MATH 205 – Calculus I **Exam #3**

Name: ______________

Instructions: Though calculators can be used for the entire exam, all problems require you to show your work. Any answer without proper justification will receive ***ZERO*** credit. Only **EXACT** answers will receive full credit unless otherwise noted. Proper Interval Notation must be used to receive credit.

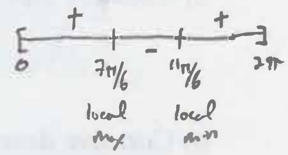
Questions 1 – 19: 4 points each, questions 20 – 23: 4 points each, questions 24 – 27: 6 points each. 11c

For questions 1 – 6:

Given $g(x) = \frac{1}{2}x - \cos x$, on $[0, 2\pi]$, determine each of the following:

1. The local extrema values and where they occur. $g'(x) = 1/2 + \sin x$

$g'(x) = 0$: $\sin x = -1/2$ $x = \sin^{-1}(-1/2)$ $x = \frac{7\pi}{6}, 11\pi/6$



a) Local maximum(s):

local max of $\frac{7\pi}{12} + \frac{\sqrt{3}}{2}$ at $x = \frac{7\pi}{6}$

≈ 2.699

b) Local minimum(s):

local min of $\frac{11\pi}{12} - \frac{\sqrt{3}}{2}$ at $x = \frac{11\pi}{6}$

≈ 2.014

2. The absolute extrema values and where they occur. $g(0) = -1$ $g(2\pi) = \pi - 1$

a) Absolute minimum of -1 at $x = 0$

b) Absolute maximum of $\frac{7\pi}{12} + \frac{\sqrt{3}}{2}$ at $x = 7\pi/6$

3. Intervals on which $g(x)$ is:

a) Increasing: $[0, 7\pi/6) \cup \left(\frac{11\pi}{6}, 2\pi\right]$

b) Decreasing: $(7\pi/6, 11\pi/6)$

12

For questions 1 – 6:

Given $g(x) = \frac{1}{2}x - \cos x$, on $[0, 2\pi]$, determine each of the following:

$g''(x) = \cos x$

$g''(x) = 0 :$ $\cos x = 0$

$x = \pi/2,\ 3\pi/2$

$+$ $\quad$ $-$ $\quad$ $+$

$0 \quad \pi/2 \quad 3\pi/2 \quad 2\pi$

4. The location of any points of inflection:

$x = \pi/2,\ 3\pi/2$

5. Intervals on which $g(x)$ is

a) Concave Up: $[0, \pi/2) \cup (3\pi/2, 2\pi]$

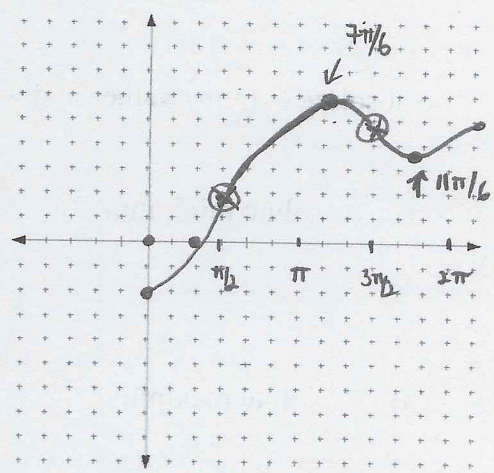
b) Concave down: $(\pi/2, 3\pi/2)$

6. Sketch the curve clearly showing
   a) the intercepts
   b) local and absolute extrema
   c) Inflection points and concavity

Calc

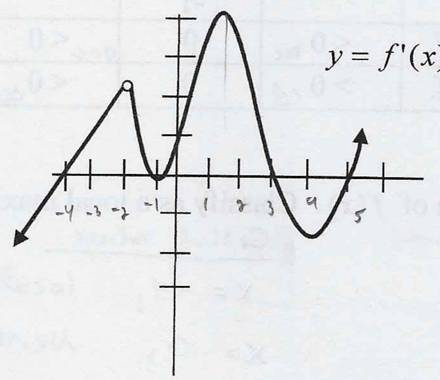
x- Intercepts: $(1.03, 0)$

y-Intercept: $(0, -1)$



⊗= Inflection pt's

For questions 7 through 14, refer to the graph of $y = f'(x)$, the ***DERIVATIVE*** of $f(x)$, show below. $f(x)$ is continuous for all real numbers. Once again, this is the graph of the **DERIVATIVE!** {Each tick mark equals 1 unit}



7. Find all critical points of the **original function** $f(x)$. $f'(x) = 0$ or undefined

$X = -4, -2, -1, 3, 5$

8. Estimate the intervals over which the **original function** $f(x)$ is increasing. $f'(x) > 0$

$(-4,-2) \cup (-2,-1) \cup (-1,3) \cup (5,\infty)$

9. Estimate the intervals over which the **original function** $f(x)$ is decreasing. $f'(x) < 0$

$(-\infty,-4) \cup (3,5)$

10. Estimate the intervals over which the **original function** $f(x)$ is concave up. $f'(x)$ increasing

$(-\infty,-2) \cup (-1,1.5) \cup (4,\infty)$

11. Estimate the intervals over which the **original function** $f(x)$ is concave down. $f'(x)$ decreasing

$(-2,-1) \cup (1.5,4)$

12. Estimate the x-coord. of all local maximum points of the **original function** $f(x)$. $f' = 0$ or undef

$+ \to -$

$x = 3$

13. Estimate the x-coord. of all the local minimum pts of the **original function** $f(x)$. $f'(x) = 0$ or undf

$- \to +$

$X = -4, 5$

14. Estimate the x-coordinates of all inflection points of the **original function** $f(x)$. $f''(x) = 0$ or undf

sign change

$X = -2, -1, 1.5, 4$

For questions 15 – 19, use the following information for $f(x)$. The domain of $f(x)$ is all real numbers.

x	$x<-4$	-4	$-4<x<0$	0	$0<x<3$	3	$x>3$
$f(x)$		5		-1		-2	
$f'(x)$	>0 Inc	Undef.	<0 Dec	0	Dec <0	Undef.	Incr >0
$f''(x)$	>0 CC↑	Undef.	>0 CC↑	0	<0 CC↓	Undef.	>0 CC↑

15. Find all the critical values of $f(x)$. Classify as a local maximum, minimum or neither. $f'(x)=0$ or undef

Critical values

$x=-4$, local maximum

$x=0$, Neither

$x=3$, local Minimum

16. Determine all intervals on which $f(x)$ is increasing and all intervals on which $f(x)$ is decreasing.

Incr: $(-\infty,-4)\cup(3,\infty)$

Decr: $(-4,3)$

17. Find all the inflection points of $f(x)$. $f''=0$ or undefined & sign change

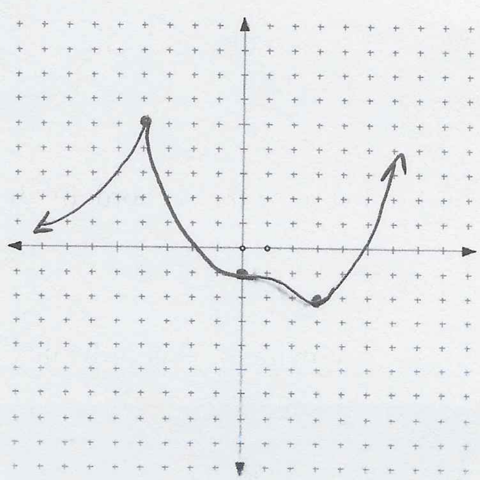
Inflection points at $x=0$ and $x=3$

18. Determine all intervals on which $f(x)$ is concave up and all intervals on which $f(x)$ is concave down.

CC↑: $(-\infty,-4)\cup(-4,0)\cup(3,\infty)$

CC↓: $(0,3)$

19. Sketch a function $f(x)$ for the above information



20. a) Determine $L(x)$ for $g(x) = e^x + \sec x$ at $x = 0$.

slope: $g'(x) = e^x + \sec x \tan x$ pt: $g(0) = e^0 + \sec(0)$ $L(x) = 2 + (1)(x-0)$

$g'(0) = 1 + 0 = 1$ $= 2$ $\boxed{L(x) = x+2}$

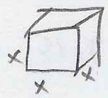
$(0,2)$

b) use $L(x)$ to approximate $g(x)$ at $x = 0.3$

$g(.3) \approx L(.3) = (.3)+2$

$= 2.3$

21. Use differentials to approximate the change in the surface area of a cube when the length of one of its edges changes from $x = 7$ inches to $x = 7.02$ inches.



Surface Area $= 6x^2$ $dx = 7.02 - 7 = .02$ $x = 7$

$S(x) = 6x^2$ $dS = (12x)dx = (12 \cdot 7)(.02)$

$S'(x) = 12x$ $= 1.68 \text{ in}^2$

22. State each of the following theorems:

a) Rolle's Theorem:

If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) and $f(a) = f(b)$ then there exists at least one c on (a,b) such that $f'(c) = 0$.

b) The Mean Value Theorem:

If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) then there exists at least one c on (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

23. Determine the value(s) of x, if any exist, that satisfy the Mean Value Theorem for $f(x) = x^{\frac{2}{3}} + 2$ on the interval [0, 8].

$f(x)$ cts on $[0,8]$? Yes

$f(x)$ diff on $(0,8)$? Yes

$$\frac{f(b)-f(a)}{b-a} = \frac{(8^{2/3}+2)-(0^{2/3}+2)}{8-0} = \frac{1}{2}$$

$f'(x) = \frac{2}{3}x^{-1/3}$

$\therefore \frac{2}{3}x^{-1/3} = \frac{1}{2}$

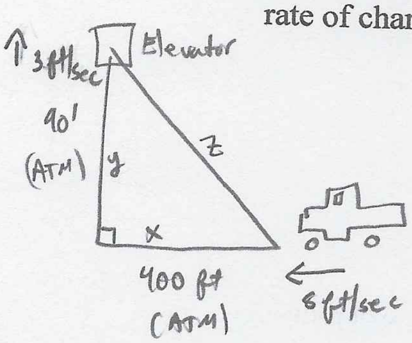
$x^{-1/3} = 3/4$

$x^{1/3} = 4/3$

$x = (4/3)^3 = 64/27$

$\boxed{x = 64/27}$

24. At a certain instant, an elevator on a construction site is 90 feet above ground and going up at a rate of 3 feet/second. At the same instant, a truck is 400 feet from the foot of the elevator shaft and is driving directly toward it at 8 feet/second. Find the rate of change of the distance between the truck and the elevator at this time.



WANT: dz/dt

Have: $y = 90\text{ ft (ATM)}$
$x = 400\text{ ft (ATM)}$
$z = 410\text{ ft (ATM)}$

$dx/dt = -8\text{ ft/sec}$
$dy/dt = 3\text{ ft/sec}$

Relationship
$x^2 + y^2 = z^2$

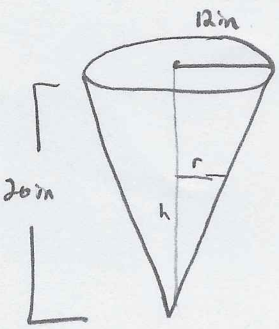
$2x\,dx/dt + 2y\,dy/dt = 2z\,dz/dt$

$(400)(-8) + (90)(3) = 410(dz/dt)$

$dz/dt = -2930/41 \approx -7.15$

The distance between the truck and the elevator is decreasing at 7.15 ft/sec at this moment.

25. A cone of ice with a height of 20 inches and a radius of 12 inches is melting and leaking through a hole in the bottom of its container at a rate of 1.2 in^3 per minute. Determine the rate at which the radius is changing with respect to time when the radius is 3 inches. **{$V_{cone} = 1/3\,\pi r^2 h$}**

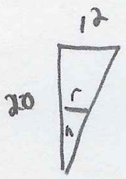


WANT dr/dt

Have: $dv/dt = -1.2\text{ in}^3/\text{min}$
$r = 3\text{ in (ATM)}$

$V = \frac{1}{3}\pi r^2\left(\frac{5}{3}r\right)$

$V = \frac{5\pi}{9} r^3$



$\frac{20}{12} = \frac{h}{r}$

$h = 5/3\, r$

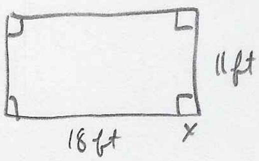
$dv/dt = \frac{5\pi}{3} r^2\, dr/dt$

$-6/5 = \frac{5\pi}{3}(3)^2\, dr/dt$

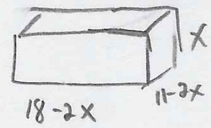
$dr/dt = \frac{-2}{25\pi} \approx -.0255$

The radius is decreasing at .0255 in/min when the radius is 3 in.

26. **Classic Calculus:** Suppose you are to make a large holding bin for shipping peanuts (yes, the Styrofoam ones) out of a 11-ft by 18-ft piece of cardboard by cutting equal squares from each corner and folding up the sides. What dimensions maximize the volume of the bin? {Dimensions can be rounded at 3 decimal places}



$\Rightarrow$



Objective Function:

$V = (18-2x)(11-2x)(x)$

$V = 4x^3 - 58x^2 + 198x$

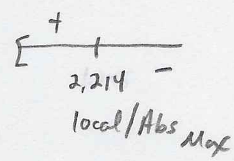
let x = length of square cut out (ft)

$V'(x) = 12x^2 - 116x + 198$

$V = l \cdot w \cdot h$

$x = 2.214,\ \cancel{7.453}$

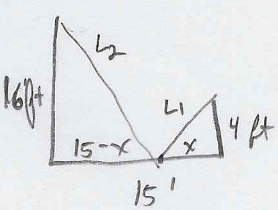
D_x: $[0, 5.5]$



To maximize the volume, the box should be 2.214 in x 6.572 in x 13.572 in

27. Two vertical poles, one 4 ft high and the other 16 ft high, stand 15 feet apart on a flat field. A worker wants to support both poles by running rope from the ground to the top of each post. If the worker wants to stake both ropes in the ground at the same point, where should the stake be placed to use the least amount of rope? {Round to two decimals places}

let x = distance from the base of the 4 ft pole where the stake is placed $[0, 15]$



let L_1, L_2 = length of wire from stake to top of 4 ft & 16 ft poles respectively [ft]

WANT TO Minimize $L = L_1 + L_2$

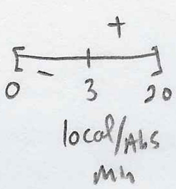
$L = \sqrt{(16)^2 + (15-x)^2} + \sqrt{x^2 + 4^2}$ (objective function)

$L' = \frac{1}{2\sqrt{(16)^2 + (15-x)^2}}\left(2(15-x)(-1)\right) + \frac{1}{2\sqrt{x^2+16}}(2x)$

Find crit pts: $0 = \frac{x}{\sqrt{x^2+16}} - \frac{(15-x)}{\sqrt{16^2 + (15-x)^2}} \Rightarrow x\sqrt{16^2 + (15-x)^2} = (15-x)\left(\sqrt{x^2+16}\right)$

$\Rightarrow x^2\left(16^2 + (15-x)^2\right) = (15-x)^2(x^2+16)$

$\Rightarrow x^2(16)^2 + \cancel{x^2(15-x)^2} = \cancel{x^2(15-x)^2} + (15-x)^2(16)$



$\Rightarrow 16x^2 = (15-x)^2 \Rightarrow 15x^2 + 30x - 225 = 0$

$x = 3, \cancel{-5}$

The stake should be placed 3 ft from the base of the 4 foot pole to minimize wire length.