4.7 L’Hopital’s Rule

MATH 205

Review of Limits

- A limit of a function, whether at a point or at $\pm\infty$, is the ***PREDICTED*** output of the function.
- For continuous functions, $\lim_{x \to a} f(x) = f(a)$
- For a limit to exist, the right-hand and left-hand limits must exist and be equal.
- Though we use graphs and tables to help recognize limits, we need to be able to algebraically determine limits.
- $\lim_{x \to 0^+} \frac{1}{x} = \infty$ and $\lim_{x \to \infty} \frac{1}{x} = 0$

Indeterminant Form

1. Determine $\lim\limits_{x \to 0} \dfrac{\tan 2x}{\ln(1+x)}$

- Substitution leads to 0/0
  - 0/0 is called an "Indeterminant Form"

- Lucky for us, there exists a method for dealing with limits of indeterminant form.

- Point of interest: The definition of the derivative as a limit of the difference quotient is an indeterminant form:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

L'Hopital's Rule

- Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I if $x \neq a$.

  If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then

  $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

  assuming that the limit on the right side exists.

- This means we can find the limit of certain indeterminant forms by finding the limit of the quotient of the derivatives

Determine the following limit

2. $\lim_{x \to 1} \frac{\ln x^2}{x^2 - 1}$

Determine the following limit

3. $\lim_{x \to 0} \frac{\sqrt{x+1} - 1 - x/2}{x^2}$

Determine the following limit

- 4. $\lim_{x \to 0^-} \frac{\sin x + \tan x}{e^x + e^{-x} - 2}$

Now for something completely different! (or is it?)

$$5.\quad \lim_{x\to\infty} \frac{e^{-x}}{x^{-1}}$$

- Ok, so taking the derivatives just makes it more complicated. Lets try re-writing the problem.

$$5a.\quad \lim_{x\to\infty} \frac{x}{e^{x}}$$

- Now the issue is a form of $\pm\infty/\pm\infty$ not 0/0.
  - Well, as you might guess, L'Hopital's rule works for this form also.

- If we run across other indeterminant forms, such as $0 \bullet \infty$ or $\infty - \infty$, we will need to algebraically manipulate the expression to get 0/0 or $\pm\infty/\pm\infty$ before apply L'Hopital's Rule.

Indeterminant Forms

6. $\lim_{x \to 0^+} \frac{\ln x}{\cot x}$

Indeterminant Forms

7. $\lim\limits_{x \to \frac{\pi}{2}^-} \left(\tan x \cdot \ln(\sin x) \right)$

Indeterminant Forms

8. $\lim_{x \to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$

Indeterminant Powers

- When faced with limits of indeterminant forms involving powers, such as 1^{∞}, 0^0, or ∞^0, we can apply L'Hopital's rule if we first take the logarithm of the function.

- Remember to put your result back into exponential form to determine the desired limit.

Indeterminant Powers

9. $\lim_{x \to 0^+} (\sin x)^x$

Indeterminant Powers

- 10. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

Indeterminant Powers

- 11. $\lim_{x \to \frac{\pi}{2}^-} (\tan x)^{\cos x}$

Growth Rates of Functions

- Suppose f and g are functions with

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$$

  Then f grows faster than g, $(g << f)$, as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \text{ or } \lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

- The functions f and g have comparable growth rates if $\lim_{x\to\infty} \frac{g(x)}{f(x)} = M$, where $0 < M < \infty$

Relative Rates of Growth

- The following functions are ordered according to increasing growth rates as $x \to \infty$
  for all positive real numbers p, q, r, s, and $b > 1$

$$\ln^q x \quad << x^p \quad << x^p \ln^r x \quad << x^{p+s} \quad << b^x \quad << x^x$$

- Show $x^p << b^x$