4.1 Maxima and Minima (a.k.a Extrema)

MATH 205

What the heck are extrema?

- When studying functions, determining the high and low points is of major importance.
- These high and low points are called the extrema.
- Extrema can either be local (relative) **or** absolute (global).

Absolute Maxima and Minima

- Let f be a function defined on an interval I containing c:

Then f has an **absolute maximum** value on I at a point c if $f(c) \geq f(x)$ for all x in I

and

an **absolute minimum** value on I at a point c if $f(c) \leq f(x)$ for all x in I.

Let $f(x) = \sin x$

- Determine the absolute extrema of $f(x)$ given the following domains:

1. $(-\infty, \infty)$
2. $(-\pi/2, \pi/2]$
3. $[-\pi/3, 7\pi/6]$
4. $(0, 2\pi)$

The Extreme Value Theorem

The Extreme Value Theorem:

If f is continuous on $[a, b]$, then f has both an absolute maximum value (M) and absolute minimum value (m) in $[a, b]$.

{Proof is beyond the scope of this class}

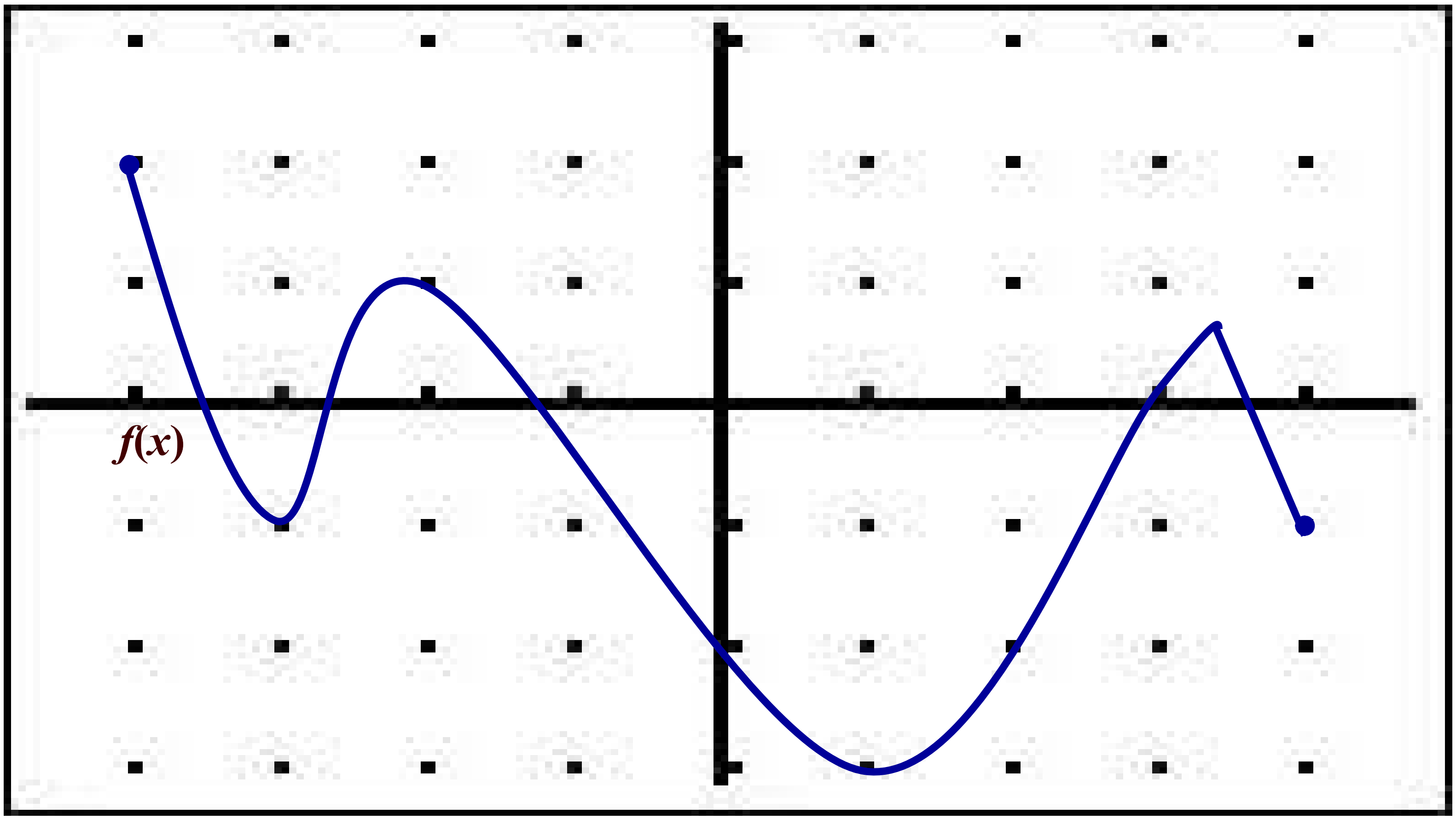
Local Maxima and Minima

- Suppose I is an interval on which f is defined and c is an interior point of I.

- A function, f, has a local maximum at an interior point, c, of its domain if $f(c) \geq f(x)$ for all x in some open interval (neighborhood) containing c.

- A function, f, has a local minimum at an interior point, c, of its domain if $f(c) \leq f(x)$ for all x in some open interval (neighborhood) containing c.

Determine all the extrema!



The First Derivative Theorem for Local Extreme Values

- If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0$$

- This does not mean every local extrema occur where $f'(c) = 0$
- The converse does not always hold. Just because $f'(c) = 0$ does not mean a local maximum or minimum exists

Definition: Critical Point

- An interior point c of the domain of a function f where $f'(c) = 0$ or $f'(c)$ is undefined is a critical point of f.
  - Critical points are candidates for local extrema

- The only places a function can have extreme values are:
  1. Interior points where the derivative equals zero
  2. Interior points where the derivative is undefined
  3. Endpoints of the domain of the function (Can only be Absolute)

Finding Extrema on a Closed Interval

- If you are determining the absolute (global) extrema for a continuous function on [a, b], you need to:
  1. Evaluate the function at all critical points and endpoints.
  2. The least of these values will be the absolute minimum and the greatest will be the absolute maximum.

Determine All the Extrema

1. $f(x) = x^3 - 7x^2 - 5x + 10 \text{ on } [0, 8]$

Determine All the Extrema

2. $g(x) = \dfrac{x}{x^2 + 1}$

Determine All the Extrema

3. $k(x) = x^{2/3}\left(x^2 - 9\right)$

Determine All the Extrema

4. $m(x) = x^{\frac{2}{3}} \cos x \text{ for } -2\pi \le x \le 2\pi$