Online Matching with Stochastic Rewards: Advanced Analyses Using Configuration Linear Programs

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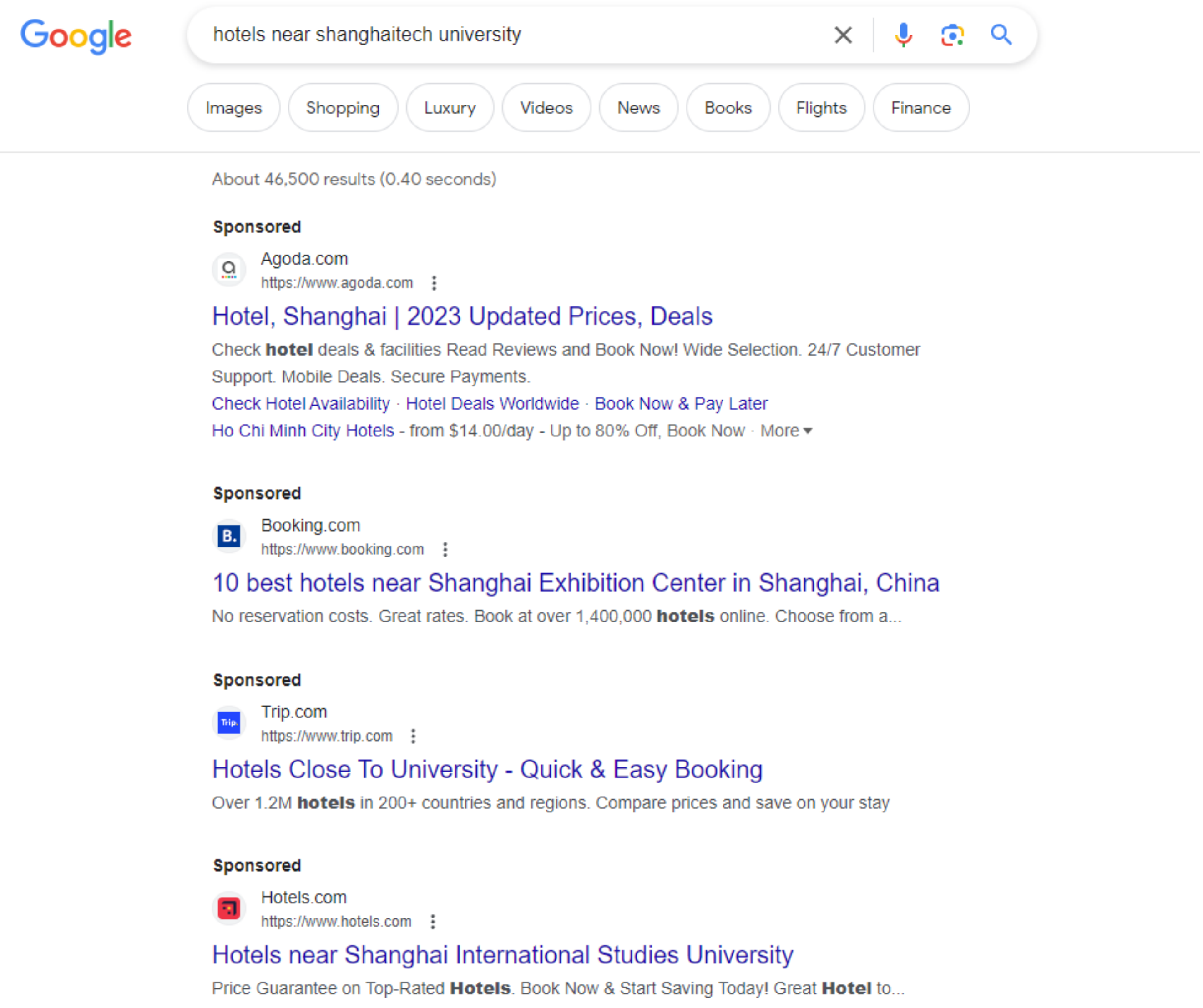
Online Advertising



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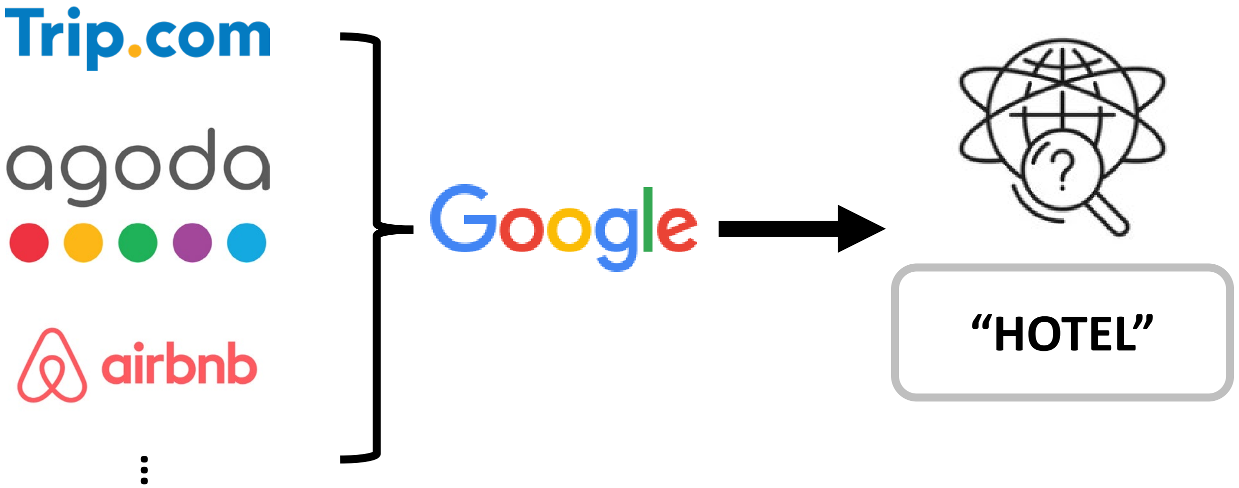
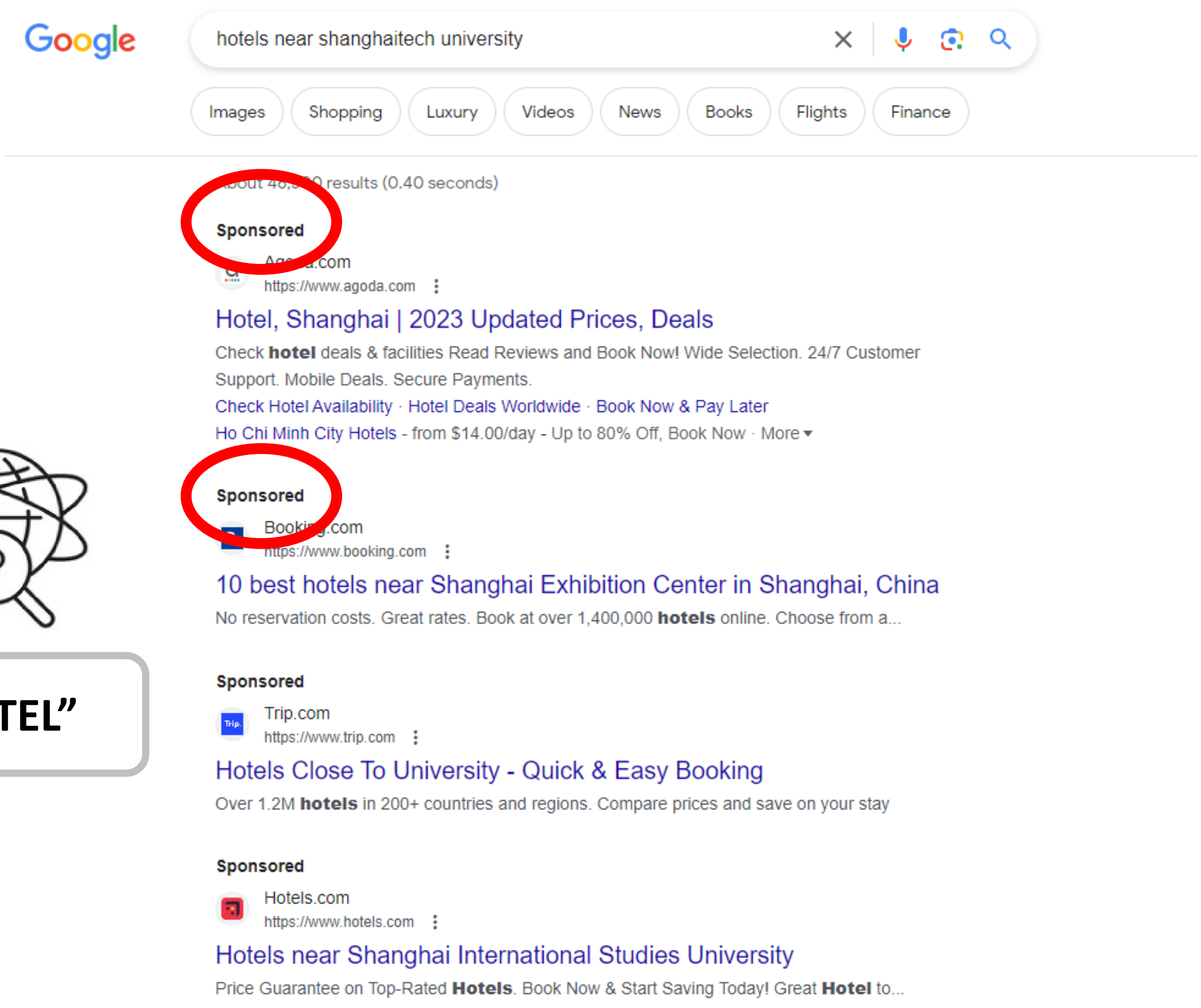




Online Bipartite Matching

[Karp, Vazirani and Vazirani 1990]

- A bipartite graph $G = (U, V, E)$:

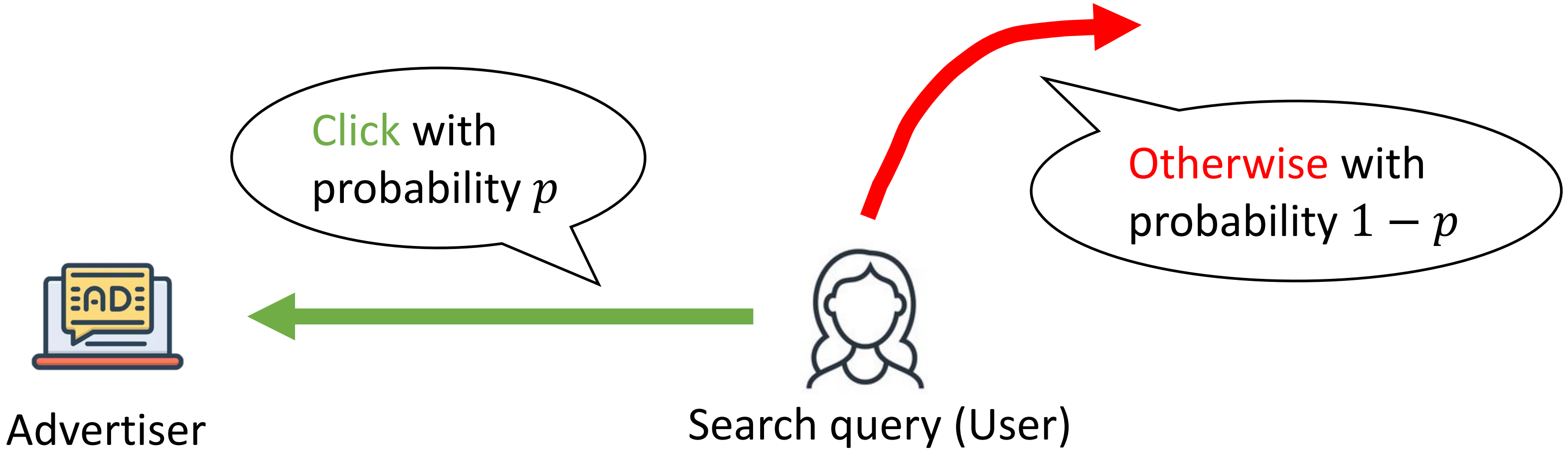
  advertiser search query

  - $u \in U$ is known upfront, $v \in V$ arrives online
  - When v arrives, its adjacent edges are revealed
  - Must irrevocably decide how to match v

- **Goal:** maximize the cardinality

Real-world Applications

- **Pay-per-click:** the advertiser pays only if the user **clicks** the ad
- **Click-through-rate:** an estimate of the probability an ad will be clicked



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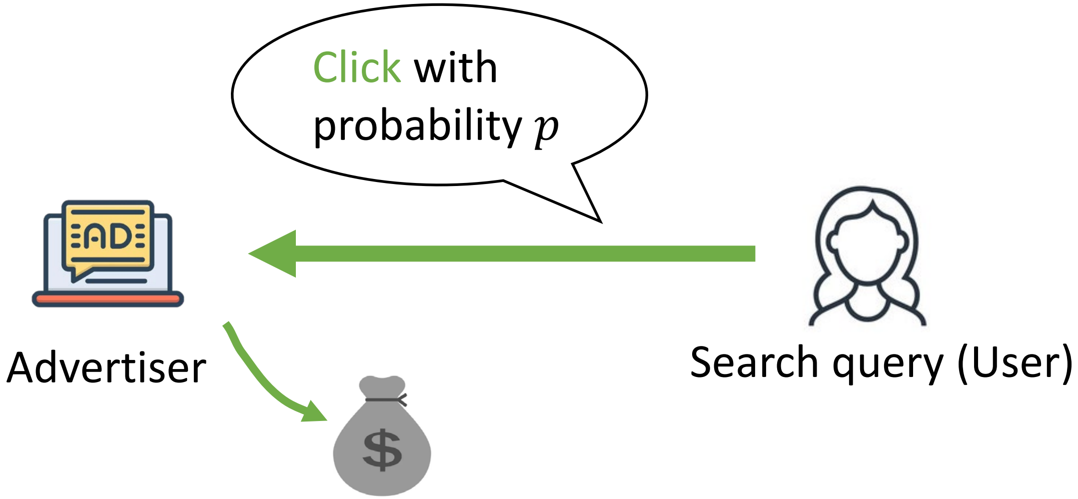

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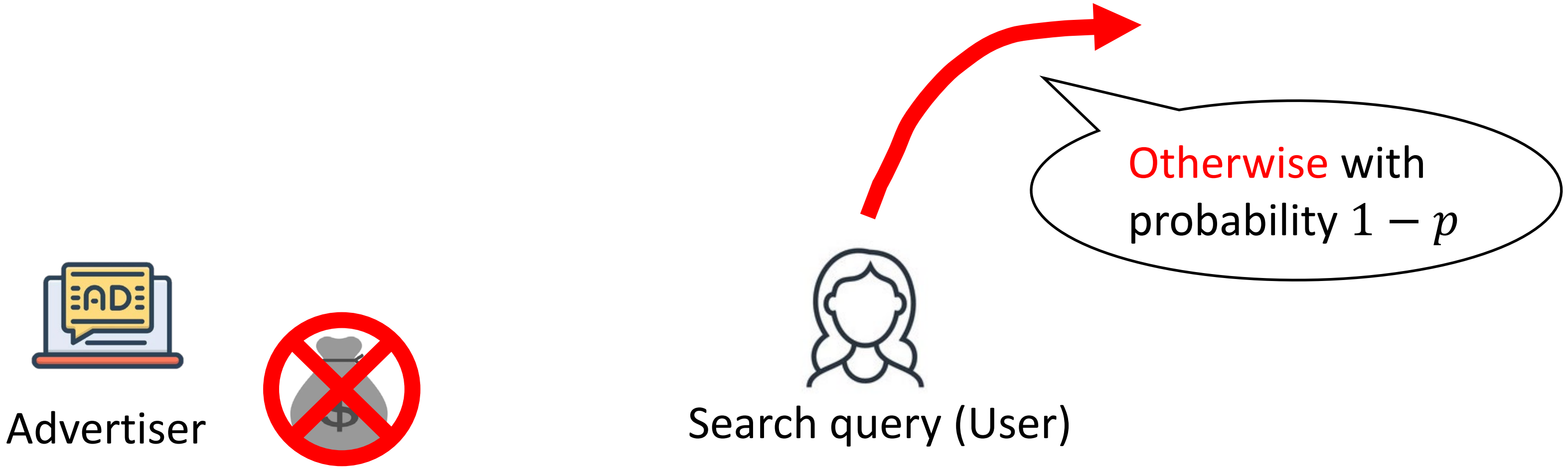

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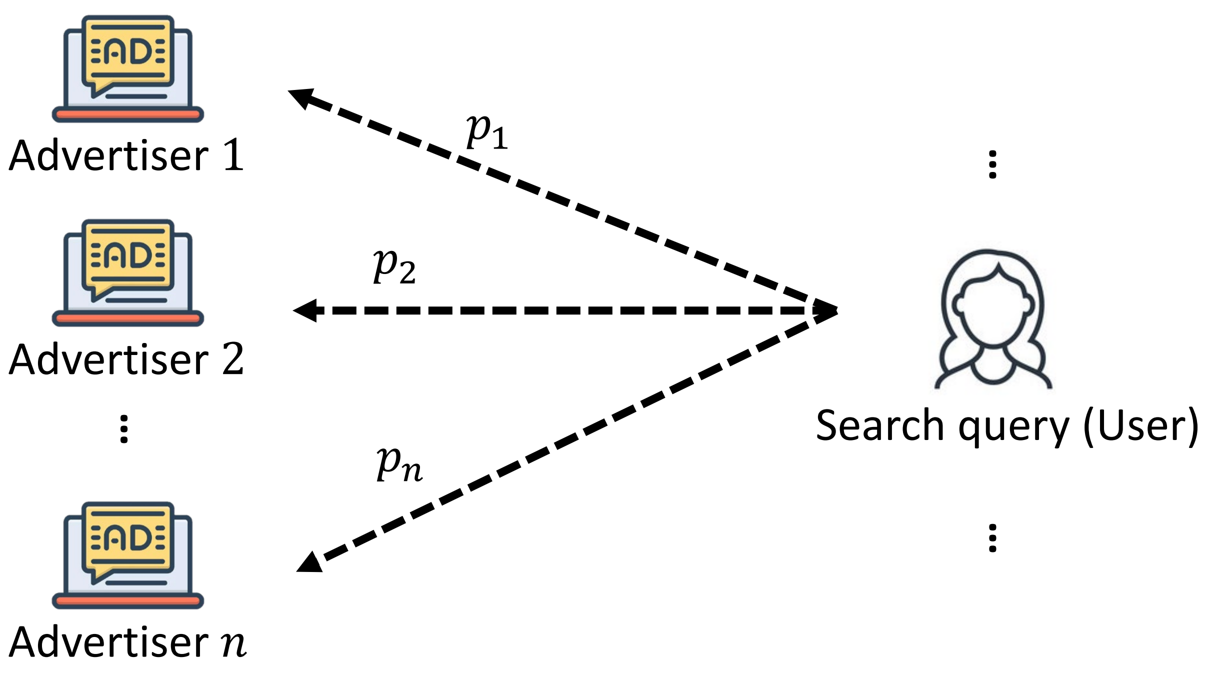

Online Matching with Stochastic Rewards

[Mehta and Panigrahi 2012]

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  - If v is matched to u, u succeeds with p_{uv}
  - If fails, u is still available for future match, but v can not get matched again

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- **Goal:** maximize the expected number of successful offline vertices

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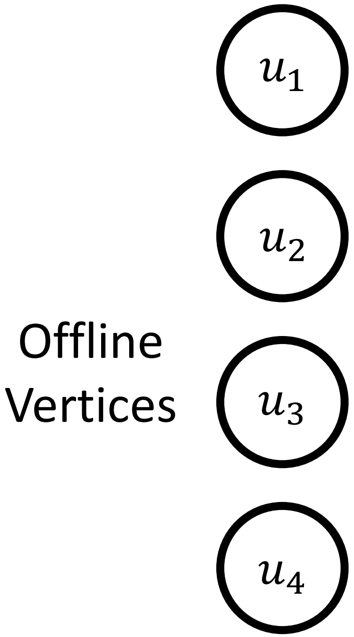

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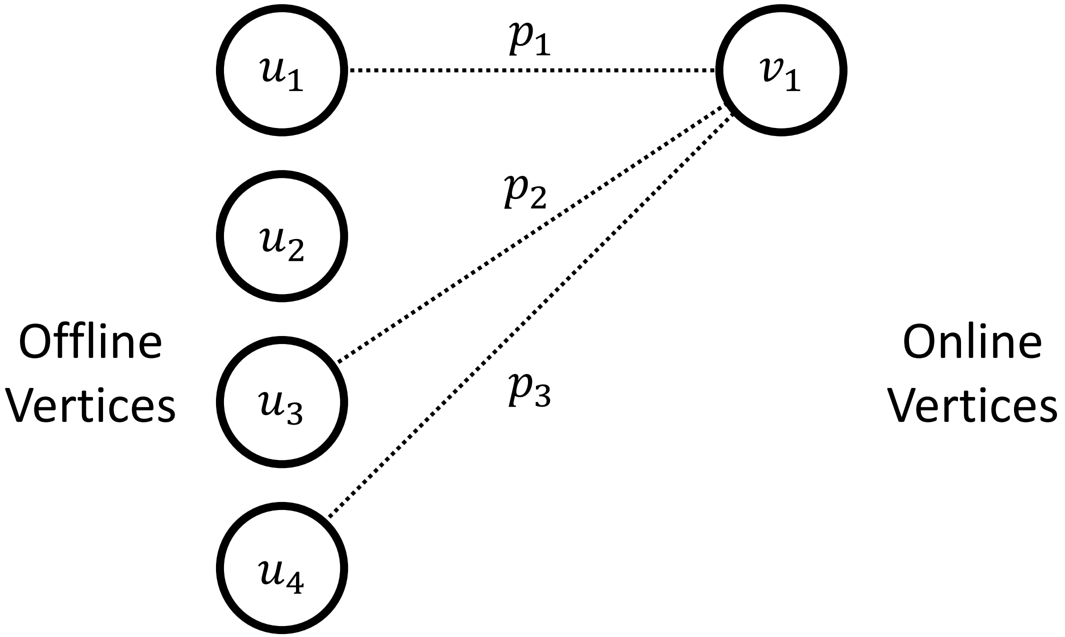

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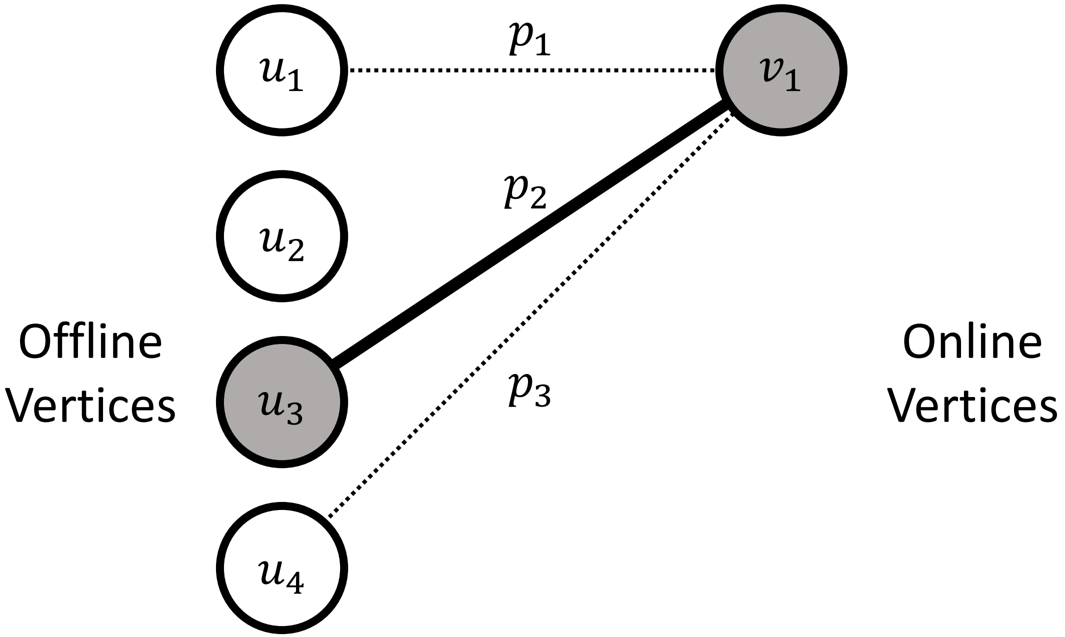

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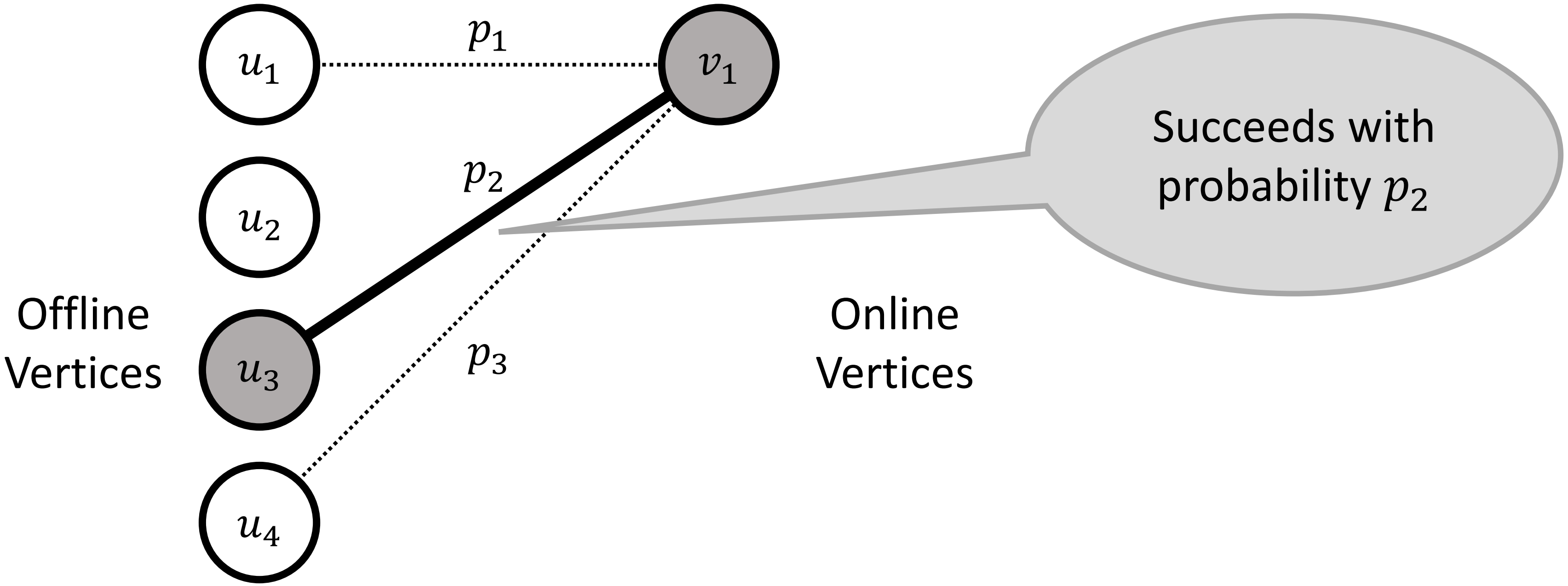

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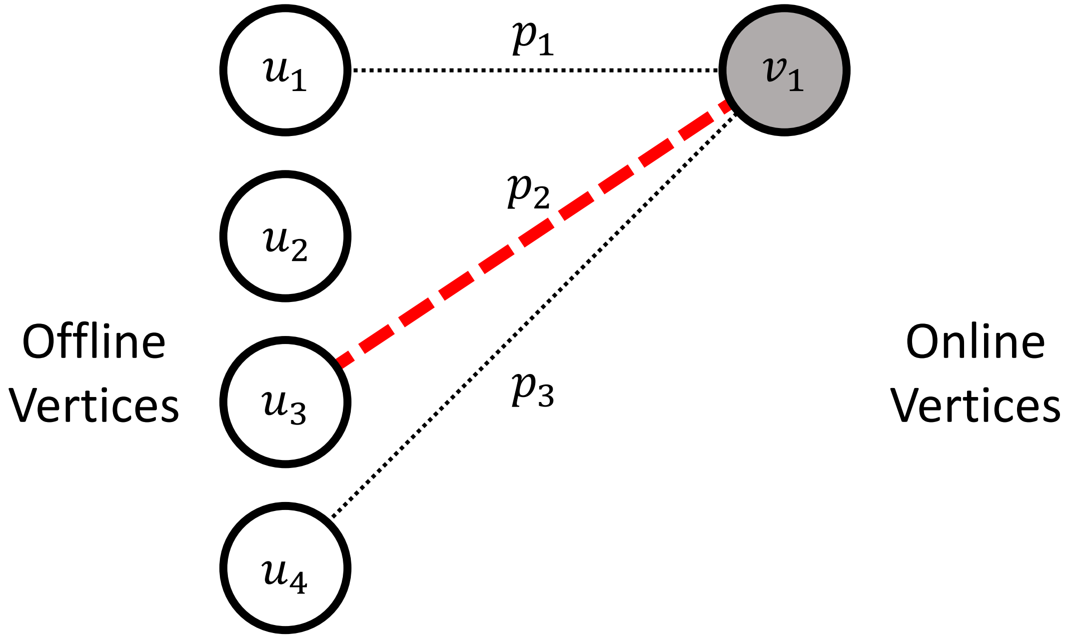

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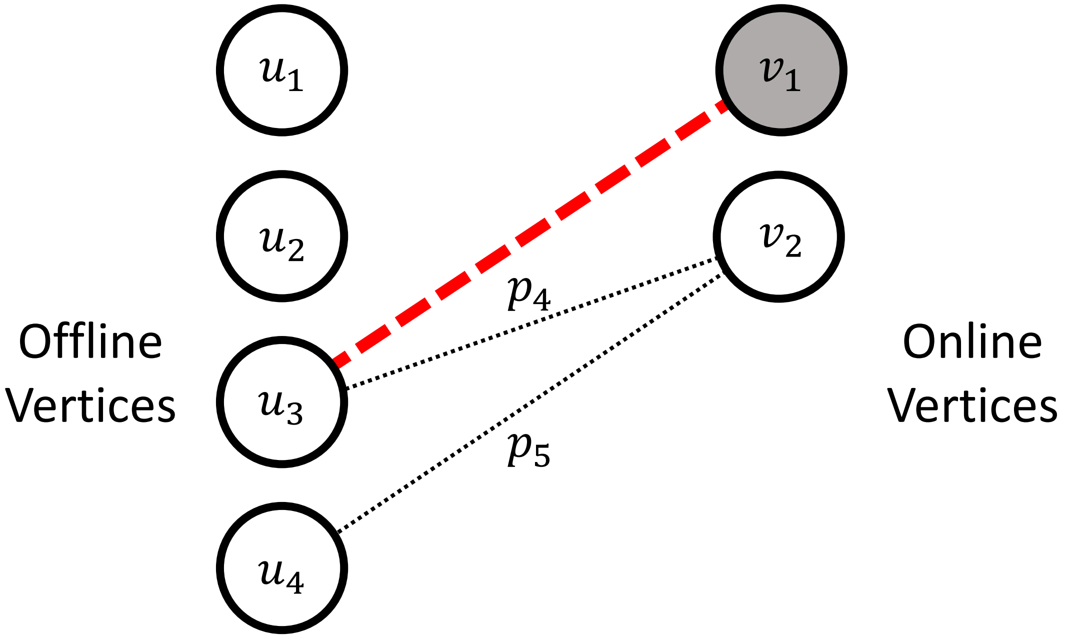

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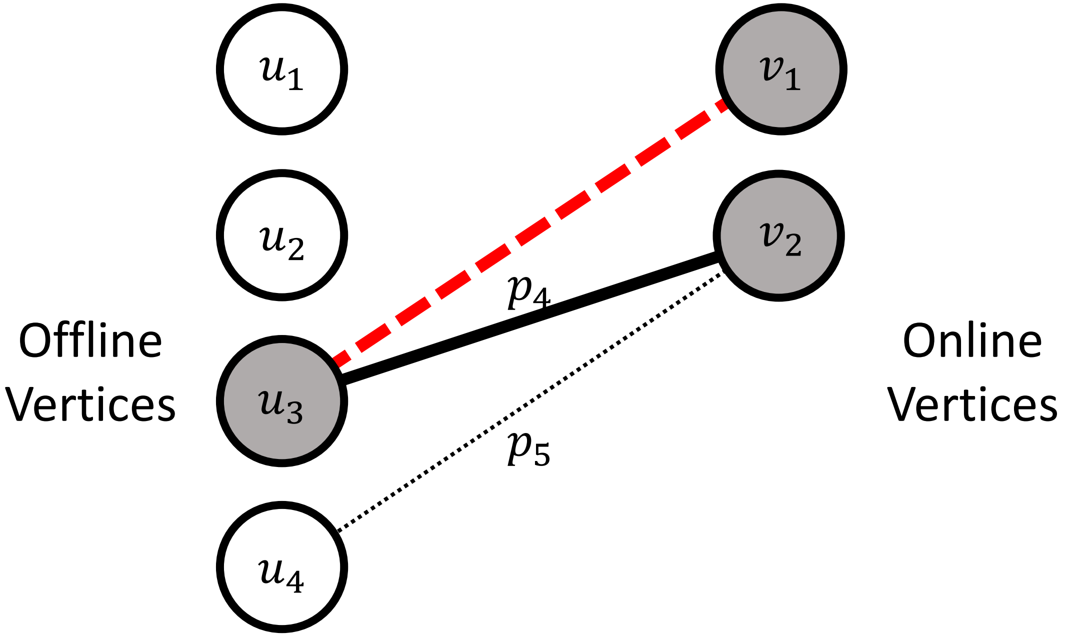

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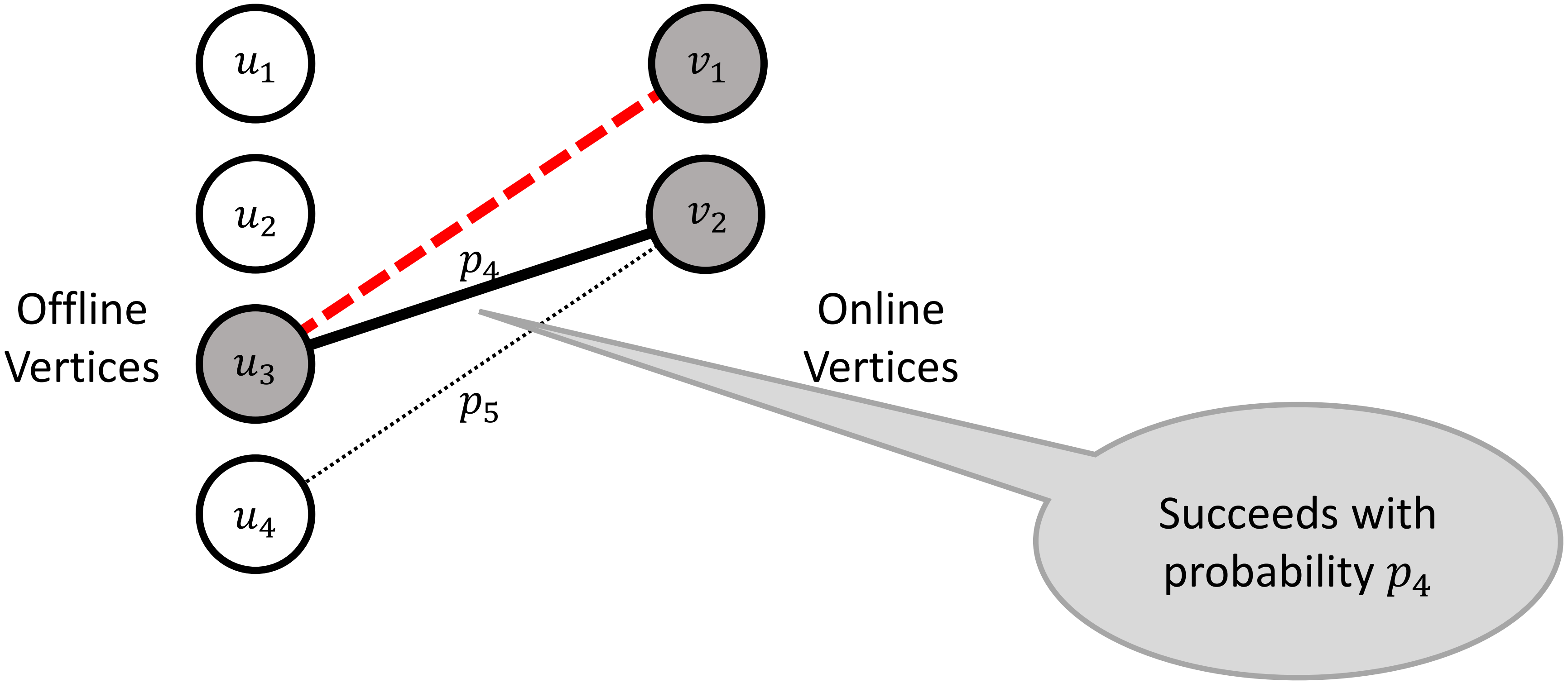

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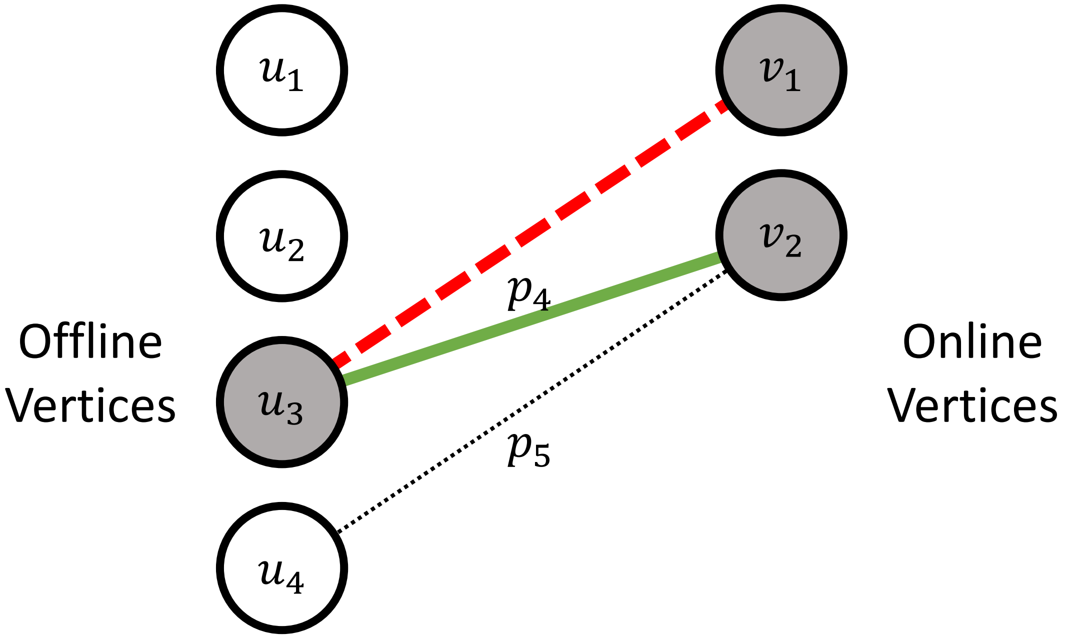

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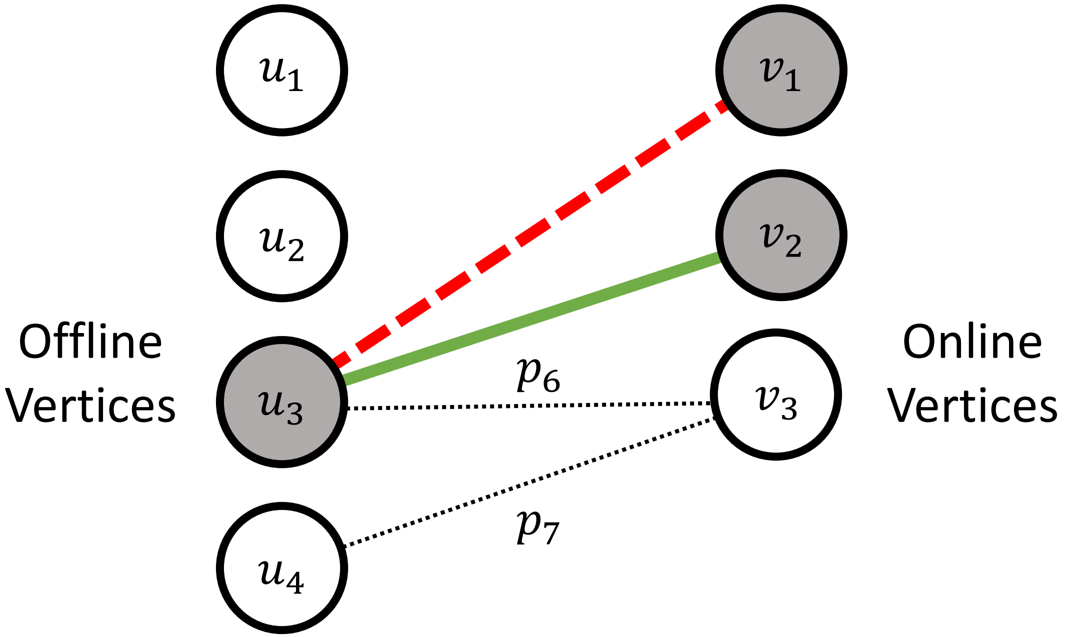

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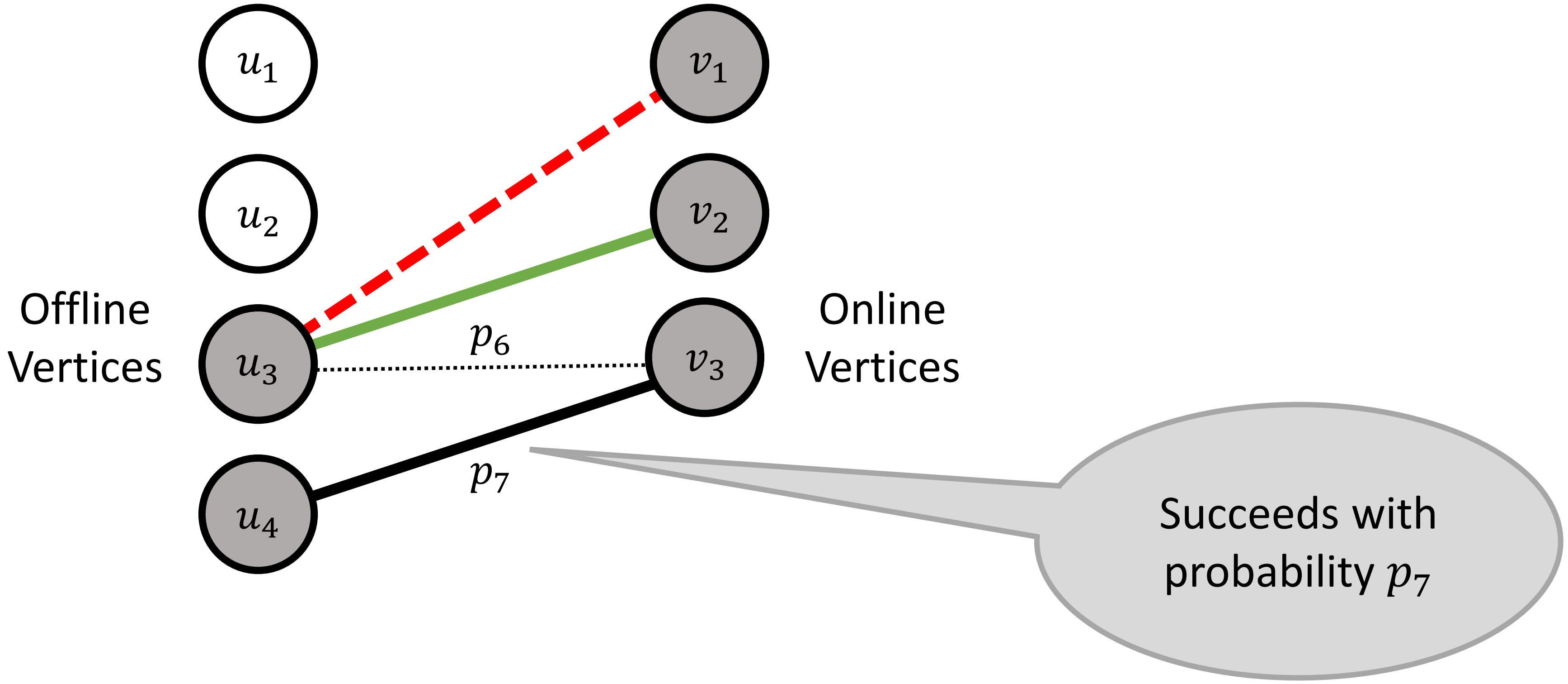

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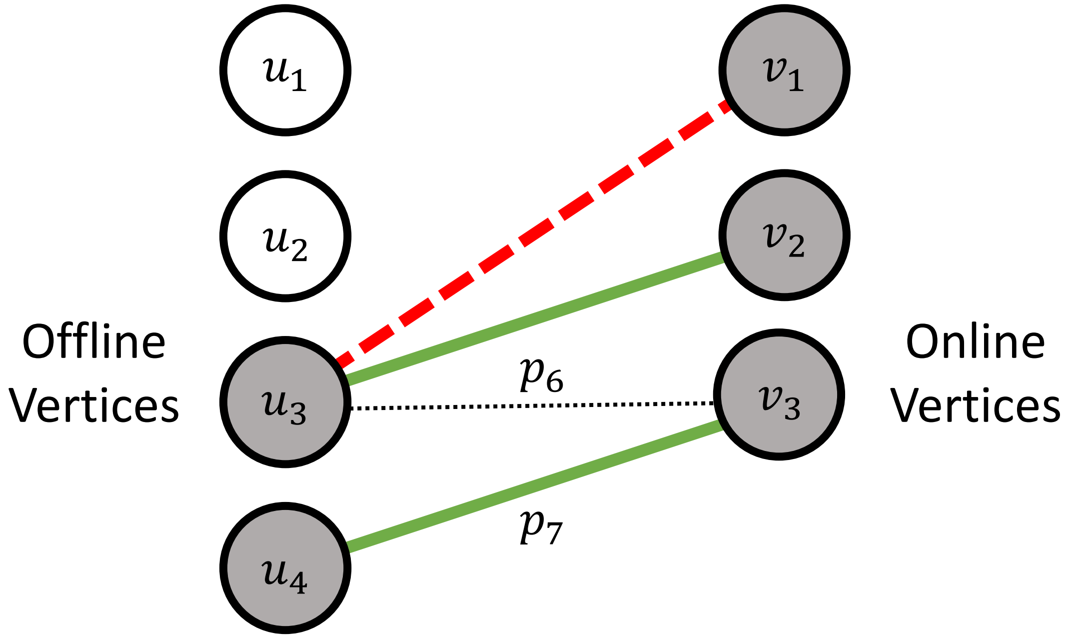

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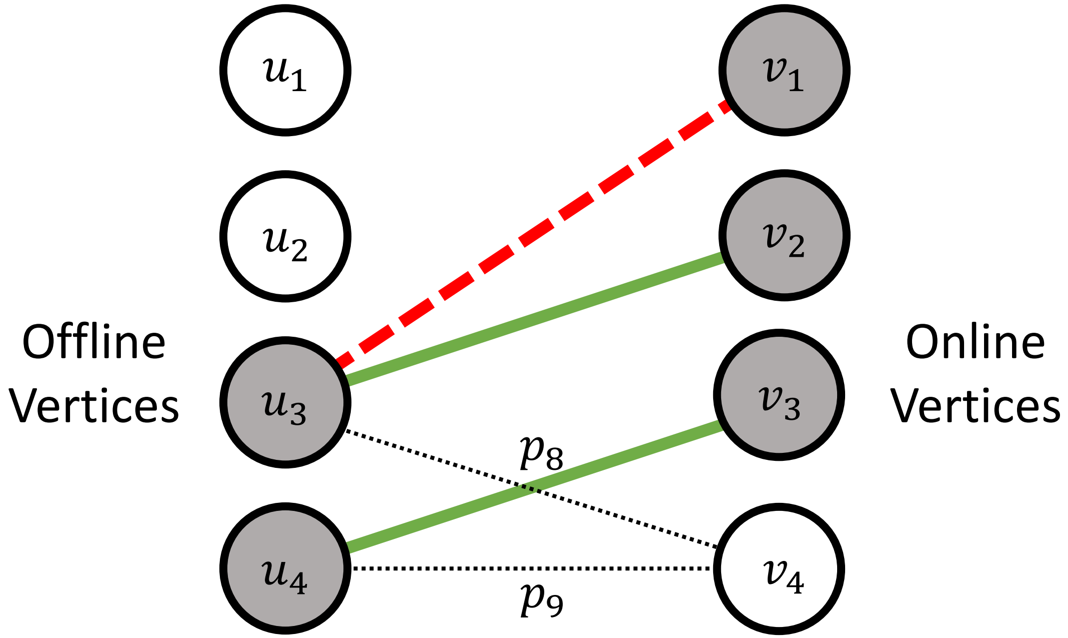

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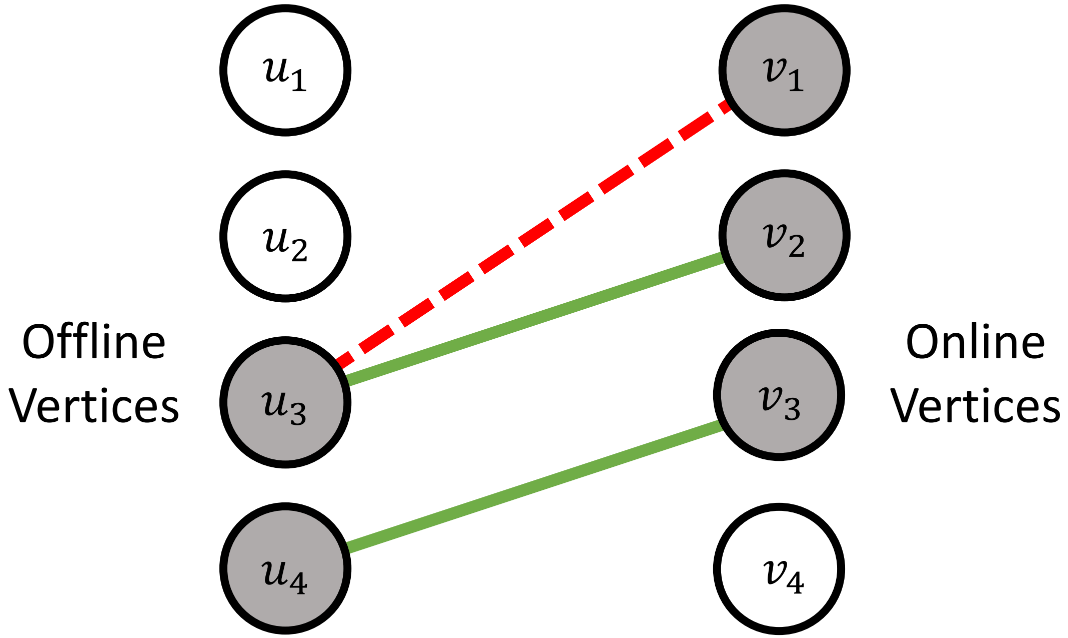

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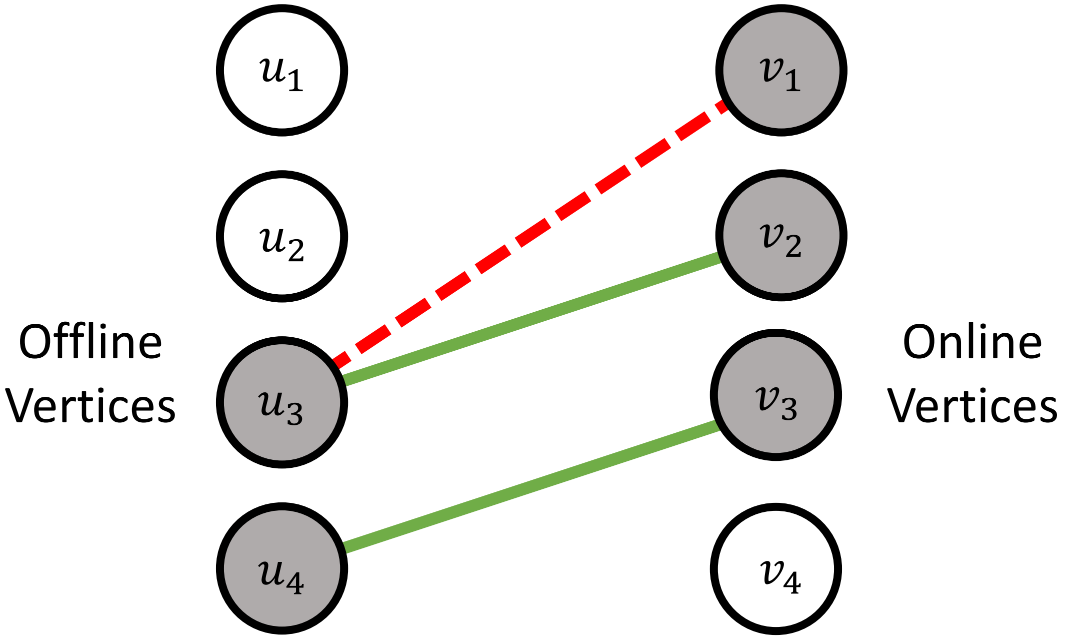

Successful Offline Vertices: 2

Alternative Viewpoint ($p \to 0$)

- Stochastic Budgets

Alternative Viewpoint ($p \to 0$)

- Stochastic Budgets:
  - At the beginning, $u \in U$ draws a budget $\theta_u \sim \mathrm{Exp}(1)$
  - Initialize u's load ℓ_u to 0
  - If match v to u, increase ℓ_u by p_{uv}
  - θ_u is not realized to algorithm until ℓ_u exceeds it
  - **Goal:** maximize $\sum_{u \in U} \min\{\ell_u, \theta_u\}$

Competitive Ratio

- The competitive ratio (CR) of a (randomized) online algorithm is

$$\mathrm{CR} = \min_{G(U,V,E)} \frac{\mathbb{E}[\mathrm{ALG}(G)]}{\mathrm{OFFLINE}(G)}$$

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$$\mathrm{CR} = \min_{G(U,V,E)} \frac{\mathbb{E}[\mathrm{ALG}(G)]}{\underline{\mathrm{OFFLINE}}(G)}$$

?

- What's known to OFFLINE?
  - **Online:** future arrivals?
  - **Stochastic rewards:** match succeeds or not?

Benchmarks

- Offline algorithm knows G and $(p_{uv})_{(u,v)\in E}$ in a priori

Benchmarks

- Offline algorithm knows G and $(p_{uv})_{(u,v)\in E}$ in a priori
- Two offline problems ($p \to 0$):

Offline **non-stochastic** optimum (**OPT**)
[Mehta and Panigrahi 2012]
Offline stochastic budgets with $\boldsymbol{\theta_u = 1}$
- u gains deterministic p_{uv} if match v to u
- u gains at most 1
- Goal: maximize the total gain among U

Offline **stochastic** optimum (**S-OPT**)
[Goyal and Udwani 2020]
Offline stochastic budgets with $\boldsymbol{\theta_u \sim \mathrm{Exp}(1)}$
- The match succeeds with p_{uv} if match v to u
- u can not get matched again if succeeds
- Goal: maximize the number of successes

Existing Algorithms

- Ranking:
  - At the beginning of the algorithm, sample a random seed $\rho_u \sim U[0,1]$ independently for each offline vertex u
  - On the arrival of v, match v to unsuccessful neighbor with the lowest ρ_u
- Balance (Equal Probabilities):
  - On the arrival of v, match v to the unsuccessful neighbor with the least fail attempts

Our Results

- Equal probabilities: If $(u, v) \in E, p_{uv} = p$
- Vanishing probabilities: If $(u, v) \in E, p_{uv} \to 0$

Our Results: Ranking

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Ranking	OPT		S-OPT	
	Vanishing	Non-vanishing	Vanishing	Non-vanishing
Equal	0.534 [MP12] $\to 0.572$	0.534 [MP12] $\to 0.572$	$1 - 1/e$ [GU20]	$1 - 1/e$ [GU20]
Unequal	???	???	???	???

[MP12]: Online matching with stochastic rewards. (FOCS 2012)
[GU20]: Online matching with stochastic rewards: Optimal competitive ratio via path-based formulation. (EC 2020)

Our Results: Balance

- Equal probabilities: If $(u, v) \in E, p_{uv} = p$
- Vanishing probabilities: If $(u, v) \in E, p_{uv} \to 0$

Balance	OPT		S-OPT	
	Vanishing	Non-vanishing	Vanishing	Non-vanishing
Equal	0.576 [HZ20]	0.5 [MP12]	0.596 [GU20] $\to$ 0.613	0.5 [GU20]
Unequal	0.572 [HZ20]	???	0.596 [GU20] $\to$ 0.611	0.5 [GU20]

[MP12]: Online matching with stochastic rewards. (FOCS 2012)
[HZ20]: Online primal dual meets online matching with stochastic rewards: configuration LP to the rescue. (STOC 2020)
[GU20]: Online matching with stochastic rewards: Optimal competitive ratio via path-based formulation. (EC 2020)

Randomized Primal Dual

[Devanur, Jain and Kleinberg 2013]

- **Standard Matching LP**

StdLP:

$$
\begin{array}{lll}
\text{maximize} & \sum_{(u,v)\in E} p_{uv} \cdot x_{uv} & \\
\text{subject to} & \sum_{v:(u,v)\in E} p_{uv} \cdot x_{uv} \leq 1 & \forall u \in U \\
 & \sum_{u:(u,v)\in E} x_{uv} \leq 1 & \forall v \in V \\
 & x_{uv} \geq 0 & \forall (u,v) \in E
\end{array}
$$

StdDual:

$$
\begin{array}{lll}
\text{minimize} & \sum_{u\in U} \alpha_u + \sum_{v\in V} \beta_v & \\
\text{subject to} & p_{uv} \cdot \alpha_u + \beta_v \geq p_{uv} & \forall (u,v) \in E \\
 & \alpha_u, \beta_v \geq 0 & \forall u \in U, \forall v \in V
\end{array}
$$

Randomized Primal Dual

[Devanur, Jain and Kleinberg 2013]

- Dual constraints in Matching LP [DJK13]:

$$p_{uv} \cdot \mathbb{E}[\text{gain of } u] + \mathbb{E}[\text{gain of } v] \geq \Gamma \cdot p_{uv}$$

Weaker Dual Constraints

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- An amortized among online vertex set S [HZ20]:

$$\mathbb{E}[\text{gain of } u] + \sum_{v \in S} \mathbb{E}[\text{gain of } v] \geq \Gamma \cdot \mathbf{Pr}[u \text{ succeeds}]$$

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Dual constraints in Configuration LP

Weaker Dual Constraints

- Against OPT: $\forall u \in U, S \subseteq N_u$,

$$\mathbb{E}[\text{gain of } u] + \sum_{v \in S} \mathbb{E}[\text{gain of } v] \geq \Gamma \cdot \mathbf{Pr}[u \text{ succeeds}]$$

- Against S-OPT: $\forall u \in U, S \subseteq N_u$,

$$\mathbb{E}[\text{gain of } u] + \sum_{v \in S} \mathbf{Pr}[u \text{ is available to } v] \cdot \mathbb{E}[\text{gain of } v]$$
$$\geq \Gamma \cdot \mathbf{Pr}[u \text{ succeeds}]$$

Ranking: Dual Updates

- **Ranking:** the rank $\rho_u \sim U[0,1]$ and $p_{uv} = p, \forall (u,v) \in E$
- **A usual plan:** if algorithm matches v to u,
  - Split the gain of p based on the rank ρ_u the non-decreasing function $g: [0,1] \rightarrow [0,1]$
  - Increase α_u by $p \cdot g(\rho_u)$
  - Set β_v as $p \cdot \left(1 - g(\rho_u)\right)$

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Against OPT: $\forall u \in U, S \subseteq N_u$,

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Ranking: Dual Updates

- **Ranking:** the rank $\rho_u \sim U[0,1]$ and $p_{uv} = p, \forall (u,v) \in E$
- **Our plan:**
  - Split the gain of joint outcome of u and **all its neighbors**

Against OPT: $\forall u \in U, S \subseteq N_u$,

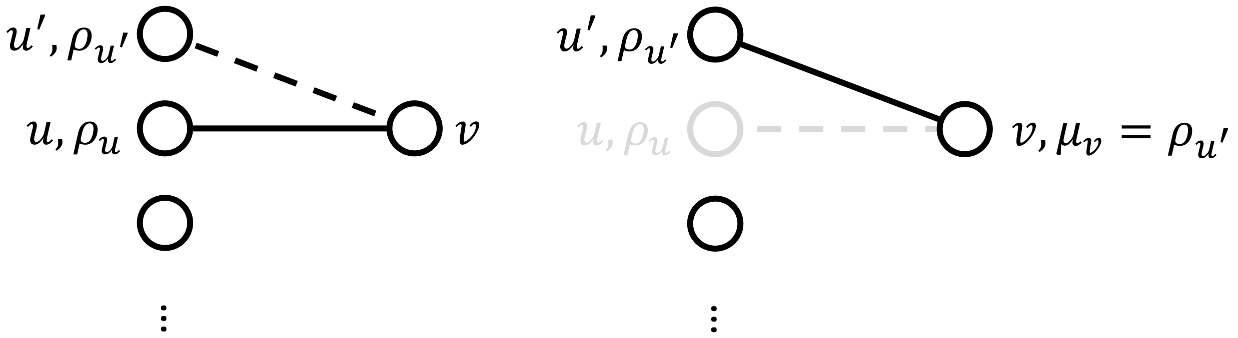
$$\mathbb{E}[\text{gain of } u] + \sum_{v \in S} \mathbb{E}[\text{gain of } v] \geq \Gamma \cdot \mathbf{Pr}[u \text{ succeeds}]$$

Ranking: Analysis with Dual Fitting

- Consider arbitrary online vertex u and its neighbors N_u
- Fix the ranks of offline vertices except u: $\boldsymbol{\rho}_{-u}$

Ranking: Analysis with Dual Fitting

- An imaginary run with vertex u removed
- Define online vertex v's **critical rank** μ_v as:
  - If v is matched to u', $\mu_v = \rho_{u'}$
  - If v is not matched, $\mu_v = 1$



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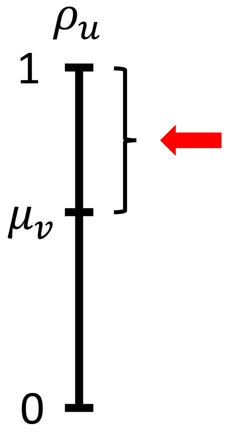
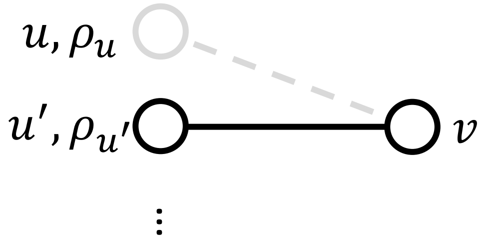
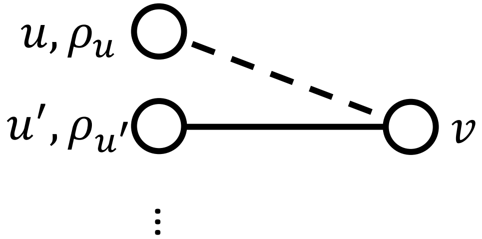
- Let $N_u(\rho_u)$ be the set of u's neighbors whose critical rank $\geq \rho_u$
- $N_u(\rho_u)$ is the set of vertices matched to u if u never succeeds







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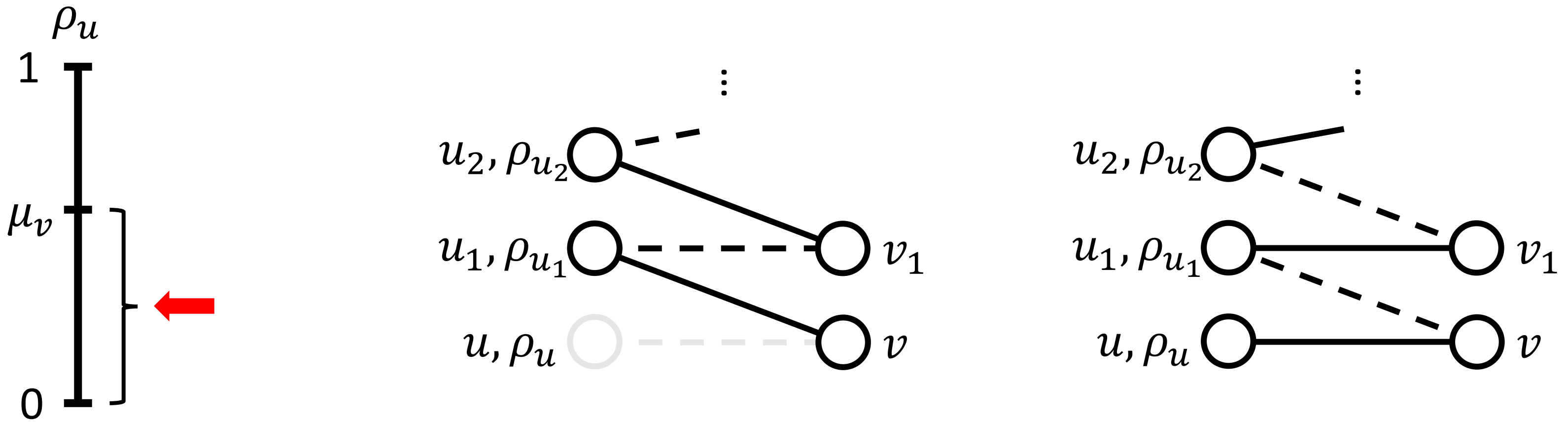

Ranking: Analysis with Dual Fitting

Contribution of $\boldsymbol{\alpha_u}$

- The probability u succeeds: $1 - (1-p)^{|N_u(\rho_u)|}$
- Thus,

$$\mathbb{E}_{\rho_u}[\alpha_u | \boldsymbol{\rho}_{-u}] = \int_0^1 \left(1 - (1-p)^{|N_u(\rho_u)|}\right) g(\rho_u)\, \mathrm{d}\rho_u$$

Ranking: Analysis with Dual Fitting

Contribution of $\boldsymbol{\beta_v}$

- β_v is at least $p\big(1 - g(\mu_v)\big)$
- For $\rho_u < \mu_v$, if u is available, v would match to u
  - This happens with probability $(1 - p)^{|\underline{N_u(\rho_u, v)}|}$

    Denotes the subset of $N_u(\rho_u)$ in which vertex arrive before v
  - β_v increases by $p\big(1 - g(\rho_u)\big) - p\big(1 - g(\mu_v)\big) = p\big(g(\mu_v) - g(\rho_u)\big)$

Ranking: Analysis with Dual Fitting

Contribution of $\boldsymbol{\beta_v}$

$$\mathbb{E}_{\rho_u}\left[\beta_v | \boldsymbol{\rho}_{-u}\right]$$

$$\geq p\left(1 - g(\mu_v) + \int_0^{\mu_v} (1-p)^{|N_u(\rho_u, v)|}\big(g(\mu_v) - g(\rho_u)\big)\,\mathrm{d}\rho_u\right)$$

Ranking: Analysis with Dual Fitting

- Expected gain from α_u is $\int_0^1 \left(1-(1-p)^{|N_u(\rho_u)|}\right) g(\rho_u)\,\mathrm{d}\rho_u$
- Expected gain from β_v is at least

$$p\left(1-g(\mu_v)+\int_0^{\mu_v}(1-p)^{|N_u(\rho_u,v)|}\left(g(\mu_v)-g(\rho_u)\right)\mathrm{d}\rho_u\right)$$

- Non-stochastic Benchmark

$$\mathbb{E}[\alpha_u+\textstyle\sum_{v\in S}\beta_v] \geq \Gamma\cdot\min\{\textstyle\sum_{v\in S} p_{uv}\,,1\}$$

Ranking: Analysis with Dual Fitting

- Find optimal value of Γ and function g satisfying:
  - $\alpha_u \geq \int_0^1 \left(1 - (1-p)^{|N_u(\rho_u)|}\right) g(\rho_u)\, \mathrm{d}\rho_u$
  - $\beta_v \geq p\left(1 - g(\mu_v) + \int_0^{\mu_v} (1-p)^{|N_u(\rho_u, v)|} \left(g(\mu_v) - g(\rho_u)\right) \mathrm{d}\rho_u\right)$
  - $\mathbb{E}[\alpha_u + \sum_{v \in S} \beta_v] \geq \Gamma \cdot \min\{\sum_{v \in S} p_{uv}, 1\}$

$$g(\rho) = \begin{cases} \min\left\{\dfrac{c}{e-(e-1)\rho}, 1 - \dfrac{1}{e}\right\}, & 0 \leq \rho < 1 \\ 1, & \rho = 1 \end{cases}$$

$c \approx 1.161$

$\Gamma = 0.572$

Balance

Structural Lemmas in [HZ20]

Stochastic Configuration LP

Equal: 0.613
Unequal: 0.611

against S-OPT

Summary

- Online primal-dual analysis of Ranking based on Configuration LP
  - Improve the competitive ratio from 0.534 to 0.572
- Stochastic benchmark
  - A new Stochastic Configuration LP
  - Improve the ratio to 0.611 (0.613 for equal probabilities) in vanishing case

Thank you!