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EC330 HW2 : Asymptotic Comparison

	$f(n)$	$g(n)$	
1) a)	$n-1$	$n-330$	$\Theta(g)$
b)	$n^{2/3}$	$n^{1/2}$	$\Omega(g)$
c)	$330n+\log n$	$n+(\log n)^2$	$\Theta(g)$
d)	$n\log n$	$330n(\log 330n)$	$\theta(g)$
e)	$\log 330n$	$\log n$	$\Theta(g)$
f)	$330\log n$	$\log(n^3)$	$\Theta(g)$
g)	$n^{1.01}$	$n\log^2 n$	$\Omega(g)$
h)	$n^2/\log n$	$n(\log n)^2$	$\Omega(g)$
i)	$(\log n)^{\log n}$	$n/\log n$	$\Omega(g)$
j)	$\sqrt{n}$	$(\log n)^3$	$\Omega(g)$
k)	$n^{1/2}$	$5^{\log_2 n}$	$O(g)$
l)	2^n	$n2^n$	$\Omega(g)$
m)	$n!$	2^n	$\Omega(g)$
n)	$(\log n)^{10}$	$n^{0.1}$	$O(g)$
o)	$\sum_{i=1}^{n} i^k$	n^{k+1}	$\Theta(g)$

(a) $\lim_{n\to\infty} \frac{n-330}{n-1} = \frac{\infty}{\infty}$ L'Hôpital

$\lim_{n\to\infty} \frac{1}{1} = 1 \neq 0 \neq \infty$ $\quad f=\Theta(g)$

(b) $\lim_{n\to\infty} \frac{n^{1/2}}{n^{2/3}}$ since $2/3 > 1/2$

$f=\Omega(g)$

(c) $\lim_{n\to\infty} \frac{330n+\log n}{n+(\log n)^2} = \frac{\infty}{\infty}$ $\quad u=(\log n)^2$ $\quad du = \frac{1}{n}$

$\lim_{n\to\infty} \frac{330+\frac{1}{n}}{1+\frac{2\log n}{n}}$ $\quad \frac{d}{dx}(u^2) = 2u\,du = 2(\log n)\cdot\frac{1}{n} = \frac{2\log n}{n}$

$\lim_{n\to\infty} = \frac{330n+1}{x+2\log(x)}$ $\frac{d}{dx}$

$\lim_{n\to\infty} \frac{330}{1+\frac{2}{n}} = 330 \neq 0 \neq \infty$ so $f=\Theta(g)$ $\quad 330\cdot\frac{1}{330n}$

(d) $\lim_{n\to\infty} \frac{330n(\log 330n)}{n\log n}$ $\frac{d}{dx}$ $\rightarrow$ ~~$330\lim$~~ $\frac{d}{dn}(n)$ $\frac{d}{dn}(\log 330n)$

$330\left(1\cdot\log 330n + n\cdot\frac{1}{n}\right)$

$330\cdot\lim_{n\to\infty} \frac{(\log(330n)+1)}{\log n+1}$

$= 330(\ln(330n)+1)$

$n\log n \rightarrow 1\cdot\log n + \frac{1}{n}\cdot n$

$= \log n + 1$

$\frac{d}{dn}$ again

2) Asymptotic Analysis

prove that $(\log n)^{\log n} = O(2^{(\log n)^2})$

$330\cdot\lim_{n\to\infty} \frac{\log(330n)+1}{\log n+1}$

to

$330\cdot\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}} = 1\cdot 330$

$\neq 0 \neq \infty$

$f=\Theta(g)$

(e) $\lim_{n\to\infty} \frac{\log n}{\log 330n} = \frac{\infty}{\infty} \quad \frac{d}{dx} \Rightarrow \quad \log n \Rightarrow \frac{1}{n}, \quad \log 330n \Rightarrow \frac{1}{n} \quad \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}} = 1 \neq 0 \neq \infty \quad \underline{f = \Theta(g)}$

(f) $\lim_{n\to\infty} \frac{\log(n^3)}{330\log n} = \frac{\infty}{\infty} \quad \log(n^3) \Rightarrow u = n^3, \; du = 3n^2 dn \quad \log u \Rightarrow \frac{1}{u} du = \frac{1}{n^3} \cdot 3n^2 = \frac{3}{n}$

$330 \log n \Rightarrow 330 \cdot \frac{1}{n}$

$\hookrightarrow \lim_{n\to\infty} \frac{\frac{3}{n}}{\frac{1}{n} \cdot 330} = \frac{3}{330} \neq 0 \neq \infty \quad \underline{f = \Theta(g)}$

(g) $\lim_{n\to\infty} \frac{n \log^2 n}{n^{1.01}} = \lim_{n\to\infty} \frac{\log^2 n}{n^{0.01}} = \frac{\infty}{\infty} \quad \frac{d}{dx}: \quad \log^2(n)' \Rightarrow 2\log(n) \cdot \frac{1}{n} \quad n^{0.01} \Rightarrow 0.01 n^{-0.01}$

$\lim_{n\to\infty} \frac{2\log(n) \cdot \frac{1}{n}}{0.01 n^{-0.01}}$ top grows at a rate of $\infty \cdot \frac{1}{\infty}$

bottom grows at a rate of $\frac{1}{\infty}$

$\hookrightarrow \lim_{n\to\infty} \frac{\frac{2\log(n)}{n}}{\frac{0.01}{n^{0.99}}} = \frac{200 \log(n)}{n^{0.01}} = \frac{\infty}{\infty} \quad \frac{d}{dx}: \quad 200 \log n \Rightarrow \frac{200}{n} \quad n^{0.01} \Rightarrow \frac{0.01}{n^{0.99}}$

$\lim_{n\to\infty} \frac{\frac{200}{n}}{\frac{0.01}{n^{0.99}}} = \frac{20000}{n^{0.01}} = 0 \quad \underline{f = \Omega(g)}$

(h) $\lim_{n\to\infty} \frac{n(\log n)^2}{n^2/\log n} \div \frac{\frac{n}{\log n}}{\frac{n}{\log n}} \Rightarrow \lim_{n\to\infty} \frac{(\log n)^3}{n} \quad \frac{\infty}{\infty} \quad \frac{d}{dx} \cdot \lim_{n\to\infty} \frac{3\log^2(n) \cdot \frac{1}{n}}{1}$

$\frac{d}{dx}$ again $\hookrightarrow \lim_{n\to\infty} \frac{6\log(n) \cdot \frac{1}{n}}{1}$

$\frac{d}{dx}$ again: $\lim_{n\to\infty} \frac{\frac{6}{n}}{1} = 0 \quad \underline{f = \Omega(g)}$

(i) $\lim_{n\to\infty} \frac{n/\log n}{(\log n)^{\log n}} \quad a^x = e^{\ln(a^x)} = e^{x \ln(a)}$

$a = \log^n \quad x = \log^n$

$\log n^{\log n} = e^{\log n(\log(\log n))}$

$\lim_{n\to\infty} \frac{\frac{n}{\log n}}{e^{\log n(\log(\log n))}} = \lim_{n\to\infty} \frac{n}{\log n (n^{\log(\log n)})} = \lim_{n\to\infty} \frac{n^{1-\log(\log n)}}{\log n} = \frac{0}{\infty} = 0$

$e^{\log n(\log(\log n))}$

$= (e^{\log n})^{\log(\log n)}$

$= n^{\log(\log n)}$

$\underline{f = \Omega(g)}$

$\frac{d}{dx}$ refers to L'Hôpital rule

(j) $\lim_{n\to\infty} \frac{(\log n)^3}{\sqrt{n}} = \frac{\infty}{\infty}$ $\frac{d}{dx}$: $\log(n)^3 \Rightarrow \frac{3(\log n)^2}{n}$ $\lim_{n\to\infty} \frac{\frac{3\log^2 n}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{3\log^2 n}{n} \cdot 2\sqrt{n}$ $\frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$

$\sqrt{n} \Rightarrow \frac{1}{2\sqrt{n}}$

$\lim_{n\to\infty} \frac{24\log(n)}{\sqrt{n}} = \frac{12\log(n)}{n} \cdot 2\sqrt{n} \Leftarrow \frac{d}{dx}$ again $\Leftarrow \lim_{n\to\infty} \frac{6\log^2(n)}{\sqrt{n}}$

↳ $\frac{d}{dx}$ again: $\lim_{n\to\infty} \frac{24}{n} \cdot 2\sqrt{n} = \lim_{n\to\infty} \frac{48}{\sqrt{n}} = 0$ $f = \Omega(g)$

(k) $\lim_{n\to\infty} \frac{5^{\log_2 n}}{n^{1/2}}$ ~~$a^x = e^{\ln(a^x)}$~~ $= e^{x \cdot \ln(a)}$

~~$a = 5$ $x = \log_2 n$~~

~~↳ $e^{\log_2 n \ln 5}$~~ $= (e^{\log_2 n})^{\ln 5}$

$A^{\log_B C} = (C^{\log_C A})^{\log_B C}$

$A = C^{\log_C A}$

$= C^{\log_C A \cdot \log_B C}$

$\log_b X = \frac{\log X}{\log b}$

$= C^{\frac{\log A}{\log C} \cdot \frac{\log C}{\log B}}$

$= C^{\frac{\log A}{\log B}}$

$= C^{\log_B A}$

$\lim_{n\to\infty} \frac{n^{2.32}}{n^{1/2}}$ ← grows faster so

$f = O(g)$

$5^{\log_2 n}$

$= n^{\log_2 5}$ ← $C = n$, $B = 2$, $A = 5$

$\approx n^{2.32}$

(l) $\lim_{n\to\infty} \frac{n2^n}{3^n} = \frac{\infty}{\infty}$ $\lim_{n\to\infty} \frac{2^n + n^2 2^{n-1}}{n3^{n-1}}$ $\lim_{n\to\infty} n\left(\frac{2}{3}\right)^n$ $r \leq 1$

← geometric series known to converge

$\frac{d}{dx}$: $n \cdot 2^n \to 2^n + n2^{n-1} \cdot n$

$3^n \to n3^{n-1}$

$\frac{d}{dx}$: $\frac{n2^{n-1} + 2n(2^{n-1}) + (n-1)2^{n-2} \cdot n^2}{3^{n-1} + (n-1)3^{n-2} \cdot n}$

$n2^n$: 0 2 8 24 64 160

3^n: 1 3 9 27 81 243 ← growing faster so $\lim = 0$ $f = \Omega(g)$

(m) $\lim_{n\to\infty} \frac{2^n}{n!} =$ ~~$\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!}$~~ from calculus II we learned a^n grows slower than $n!$ thus limit $= 0$

$f = \Omega(g)$ Ratio test

(n) $\lim_{n\to\infty} \frac{n^{0.1}}{(\log n)^{10}}$ $\frac{\infty}{\infty}$ $\frac{d}{dx}$: $\frac{\frac{0.1}{n^{0.9}}}{\frac{10\log^9 n}{n}}$ $= \lim_{n\to\infty} \frac{0.1 n^{0.1}}{10\log^9 n} = \frac{\infty}{\infty}$ $\frac{d}{dx}$ again $\frac{\frac{0.1 \cdot 0.1}{n^{0.9}}}{\frac{0.9\log^8 n}{n}} = \frac{0.01 n^{0.1}}{10 \cdot 9 \cdot \log^8 n}$

$f = O(g)$

these steps will continue until the denominator becomes $10!$ while the numerator becomes $(0.1)^{10} n^{0.1}$ at which point

$\lim_{n\to\infty} \frac{(0.1)^{10} n^{0.1}}{10!} = \infty$

(c) $\lim\limits_{n\to\infty} \dfrac{n^{k+1}}{\sum_{i=1}^{n} i^k}$ $\dfrac{\infty}{\infty}$

$\infty^{k+1} > c_1 + c_2 \cdots + \infty^k$

~~Since ∞^{k+1} is multiplied by an additional infinity, $f = O(g)$ at least~~

$1^{k+1}, 2^{k+1}, \ldots \infty^{k+1}$

$\sum\limits_{i=1}^{n} i^k =$ $1^k + 2^k + 3^k \ldots + \infty^k$

$1^k, 1^k + 2^k, 1^k\, 2^k + 3^k$

$\hookrightarrow S_n = \hat{\mathcal{E}}$

The given functions cannot be simplified further then the given states. Both functions grow at a rate of $O(n)$ there fore $\underline{f = \Theta(g)}$

$\hookrightarrow$ running time is linearly linked with n.

2) $(\log n)^{\log n} = O(2^{(\log n)^2})$

~~• For this proof log is assumed to have base~~ e

exponential rules proven so far:

implies

$A^{\log_B C} = (C^{\log_C A})^{\log_B C}$ so $A^{\log_B C} = C^{\log_B A}$

$\lim\limits_{n\to\infty} \dfrac{2^{(\log n)^2}}{(\log n)^{\log n}} = \infty$

$= C^{\log_C A \cdot \log_B C}$

$= C^{\frac{\log A}{\log C} \cdot \frac{\log C}{\log B}}$

$2^{(\log n)^2} = 2^{\log n \cdot 2}$

$= C^{\log_B A}$

For the numerator: $\log = \log_e$

For the denominator:

$2^{(\log_e n)^2} = 2^{\log_e n} \cdot 2^{\log_e n}$

$a^x = e^{\ln(a^x)}$ $a = \log n$ $x = \log n$

$= e^{x \ln(a)}$

$\log n^{\log n} = e^{\log n (\log(\log n))}$

$2^{\log_e n} = (n^{\log_n 2})^{\log_e n}$ $\hookrightarrow = 2$

$(e^{\log n})^{\log(\log n)}$ $= n$

$= n^{\log_n 2 \cdot \log_e n}$

$2^{\log n} * 2^{\log n}$

$= n^{\frac{\log_e 2}{\log_e n} \cdot \frac{\log_e n}{\log_e e}}$

$= n^{\log 2} * n^{\log 2}$

$= n^{\log(\log n))}$

$= n^{\log_e 2 / \log_e e}$ $\log_e e = 1$

$= n^{\log_e 2}$

$\hookrightarrow \lim\limits_{n\to\infty} \dfrac{n^{\log 2 + \log 2}}{n^{(\log(\log n))}} = n^{\log 2 + \log 2 - (\log(\log n))}$

$a = n$

$\hookrightarrow a^x = e^{x \ln(a)}$

$x = \log 2 + \log 2 - (\log(\log n))$

constant

$= \lim\limits_{n\to\infty} e^{(2\log 2 - \log(\log n))(\ln(n))}$

$= \lim\limits_{n\to\infty} e^{(\ln(n)\, 2\log 2 - \ln(n)(\log(\log n)))}$

Restart

2) $\lim_{n\to\infty} \frac{2^{(\log n)^2}}{(\log n)^{\log n}} = \lim_{n\to\infty} 2^{(\log n)^2}(\log n)^{-\log n}$

$a^x = e^{\ln(a^x)} = e^{x\ln(a)}$ $\quad a = 2, \log n$

$x = \cancel{2}(\log n)^2, -\log n$

$\lim_{n\to\infty} e^{\cancel{2}(\log n)^2(\log(2))} \cdot e^{-\log n(\log(\log(n)))}$

$= \lim_{n\to\infty} e^{\cancel{2}(\log n)^2(\log(2)) - \log(n)(\log(\log(n)))}$

$= e^{\lim_{n\to\infty} \cancel{2}(\log n)^2(\log 2) - \log n(\log(\log(n)))}$

$= e^{\lim_{n\to\infty}\left(\left(1 - \frac{\log(\log(n))}{\log(2)(\log(n))}\right)(\log n)^2(\log(2))\right)}$

$= e^{\left(\lim_{n\to\infty} 1 - \frac{1}{\log 2}\cdot \lim_{n\to\infty}\left(\frac{\log(\log(n))}{\log(n)}\right)\right)\left(\lim_{n\to\infty}(\log n)^2\right)\left(\lim_{n\to\infty}\log(2)\right)}$

$\frac{d}{dx}$: $\quad u = \log(n)$ $\quad \frac{d}{dx}$: $\log n \to \frac{1}{n}$

$du = \frac{1}{n}dn$

$\frac{d}{du}\log(u) = \frac{1}{u}\cdot\frac{1}{n} = \frac{1}{n\log n}$ $\qquad \frac{\frac{1}{n\log n}}{\frac{1}{n}} = \frac{1}{\log n}$

$= e^{\left(1 - \frac{1}{\log 2}\cdot \lim_{n\to\infty}\left(\frac{1}{\log n}\right)^{\to 0}\right)\cdot \lim_{n\to\infty}(\log n)^2\cdot\log(2)}$

$= e^{(1)\lim_{n\to\infty}(\log n)^2\cdot\log(2)}$

goes to infinity

therefore

so $\lim_{n\to\infty} \frac{2^{(\log n)^2}}{(\log n)^{\log n}} = \infty$

$(\log n)^{\log n} = O\left(2^{(\log n)^2}\right)$