5.3 The Fundamental Theorem of Calculus

MATH 205

So, what is a "Fundamental" theorem?

- A fundamental theorem is a theorem that lies at the core of the mathematics being studied.

- Fundamental Theorem of Arithmetic:
  - The prime factorization of any natural number greater than 1 is unique.

- Fundamental Theorem of Algebra:
  - An n^{th}-degree polynomial has exactly n roots.

Velocity Again?

- Given a velocity function or velocity graph, how could you determine the displacement of an object?
  - Later, we will deal with the total distance traveled.

- Suppose we are interested in the interval [0, 10] and $v(t)$ is in m/sec. Determine the displacement of an object given:

i. $v(t) = 7$

ii. $v(t) = 6t$

iii. $v(t) = -2t + 8$

iv. $v(t) = -t^2 + 4$

The Fundamental Theorem of Calculus: Part 2

- Also know as the Fundamental Theorem of the Integral of the Derivative
  - {A.K.A. The Net Change Theorem}

- If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$ then,

$$\int_a^b f(x)dx = F(b) - F(a)$$

- Allows for the calculation of a definite integral without having to explicitly calculate a Riemann Sum.

Evaluate the Definite Integral

1. $\int_{-2}^{1} x^5 \, dx$

2. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\theta \, d\theta$

Evaluate the Definite Integral

3. $\displaystyle\int_{e^4}^{e^{10}} \frac{1}{t}\,dt$

4. $\displaystyle\int_{-27}^{-1} (8\sqrt[3]{x} + 7)\,dx$

Evaluate the Definite Integral

5. $$\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$$

Evaluate the Definite Integral

6. $\int_0^1 \left(8x^3 + \sec^2 x - \ln(8)8^x\right)dx$

Definite Integral as a Function

- Let $F(x) = \int_{\frac{\pi}{2}}^{x} \sin t \, dt$, evaluate $F(x)$ at

$$x = \frac{\pi}{2}, \ \frac{3\pi}{4}, \ \pi, \ \frac{3\pi}{2}, \ 2\pi, \ \frac{\pi}{4}.$$

- $F(x)$ is accumulating the area under $f(t) = \sin t$ from $t = \frac{\pi}{2}$ to $t = x$

The Fundamental Theorem of Calculus: Part 1

- Also know as the Fundamental Theorem of the Derivative of the Integral

- If f is continuous on $[a, b]$, then $A(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$A'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x) \qquad A'(x) = \frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x))g'(x)$$

- The instantaneous rate of change of an area (accumulation) function at x_1 is the height of the function at x_1 $\{f(x_1)\}$

Determine each of the following:

13. $\dfrac{d}{dx}\displaystyle\int_{-3}^{x} t^2\,dt$

14. $\dfrac{d}{dx}\displaystyle\int_{4}^{x} \sin^2 t\,dt$

Determine each of the following:

15. $\dfrac{d}{dy}\displaystyle\int_0^y e^x \cos x\,dx$

16. $\dfrac{d}{dx}\displaystyle\int_x^9 \frac{5t}{t^3 - 2t + 7}\,dt$

Determine each of the following:

17. $\dfrac{d}{dx}\displaystyle\int_{9}^{x^2} \tan\theta \, d\theta$

18. $\dfrac{d}{dw}\displaystyle\int_{\sin w}^{w^2} \ln\left(\dfrac{x}{1-x^2}\right) dx$

And now the power of the FTC: Part 1!

19. Determine a function, $y = f(x)$, with a domain of $[0, \infty)$ whose derivative is $$\frac{dy}{dx} = \sqrt{xe^x}$$ and satisfies the condition $f(5) = 4$.

20. Determine a function, $y = f(x)$, with a domain of $(0, \pi/2)$ whose derivative is $$\frac{dy}{dx} = \frac{\cos^2(x-7)}{\tan^3 x^2}$$ and satisfies the condition $f(\pi/4) = 8$.