MATH 205 – Calculus I **Exam #3**

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Instructions: Though calculators can be used for the entire exam, all problems require you to show your work. Any answer without proper justification will receive ***ZERO*** credit. Only **EXACT** answers will receive full credit unless otherwise noted. Proper Interval Notation must be used to receive credit.

Questions 1 – 19: 4 points each, questions 20 – 23: 4 points each, questions 24 – 27: 6 points each.

For questions 1 – 6:

Given $g(x) = \frac{1}{2}x - \cos x$, on $[0, 2\pi]$, determine each of the following:

$g'(x) = \frac{1}{2} + \sin x$

$0 = \frac{1}{2} + \sin x$

$-\frac{1}{2} = \sin x$

$\frac{7\pi}{6} = x$

$\frac{11\pi}{6} = x$

1. The local extrema values and where they occur.

$\frac{1}{2}\left(\frac{7\pi}{6}\right) - \cos\left(\frac{7\pi}{6}\right)$

$\frac{7\pi}{12} + \sqrt{3}/2$

$\frac{1}{2}\left(\frac{11\pi}{6}\right) - \cos\left(\frac{11\pi}{6}\right)$

$\frac{11\pi}{12} - \frac{\sqrt{3}}{2}$

a) Local maximum(s):

Local max of $\frac{7\pi}{12} + \frac{\sqrt{3}}{2}$ at $x = \frac{7\pi}{6}$

b) Local minimum(s):

Local min of $\frac{11\pi}{12} - \frac{\sqrt{3}}{2}$ at $x = \frac{11\pi}{6}$

2. The absolute extrema values and where they occur.

$g(0) = \frac{1}{2}(0) - \cos(0)$

$g(0) = -1$

$g(2\pi) = \frac{1}{2}(2\pi) - \cos(2\pi)$

$g(2\pi) = \pi - 1$

≈ 2.14

a) Absolute minimum

Absolute min of -1 at $x = 0$

b) Absolute maximum

Absolute max of $\frac{7\pi}{12} + \frac{\sqrt{3}}{2}$ at $x = \frac{7\pi}{6}$

3. Intervals on which $g(x)$ is:

1 5 6

0 $+$ $\frac{7\pi}{6}$ $-$ $\frac{11\pi}{6}$ $-$ 2π

a) Increasing:

$g(x)$ is increasing from $\left[0, \frac{7\pi}{6}\right) \cup \left(\frac{11\pi}{6}, 2\pi\right]$

b) Decreasing:

$g(x)$ is decreasing from $\left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$

$g''(x) = \cos x$

$0 = \cos x$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

For questions 1 – 6:
Given $g(x) = \frac{1}{2}x - \cos x$, on $[0, 2\pi]$, determine each of the following:

4. The location of any points of inflection:

g(x) has points of inflection when $x = \frac{\pi}{2}, \frac{3\pi}{2}$

0 2 5

$+ \quad \frac{\pi}{2} \quad - \quad \frac{3\pi}{2} \quad +$

5. Intervals on which $g(x)$ is

a) Concave Up:

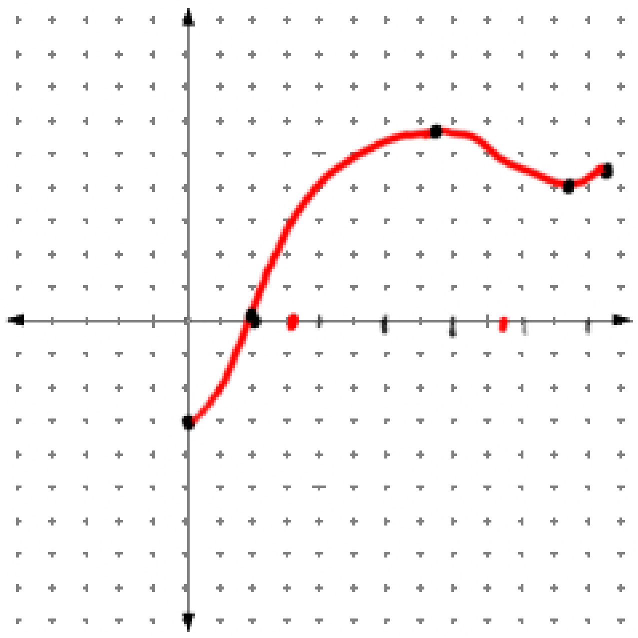
g(x) is concave up from $[0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi]$

b) Concave down:

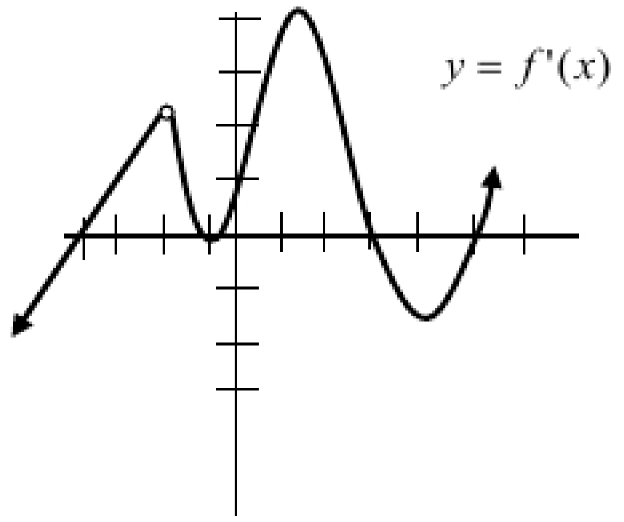
g(x) is concave down from $(\frac{\pi}{2}, \frac{3\pi}{2})$

Scale .5

6. Sketch the curve clearly showing
   a) the intercepts
   b) local and absolute extrema
   c) Inflection points and concavity



For questions 7 through 14, refer to the graph of $y = f'(x)$, the ***DERIVATIVE*** of $f(x)$, show below. $f(x)$ is continuous for all real numbers. Once again, this is the graph of the **DERIVATIVE!** {Each tick mark equals 1 unit}



7. Find all critical points of the **original function** $f(x)$.

$f(x)$ has critical points when $x = -4, -1, 3, 5$

8. Estimate the intervals over which the **original function** $f(x)$ is increasing.

$f(x)$ is increasing from $(-4,-2) \cup (-2,3) \cup (5,\infty)$

9. Estimate the intervals over which the **original function** $f(x)$ is decreasing.

$f(x)$ is decreasing from $(-\infty,-4) \cup (3,5)$

10. Estimate the intervals over which the **original function** $f(x)$ is concave up.

$f(x)$ is concave up from $(-\infty,-2) \cup (-1,1) \cup (4,\infty)$

11. Estimate the intervals over which the **original function** $f(x)$ is concave down.

$f(x)$ is concave down from $(-2,-1) \cup (1,4)$

12. Estimate the x-coord. of all local maximum points of the **original function** $f(x)$.

$f(x)$ has local max's when $x = 3$

13. Estimate the x-coord. of all the local minimum pts of the **original function** $f(x)$.

$f(x)$ has local min's when $x = -4, 5$

14. Estimate the x-coordinates of all inflection points of the **original function** $f(x)$.

$f(x)$ has points of inflection when $x = -2, -1, 1, 4$

For questions 15 – 19, use the following information for $f(x)$. The domain of $f(x)$ is all real numbers.

x	$x < -4$	-4	$-4 < x < 0$	0	$0 < x < 3$	3	$x > 3$
$f(x)$		5		-1		-2	
$f'(x)$	> 0	Undef.	< 0	0	< 0	Undef.	> 0
$f''(x)$	> 0	Undef.	> 0	0	< 0	Undef.	> 0

15. Find all the critical values of $f(x)$. Classify as a local maximum, minimum or neither.

Local max at 5 when $x=-4$
Local min at -2 when $x=3$
No max/min at -1 when $x=0$

16. Determine all intervals on which $f(x)$ is increasing and all intervals on which $f(x)$ is decreasing.

$f(x)$ is increasing from $(-\infty,-4) \cup (3,\infty)$
$f(x)$ is decreasing from $(-4,0) \cup (0,3)$

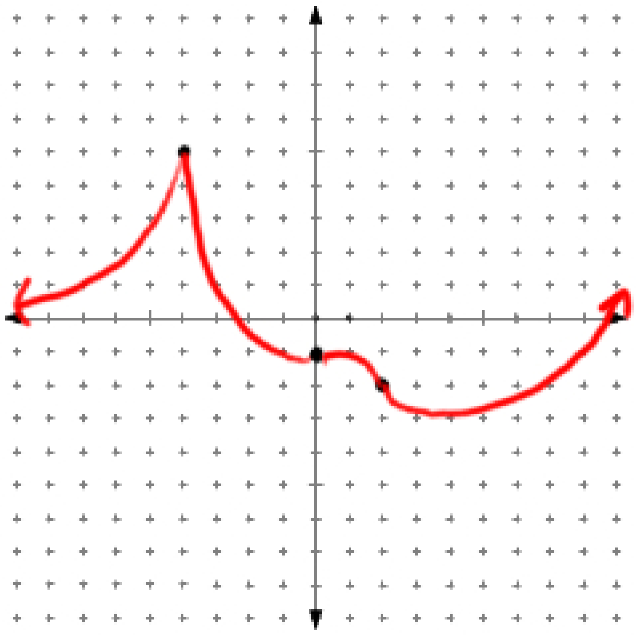
17. Find all the inflection points of $f(x)$.

$f(x)$ has points of inflection at $(0,-1)$ & $(3,-2)$

18. Determine all intervals on which $f(x)$ is concave up and all intervals on which $f(x)$ is concave down.

$f(x)$ is concave up from $(-\infty,-4) \cup (-4,0) \cup (3,\infty)$
$f(x)$ is concave down from $(0,3)$

19. Sketch a function $f(x)$ for the above information



$f(a)+f'(a)(x-a)$

20. a) Determine $L(x)$ for $g(x) = e^x + \sec x$ at $x = 0$.

$g'(x)=e^x+\sec(x)\tan(x)$

$g(0)=2$

$g'(0)=1$

$2+1(x-0)$

$L(x)=2+x$

b) use $L(x)$ to approximate $g(x)$ at $x = 0.3$

$L(0.3)=2+0.3$

$L(0.3)=2.3$

21. Use differentials to approximate the change in the surface area of a cube when the length of one of its edges changes from $x = 7$ inches to $x = 7.02$ inches.

$dx=.02$

$S_{area}=6x^2$

$S'_{area}=12x$

$dy=f'(x)dx$

$dy=12(7)(.02)$

$dy=1.68$

22. State each of the following theorems:

a) Rolle's Theorem:

If $f(x)$ is differentiable on $[a,b]$, $f(x)$ is continuous on (a,b), & $f(a)=f(b)$ } then there exists some point c on (a,b) $f'(c)=0$

b) The Mean Value Theorem:

If $f(x)$ is differentiable on $[a,b]$, & $f(x)$ is continuous on (a,b) } then there exists some point c on (a,b) where

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

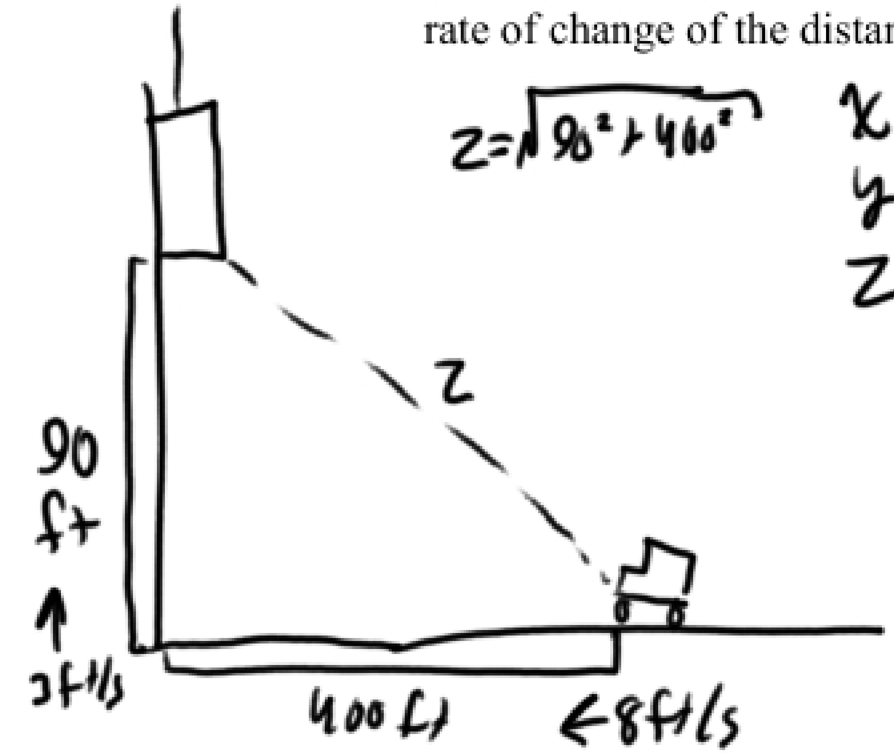
23. Determine the value(s) of x, if any exist, that satisfy the Mean Value Theorem for $f(x) = x^{\frac{2}{3}} + 2$ on the interval [0, 8].

$f(x)$ cts $[0,8]$? yes

$f(x)$ diff $(0,8)$? no - cusp

$f(x)$ does not meet the qualifications of the MVT for it to be applied.

24. At a certain instant, an elevator on a construction site is 90 feet above ground and going up at a rate of 3 feet/second. At the same instant, a truck is 400 feet from the foot of the elevator shaft and is driving directly toward it at 8 feet/second. Find the rate of change of the distance between the truck and the elevator at this time.



$z=\sqrt{90^2+400^2}$

$x=90 \quad dx=3$

$y=400 \quad dy=8$

$z=410$ ftm

$\sqrt{x^2+y^2}=z$

$\frac{1}{2}(x^2+y^2)^{-1/2}(2x\,dx+2y\,dy)=dz$

$\frac{2x\,dx+2y\,dy}{2\sqrt{x^2+y^2}}=dz$

$\frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}=dz$

want dz

$\frac{90(3)+400(8)}{\sqrt{90^2+400^2}}=dz$

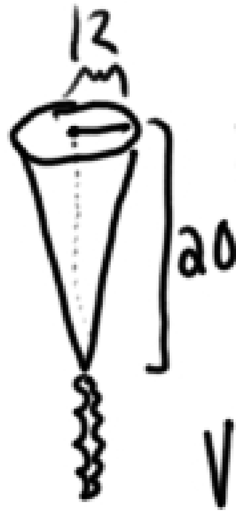
$\frac{270+3200}{410}=dz$

$\frac{3470}{410}=dz$

$8.46\approx dz$

The truck is approaching the elevator at a rate of 8.46ft/s.

25. A cone of ice with a height of 20 inches and a radius of 12 inches is melting and leaking through a hole in the bottom of its container at a rate of 1.2 in^3 per minute. Determine the rate at which the radius is changing with respect to time when the radius is 3 inches. **{$V_{cone} = 1/3\ \pi r^2 h$}**



$r=12$

$h=20$

need $\frac{dr}{dt}$

$\frac{dV}{dt}=-1.2\text{in}^3/\text{m}$

$V=3015.928$

$3v=\pi r^2 h$

$\frac{3v}{\pi r^2}=h$

$3v=\pi r^2 h$

$\sqrt{\frac{3v}{h\pi}}=r$

$V=\frac{1}{3}(\pi)(3)^2(5)$

$V=47.124$

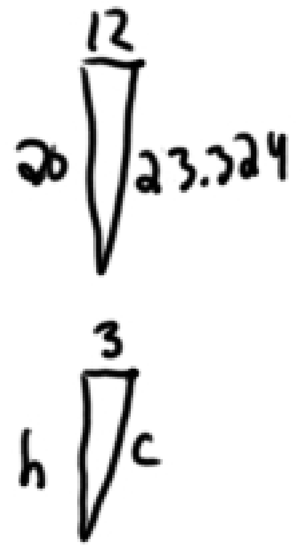
$\frac{dh}{dt}=\frac{(\pi r^2)3\frac{dv}{dt}-(3v)(2\pi r)}{\pi r^4}$

$\frac{dh}{dt}=-10.073$

$\frac{dr}{dt}=\frac{1}{2\sqrt{\frac{3v}{h\pi}}}\left(\frac{h\pi(3)\frac{dv}{dt}-3v\frac{dh}{dt}}{(h\pi)^2}\right)$

$\frac{dr}{dt}=-1$

The rate at which the radius is changing when the radius is 3 inches is -1 in/min while the height is at 5 in.

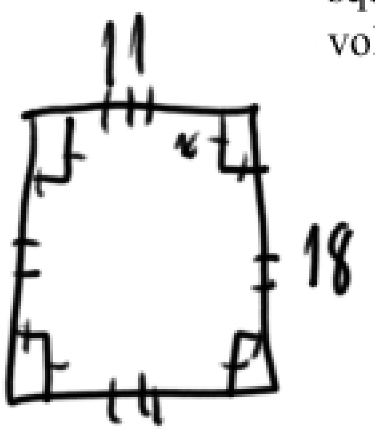


$\frac{12}{20}=\frac{3}{h}$

$12h=60$

$h=5$

26. **Classic Calculus:** Suppose you are to make a large holding bin for shipping peanuts (yes, the Styrofoam ones) out of a 11-ft by 18-ft piece of cardboard by cutting equal squares from each corner and folding up the sides. What dimensions maximize the volume of the bin? {Dimensions can be rounded at 3 decimal places}



$D[0, 5.5]$

$L = 6.572$
$W = 13.572$
$H = 2.214$

$L = 11 - 2x$
$W = 18 - 2x$
$h = x$

$V_{box} = Lwh$
$V = (11-2x)(18-2x)(x)$
$V = (11-2x)(18x - 2x^2)$
$V = 4x^3 - 58x^2 + 198x$
$V' = 12x^2 - 116x + 198$

$x = \dfrac{116 \pm \sqrt{-116^2 - 4(12)(198)}}{24}$

$x = \dfrac{116 \pm \sqrt{13456 - 9504}}{24}$

$x = \dfrac{116 \pm \sqrt{3952}}{24}$

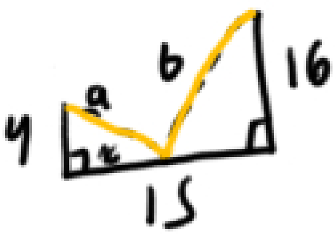
$x = \dfrac{116 \pm 4\sqrt{247}}{24}$

$x = 2.214$ & ~~$x = 7.453$~~

$V(2.214) = 197.478$

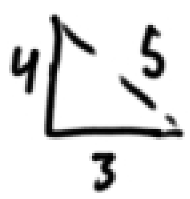
To maximize the volume of the container at 2.214 ft³, the length, width, and height must measure 6.572 ft, 13.522 ft, & 2.214 ft respectively.

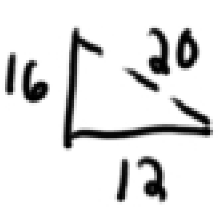
27. Two vertical poles, one 4 ft high and the other 16 ft high, stand 15 feet apart on a flat field. A worker wants to support both poles by running rope from the ground to the top of each post. If the worker wants to stake both ropes in the ground at the same point, where should the stake be placed to use the least amount of rope? {Round to two decimals places}



$\dfrac{4}{x} = \dfrac{16}{15-x}$

$60 - 4x = 16x$
$60 = 20x$
$3 = x$





For a minimum amount of rope to be used, the stake must be 3 ft from the small pole yielding a rope length of 25 ft, with 5 ft going from the small pole to the stake and 20 ft of rope going from the tall pole to the stake.