))}}{r_i!})(\prod_j \frac{\beta^\alpha}{\Gamma(\alpha)} c_j^{\alpha-1} e^{\beta c_j})
\end{aligned} \tag{10}
$$

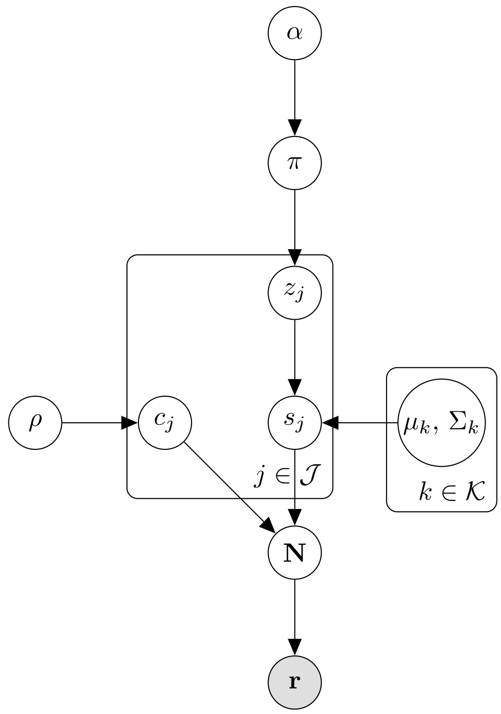
$$
\begin{aligned}
\log P(\mathbf{c},\mathbf{s}|\mathbf{r}) &= \sum_i [-\log r_i! + r_i \log(\phi(\sum_j (f_i(s_j)c_j))) - (\phi(\sum_j (f_i(s_j)c_j)))]] \\
&\quad + \sum_j [(\alpha-1)\log c_j - \beta c_j - \log(\beta^{-\alpha}\Gamma(\alpha))]
\end{aligned} \tag{11}
$$

$$
Q(\mathbf{c},\mathbf{s}|\mathbf{r}) = Q(\mathbf{c}|\mathbf{r})Q(\mathbf{s}|\mathbf{r}) \tag{12}
$$

$$
\begin{aligned}
\log Q(\mathbf{c}|\mathbf{r}) &= \langle \log P(\mathbf{c},\mathbf{s}|\mathbf{r})\rangle_{Q(\mathbf{s}|\mathbf{r})} \\
&= \sum_i [r_i \langle \log(\phi(\sum_j (f_i(s_j)c_j)))\rangle_{Q(\mathbf{s}|\mathbf{r})} - \langle \phi(\sum_j (f_i(s_j)c_j)))\rangle_{Q(\mathbf{s}|\mathbf{r})}] \\
&\quad + \sum_j [(\alpha-1)\log c_j - \beta c_j - \log(\beta^{-\alpha}\Gamma(\alpha))]
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\log Q(\mathbf{s}|\mathbf{r}) &= \langle \log P(\mathbf{c},\mathbf{s}|\mathbf{r})\rangle_{Q(\mathbf{c}|\mathbf{r})} \\
&= \sum_i [r_i \langle \log(\phi(\sum_j (f_i(s_j)c_j)))\rangle_{Q(\mathbf{c}|\mathbf{r})} - \langle \log(\phi(\sum_j (f_i(s_j)c_j)))c_j\rangle_{Q(\mathbf{c}|\mathbf{r})}]
\end{aligned} \tag{14}
$$

For later: consider with relative contrast (Dirichlet?)



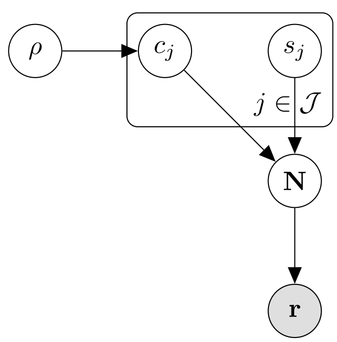
$s_j|z_j \sim \mathcal{N}(\mu_{z_j}, \Sigma_{z_j})$
$N_{ij}|\{s_j\} \sim \text{Poisson}(c_j f_i(s_j))$
$r_i|N_{ij} \sim \delta(r_i - z_j N_{ij})$
$\pi \sim \text{Dir}(\alpha)$
$c \sim \text{Dir}(\rho)$

$$\begin{aligned}
\log Q(z) &= \langle \log P(z, s, c, \mu, \Sigma, N, r) \rangle_{Q(s)Q(N)Q(c)Q(\mu,\Sigma)} \\
&= \langle \log(\text{Mult}(z; \pi)\text{Dir}(\pi; \alpha) \prod_{j=1}^{J} \mathcal{N}(s_j; \mu_{z_j}, \Sigma_{z_j})\text{Dir}(c, \rho) ... \\
&\quad \prod_{j=1}^{J}((\delta(r_i - z_j N_{ij})\text{Poisson}(c_j f_i(s_j))) ... \\
&\quad \prod_{k=1}^{K} \mathcal{N}(\mu_k; m_k, (\beta_0 \Sigma_k))\text{Wi}(\Sigma_k^{-1}; \mathbf{L}_0, \nu_0)) \rangle_{Q(s)Q(N)Q(c)Q(\mu,\Sigma)}
\end{aligned} \tag{15}$$

$$\begin{aligned}\log Q(s) &= \langle \log \mathcal{N}(\mu_{z_j}, \Sigma_{z_j}) \rangle_{Q(z)Q(\mu,\Sigma)} + \sum_{i=1}^{I} \langle N_{ij} \log f_i(s_j) \rangle_{Q(N)} \\ &= -(\langle (s_j - \mu_j)^T \Sigma_{z_j}^{-1} (s_j - \mu_j) \rangle_{Q(z)Q(\mu,\Sigma)} + \sum_{i=1}^{I} \langle N_{ij} \rangle_{Q(N)} \log f_i(s_j)) \\ &= -(\sum_{k=1}^{K} Q(z_j = k) \langle (s_j - \mu_j)^T \Sigma_{z_j}^{-1} (s_j - \mu_j) \rangle_{Q(\mu,\Sigma)} - \langle N_{ij} \rangle_{Q(N)} \frac{||(s_j - s_i^{\text{pref})}||^2}{L}) \end{aligned} \tag{16}$$

$$\begin{aligned}\log Q(c) &= \langle \sum_j (\rho_j - 1) \log c_j \rangle + \sum_{ij} \langle N_{ij} \log c_j \rangle \\ &= \sum_j [\log c_j][\rho_j - 1 + \sum_i \langle N_{ij} \rangle \end{aligned} \tag{17}$$

Redoing above with this graphical model:



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The tuning curves are von Mises: $f_i(s_j) = e^{\kappa(\cos(s_j - s_i^{\text{pref}}) - 1)}$
N_{ij} is the response of neuron i to grating j
$N_{ij} | \{s_j\} \sim \text{Poisson}(c_j f_i(s_j))$
$r_i | N_{ij} \sim \delta(r_i - z_j N_{ij})$

$\pi \sim \mathrm{Dir}(\alpha)$
$c \sim \mathrm{Dir}(\rho)$
$s \sim \mathrm{Unif}(0, \pi)$

$$\begin{aligned} P(\mathbf{c}, \mathbf{s}|\mathbf{r}) &= \sum_N P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) = \sum_N P(\mathbf{r}|\mathbf{N}) P(\mathbf{N}|\mathbf{c}, \mathbf{s}) P(\mathbf{c}) P(\mathbf{s}) \\ &\propto \prod_i (\delta(r_i - \sum_j N_{ij}) \prod_j \frac{(f_i(s_j)c_j)^{N_{ij}} e^{-f_i(s_j)c_j}}{N_{ij}!})(\prod_j \frac{1}{\beta(\rho)} c_j^{\rho-1}) \end{aligned} \tag{18}$$

For the variational inference, we need

$$\begin{aligned} \log P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) = \sum_i [\log(\delta(r_i - \sum_j N_{ij}))] - \sum_j [-\log N_{ij}! + N_{ij} \log(f_i(s_j)c_j)] \\ + \sum_j [(\rho_j - 1) \log c_j] \end{aligned} \tag{19}$$

Then we approximate this using a factorized distribution:

$$Q(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) = Q(\mathbf{N}|\mathbf{r}) Q(\mathbf{c}|\mathbf{r}) Q(\mathbf{s}|\mathbf{r}) \tag{20}$$

$$\begin{aligned} \log Q(\mathbf{N}|\mathbf{r}) &= \langle \log P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) \rangle_{Q(\mathbf{c}|\mathbf{r})Q(\mathbf{s}|\mathbf{r})} \\ &= \sum_i [\log(\delta(r_i - \sum_j N_{ij})) - \sum_j [N_{ij} \langle \log(f_i(s_j))c_j \rangle_{Q(\mathbf{c}|\mathbf{r})Q(\mathbf{s}|\mathbf{r})} - \log N_{ij}!]] \end{aligned} \tag{21}$$

$$\begin{aligned} \log Q(\mathbf{c}|\mathbf{r}) &= \langle \log P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) \rangle_{Q(\mathbf{N}|\mathbf{r})Q(\mathbf{s}|\mathbf{r})} \\ &= \sum_j [\log c_j (\rho_j - 1 + \sum_i \langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})})] \end{aligned} \tag{22}$$

$$
\begin{aligned}
\log Q(\mathbf{s}|\mathbf{r}) &= \langle \log P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) \rangle_{Q(\mathbf{N}|\mathbf{r})Q(\mathbf{c}|\mathbf{r})} \\
&= \sum_{ij} [\langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})} \log(f_i(s_j)) - f_i(s_j) \langle c_j \rangle_{Q(\mathbf{c}|\mathbf{r})}] = \sum_{ij} [\langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})} \log(f_i(s_j))]
\end{aligned} \tag{23}
$$

So $Q(\mathbf{N}|\mathbf{r})$ is multinomial, $Q(\mathbf{c}|\mathbf{r})$ is Dirichlet and $Q(\mathbf{s}|\mathbf{r})$ is von Mises

$$
Q(\mathbf{N}|\mathbf{r}) = \prod_i \delta(r_i - \sum_j N_{ij}) r_i! \prod_j \frac{\langle (f_i(s_j) c_j) \rangle^{N_{ij}}_{Q(\mathbf{c}|\mathbf{r})Q(\mathbf{s}|\mathbf{r})}}{N_{ij}!} \tag{24}
$$

$$
Q(\mathbf{c}|\mathbf{r}) = \prod_j \mathrm{Dir}(c_j; \rho_j - 1 + \sum_i \langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})}) \tag{25}
$$