Exercises in the Locale of Reals

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Considering set

$$U = \{y \in \mathbb{Q}\, |0 < y < 1\}$$

and supposing some experiment t is deemed a success if it falls within U.

If additionally there is some uncertainty of the accuracy of the measurement of t that can be defined by a tolerance factor $\varepsilon > 0$ such that

$$t - \varepsilon < T < t + \varepsilon$$

describes the experiment being accepted as successful. Given the set U what value of ε guarantees our experiment t as being successful? For this to be the case $T \in U$, and so $0 < T < 1$. From here we can see that

$$t - \varepsilon = 0 \qquad \text{and} \qquad t + \varepsilon = 1$$

and so either $\varepsilon = t$ or $\varepsilon = 1 - t$ depending on which is more relevant to our experiment.

$$\varepsilon = min\{t, 1 - t\}$$

Definition 1. Let $a, r \in \mathbb{Q}$ with $r > 0$. The *open intervlal* of *radius* r and center a is the set

$$B_r(a) = \{x \in \mathbb{Q} \mid a - r < x < a + r\}$$

.

Proposition 2. *Let $a, r \in \mathbb{Q}$ with $r > 0$. If $t \in B_r(a)$, then there exists $\varepsilon > 0$ such that*

$$B_\varepsilon(t) \subseteq B_r(a)$$

Proof. Generically we write

$$B_\varepsilon(t) = \{x \in \mathbb{Q} \mid t - \varepsilon < x < t + \varepsilon\}$$

and given $t \in B_r(a)$ then

$$a - r + \varepsilon < x < a + r - \varepsilon$$

if we chose $\varepsilon = r/2$

$$a - r/2 < x < a + r/2$$

as

$$B_r(a) = \{x \in \mathbb{Q} \mid a - r < x < a + r\}$$

and so

$$B_\varepsilon(t) \subseteq B_r(a)$$

□

The above tells us that for any experimental result $t \in B_r(a)$ we can define an accuracy $\varepsilon > 0$ that ensures the experiment was a success. It is key that the sets we are talking about are open. If we defined our experiment with closed intervals $0 \leq T \leq 1$ then this results in a different situation. We can see this by letting $t = 0$ and noticing that there exists no $\varepsilon > 0$ that satisfies $t - \varepsilon \geq 0$ and similarly for $t = 1$ there is no $\varepsilon > 0$ such that $t + \varepsilon \leq 1$.

Definition 3. A subset $U \subseteq \mathbb{Q}$ of rational numbers is *open* if for all $t \in U$ there exists an $\varepsilon > 0$ such that

$$B_\varepsilon(t) \subseteq U$$

The value of the *open* set of U is that we can say with confidence that a rational value lies within it given some error $\varepsilon > 0$. This is not necessarily so with *non-open* sets, as there are situations (as expressed previously) where we cannot say this for certain.

Theorem 4. *The set $\mathbb{Q}$ is open, as is the empty set $\emptyset$. The intersection of any two open subsets of $\mathbb{Q}$ is also open. In fact, the intersection of any finite number of open sets in $\mathbb{Q}$ is open (with the case of zero open sets yeilding $\mathbb{Q}$). The union of any collection of open subsets is open (with the emption collection yeilding $\emptyset$).*

Proof. Let $\{O_i\}_{i \in I}$ be a collection of *open subsets* with I index set and

$$\bigcup_{i \in I} O_i$$

be the union of all I *open subsets.* If

$$x \in \bigcup_{i \in I} O_i$$

then we note that $x \in O_i$, namely x is in some specific *open subset.*
As O_i is *open*, then by definition there exists an $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subseteq O_i$$

and it follows that

$$B_\varepsilon(x) \subseteq \bigcup_{i \in I} O_i$$

. This holds for *all* sets in $\{O_i\}_{i\in I}$, and so $\bigcup_{i\in I} O_i$ is *open*.

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Alternatively, we can note that the *collection of open sets* $\{O_i\}_{i\in I}$ is a *poset*, which we can name L, and has an ordering $\subseteq$. Now we can say that $\wedge = \cap$ and $\vee = \cup$.

I'd like to follow this approach, but I think I need to justify that there exists a unique $\subseteq$-smallest $j \in \{O_i\}_{i\in I}$, to make sure our $O_i \vee O_j$ has a unique solution. Our definition doesn't have any such guarantees.

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Proof. The *empty set* $\emptyset$ is *open* as $\forall t \in \emptyset$ results in no elements, and so is *vacuously true*.

$$\bigcup_{i\in\emptyset} O_i = \emptyset$$

□

Proof. For *finite intersections*,

$$x \in \bigcap_{i\in I}^{n} O_i \implies x \in O_n$$

where O_n represents the *smallest subset*, then

$$\forall x \in \bigcap_{i\in I}^{n} O_i \exists r : B_r(x) \subseteq O_i$$

if we write $\{r_i\}_{i\in I}$ to be the collection of radii r, we can take

$$\varepsilon := min\{\{r_i\}_{i\in I}\}$$

to be the smallest, then *for all* x

$$B_\varepsilon(x) \subseteq \bigcap_{i\in I}^{n} O_i$$

, which proves the *intersection of any finite number of open sets* in $\mathbb{Q}$ is *open*. □

Proof. If we have no open sets $\{\}_{i\in I}$, then

$$\bigcap_{i\in\emptyset} O_i = \mathbb{Q}$$

. □

These properties of open sets come from Definition 3 of subsets of rational numbers, as we can always provide a smaller ε as needed.

Now if we write $\Omega(\mathbb{Q})$ for the collection of all open subsets of $\mathbb{Q}$, ordered by inclusion we get a poset $(\Omega(\mathbb{Q}), \leq)$. This has arbitrary joins and finite meets, as per the above definition, and so forms a lattice.