1 Model

My first try was with the following differential equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad N(0) = N_0$$

Which admits the following solution:

$$N(t) = \frac{K}{1 + \left(\frac{K-P_0}{P_0}\right)e^{-rt}}$$

That has a flex (i.e. the max of its derivative) in:

$$t^* = \frac{1}{r}\ln\left(\frac{K-P_0}{P_0}\right)$$

To use a linear regression, the differential equation can be discretized into:

$$N_{t+1} = \alpha_1 N_t + \alpha_2 N_t^2$$

So the parameter can be found using:

$$r = \alpha_1 - 1, \quad K = \frac{1-\alpha_1}{\alpha_2}, \quad P_0 = 1$$

Recently, I found that:

$$N_{t+1} = \alpha_1 N_t + \alpha_2 N_t^{1+\varepsilon} \tag{1}$$

With $\varepsilon \in (0,1)$ would work much better to reproduce the right-hand skewdness of the contagion curve.

This equation too has a close form solution, being a Bernoulli differential equation:

$$V = N^{-\epsilon}, \qquad \frac{\dot{V}}{V} = -r\epsilon\frac{\dot{N}}{N}, \quad \Rightarrow \quad \dot{V} = -r\epsilon V + \frac{r\epsilon}{K}.$$

Which has close form solution:

$$V(t) = K^{-1} + \left(V_0 - K^{-1}\right)e^{-r\epsilon t} \quad \Rightarrow \quad N(t) = \frac{1}{\left(K^{-1} + \left(N_0^{-\epsilon} - K^{-1}\right)e^{-r\epsilon t}\right)^{\epsilon}}.$$

The maximum of its derivative is determined by solving:

$$\ddot{N} = -\frac{V^{-\frac{1+\epsilon}{\epsilon}}}{\epsilon}\left(\ddot{V} - \frac{1+\epsilon}{\epsilon}\frac{\dot{V}^2}{V}\right)$$

Where:

$$\dot{V} = -r\epsilon\left(V_0 - K^{-1}\right)e^{-r\epsilon t}, \qquad \ddot{V} = (r\epsilon)^2\left(V_0 - K^{-1}\right)e^{-r\epsilon t}.$$

In terms of finite differences:

$$V_{t+1} = \beta_0 + \beta_1 V_t,$$

where:

$$\beta_0 = \frac{r\epsilon}{K}, \quad \beta_1 = r\epsilon.$$

2 Estimation 1

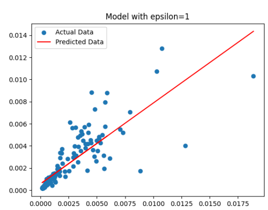
2.1 Preliminary Regression

Let's regress the following model with OLS:

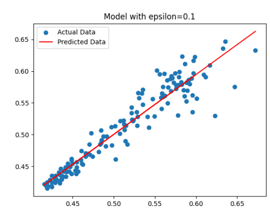
$$V_{t+1} = \beta_0 + \beta_1 V_t \tag{2}$$

where $V_t = N_t^{-\epsilon}$, for different values of ϵ.

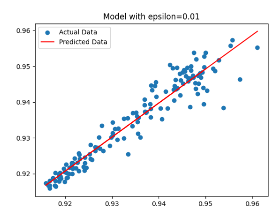
ϵ	β_0	β_1
1	5.61E-04	7.31E-01
0.1	2.93E-02	9.42E-01
0.01	4.54E-02	9.51E-01
0.005	4.65E-02	9.52E-01
0.001	4.74E-02	9.52E-01
0.0005	4.75E-02	9.52E-01
0.0001	4.76E-02	9.52E-01



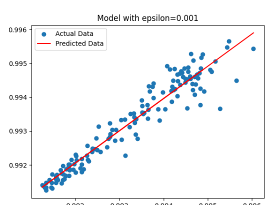
(a) Caption for plot 1



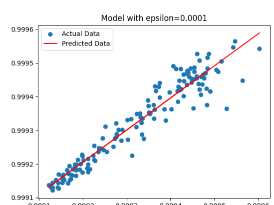
(b) Caption for plot 2



(c) Caption for plot 3



(d) Caption for plot 4



(e) Caption for plot 5

Figure 1: Overall caption for the figure

2.2 GMM Regression

Let's consider the model:

$$N_t = \alpha_1 N_{t-1} + \alpha_2 N_{t-1}^{1+\varepsilon} + U_t \tag{3}$$

For a dataset of size T. We use GMM estimation, and since we have 3 parameters, we need to use 3 moment conditions. We consider different UCMR:

- Assuming U_t is orthogonal to N_{t-1} and N_{t-2} every period.
- Assuming U_t is orthogonal to N_{t-1} and N_{t-1}^{ϵ} every period.

2.3 Case 1

Let's assume that U_t is exogenous to N_{t-1} and N_{t-2} every period:

$$E[U_t|N_{t-2}, N_{t-1}] = 0 \tag{4}$$

Which implies the following unconditional moment restriction:

$$E \begin{bmatrix} 1 \\ N_{t-1} \\ N_{t-2} \end{bmatrix} U_t = 0 \tag{5}$$

In canonical form:

$$E\, g(N_{t-2}, N_{t-1}, N_t, \theta) = 0, \tag{6}$$

Where:

$$g(N_{t-2}, N_{t-1}, N_t, \theta) = \begin{bmatrix} 1 \\ N_{t-1} \\ N_{t-2} \end{bmatrix} \left(N_t - \alpha_1 N_{t-1} - \alpha_2 N_{t-1}^{1+\varepsilon}\right), \qquad \theta = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \varepsilon \end{bmatrix} \tag{7}$$

Bookkeeping:

- The estimator is just identified:

$$\dim \theta = \dim g = 3 \tag{8}$$

- Function g is non-linear in the parameters.
- Its Jacobian is:

$$D(\theta) = \frac{\partial g}{\partial \theta} = \begin{bmatrix} \frac{\partial g}{\partial \alpha_1} & \frac{\partial g}{\partial \alpha_2} & \frac{\partial g}{\partial \varepsilon} \end{bmatrix} = - \begin{bmatrix} N_{t-1} & N_{t-1}^{1+\varepsilon} & \alpha_2 N_{t-1}^{1+\varepsilon} \ln N_{t-1} \\ N_{t-1}^2 & N_{t-1}^{2+\varepsilon} & \alpha_2 N_{t-1}^{2+\varepsilon} \ln N_{t-1} \\ N_{t-1} N_{t-2} & N_{t-1}^{1+\varepsilon} N_{t-2} & \alpha_2 N_{t-1}^{1+\varepsilon} N_{t-2} \ln N_{t-1} \end{bmatrix} \tag{9}$$

The estimator $\hat{\theta}$ is the minimizer:

$$\hat{\theta} = \arg\min_{\theta} \; E g^T(N_{t-2}, N_{t-1}, N_t, \theta)\, S\, E g(N_{t-2}, N_{t-1}, N_t, \theta) \tag{10}$$

for some weighting matrix S. Result:

$$\theta = \begin{bmatrix} 8.00 & -6.95 & 8.69\, e - 04 \end{bmatrix} \tag{11}$$

2.4 Case 2

Let's assume that U_t is exogenous to N_{t-1} every period:

$$E[U_t|N_{t-1}] = 0 \tag{12}$$

Which implies the following unconditional moment restriction:

$$E \begin{bmatrix} 1 \\ N_{t-1} \\ N^{\epsilon}_{t-1} \end{bmatrix} U_t = 0 \tag{13}$$

In canonical form:

$$E\, g(N_{t-1}, N^{\epsilon}_{t-1}, N_t, \theta) = 0, \tag{14}$$

This time we find:

$$\theta = \begin{bmatrix} 8.00 & -6.95 & 8.69\, e-04 \end{bmatrix} \tag{15}$$

2.5 Numerical procedure

2.5.1 Try 1

1. Start by generating a random weighting matrix S_0.
2. Make a 3×3 grid of values for the parameters $(\alpha_1, \alpha_2, \varepsilon)$.
3. Compute the sample equivalent of $m(\theta) = Eg(N_t, N_{t+1}, \theta)$ for all points in the grid.
4. Take $\hat{\theta}^{(1)}$ for which the target function $Eg(N_t, N_{t+1}, \theta)\, S\, Eg^T(N_t, N_{t+1}, \theta)$ is minimum.
5. Estimate the variance as the sample equivalent of $Eg(N_t, N_{t+1}, \theta)\, g^T(N_t, N_{t+1}, \theta)$ for $\theta = \hat{\theta}(1)$:
$$\hat{V}^{(1)} = \frac{1}{T-1} \sum_{t=1}^{T-1} g(N_t, N_{t+1}, \hat{\theta}^{(1)}) g^T(N_t, N_{t+1}, \hat{\theta}^{(1)}) \tag{16}$$
6. Define a new weighting matrix $S^{(1)} = \left(\hat{V}^{(1)}\right)^{-1}$.
7. Iterate until convergence, i.e., until $\left|\hat{\theta}^{(t+1)} - \hat{\theta}^{(t)}\right|_i < \Delta$ for each component $i = 1, 2, 3$.
8. Try with different starting matrices S_0 and different grids $(\alpha_1, \alpha_2, \varepsilon)$.

2.5.2 Try 2

1. Start by generating a random weighting matrix S_0.
2. Numerically solve the FOCs for θ:
$$\hat{D}(\theta)\, S_0\, \hat{m}(\theta) = 0 \tag{17}$$
the resulting $\hat{\theta}^{(1)}$ minimize the target function $Em\theta)\, S\, m(\theta)$.
3. Estimate the variance as the sample equivalent of $Eg(N_t, N_{t+1}, \theta)\, g^T(N_t, N_{t+1}, \theta)$ for $\theta = \hat{\theta}(1)$:
$$\hat{V}^{(1)} = \frac{1}{T-1} \sum_{t=1}^{T-1} g(N_t, N_{t+1}, \hat{\theta}^{(1)}) g^T(N_t, N_{t+1}, \hat{\theta}^{(1)}) \tag{18}$$
4. Define a new weighting matrix $S^{(1)} = \left(\hat{V}^{(1)}\right)^{-1}$.
5. Iterate until convergence, i.e., until $\left|\hat{\theta}^{(t+1)} - \hat{\theta}^{(t)}\right|_i < \Delta$ for each component $i = 1, 2, 3$.
6. Try with different starting matrices S_0 and different grids $(\alpha_1, \alpha_2, \varepsilon)$.

For P-0 consider the stationary distribution as $t \to \infty$ and estimate t* as an average with that distribution