Definition 1. The set $\mathcal{I}$ is defined to be all the intervals of rational numbers (a, b) s.t. $a < b$. A relation $\sqsubseteq$ is defined so that if $(a, b) \sqsubseteq (c, d)$ then $c \leq a$ and $b \leq d$.

We can see that $\sqsubseteq$ is an ordering of $\mathcal{I}$ by the fact that it abides by:

Definition 2. A non strict partial ordering is defined by:

- Reflexivity: $(a, b) \sqsubseteq (a, b)$.
- Antisymmetry: If $(a, b) \sqsubseteq (c, d)$ and $(c, d) \sqsubseteq (a, b)$ then $(a, b) = (c, d)$.
- Transitivity: If $(a, b) \sqsubseteq (c, d)$ and $(c, d) \sqsubseteq (e, f)$ then $(a, b) \sqsubseteq (e, f)$.

Problem 3. Is $\sqsubseteq$ an ordering of $\mathcal{I}$?

Proof. Let $x \in \mathbb{Q}$ so that $x \leq x \implies x = x$, and so reflexivity holds.

Let $x, y \in \mathbb{Q}$ so that $x \leq y$ and $y \leq x \implies x = y$, and so antisymmetry holds.

Let $x, y, z \in \mathbb{Q}$ so that $x \leq y \leq z$ if $x \leq y$ and $y \leq z$ then $x \leq z$, and so transitivity holds.

And so $\sqsubseteq$ is an ordering, and therefore $(\mathcal{I}, \sqsubseteq)$ is a poset. □

Definition 4. A poset $(\mathcal{L}, \leq)$ is called a *lattice* if for any two-element subset $\{a, b\} \subseteq L$ has both *joins* $a \vee b$ and *meets* $a \wedge b$.

A lattice is said to be a *complete lattice* if every subset $S \subseteq \mathcal{L}$ has both all meets $\bigwedge S$ and all joins $\bigvee S$.

A lattice where there are finite meets, all joins and where *meets distribute over all joins*:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

or more precisely, for all $x, \{y_i\}_i$in $\mathcal{L}$:

$$x \wedge (\bigvee_i y_i) \leq \bigvee_i (x \wedge y_i)$$

is said to be a *frame*.

Problem 5. If $\mathcal{I}$ is a poset is it a *lattice* and if so is it a *complete lattice* or even a *frame*?

Proof. Let $S \subseteq (\mathcal{I}, \sqsubseteq)$ so that $S = (a, b)$

As $a < b$, we have $a \vee b = b$ and $a \wedge b = a$. This holds for any arbitrary interval $(a, b) \in \mathcal{I}$, and so $(\mathcal{I}, \sqsubseteq)$ is a *complete lattice*.

Let $T \subseteq S$ and $T = (a', b')$ so that $(a \leq a')$ and $(b' \leq b)$, so that we have $a \leq a' < b' \leq b$, then

$$a \wedge (b \vee b') = a \wedge \top = a$$

$$(a \vee b) \wedge (a \vee b') = b \wedge b' = \bot$$

. So meets do not distribute over joins, meaning this is not a frame. □