5.2 The Definite Integral

MATH 205

What is a definite integral?

- The definite integral can be thought of as:

1) Algebraically: The accumulated value of a function

   or

2) Geometrically: The (*signed*) area under a curve.
   - The net area bounded by a function and the x-axis.

Riemann Sums and the Definite Integral

- If we allow the number of rectangles to increase without bound (an infinite number), the width of each rectangle approaches zero.

- The limit of a Riemann Sum as the largest subinterval, $||P||$, goes to zero is the Definite Integral

- The limit of a Riemann sum as the number of rectangles approaches infinity is the Definite Integral (if the limit exists).

- The Definite Integral is a number!

The Definite Integral

- A function f on $[a, b]$ is integrable on $[a, b]$ if $\lim_{\Delta x_k \to 0} \sum_{k=1}^{\infty} f(x_k^*)\Delta x_k$ exists.
  - This limit is the definite integral of f from a to b.

- The symbol for the Definite Integral of f with respect to x on $[a, b]$ is: $\int_a^b f(x)dx$

where: a is the lower limit of integration

b is the upper limit of integration

$f(x)$ is the integrand

the differential dx is the change in x

- A function continuous on $[a, b]$ or *bounded* on $[a, b]$ with a finite number of discontinuities is integrable on $[a, b]$

Partition Notation Morphed into Integral Notation!

1. Express $\lim_{\Delta x_k \to 0} \sum_{k=1}^{n} {x_k^*}^2 \Delta x_k$ as a definite integral of f partitioned on [0, 2].

2. Express $\lim_{\Delta x_k \to 0} \sum_{k=1}^{n} \cos x_k^* \Delta x_k$, as a definite integral of f partitioned on [0, 2π].

3. Express $\lim_{\Delta x_k \to 0} \sum_{k=1}^{n} (e^{x_k^*} + {x_k^*}^4) \Delta x_k$, as a definite integral of f partitioned on [-6, 12].

Indefinite vs. Definite

- The Indefinite Integral returns a family of antiderivatives.

- The Definite Integral returns a number (the signed (net) area under a curve or the accumulated amount over an interval)

Evaluate the Definite Integral

4. $\int_{-9}^{12}\left(\frac{x}{3}+5\right)dx$

Evaluate the Definite Integral

5. $\int_{-4}^{7} |x+2| \, dx$

Evaluate the Definite Integral

6. $\int_{-5}^{0} \sqrt{25 - x^2}\, dx$

Evaluate the Definite Integral

7. $\int_1^3 2x^2\,dx$

A couple of ideas

- Determine $\int_a^b c\,dx$ for any constant c.
- Determine $\int_a^b x\,dx$
- Determine $\int_a^b x^2\,dx$

Properties of Definite Integrals

- If f, g are integrable on $[a, b]$ then,

1. Order of Integration : $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. Zero Width Interval : $\int_a^a f(x)dx = 0$

3. Constant Multiple : $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

4. Sum/Difference : $\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Properties {Continued}

5. Additivity: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

6. Max-Min Inequality: If f has a maximum value (max f) and minimum value (min f) on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)dx \le \max f \cdot (b-a)$$

7. If $f(x) \ge g(x)$ on $[a,b]$, then $\int_a^b f(x)dx \ge \int_a^b g(x)dx$

Practice

- Suppose $\int_{-3}^{2} g(x)dx = 8$, $\int_{-3}^{2} k(x)dx = 5$, $\int_{2}^{8} g(x)dx = -3$ determine each of the following:

8. $\int_{2}^{-3} k(x)dx$

9. $\int_{-3}^{8} g(x)dx$

10. $\int_{-3}^{2} \left(g(x) - k(x)\right)dx$

11. $\int_{11}^{11} g(x)dx$