1 Anatomy of a typical financial model

** This probably needs to go in a second paper but is here as a reminder to talk about things in relation of financial models. **

1. Use data analysis & exploration of statistical features to identify candidate models.
2. Estimate parameters for the candidate model.
3. Check model feasibility through examination of residuals.
4. Validate model with out-of-sample data.

A standard practice is to simplify the complex nature of markets into a more palatable form, such as time series of returns. [time series log returns formula]

The next step is to hone in on some features that could be used to explain the behavior of this simplified view.

These features are then applied into a mathematical model that explain or forcast the behavior of the system. [time series forcasting using cointegration for pairs trading]

** Look at the entropy of this process. We should have something that starts at a OB and ends with a vector of AR returns. It is likely that this procedure has a large loss of information. **

2 Trading as a category

In this section we introduce the concept of trading as a categorical concept. ** expand on this when we have more **.

2.1 Trading Positions

We start with a primitive notion of a trading position

$$\psi : [0, \infty] \times [0, \infty] \to \{\top, \bot\}$$

as a willingness to exchange a quantity q_b of asset b for q_a of asset a

$$\psi(q_a, q_b) = \begin{cases} \top & \text{trader } \boldsymbol{agrees} \text{ to swap } q_a \text{ for } \mathrm{q_b} \\ \bot & \text{trader is } \boldsymbol{not\ willing} \text{ to make the swap} \end{cases}$$

. An order book can be thought of as a collection of trading positions so that the ordering

$$\psi_1(q_1, q_2) \geq \psi_2(q_3, q_4) \quad \implies \frac{q_1}{q_2} \geq \frac{q_3}{q_4}$$

for buyers, and

$$\psi_1(q_1, q_2) \leq \psi_2(q_3, q_4) \quad \implies \frac{q_1}{q_2} \leq \frac{q_3}{q_4}$$

Table 1: ψ example

buy amount	sell amount	ψ
2	1	$\top$
4	2	$\top$
5	3	$\top$
5	2	$\bot$
6	3	$\top$
10	5	$\bot$

for sellers implies an ordering by price. However our trading position is able to provide more and allows for a comparison of quantities directly

$$\psi_1 \geq \psi_2 \iff \exists q_1, q_2 \,.\, \psi_1(q_1, q_2) = \top \quad \text{and} \quad \psi_2(q_1, q_2) = \bot \tag{1}$$

. This means that two trading positions may be price equivalent but they can still be compared relative to their exchangeable quantities.

Theorem 1. *A trade executes between commutative trading positions ψ_A, ψ_B if there exists a quantity that satisfy both trading positions.*

Proof. Let $\psi_1 : A \to B$ admit a trading position at $\psi_1(a_1, b_1) = \top$ and $\psi_2 : B \to A$ similarly at $\psi_2(b_2, a_2) = \top$, where $a_1, a_2 \in |A|$ and $b_1, b_2 \in |B|$ are the relative quantities of assets A, B.

If we can find $a \in |A|$ and $b \in |B|$ such that

$$a \geq a_1 \quad \text{and} \quad a \leq a_2$$

$$b \geq b_2 \quad \text{and} \quad b \leq b_1$$

which results in

$$\psi_1(a, b) \geq \psi_1(a_1, b_1) \qquad \psi_2(b, a) \leq \psi_2(b_2, a_2)$$

$$\psi_1(a, b) = \top \qquad \psi_2(b, a) = \top$$

or in other words a trade being executed. $\square$

Table 1 shows an example of a buy trading position, which reinforces the notion that ψ is a both sensitive to price and quantity.

Now lets consider combining trading positions. We'll introduce a shorthand asset quantity parameter $\alpha = (a_1, a_2)$ that describes a pair of quantities, a_1 to buy and a_2 to sell. This is used to describe a trading position

$$\alpha \mapsto \psi_\alpha \implies \psi(a_1, a_2) = \top$$

. We can then say if two quantities s_1, s_2 are accepted by the trading position ψ_α that implies the price is better or equal to α

$$\psi_\alpha(s_1, s_2) = \top \implies \frac{s_1}{s_2} \geq \alpha$$

. Given a second trading position ψ_β we can write

$$\psi_{\alpha+\beta} = \psi_\alpha + \psi_\beta$$

and

$$\psi_{\alpha+\beta}(s_1, s_2) = \top \implies \frac{s_1}{s_2} \geq \alpha + \beta$$

. Suppose we split appart a quantity $s_2 = s_2' + s_2''$ then we can see that

$$\psi_\alpha(s_1, s_2') = \top \qquad \text{and} \qquad \psi_\beta(s_1, s_2'') = \top$$

as $s_2' < s_2$ and $s_2'' < s_2$. However the same is not true for $s_1 = s_1' + s_1''$ as

$$\psi_\alpha(s_1', s_2) \leq \top \qquad \text{and} \qquad \psi_\beta(s_1'', s_2) \leq \top$$

. ** does this make sense?? **

Definition 2. The combination of two trading positions ψ_α and ψ_β is given by

$$\psi_\alpha + \psi_\beta : [0, \infty] \times [0, \infty]^{op} \to \mathscr{B}$$

where

$$\psi_\alpha + \psi_\beta(s_1, s_2) = \top$$

if there exists component quantities that result in trading positions

$$\exists s_1 = s_1' + s_1'' \, . \, \psi_\alpha(s_1', s_2') = \top$$
$$\exists s_2 = s_2' + s_2'' \, . \, \psi_\beta(s_1'', s_2'') = \top$$

.

Using this combination we can validate that trading positions behave as expected.

Example 3. Lets look at a trading position where one quantity is zero, $\psi_0(0, s_2) = \top$. So if we combine this trading position with a second one ψ_1

$$\psi_0(0, s_2) + \psi_1(s_1, s_2) = \top \implies \exists s_1 = s_1' + s_1'' \, . \, \psi_1(s_1', s_2') = \top$$

but seeing as $s_1 \in [0, \infty]$, the only possible values for $s_1' = s_1'' = 0$, meaning that $\psi_1(s_1', s_2') = \top$ for any s_2'. In other words, we can always give things away for free.

On the other extreme let's try to get something for free. Lets assume $\psi_0(s_1, 0) = \top$. In this case we need to find $\exists s_2 = s_2' + s_2'' = 0$ and again the only option is to have $s_2' = s_2'' = 0$. And so we get $\psi_1(s_1', 0) = \bot$ for $s_1' > 0$, meaning we cannot give nothing unless we agree to get nothing in return.

So far we have highlighted trivial algebraic properties of trading positions but now we introduce a categorical view of trading positions. We will use standard categorical notation for composition $\psi_1 \circ \psi_2(a, b) = \top$ to mean the same as $\psi_2(b, \psi_1(a, b))$. The composition of trading positions becomes a meet (or infimum) of quantities that share a common buy and sell amount

$$\psi_1 \circ \psi_2(a, b) = \bigvee_s \psi_1(a, s) \wedge \psi_2(s, b)$$

giving us a categorical version of our trading position comparison equation (1)

$$\psi_1 \circ \psi_2(a, b) = \top \iff \exists s \,.\, \psi_1(a, s) = \top \;\wedge\; \psi_2(s, b) = \top$$

. Now lets look closer at our definition of a trading position. We can define a category of positions **Pos** with objects as asset positions $\mathbf{Pos} \in P_A, P_B$ and arrows as trading positions ψ. Here $P_A = (a, b)$, $P_B = (b, a)$ are the asset positions, with quantities $a \in A$, $b \in B$, such that $\psi_{P_A} = \top$ and $\psi_{P_B} = \top$.

Seeing as trading positions exhibit a contravariant relation along P_A and P_B, a trading position ψ is a profunctor

$$\psi : P_A \nrightarrow P_B$$

, more precisely this is a Cartesian product of asset positions to a set

$$\psi : P_B^{op} \times P_A \to \mathscr{B} \tag{2}$$

where $\mathscr{B} = \{\bot, \top\}$ with ordering $\bot \leq \top$.

2.2 Quantales

Definition 4. A quantale Q is a complete lattice $\mathcal{L}$, that admit joins $\bigvee$ and meets $\bigwedge$ and has an associative binary operator $\otimes : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$, that commutes with joins: $x \otimes (\bigvee S) = \bigvee (x \otimes S)$ and $(\bigvee S) \otimes x = \bigvee (S \otimes x)$.

In [** ref diagrams of quantales I.Weiss **] the notion of a *Lawvere Quantale* is further developed as a category Q with objects $\mathrm{ob}(Q) = [0, \infty]$ and arrows $x \to y$ where $x \geq y$. We have developed the category $\mathbf{Prof}(\mathbf{Pos})$ as a candidate for trading positions. It is a good category, but it could be better. We can improve on it by noting that asset positions are equivalent to $[0, \infty]$ in the following way

$$P_A = P_B = Q^{op} = ([0, \infty]^{op})^{op} = [0, \infty]$$

. The Q^{op} comes from the need to describe our arrows in the opposite direction as the *Lawvere Quantale*.

This means we can form our trading position (2) using $[0, \infty]$

$$\psi : [0, \infty]^{op} \times [0, \infty] \to \mathscr{B}$$

Definition 5. The quantale $Q = (\mathcal{L}, \otimes, I)$ with joins $\bigvee$ form a category of trading positions $\mathbf{Qnt}(\mathbf{Pos})$ where trading positions ψ_i are the hom-set

$$\psi_i \in \mathbf{Qnt}(Q, Q)$$

and an individual trading position ψ is the arrow between quantales

$$\psi : Q \to Q$$

.

The above definition provides to following behaviors:

1. $\psi(I) = I$
2. $\psi(a \otimes b) = \psi(a) \otimes \psi(b)$
3. $\psi(\bigvee_{s \in S} s) = \bigvee_{s \in S} \psi(s)$

Theorem 6. **Tra** *is a category enriched in* **Qnt***.*