4.6 Linear Approximations and Differentials

MATH 205

Linear Approximation

- The idea behind linear approximation is to use a linear function to approximate the value of a more complicated function at a given input.

- This relies on the concept of local linearization.
  - If $f(x)$ is a well-behaved (smooth) curve at a point P, then the curve approaches its tangent line at P.

Linear Approximation

- Suppose f is differentiable on an interval I containing the point c. The Linear Approximation of f at c is the linear function:

$$y = f(c) + f'(c)(x - c)$$

for all x in I.

Given $f(x) = \ln(1 + x)$

1. Determine the tangent line approximation of f at $c = 0$.
2. Approximate the value of $f(0.12)$.
3. What is the percent error in the approximation.

Linear Approximation

- $f(x) = \cos x$

$$\sqrt{62.5}$$

Differentials

- Δy represents the change in $y = f(x)$ when x changes from a to $a + \Delta x$.
  - Exactly $\Delta y = f(a + \Delta x) - f(a)$
- How does this correspond to a change in the linear approximation as x changes from a to $a + \Delta x$?
  - $\Delta L = f'(a)\Delta x$
- To denote the difference between Δy and ΔL , two new variables, dy and dx, called differentials are used.
  - $\Delta L = dy$ and $\Delta x = dx$

Differentials

Let $y=f(x)$ be differentiable on an open interval containing x.

- The **differential of *x***, denoted dx, is any nonzero real number.
- The **differential of *y***, denoted dy, is $dy = f'(x)dx$

3. Given $y = x^2$. Find dy when $x = 1$ and $dx = 0.01$. Compare this value with Δy for $x = 1$ and $\Delta x = 0.01$.

4. Given $y = \sqrt{x}$. Find dy when $x = 4$ and $dx = 0.2$. Compare this value with Δy for $x = 4$ and $\Delta x = 0.2$.

Differentials

7. Approximate the change in $y = 2x^7 - 6x^3 + 5x - 7$ when x changes from 1 to 0.8.
8. Approximate the change in the volume of a spherical balloon when the radius increases from 2 to 2.15.
9. Given $f(x) = 5\tan^{-1} x$, use differentials to approximate the change in f when x changes from 1 to 1.02.

Recap

- To approximate f near $x = a$, use the linearization:

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

- To approximate the change in f corresponding to a small change in x, use differentials:

$$\Delta y = f(x + \Delta x) - f(x) \approx dy = f'(x)dx$$