THE LOCALE OF REALS

The intention of this note is to provide a self-contained account of the locale of reals without assuming topology or category theory. In fact, the locale of reals is used to motivate lattice theory, and by proxy to category theory.

As background knowledge we assume the natural numbers $\mathbb{N}$ are given together with addition, multiplication, equality, and $\leq$. We note that all of these are decidable. We further assume $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{Q}_+$ have been constructed (we'll make this very precise later), and the operations have been extended.

Logic discussion: Consider the set $U = \{y \in \mathbb{Q} \mid 0 < y < 1\}$. Suppose that the result of some (physical) experiment is a rational number t. The experiment is deemed a success if t falls within the set U, namely if $0 < t < 1$. However, the nature of the experimental measurement is that the outcome is not entirely accurate. Suppose further that one can control for the accuracy of the measurements ahead of time. In more detail, once a tolerance $\varepsilon > 0$ is chosen, the experiment can be done in such a way that the actual outcome t is guaranteed to be correct to within ε. In symbols, if T is the 'true' outcome of the experiment (perhaps under ideal conditions), then $t - \varepsilon < T < t + \varepsilon$ is guaranteed.

Now, suppose that the tolerance ε is chosen, the experiment is performed, and the measurement results in $1/2$. How small should ε be in order to conclude with absolute confidence that the experiment was a success? The answer is that $\varepsilon = 1/2$, or any positive rational number smaller than $1/2$ suffices. Indeed, if $\varepsilon = 1/2$, then observing $t = 1/2$ ensures that

$$0 = 1/2 - 1/2 < t - \varepsilon < T < t + \varepsilon < 1/2 + 1/2 = 1$$

so indeed there can be no doubt that

$$0 < T < 1,$$

namely $T \in U$.

Not much changes if the question is altered to: given that the experiment resulted in the value t that falls in U, how small must ε be to ensure the experiment was successful? The answer is

$$\varepsilon = \min\{1 - t, t\}$$

as can be verified (exercise!).

In fact, there is nothing unique here to the numbers 0 and 1.

Definition 1. Let $a, \varepsilon \in \mathbb{Q}$ with $r > 0$. The *open interval* of *radius* r and *centre* a is the set

$$\mathrm{B}_r(a) = \{x \in \mathbb{Q} \mid a - r < x < a + r\}.$$

Proposition 2. *Let $a, r \in \mathbb{Q}$ with $\varepsilon > 0$. If $t \in \mathrm{B}_\varepsilon(a)$, then there exists $\varepsilon > 0$ such that*

$$\mathrm{B}_\varepsilon(t) \subseteq \mathrm{B}_r(a).$$

Proof. Exercise!! □

Thinking about the measurement out of an experiment, the result above says that if success is synonymous with "the true result T satisfies $T \in \mathrm{B}_r(a)$", then for any possible experimental result $t \in \mathrm{B}_r(a)$ there exists a tolerance level $\varepsilon > 0$ such that if t is observed, then the experiment was surely successful.

If we now change the success criterion slightly, things change drastically. Suppose that success is synonymous with "the true result T satisfies $0 \leq T \leq 1$". Show that no tolerance $\varepsilon > 0$ exists that guarantees that an observed value $t = 0$ ensures the experiment was successful.

This qualitative difference is important.

Definition 3. A subset $U \subseteq \mathbb{Q}$ of rational numbers is *open* if for any $t \in U$ there exists $\varepsilon > 0$ such that

$$\mathrm{B}_\varepsilon(t) \subseteq U.$$

So, the quality of a set U being open is that if success of an experiment is synonymous with "the true result belongs to U", then any actual result that indeed belongs to U, can in fact be conclusive evidence of success, provided the error was controlled well enough. Stated differently, any value that indicates success can serve as observed evidence. Sets that are not open do not enjoy that property in the sense that at least one value that indicates success can never serve as observed evidence, no matter how small the error threshold was set.

We thus take the open subsets of $\mathbb{Q}$ very seriously.

We observe some of their stability properties.

Theorem 4. *The set $\mathbb{Q}$ is open, as is the empty set $\emptyset$. Moreover, the intersection of any two open subsets is open. In fact, the intersection of any finite number of open sets is open (the case of zero open sets yields $\mathbb{Q}$). The union of any collection of open subsets is open (the empty collection yields $\emptyset$).*

We write $\Omega(\mathbb{Q})$ for the collection of all open subsets of $\mathbb{Q}$, ordered by inclusion. It is a poset and, by the above result, it has arbitrary joins and finite meets. Generally, open sets can be very wild. We can rather systematically manufacture all of them.

Theorem 5. *A subset of $\mathbb{Q}$ is open if, and only if, it is a union of open intervals.*

Proof. Suppose that $U \subseteq \mathbb{Q}$ is open. Then, by definition, for each $t \in U$ there exists a positive rational $\varepsilon_t > 0$ (that depends on t, hence the subscript) such that

$$\mathrm{B}_\varepsilon(t) \subseteq U.$$

Thus

$$\bigcup_{t \in U} \mathrm{B}_\varepsilon(t) = U.$$

Conversely, suppose that $\{\mathrm{B}_{r_i}(x_i)\}_{i \in I}$ is a collection, indexed by a set I, of open intervals. Then the union

$$\bigcup \mathrm{B}_{r_i}(x_i)$$

is an open set (can you justify this succinctly?) $\square$

Let us take this criterion in order to represent open subsets of $\mathbb{Q}$, as follows. Each open interval $\mathrm{B}_r(a)$ is completely determined by its end-point $a-r$ and $a+r$. Conversely, given any two rational numbers $b < c$, the set $\{x \in \mathbb{Q} \mid b < x < c\}$ is precisely the open interval $\mathrm{B}_{\frac{c-b}{2}}(\frac{b+c}{2})$, which we denote by $\mathrm{I}(b, c)$. Consider now the set $\mathcal{I} = \{(b, c) \in \mathbb{Q} \times \mathbb{Q} \mid b < c\}$; it is the set of all names of all open intervals.

Any subset $S \subseteq \mathcal{I}$ thus serves as instructions for building an open set. In more detail, we have a function

$$\Psi \colon \mathcal{P}(\mathcal{I}) \to \Omega(\mathbb{Q})$$

given by

$$\Psi(S) = \bigcup_{(b,c)\in S} \mathrm{I}(b,c).$$

Theorem 6. *The function Ψ is onto.*

Proof. Exercise. □

Show explicitly that Ψ is not injective. Let's contemplate what that implies. We think of $S \subseteq \mathcal{I}$ as a presentation of the open set $\Psi(S)$. The lack of injectivity says that the same open set may have many different presentations.

Exercise: What is the minimal/maximal number of presentations that open sets may have?

We would like to reduce some of the redundancy in the presentations.

Suppose $S_1, S_2 \in \mathcal{I}$ and assume $S_1 \subseteq S_2$. It is then clear that $\Psi(S_1) \subseteq \Psi(S_2)$. We would like to find some conditions on the sets S_1 and S_2 that would imply equality.

Theorem 7. *For $S_1 \subseteq S_2$ as above, if any of the following conditions hold, then $\Psi(S_1) = \Psi(S_2)$.*

(1) *If $(a,b) \in S_1$, then $(c,d) \in S_2$ for all $c > a$ and $d < b$.*
(2) *If $(a,b),(c,d) \in S_1$ and $c < d$, then $(a,d) \in S_2$.*
(3) *If $(a,b) \in S_1$ for all $a > \alpha$ and all $b < \beta$, then $(\alpha,\beta) \in S_2$.*

Proof. Exercise. □

Exercise: Think of the definition we gave today of the locale of reals. The theorem above says that the things in that locale are in some sense the maximal things in $\mathcal{I}$. Try to give a precise formalisation of that.

Note how the locale of reals we defined is a subset of $\mathcal{I}$. The motivation to restrict to that subset was to reduce redundancy in the presentation. That redundancy was measured by the lack of injectivity of Ψ. Is Ψ, when restricted to the locale of reals, injective?