Question 3:

Solve the following questions from the Discrete Math zyBook:

a. Exercise 4.1.3, sections b, c

Which of the following are functions from $\mathbb{R}$ to $\mathbb{R}$? If f is a function, give its range.

4.1.3.b. $f(x) = \frac{1}{x^2-4}$

- This is not a function from R to R. If x = 2, you'll end up dividing by 0, which does not hit the target of R.

4.1.3.c. $f(x) = \sqrt{x^2}$

- This is a function. The square root of any real number squared will equal the original real number. And since any number squared will equal its positive value, the range of f is all non-negative numbers.

b. **Exercise 4.1.5, sections b, d, h, i, l**

Express the range of each function using roster notation.

4.1.5.b. Let $A = \{2, 3, 4, 5\}$

$f: A \rightarrow \mathbb{Z}$, such that $f(x) = x^2$

- Range of $f = \{4, 9, 16, 25\}$

4.1.5.d. $f: \{0, 1\}^5 \rightarrow \mathbb{Z}$. For $x \in \{0, 1\}^5$, $f(x)$ is the number of 1's that occur in x. For example $f(01101) = 3$, because there are three 1's in the string "01101".

- Range of $f = \{0, 1, 2, 3, 4, 5\}$

4.1.5.h. Let $A = \{1, 2, 3\}$.

$f: A \times A \rightarrow \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (y, x)$.

- Range of $f = \{(1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2), (1, 3), (2, 3), (3, 3)\}$

4.1.5.i. Let $A = \{1, 2, 3\}$.

$f: A \times A \rightarrow \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (x, y + 1)$.

- Range of $f = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

4.1.5.l. Let $A = \{1, 2, 3\}$

$f: P(A) \rightarrow P(A)$. For $X \subseteq A$, $f(X) = X - \{1\}$.

- Range of $f = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$

<u>Question 4:</u>

I. Solve the following questions from the Discrete Math zyBook:

a. **Exercise 4.2.2, sections c, g, k**

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

4.2.2.c. $h: \mathbb{Z} \to \mathbb{Z}. h(x) = x^3$

- This is one-to-one, but not onto.
- Since the target is all integers, that includes prime numbers. No prime numbers will be the result of x^3 when the domain is all integers. Since not all of the values in the target are mapped, this is not onto.

4.2.2.g. $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z},\ f(x, y) = (x + 1, 2y)$

- This function is one-to-one, but not onto.
- Since the target is $\mathbb{Z} \times \mathbb{Z}$, then the coordinates $(x, 1)$ would be included in the target. With the domain of only integers, and $f(y) = 2y$, you can never get the result of $(x,\ 1)$. No integer value multiplied by 2 can equal 1.

4.2.2.k. $f: \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+,\ f(x, y) = (2^x + y)$

- This function is neither one-to-one nor onto.
- Since we're only looking at positive integers, the smallest values for $f(x, y)$ and 1. So, $f(1, 1) = (2^1 + 1) = (2 + 1) = 3$. Since no value can go lower, the positive integers 1 and 2, which are both in the target, will not be mapped to, meaning this isn't onto.
- This isn't one-to-one because $f(2, 1)$ and $f(1, 3)$ both equal 5.

b. **Exercise 4.2.4, sections b, c, d, g**

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

4.2.4.b. $f: \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.

- This is neither one to one, or onto.
- It's not one-to-one because $f(000) = 100$ and $f(100) = 100$.
- It's not onto because the four values that start with a 0 in the range never get used.

4.2.4.c. $f: \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example $f(011) = 110$.

- This is both one to one and onto.

4.2.4.d. $f: \{0, 1\}^3 \to \{0, 1\}^4$. The output of f is obtained by taking the input string and adding an extra copy of the first bit of the string. For example, $f(100) = 1001$.

- This is one-to-one, but not onto.
- There are 8 values in the domain and 16 in the target. Since it's one to one, only 8 of the values in the target are going to get used. Two examples would be 1000 and 0001. Since the last bit will always match the first bit, these two will never be used, despite being in the target.

4.2.4.g. Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $B = \{1\}$. $f: P(A) \to P(A)$. For $X \subseteq A$, $f(X) = X - B$.

- This is neither one-to-one or onto.
- This is not one-to-one because $f(\emptyset) = \emptyset$ and $f(1) = \emptyset$.
- This is not onto because any of the sets in the powerset P(A) that have 1 as an element will not have anything mapped to them.

II. Give an example of a function from the set of integers to the set of positive integers that is:

a. one-to-one, but not onto.
   - $f: \mathbb{Z} \to \mathbb{Z}^+, f(x) = \{(x > 0: 2x + 1) \wedge (x <= 0: -2x + 2)\}$

b. onto, but not one-to-one.
   - $f: \mathbb{Z} \to \mathbb{Z}^+, f(x) = |x| + 1$

c. one-to-one and onto.
   - $f: \mathbb{Z} \to \mathbb{Z}^+, f(x) = \{(x > 0: 2x) \wedge (x <= 0: -2x + 1)\}$

d. neither one-to-one nor onto
   - $f: \mathbb{Z} \to \mathbb{Z}^+, f(x) = x^2 + 1$

Question 5:
Solve the following questions from the Discrete Math zyBook:

a. **Exercise 4.3.2, sections c, d, g, i**
For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

4.3.2.c. $f: \mathbb{R} \rightarrow \mathbb{R}. f(x) = 2x + 3$
- **The inverse is well defined**
- $f^{-1}(x) = \frac{x-3}{2}$

4.3.2.d. Let A be defined to be the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$
$f: P(A) \rightarrow \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$
For $X \subseteq A, f(X) = |X|$. Recall that for a finite set A, $P(A)$ denotes the power set of A which is the set of all subsets of A.
- **The inverse is not well-defined.**

4.3.2.g. $f: \{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example, $f(011) = 110$
- **The inverse is well-defined.**
- The output of f^{-1} is obtained by taking the input string and reversing the bits. For example, $f^{-1}(110) = 011$.

4.3.2.i. $f: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}, f(x, y) = (x + 5, y - 2)$
- **The inverse is well-defined.**
- $f^{-1}(y, x) = (y - 5, x + 2)$

b. **Exercise 4.4.8, sections c, d**
The domain and target set of functions f, g, and h are $\mathbb{Z}$. The functions are defined as:
$f(x) = 2x + 3$
$g(x) = 5x + 7$
$h(x) = x^2 + 1$

Give an explicit formula for each function given below.

4.4.8.c. $f \circ h$
- $(f \circ h)(x) = f(h(x)) = 2(x^2 + 1) + 3$
- $(f \circ h)(x) = f(h(x)) = 2x^2 + 5$

4.4.8.d. $h \circ f$

- $(h \circ f)(x) = (h(f(x)) = (2x + 3)^2 + 1$
- $(h \circ f)(x) = (h(f(x)) = (2x + 3)(2x + 3) + 1$
- $(h \circ f)(x) = (h(f(x)) = 4x^2 + 6x + 6x + 9 + 1$
- $(h \circ f)(x) = (h(f(x)) = 4x^2 + 12x + 10$
- $(h \circ f)(x) = (h(f(x)) = 2(2x^2 + 6x + 5)$

c. **Exercise 4.4.2, sections b-d**

Consider three functions f, g, and h, whose domain and target are $\mathbb{Z}$. Let

$f(x) = x^2$ $g(x) = 2^x$ $h(x) = \lceil \frac{x}{5} \rceil$

4.4.2.b. Evaluate $(f \circ h)(52)$

- $h(52) = \lceil \frac{52}{5} \rceil = \lceil 10.4 \rceil = 11$
- $f(11) = 11^2 = 121$
- $(f \circ h)(52) = 121$

4.4.2.c. Evaluate $(g \circ h \circ f)(4)$

- $f(4) = 4^2 = 16$
- $h(16) = \lceil \frac{16}{5} \rceil = \lceil 3.2 \rceil = 4$
- $g(4) = 2^4 = 16$
- $(g \circ h \circ f)(4) = 16$

4.4.2.d. Give the mathematical expression for $h \circ f$

- $h \circ f = \lceil \frac{x^2}{5} \rceil$

d. **Exercise 4.4.6, sections c-e**

Define the following functions f, g, and h:

- $f : \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$ and $f(110) = 110$.
- $g : \{0,1\}^3 \to \{0,1\}^3$. The output of g is obtained by taking the input string and reversing the bits. For example, $g(011) = 110$.
- $h : \{0,1\}^3 \to \{0,1\}^3$. The output of h is obtained by taking the input string x, and replacing the last bit with a copy of the first bit. For example, $h(011) = 010$.

4.4.6.c. What is $(h \circ f)(010)$?

- **111**

4.4.6.d. What is the range of $(h \circ f)$

- **{101, 111}**

4.4.6.e. What is the range of $(g \circ f)$

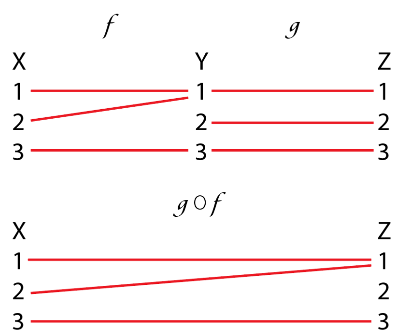
- **{001, 011, 101, 111}**

e. **Extra Credit: Exercise 4.4.4, sections c, d**

4.4.4.c. Is it possible that f is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

- a. No, this is not possible. For $g \circ f$ to be one-to-one, the target of g needs to be at least as large as the domain of f. If the target of g is smaller than the domain of f, it's not possible for it to be one-to-one simply by the lack of targets. If f is not one-to-one it implies that at least two values in the domain of f map to the same value in the domain of g (which is also the target of f). Wherever that value maps to will result in both of the values from the domain of f mapping to in the target of g and would prevent $g \circ f$ being one-to-one. Diagram below:

f: X --> Y
g: Y --> Z



4.4.4.d. Is it possible that g is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

- Yes, this is possible.

f: X --> Y
g: Y --> Z