Exercises in the Locale of Reals

March 16, 2022

Considering set
$$U = \{y \in \mathbb{Q} \,|0 < y < 1\}$$
and supposing some experiment t is deemed a success if it falls within U.

If additionally there is some uncertainty of the accuracy of the measurement of t that can be defined by a tolerance factor $\varepsilon > 0$ such that

$$t - \varepsilon < T < t + \varepsilon$$

describes the experiment being accepted as successful. Given the set U what value of ε guarantees our experiment t as being successful? For this to be the case $T \in U$, and so $0 < T < 1$. From here we can see that

$$t - \varepsilon = 0 \qquad \text{and} \qquad t + \varepsilon = 1$$

and so either $\varepsilon = t$ or $\varepsilon = 1 - t$ depending on which is more relevant to our experiment.
$$\varepsilon = min\{t, 1 - t\}$$

Definition 1. Let $a, r \in \mathbb{Q}$ with $r > 0$. The *open intervlal* of *radius* r and center a is the set

$$B_r(a) = \{x \in \mathbb{Q} \mid a - r < x < a + r\}$$

.

Proposition 2. *Let $a, r \in \mathbb{Q}$ with $r > 0$. If $t \in B_r(a)$, then there exists $\varepsilon > 0$ such that*
$$B_\varepsilon(t) \subseteq B_r(a)$$

Proof. Generically we write

$$B_\varepsilon(t) = \{x \in \mathbb{Q} \mid t - \varepsilon < x < t + \varepsilon\}$$

and given $t \in B_r(a)$ then

$$a - r + \varepsilon < x < a + r - \varepsilon$$

if we chose $\varepsilon = r/2$

$$a - r/2 < x < a + r/2$$

as

$$B_r(a) = \{x \in \mathbb{Q} \mid a - r < x < a + r\}$$

and so

$$B_\varepsilon(t) \subseteq B_r(a)$$

□

The above tells us that for any experimental result $t \in B_r(a)$ we can define an accuracy $\varepsilon > 0$ that ensures the experiment was a success. It is key that the sets we are talking about are open. If we defined our experiment with closed intervals $0 \leq T \leq 1$ then this results in a different situation. We can see this by letting $t = 0$ and noticing that there exists no $\varepsilon > 0$ that satisfies $t - \varepsilon \geq 0$ and similarly for $t = 1$ there is no $\varepsilon > 0$ such that $t + \varepsilon \leq 1$.

Definition 3. A subset $U \subseteq \mathbb{Q}$ of rational numbers is *open* if for all $t \in U$ there exists an $\varepsilon > 0$ such that

$$B_\varepsilon(t) \subseteq U$$

The value of the *open* set of U is that we can say with confidence that a rational value lies within it given some error $\varepsilon > 0$. This is not necessarily so with *non-open* sets, as there are situations (as expressed previously) where we cannot say this for certain.

Theorem 4. *The set $\mathbb{Q}$ is open, as is the empty set $\emptyset$. The intersection of any two open subsets of $\mathbb{Q}$ is also open. In fact, the intersection of any finite number of open sets in $\mathbb{Q}$ is open (with the case of zero open sets yeilding $\mathbb{Q}$). The union of any collection of open subsets is open (with the emption collection yeilding $\emptyset$).*

Proof. Let $\{O_i\}_{i \in I}$ be a collection of *open subsets* with I index set and

$$\bigcup_{i \in I} O_i$$

be the union. If

$$x \in \bigcup_{i \in I} O_i$$

then we note that $\exists O \in \bigcup_{i \in I} O_i; x \in O$, namely x is in some specific *open subset.*

Let $\varepsilon > 0$.

As O is *open*, then

$$B_\varepsilon(x) \subseteq O$$

is *open*, and it follows that

$$B_\varepsilon(x) \subseteq \bigcup_{i \in I} O_i$$

is also *open*. This holds for *all* $O \in \{O_i\}_{i \in I}$ and so, $\bigcup_{i \in I} O_i$ is *open*. □

Proof. The *empty set* $\emptyset$ is *open* as $\forall t \in \emptyset$ results in no elements, and so is *vacuously true*.

$$\bigcup_{i \in \emptyset} O_i = \emptyset$$

□

Proof. For *finite intersections of open subsets*

$$x \in \bigcap_{i \in I}^{n} O_i \implies x \in O_{min}$$

where O_{min} represents the *smallest open subset* of the intersection, then

$$\forall x \in \bigcap_{i \in I}^{n} O_i \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq O_{min}$$

and

$$B_\varepsilon(x) \subseteq O_{min} \subseteq \bigcap_{i \in I}^{n} O_i \implies B_\varepsilon(x) \subseteq \bigcap_{i \in I}^{n} O_i$$

, which proves the *intersection of any finite number of open sets* in $\mathbb{Q}$ is *open*. □

Proof. If we have no open sets $\{\}_{i \in I}$, then

$$\bigcap_{i \in \emptyset} O_i = \mathbb{Q}$$

. □

These properties of open sets come from Definition 3 of subsets of rational numbers, as we can always provide a smaller ε as needed.

Now if we write $\Omega(\mathbb{Q})$ for the collection of all open subsets of $\mathbb{Q}$, ordered by inclusion we get a poset $(\Omega(\mathbb{Q}), \leq)$. This has arbitrary joins and finite meets, as per the above definition, and so forms a lattice.

Theorem 5. *A subset of* $\mathbb{Q}$ *is open if, and only if, it is a union of open intervals.*

Proof. Let $U \subseteq \mathbb{Q}$ be open. Then by definition for each $t \in U$ there exists a positive rational $\varepsilon_t > 0$ (that depends on t) such that

$$B_\varepsilon(t) \subseteq U$$

and

$$\bigcup_{t \in U} B_\varepsilon(t) = U$$

. This result seems counter-intuitive, as it seems to suggest the union all $t \in U$ with the addition of some *ball* $B_\varepsilon(t) = \{x \in \mathbb{Q} \mid t - \varepsilon < x < t + \varepsilon\}$, still results in U. But at the very heart of this seeming contradiction, is the core feature of an *open set*: there is always room to contain some $\varepsilon > 0$ within the set.

Now if we start with a collection of open intervals $\{B_{r_i}(x_i)\}_{i \in I}$, then the union

$$\bigcup_{i \in I} B_{r_i}(x_i)$$

is also an open set. This is easier to digest, as each $B_{r_i}(x_i)$ is open, and the union is just the largest set that represents all of them in which the interval bounds are the minimum and maximum of all intervals in $B_{r_i}(x_i)$.

** If I can form a poset from this collection then I would write:

If we write $\Omega(U)$ to be the collection of open subsets in U, and if we order this by inclusion $\subseteq$, then we have a powerset $\mathcal{P} = (\Omega(U), \subseteq)$. The union of all open balls is then equivalent to the join of $\mathcal{P}$

$$\bigcup_{i \in I} B_{r_i}(x_i) = \bigvee \mathcal{P}$$

. From here I'd like to say the join is somehow open. ** $\square$

Let's write $\Omega(\mathbb{Q})$ to be the *collection of all open subsets* of $\mathbb{Q}$ ordered by inclusion.

Simultaneously we note that each open interval $B_r(a)$ is completely determined by its end-points $a-r$ and $a+r$. If we take two rational numbers $b, c \in \mathbb{Q}$, such that $b < c$, the set $\{x \in \mathbb{Q} \mid b < x < c\}$ is the open interval $B_{c-b/2}(b+c/2)$, which we donte by $I(b, c)$.

Now the set $\mathcal{I} = \{(b, c) \in \mathbb{Q} \times \mathbb{Q} \mid b < c\}$ is the *set of all names of all open intervals*. Any subset $S \subseteq \mathcal{I}$ thus serves as an instruction on building an open set:

$$\Psi : \mathcal{P}(\mathcal{I}) \to \Omega(\mathbb{Q})$$

given by

$$\Psi(S) = \bigcup_{(b,c) \in S} I(b, c)$$

.

Theorem 6. *The function* Ψ ***is*** *onto.*

Proof. $image(\Psi) = \{\Psi((a, b)) : (a, b) \in S\} = \bigcup_{(a,b) \in S} I(a, b) = \Omega((a, b)) = codomain(\Psi)$. This is *onto.* $\square$

Note however that Ψ **is not** *one-to-one* as the domain and codomain to not have the same cardinality,

$$\mid \mathcal{P}(\mathcal{I}) \mid = 2^{|(a,b)|}$$

$$\mid \Omega(\mathbb{Q}) \mid = 2^{|(a,b)|} - 2^{|closed\mathbb{Q}|}$$

$$| \mathcal{P}(\mathcal{I}) | \neq | \Omega(\mathbb{Q}) |$$

.

** Also in the category of **Set** a morphism is *epic if and only if it is onto* (surjective). Can we use this? **