Question 5:

Use the definition of Θ in order to show the following:

a. $5n^3 + 2n^2 + 3n = \Theta(n^3)$

Proof:

$f(n) = 5n^3 + 2n^2 + 3n$

$g(n) = \Theta(n^3)$

In this proof I'll show that $5n^3 + 2n^2 + 3n = \Theta(n^3)$ by showing that $c_2 * g(n) \le f(n) \le c_1 * g(n)$ for all values of $n \ge n_0$.

First we must establish c_1

$5n^3 + 2n^2 + 3n \le 5n^3 + 2n^2 + 3n$	Starting values of $f(n) \le f(n)$
$5n^3 + 2n^2 + 3n \le 6n^3$	Increase the largest co-efficient by 1, and drop the rest.
$f(n) \le 6 * n^3$	Analyze against the starting proof.
$c_1 = 6$	Final value of c_1

Next we must establish c_2

$5n^3 + 2n^2 + 3n \le 5n^3 + 2n^2 + 3n$	Starting values of $f(n) \le f(n)$
$5n^3 \le 5n^3 + 2n^2 + 3n$	Since all the values to the right of the inequality are positive, and n must be positive, this will also evaluate to true.
$4n^3 \le 5n^3 + 2n^2 + 3n$	Drop the multiple by 1 just for good measure on the left of the inequality.
$4 * n^3 \le f(n)$	Analyze against the starting proof.
$c_2 = 4$	Final value of c_2

Next we solve for n_0 for both of the constants.

Solving n_0 for $f(n) \le c_1 * g(n)$

$5n^3 + 2n^2 + 3n \le 6n^3$	Starting expression
$2n^2 + 3n \le n^3$	Subtract $5n^3$ from both sides.
$2n + 3 \le n^2$	Divide by n
$n^2 - 2n - 3 \ge 0$	Arrange to equal 0.
$(n-3)(n+1) \ge 0$	Factor the expression.
$n - 3 \ge 0 \Rightarrow n \ge 3$ $n + 1 \ge 0 \Rightarrow n \ge -1$	Evaluate both factors against zero.
Therefore, $n \ge 3$ for $f(n) \le c_1 * g(n)$	Since we're only evaluating positive values of n, we can eliminate one of the factors.

Solving n_0 for $f(n) \le c_2 * g(n)$

$4n^3 \le 5n^3 + 2n^2 + 3n$	Starting expression
$0 \le n^3 + 2n^2 + 3n$	Subtract $4n^3$ from both sides.
$0 \le n^2 + 2n + 3$	Divide by n
$n^2 + 2n + 3 \ge 0$	Flip the sides
Therefore, $n \ge 1$ for $c_2 * g(n) \le f(n)$	Since we know that all n must be positive, all possible values of n will result in the inequality being true.

And lastly, we compare the n_0 values and can conclude:

If we take $c_1 = 6$

$c_2 = 4$

$n_0 = 3$

Then, $5n^3 + 2n^2 + 3n = \Theta(n^3)$ in the form of $c_2 * g(n) \le f(n) \le c_1 * g(n)$ for all values of $n \ge n_0$.

b. $\sqrt{7n^2 + 2n - 8} = \Theta(n)$

Proof:

$f(n) = \sqrt{7n^2 + 2n - 8}$
$g(n) = \Theta(n)$

In this proof I'll show that $\sqrt{7n^2 + 2n - 8} = \Theta(n)$ by showing that $c_2 * g(n) \le f(n) \le c_1 * g(n)$ for all values of $n \ge n_0$.

First we must establish c_1

$\sqrt{7n^2 + 2n - 8} \le \sqrt{7n^2 + 2n - 8}$	Starting values of $f(n) \le f(n)$
$\sqrt{7n^2 + 2n - 8} \le \sqrt{7n^2 + 2n^2 + n^2}$	$2n^2 + n^2$ is going to be larger than $2n - 8$ for all positive values of n.
$\sqrt{7n^2 + 2n - 8} \le \sqrt{10n^2}$	Simplify the right hand side.
$\sqrt{7n^2 + 2n - 8} \le \sqrt{10}n$	Calculate square root of the right hand size.
$f(n) \le \sqrt{10} * n$	Analyze against the starting proof.
$c_1 = \sqrt{10}$	Final value of c_1

Next we must establish c_2

$\sqrt{7n^2 + 2n - 8} \le \sqrt{7n^2 + 2n - 8}$	Starting values of $f(n) \le f(n)$
$\sqrt{7n^2} \le \sqrt{7n^2 + 2n - 8}$	Since $2n - 8$ is positive for $n \ge 4$, we can ignore it for the lower bound.
$\sqrt{7}n \le \sqrt{7n^2 + 2n - 8}$	Calculate square root of the left hand side.
$\sqrt{7} * n \le f(n)$	Analyze against the starting proof.
$c_2 = \sqrt{7}$	Final value of c_2

Since we already established $n \ge 4$ while finding c_2 we can use that for the proof.

If we take $c_1 = \sqrt{10}$
$c_2 = \sqrt{7}$
$n_0 = 4$

Then, $\sqrt{7n^2 + 2n - 8} = \Theta(n)$ in the form of $c_2 * g(n) \le f(n) \le c_1 * g(n)$ for all values of $n \ge n_0$.