<u>Question 5:</u>

a) **Solve the following questions from the Discrete Math zyBook:**

1) **Exercise 1.12.2, sections b, e**

Use the rules of inference and the laws of propositional logic to prove that each argument is valid. Number each line of your argument and label each line of your proof "Hypothesis" or with the name of the rule of inference used at that line. If a rule of inference is used, then include the numbers of the previous lines to which the rule is applied.

1.12.2.b)

$$\begin{array}{l} p \rightarrow (q \land r) \\ \neg q \\ \hline \therefore \neg p \end{array}$$

$p \rightarrow (q \land r)$	Hypothesis
$\neg p \lor (q \land r)$	Conditional, 1
$(\neg p \lor q) \land (\neg p \lor r)$	Distributive, 2
$\neg p \lor q$	Simplification, 3
$p \rightarrow q$	Conditional, 4
$\neg q$	Hypothesis
$\neg p$	Modus Tollens, 5, 6

1.12.2.e)

$$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \neg q \\ \hline \therefore r \end{array}$$

$p \lor q$	Hypothesis
$\neg p \lor r$	Hypothesis
$q \lor r$	Resolution 1, 2
$\neg q$	Hypothesis
r	Disjunctive Syllogism 3, 4

2) **Exercise 1.12.3, section c**
Some of the rules of inference can be proven using the other rules of inference and the laws of propositional logic.

1.12.3.c) One of the rules of inference is Disjunctive Syllogism:

$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$$

Prove that Disjunctive Syllogism is valid using the laws of propositional logic and any of the other rules of inference besides Disjunctive Syllogism.

$p \lor q$	Hypothesis
$\neg\neg p \lor q$	Double Negation
$\neg p \rightarrow q$	Conditional Identity
$\neg p$	Hypothesis
q	Modus Ponens 3, 4

3) **Exercise 1.12.5, sections c, d**
Give the form of each argument. Then prove whether the argument is valid or invalid. For valid arguments, use the rules of inference to prove validity.

1.12.5.c)

I will buy a new car and a new house only if I get a job.
I am not going to get a job.
∴ I will not buy a new car.

j: if i get a job
c: buy new car
h: buy new house

$$\begin{array}{l} (c \land h) \rightarrow j \\ \neg j \\ \hline \therefore \neg c \end{array}$$

This argument is not valid. When $j = h = F$ and $c = T$, both of the hypotheses are T but the conclusion is False.

j	c	h	$\neg j$	$\neg c$	$\neg h$	$c \wedge h$	$(c \wedge h) \rightarrow j$
T	T	T	F	F	F	T	T
T	T	F	F	F	T	F	T
T	F	T	F	T	F	F	T
T	F	F	F	T	T	F	T
F	T	T	T	F	F	T	F
F	T	F	T	F	T	F	T
F	F	T	T	T	F	F	T
F	F	F	T	T	T	F	T

1.12.5.d)

> I will buy a new car and a new house only if I get a job.
> I am not going to get a job.
> I will buy a new house.
> ---
> ∴ I will not buy a new car.

j: if i get a job
c: buy new car
h: buy new house

$(c \wedge h) \rightarrow j$
$\neg j$
h
$\therefore \neg c$

This argument is valid, The only situation all three hypotheses are True is when $j = c = F$ and $h = T$. Proof below:

$\neg j$	Hypothesis
$(c \wedge h) \rightarrow j$	Hypothesis
$\neg(c \wedge h)$	Modus Tollens 1, 2
$\neg c \vee \neg h$	De Morgans 3
h	Hypothesis
$\neg c$	Disjunctive Syllogism 4, 5

b) Solve the following questions from the Discrete Math zyBook:

1) Exercise 1.13.3, section b

Show that the given argument is invalid by giving values for predicates P and Q over the domain $\{a, b\}$.

b)

$$\begin{array}{l} \exists x(P(x) \vee Q(x)) \\ \exists x \neg Q(x) \\ \hline \therefore \exists x P(x) \end{array}$$

	$P(x)$	$Q(x)$
a	F	T
b	F	F

In the above scenario, both hypotheses are True, but the conclusion is False as there is not a value of x where $P(x)$ is True.

2) **Exercise 1.13.5, sections d, e**

Prove whether each argument is valid or invalid. First find the form of the argument by defining predicates and expressing the hypotheses and the conclusion using the predicates. If the argument is valid, then use the rules of inference to prove that the form is valid. If the argument is invalid, give values for the predicates you defined for a small domain that demonstrate the argument is invalid.

The domain for each problem is the set of students in a class.

1.13.5.d)

Every student who missed class got a detention.
Penelope is a student in the class.
Penelope did not miss class.

Penelope did not get a detention.

$C(x)$: x missed class
$D(x)$: x got detention

$\forall x(C(x) \rightarrow D(x))$
$Penelope\ is\ a\ student\ in\ the\ class$
$\neg C(Penelope)$

$\therefore \neg D(Penelope)$

This argument is invalid. If the value of $C(x) = F$ and $D(x) = T$ for Penelope, and Penelope is within the domain of students, then all three hypotheses are True. But the conclusion $\neg D(x)$ would be False, proving this argument invalid.

1.13.5.e)

Every student who missed class or got a detention did not get an A.
Penelope is a student in the class.
Penelope got an A.

Penelope did not get a detention.

$C(x)$: x missed class
$D(x)$: x got detention
$A(x)$: x got an A

$\forall x((C(x) \vee D(x)) \rightarrow \neg A(x)$
$Penelope\ is\ a\ student\ in\ the\ class$
$A(Penelope)$

$\therefore \neg D(Penelope)$

This argument is valid for $C(x) = D(x) = F$ and $A(x) = T$. Proof below:

$\forall x((C(x) \vee D(x)) \rightarrow \neg A(x)$	Hypothesis
$Penelope\ is\ a\ student\ in\ the\ class$	Hypothesis
$(C(Penelope) \vee D(Penelope)) \rightarrow \neg A(Penelope)$	Universal Instantiation 1, 2
$A(Penelope)$	Hypothesis
$\neg\neg A(Penelope)$	Double Negation 4
$\neg(C(Penelope) \vee D(Penelope)$	Modus Tollens 3, 5
$\neg C(Penelope) \wedge \neg D(Penelope)$	De Morgan's Law 6
$\neg D(Penelope) \wedge \neg C(Penelope)$	Commutative Law 7
$\neg D(Penelope)$	Simplification 8

<u>Question 6:</u>

a) Solve the following questions from the Discrete Math zyBook:

1) **Exercise 2.4.1, section d**

Each statement below involves odd and even integers. An odd integer is an integer that can be expressed as $2k + 1$, where k is an integer. An even integer is an integer that can be expressed as $2k$, where k is an integer.

Prove each of the following statements using a direct proof.

2.4.1.d) The product of two odd integers is an odd integer.

I. Let x and y be two odd integers, we will show that $x * y$ is an odd integer
II. Since x is odd, there is an integer j such that $x = 2j + 1$. Since y is odd, there is an integer k such that $y = 2k + 1$.
III. Plug $2j + 1$ in for x and $2k + 1$ in for y into the expression $x * y$ and you get $(2j + 1)(2k + 1)$.
IV. The expression can be simplified using algebra to $4jk + 2j + 2k + 1$.
V. And further simplified to $2(2jk + j + k) + 1$.
VI. Since k and j are integers, we know that $(2jk + j + k)$ is also an integer.
VII. $x * y$ is equal to $2m + 1$ where $m = (2jk + j + k)$. Therefore the product of $x * y$ is an odd integer. ■

2) **Exercise 2.4.3, section b**

Proving algebraic statements with direct proofs

2.4.3.b) If x is a real number and $x \leq 3$, then $12 - 7x + x^2 \geq 0$.

I. Let x be a real number that is less than or equal to 3, we will show that $12 - 7x + x^2 \geq 0$.
II. So first we need to re-arrange the numbers in descending powers of x, and we get $x^2 - 7x + 12 \geq 0$.
III. Next we can factor the equation to get the result $(x - 3)(x - 4) \geq 0$.
IV. Since we know that $x \leq 3$, $(x - 3)$ is always less than or equal to zero and $(x - 4)$ is always less than zero.
V. Since the product of two negative real numbers is a positive real number, and the product of any negative real number and zero is zero, we can conclude that when x is a real number and $x \leq 3$ that $12 - 7x + x^2 \geq 0$ is True. ■

Question 7:

a) Solve the following questions from the Discrete Math zyBook:

1) **Exercise 2.5.1, section d**

   Prove each statement by contrapositive.

   2.5.1.d) For every integer n, if $n^2 - 2n + 7$ is even then n is odd.

   I. Let n be an integer. We will assume that n is even and will show that $n^2 - 2n + 7$ is odd.
   II. $n = 2k$ for some integer k.
   III. Therefore $n^2 - 2n + 7 = (2k)^2 - 2(2k) + 7$
   IV. $(2k)^2 - 2(2k) + 7 = 4k^2 - 4k + 7 = 2(2k^2 - 2k + 3) + 1$
   V. Since k is an integer, then $2k^2 - 2k + 3$ is also an integer.
   VI. Therefore since $n^2 - 2n + 7$ is equal to two times an integer plus 1, $n^2 - 2n + 7$ is odd. ■

2) **Exercise 2.5.4, section a, b**

   Prove each statement by contrapositive

   2.5.4.a) For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.

   I. Let x and y be real numbers. We will assume that x > y and show that $x^3 + xy^2 > x^2y + y^3$.
   II. Since we know that $x > y$, we also know that $x - y > 0$.
   III. Additionally we know that $x^2 + y^2 > 0$ since the square of any real number is non-negative, and the sum of non-negative numbers is also non-negative.
   IV. If we multiply the two expressions together we get: $(x^2 + y^2)(x - y) > 0 = x^3 - x^2y + xy^2 - y^3 > 0$
   V. Adding x^2y and y^3 to both sides we can prove that $x^3 + xy^2 > x^2y + y^3$. ■

   2.5.4.b) For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$.

   I. Let x and y be real numbers. We will assume that it is not true that $x > 10$ or $y > 10$ and prove that $x + y \leq 20$.
   II. The assumption that it is not true that $x > 10$ or $y > 10$ is equivalent to $x \leq 10$ and $y \leq 10$, by De Morgan's Law.
   III. By simply adding the inequalities we can prove that $x + y \leq 20$. ▌

3) **Exercise 2.5.5, section c**
Prove each statement using a direct proof or proof by contrapositive. One method may be much easier than the other

2.5.5.c) For every non-zero real number x, if x is irrational, then $\frac{1}{x}$ is also irrational.

I. Let x be a real number and $x \neq 0$. We will assume that $\frac{1}{x}$ is a rational number and prove that x is also a rational number.

II. Since we know that $\frac{1}{x}$ is rational, we know that it can be expressed as the fraction $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

III. If $\frac{1}{x} = \frac{p}{q}$ then $x = \frac{q}{p}$ by taking the reciprocals of both sides.

IV. Since p and q are both integers and $p \neq 0$ ($x \neq 0$ and $\frac{1}{x}$ is defined), x can be expressed as a fraction of integers where the denominator is not zero. Therefore, x is a rational number.■

<u>Question 8:</u>

a) Solve the following questions from the Discrete Math zyBook:

1) Exercise 2.6.6, section c, d
Give a proof for each statement.

2.6.6.c) Theorem: The average of three real numbers is greater than or equal to at least one of the numbers.

I. Let x, y, and z be real numbers. We will assume the average of three real numbers is less than each of them.
II. Let's express this as three separate statements
   A. $\frac{x+y+z}{3} < x$
   B. $\frac{x+y+z}{3} < y$
   C. $\frac{x+y+z}{3} < z$
III. Simplifying the above we get:
   A. $x + y + z < 3x = y + z < 2x$
   B. $x + y + z < 3y = x + z < 2y$
   C. $x + y + z < 3z = x + y < 2z$
IV. Since all of the inequalities are the same we can add them together to get:
$(y + z) + (x + z) + (x + y) < 2x + 2y + 2z$
V. Simplifying the left side we get:
$2x + 2y + 2z < 2x + 2y + 2z$
VI. We can see that this is a contradiction because both sides are identical and it's not possible for them to be identical while one side is less than the other. ■

2.6.6.d) Theorem: There is no smallest integer.

I. Assume there is a smallest integer called x.
II. An integer is a whole number that can either be positive, negative, or zero.
III. Due to the closure property of integers, the difference of any two integers will also be an integer.
IV. Since x is an integer and 1 is an integer, we can assume that $x - 1$ is also an integer.
V. This would create the inequality that $x - 1 < x$.
VI. This contradicts the assumption that x is the smallest integer and therefore proves the original theorem. ■

Question 9:

a) Solve the following questions from the Discrete Math zyBook:

1) **Exercise 2.7.2, section b**

Prove each statement.

2.7.2.b) If integers x and y have the same parity, then $x + y$ is even.

The parity of a number tells whether the number is odd or even. If x and y have the same parity, they are either both even or both odd.

I. **Case 1:** x and y are even.

II. If x and y are even, then x can be expressed as $2k$, for some integer k; and y can be expressed as $2j$, for some integer j.

III. We can plug the values into the expression $x + y = 2k + 2j = 2(k + j)$

IV. Since k and j are integers, we know that $(k + j)$ is also an integer.

V. $x + y$ is equal to $2m$ where $m = k + j$ is an integer. Therefore $x + y$ is even.

VI. **Case 2:** x and y are odd.

VII. If x and y are odd, then x can be expressed as $2k + 1$, for some integer k; and y can be expressed as $2j + 1$, for some integer j.

VIII. We can plug the values into the expression
$x + y = 2k + 1 + 2j + 1 = 2(k + j) + 2$

IX. Since k and j are integers, we know that $(k + j)$ is also an integer.

X. $x + y$ is equal to $2m + 2$ where $m = k + j$ is an integer. Any integer that's a multiple of 2 is even, and adding 2 will also mean it stays even.
Therefore $x + y$ is even. ■