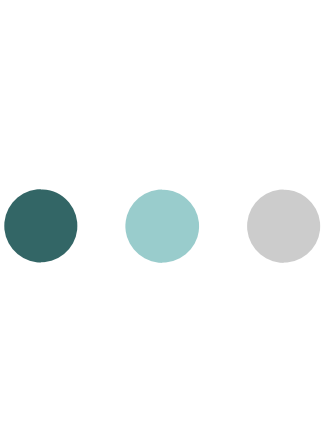


2.6 Continuity

MATH 205

What is continuity?

- Warm Fuzzy: Trace the graph without lifting the pencil.
  - Any point at which you lift the pencil is a point of discontinuity.
- Continuity and limits are intricately connected.
- There are two types of continuity:
  - Continuity at an interior point of the domain of a function
  - Continuity at an end point of the domain of a function

Mathematical Definition of Continuity at a Point

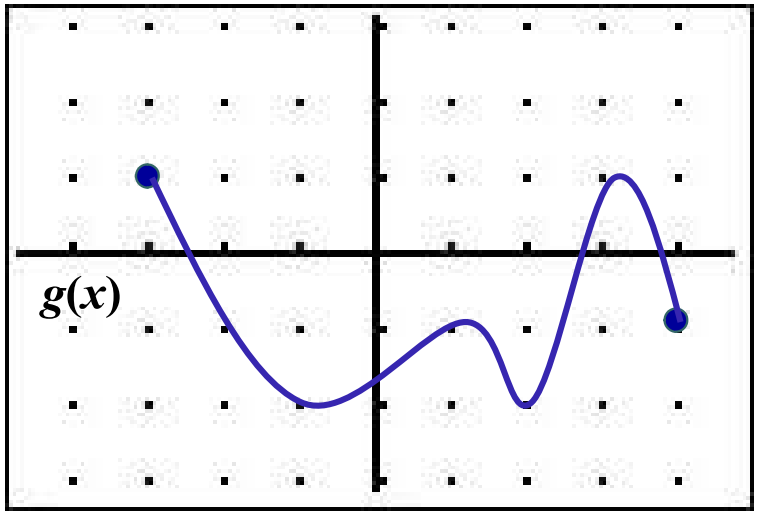
- A function, $y = f(x)$, is continuous at a point, a, in the interior of its domain if $\lim_{x \to a} f(x) = f(a)$
  - $f(a)$ is defined
  - $\lim_{x \to a} f(x)$ exists $\left\{ \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \right\}$
- A function, $y = f(x)$, is continuous at a left endpoint, a, of its domain if $\lim_{x \to a^+} f(x) = f(a)$ {right-continuous}
- A function, $y = f(x)$, is continuous at a right endpoint, b, of its domain if $\lim_{x \to b^-} f(x) = f(b)$ {left-continuous}

Continuity on an Interval

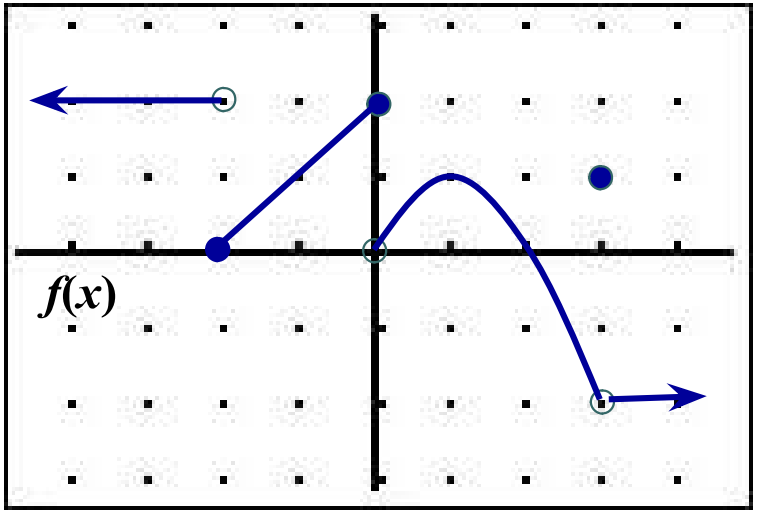
- A function *f* is said to be continuous on the interval, *I*, if it is continuous at all points of *I*. If *I* contains its endpoints, continuity on *I* means continuous from the right or the left at the endpoints.

On what interval(s) are the following functions continuous?

1.



2.



Discontinuities

- There are four ways a function can be discontinuous at a point:
  - Removable Discontinuity: Hole
  - Essential Discontinuities: Jump, Asymptotic (Infinite), and Erratic (Oscillating).
- Knowing the limit(s) at a point tells us if a discontinuity exists and, if so, what type of discontinuity it is.

Describe the type of discontinuity that exists at $x = c$

3. $\lim_{x \to c^+} f(x) = \infty$ and $\lim_{x \to c^-} f(x) = -\infty$

4. $\lim_{x \to c^-} f(x) = 10$ and $\lim_{x \to c^+} f(x) = -3$

5. $\lim_{x \to c^-} f(x) = 7$ and $\lim_{x \to c^+} f(x) = 7$ and $f(c)$ is 13

6. $\lim_{x \to c^-} f(x) = 7$ and $\lim_{x \to c^+} f(x) = 7$ and $f(c)$ is undefined

7. $\lim_{x \to c^-} f(x) =$ D.N.E. and $\lim_{x \to c^+} f(x) =$ D.N.E.

So, what is a continuous function?

a. Is $f(x) = 3x^2 + 5x - 7$ a continuous function?

b. Is $k(x) = \cos(x)$ a continuous function?

c. Is $g(x) = \dfrac{4x^3 - 5x}{11x^2 + 1}$ a continuous function?

d. Is $m(x) = \dfrac{6x^2 - 19}{x^3 - 1}$ a continuous function?

e. Is $h(x) = \tan(x)$ a continuous function?

f. Is $p(x) = [[x]]$ a continuous function?

Definition of a Continuous Function

- A continuous function is a function that is continuous at every point in its domain.
- All polynomials functions are continuous functions.
- All rational functions are continuous functions
- All six basic trigonometric functions are continuous functions.
- Exponential functions are continuous functions
- If $f(x)$ is a continuous function and has an inverse on I, then $f^{-1}(x)$ is also continuous (on the interval consisting of the points $f(x)$, where x is in I)
- A continuous function MAY NOT be continuous for all real numbers, it only has to be continuous for numbers in its domain.

Properties of Continuous Functions

- Since continuity is based on limits, the properties of continuous functions are almost identical to the properties of limits.

- If f and g are continuous at $x = a$, then the following are continuous at $x = a$:

  Sum: $f + g$ Difference: $f - g$ Product: $f * g$

  Quotient: f/g, $g \neq 0$ Constant Multiplier: $k * f$

  Powers: $f^{r/s}$, provided it is defined on an open interval containing a where r and s are integers

Ahh, but what about the compostition of functions?

- Let $f(x) = \sin(x)$ and $g(x) = 2x - 7$. Is $f(g(x))$ a continuous function?

- The composition of continuous functions, at $x = a$, will be continuous at $x = a$ if the outer function is continuous at the output of the inner function.

- If f is continuous at $x = a$, and g is continuous at $f(a)$, then $g(f(a))$ is continuous at a.

8. Determine the intervals on which $h(x) = \cos\left(\sqrt{x^2 - 81}\right)$ is continuous.

Continuous Extensions or "How do we plug the hole?"

- If a function has a removable discontinuity (a hole), we can determine a piecewise function that is an extension of the original function that "fills the hole".

9. Define $f(3)$ in a way that extends $f(x) = \dfrac{x^2 + 2x - 15}{x^2 - 9}$ to be continuous at $x = 3$.

10. Determine a piecewise function so that $g(x) = \dfrac{\sin x}{x}$ is continuous for all x.

Climb that Mountain

11. Suppose a hiker starts up Mt. Washington at 8:00 am on 13 Jul 09 and reaches the top at 4:04 pm that same day. The hiker spends the night at the summit and some time around 7:45 am the next morning starts the trek back down. If the hiker reaches the trail head at 1:31 pm, is there any moment at which the hiker is at the same altitude at the exact same time on both days?

The I.V.T.

- The Intermediate Value Theorem: A function $y = f(x)$ that is continuous on $[a, b]$ takes on every value between $f(a)$ and $f(b)$.

- A function continuous over an interval cannot have any gaps.

- A function continuous over an interval in which the function's values change sign must have at least one zero on that interval.

12. Use the I.V.T. to show $\sqrt{x^4 - x^3 + 8x^2 + 22} = 7$ has at least one solution on [0,3].