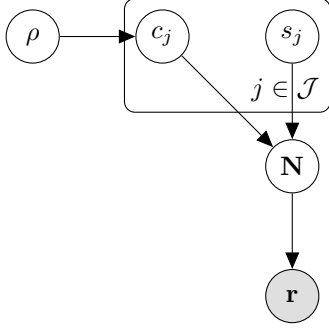# Divisive Normalization Notes

We consider the problem of demixing stimuli that are mixed due to neurons having large receptive fields or wide tuning curves.

0.1 Generative model

s_j is the jth stimulus, $\mathbf{s} \sim \text{Unif}(0, \pi)$
c_j is the contrast of the jth stimulus, $\mathbf{c} \sim \text{Dir}(\rho)$

$f_i(s)$ is the tuning curve of the ith neuron in response to a single stimulus s
Tuning curve assumptions:

- Tuning curves cover the space so $\sum_i f_i(s)$ is independent of s

- The tuning curves are Gaussian: $f_i(s_j) \sim \mathcal{N}(s_i^{\text{pref}}, \sigma_{\text{tc}}^2)$

N_{ij} is the spike count of the ith neuron elicited by the jth stimulus
$N_{ij}|s_j \sim \text{Poisson}(c_j f_i(s_j))$
r_i is the total spike count of the ith neuron in response to all stimuli
$r_i = \sum_j N_{ij}$

0.2 Inference

$$P(\mathbf{c}, \mathbf{s}|\mathbf{r}) = \sum_{\mathbf{N}} P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r})$$

$$
\begin{aligned}
P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) &= P(\mathbf{r}|\mathbf{N})P(\mathbf{N}|\mathbf{c}, \mathbf{s})P(\mathbf{c})P(\mathbf{s}) \\
&\propto \prod_i \left[\delta\Big(r_i - \sum_j N_{ij}\Big)\Big(\prod_j \frac{(f_i(s_j)c_j)^{N_{ij}}}{N_{ij}!}\Big) \right] \prod_j \frac{1}{\beta(\rho)} c_j^{\rho_j - 1}
\end{aligned}
\tag{1}
$$

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For the variational inference, we need

$$\log P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) = \sum_i \left[\log \delta\left(r_i - \sum_j N_{ij}\right) + \sum_j \left[-\log N_{ij}! + N_{ij} \log(f_i(s_j)c_j) \right] \right]$$
$$+ \sum_j (\rho_j - 1) \log c_j$$

$$(2)$$

0.3 Variational approximation

Then we approximate this using a factorized distribution:

$$Q(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) = Q(\mathbf{N}|\mathbf{r})Q(\mathbf{c}|\mathbf{r})Q(\mathbf{s}|\mathbf{r}) \tag{3}$$

$$\log Q(\mathbf{N}|\mathbf{r}) = \sum_i \log Q(\mathbf{N}_i|\mathbf{r}_i)$$

$$\log Q(\mathbf{N}_i|\mathbf{r}_i) = \langle \log P(\mathbf{N_i}, \mathbf{c}, \mathbf{s}|\mathbf{r_i}) \rangle_{Q(\mathbf{c}|\mathbf{r})Q(\mathbf{s}|\mathbf{r})}$$
$$= \log(\delta\left(r_i - \sum_j N_{ij}\right) + \sum_j N_{ij}\left(\langle \log c_j \rangle + \langle \log f_i(s_j) \rangle\right) - \sum_j \log N_{ij}!$$

$$\log Q(\mathbf{c}|\mathbf{r}) = \langle \log P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) \rangle_{Q(\mathbf{N}|\mathbf{r})Q(\mathbf{s}|\mathbf{r})}$$
$$= \sum_j [\log(c_j \rho_j - 1) + \sum_i \langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})}]$$

$$\log Q(\mathbf{s}|\mathbf{r}) = \langle \log P(\mathbf{N}, \mathbf{c}, \mathbf{s}|\mathbf{r}) \rangle_{Q(\mathbf{N}|\mathbf{r})Q(\mathbf{c}|\mathbf{r})}$$
$$= \sum_{ij} [\langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})} \log(f_i(s_j)) - f_i(s_j)\langle c_j \rangle_{Q(\mathbf{c}|\mathbf{r})}]$$
$$= \sum_{ij} \langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})} \log(f_i(s_j))$$

$$(4)$$

So we now know

$$Q(\mathbf{c}|\mathbf{r}) \sim \mathrm{Dir}(\boldsymbol{\alpha})$$
$$\alpha_j = \rho_j - 1 + \sum_i \langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})} \tag{5}$$

and

$$Q(s_j|r) \propto \frac{1}{2\sigma_{tc}^2} \sum_{i=1}^{I} \langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})} (s_j - s_i^{\text{pref}})^2$$

$$\sim \mathcal{N}(\mu_j, \tau_j)$$

$$\tau_j = \sum_{i=1}^{I} \frac{\langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})}}{2\sigma_{tc}^2} \tag{6}$$

$$\mu_j = \frac{1}{\tau_j} \sum_{i=1}^{I} s_i^{\text{pref}} \frac{\langle N_{ij} \rangle_{Q(\mathbf{N}|\mathbf{r})}}{2\sigma_{tc}^2}$$

$$Q(N_{ij}|r) \sim \text{Mult}(r_i, p_{ij})$$

$$p_{ij} = \frac{e^{\langle \log c_j \rangle + \langle \log f_i(s_j) \rangle}}{\sum_{j=1}^{J} e^{\langle \log c_j \rangle + \langle \log f_i(s_j) \rangle}} \tag{7}$$

From this we can compute:

$$\langle \log c_j \rangle_{Q(c_j)} = \Psi(\alpha_j) - \Psi(\sum_{j=1}^{J} \alpha_j) \tag{8}$$

and

$$\langle \log f_i(s_j) \rangle_{Q(s_j)} = \frac{-\langle (s_j - s_i^{\text{pref}})^2 \rangle_{Q(s_j)}}{2\sigma_{tc}^2}$$

$$= \frac{-[(s_i^{\text{pref}} - \mu_j)^2 + \tau_j^{-1}]}{2\sigma_{tc}^2} \tag{9}$$

Therefore:

$$p_{ij} = \frac{e^{\Psi(\alpha_j) - \Psi(\sum_{j=1}^{J} \alpha_j) - \frac{(s_i^{\text{pref}} - \mu_j)^2 + \tau_j^{-1}}{2\sigma_{tc}^2}}}{\sum_{j=1}^{J} e^{\Psi(\alpha_j) - \Psi(\sum_{j=1}^{J} \alpha_j) - \frac{(s_i^{\text{pref}} - \mu_j)^2 + \tau_j^{-1}}{2\sigma_{tc}^2}}} \tag{10}$$

$$\alpha_j = \rho_j - 1 + \sum_i r_i p_{ij} \tag{11}$$

$$\tau_j = \sum_{i=1}^{I} \frac{r_i p_{ij}}{2\sigma_{tc}^2} \tag{12}$$

$$\mu_j = \frac{1}{\tau_j} \sum_{i=1}^{I} s_i^{\text{pref}} \frac{r_i p_{ij}}{2\sigma_{tc}^2} \tag{13}$$

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We actually want the natural parameters so

$$\eta_j = \mu_j \tau_j = \sum_{i=1}^{I} s_i^{\mathrm{pref}} \frac{r_i p_{ij}}{2\sigma_{\mathrm{tc}}^2} \tag{14}$$

Also we don't want complex division in the neural computations so:

$$F_i(\mu_j, \alpha_j, \tau_j) = e^{\Psi(\alpha_j) - \Psi(\sum_{j=1}^{J} \alpha_j) - \frac{(s_i^{\mathrm{pref}} - \frac{\eta_j}{\tau_j})^2 + \tau_j^{-1}}{2\sigma_{\mathrm{tc}}^2}}$$
$$p_{ij} = F_i(\mu_j, \alpha_j, \tau_j) \pi_{ij} \tag{15}$$
$$\pi_{ij} = \frac{1}{\sum_{j=1}^{J} F_i(\mu_j, \alpha_j, \tau_j)}$$

And now we have update equations:

$$\frac{d\eta_j}{dt} = -\eta_j + \frac{1}{2\sigma_{\mathrm{tc}}^2} \sum_{i=1}^{I} s_i^{\mathrm{pref}} r_i p_{ij}$$

$$\frac{d\tau_j}{dt} = -\tau_j + \frac{1}{2\sigma_{\mathrm{tc}}^2} \sum_{i=1}^{I} r_i p_{ij}$$

$$\frac{dp_{ij}}{dt} = -p_{ij} + F_i(\mu_j, \alpha_j, \tau_j) \pi_{ij} \tag{16}$$

$$\frac{d\pi_{ij}}{dt} = 1 - \pi_{ij} \sum_{j=1}^{J} F_i(\mu_j, \alpha_j, \tau_j)$$

$$\frac{d\alpha_j}{dt} = -\alpha_j + \rho_j - 1 + \sum_{i} r_i p_{ij}$$