5.1 Approximating Areas Under Curves

MATH 205

Review of Sigma Notation

- Sigma notation is a shorthand method used to denote the sum of a sequence.
  - Sum of a sequence is called a series

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

  - The Greek letter sigma, Σ, means the sum
  - The lower case k is the index of summation
    - The sum starts at $k = 1$ and ends at $k = n$
  - a_k is the formula for the k^{th} term

- Determine $\displaystyle\sum_{n=1}^{20} (2n - 7)$
- Write $18 + 9 + 4.5 + 2.25 + \ldots$ in Sigma Notation

Some Common Sums

- Let n be a positive integer

$$\sum_{k=1}^{n} c = cn \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

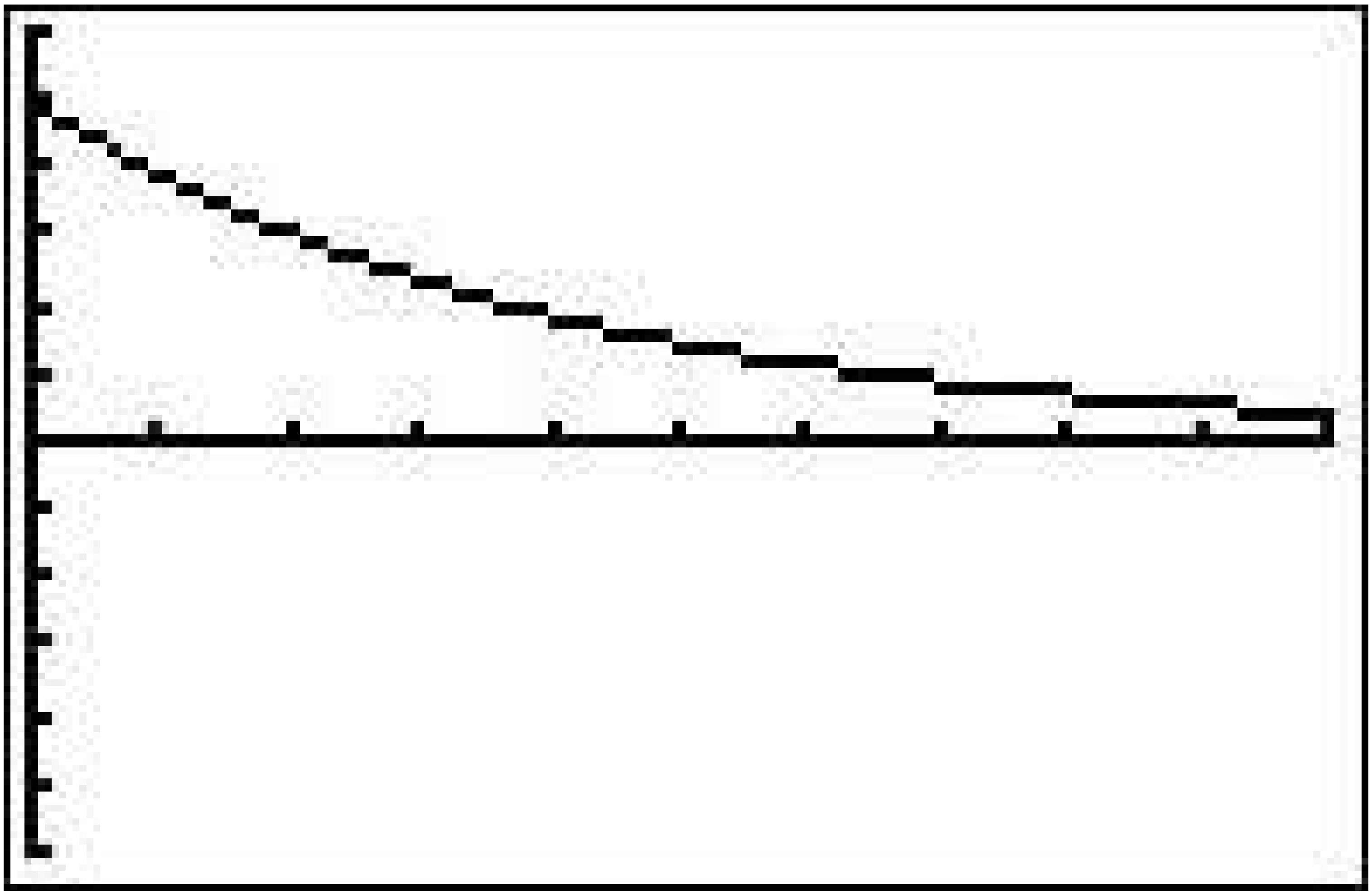
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Ecologically Sound?

Suppose the number of quarts of oil per minute, A, spilling out of a barrel at time t, t in minutes, is modeled by the equation $A = \dfrac{5}{e^{\frac{t}{4}}}$.

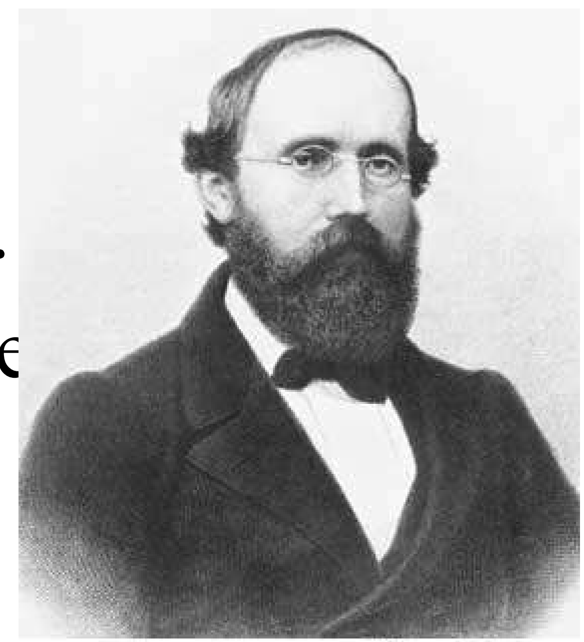
1. At what rate is the oil spilling out of the barrel the moment the leak occurs?
2. At what rate is the oil spilling out of the barrel at the two minute mark?
3. How many quarts have spilled out of the barrel after 6 minutes have passed?

So how do we find the total amount?



The Riemann Sum

- Named for the German mathematician Bernhard Riemann (1826 – 1866)
- It is a finite approximation method using known geometrical shapes.
- Though rectangles are used quite trapezoid is also commonly used.
- To create a Riemann Sum, we nee *partition* the area in question.



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Partitions

- To create a partition of $f(x)$ on $[a, b]$

a) Divide $[a, b]$ into n subintervals (not necessarily the same length) denoted by Δx_k {The width of the rectangle}.

b) In each subinterval, pick a value for x, called c_k.

c) For each subinterval, determine $f(c_k)$ {the height of the rectangle}, and draw the corresponding rectangle.

d) Sum the areas of all the rectangles: $\sum_{k=1}^{n} f(c_k)\Delta x_k$

- This is a Riemann Sum!

More on Partitions (a.k.a. Making our life easy)

- Since we can make the subintervals any length we want, lets make them uniform: $\Delta x = \dfrac{b-a}{n}$
- Since we can pick any c in the subinterval, it is common to pick only the right endpoints, left endpoints or midpoints in each interval.
  - Also, one could pick the minimum or maximum value of $f(x)$ on the subinterval.
- These are called Right, Left, or Midpoint Riemann Sums respectively.
  - Called Lower or Upper Riemann Sums respectively.

Sigma Notation and Riemann Sums

- Suppose f is defined on $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k is a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$, then the Riemann sum of f on $[a, b]$ is $\sum_{k=1}^{n} f(x_k)\Delta x_k$.
- Right Riemann Sum if $x_k = a + k\Delta x$
- Left Riemann Sum if $x_k = a + (k-1)\Delta x$
- Midpoint Riemann Sum if $x_k = a + \left(k - \frac{1}{2}\right)\Delta x$

Calculate the Riemann Sum:

4. Determine the Right Riemann Sum for $g(x) = x^2$ on [1, 3] using 6 intervals.
   - Will this be an over or under estimate?

Calculate the Riemann Sum:

5. Determine the Left Riemann Sum for $k(x) = 6\sqrt{x+1}$ on [0, 15] using 5 intervals.
   - Will this be an over or under estimate?

Calculate the Riemann Sum:

6. Determine the Midpoint Riemann Sum for $p(x) = 2^x - 8$ on $[0, 3]$ using 4 intervals.
   - Will this be an over or under estimate?

Calculate some Riemann Sums

Determine the area bounded by $f(x) = \sin(x)$ and the x-axis on $[0, \pi]$:

7) Using an upper Sum estimation with 4 rectangles.

8) Using a right Sum estimation with 4 rectangles.

Calculate some Riemann Sums

Determine the area bounded by $g(x) = \dfrac{1}{x^2 + 1}$ and the x-axis on [-3, 7]:

9) Using a lower sum estimation with 5 rectangles.

10) Using a midpoint sum estimation with 5 rectangles.

A Velocity Table

- The following table is a list of velocities (in ft/sec) for a car from time $t = 0$ seconds to $t = 60$ seconds. Determine the distance the car traveled over period.

Time	Velocity	Time	Velocity
0	0	37	37
5	7	42	47
8	12	44	56
15	18	50	67
21	23	53	76
30	29	60	88