

1 Model

My first try was with the following differential equation

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{K} \right), \quad N(0) = N_0$$

Which admits the following solution:

$$N(t) = \frac{K}{1 + \left(\frac{K-P_0}{P_0} \right) e^{-rt}}$$

That has a flex (i.e. the max of its derivative) in:

$$t^* = \frac{1}{r} \ln \left(\frac{K - P_0}{P_0} \right)$$

To use a linear regression, the differential equation can be discretized into:

$$N_{t+1} = \alpha_1 N_t + \alpha_2 N_t^2$$

So the parameter can be found using:

$$r = \alpha_1 - 1, \quad K = \frac{1 - \alpha_1}{\alpha_2}, \quad P_0 = 1$$

Recently, I found that:

$$N_{t+1} = \alpha_1 N_t + \alpha_2 N_t^{1+\varepsilon} \tag{1}$$

With $\varepsilon \in (0, 1)$ would work much better to reproduce the right-hand skewness of the contagion curve.

This equation too has a close form solution, being a Bernoulli differential equation:

$$V = N^{-\varepsilon}, \quad \frac{\dot{V}}{V} = -r\varepsilon \frac{\dot{N}}{N}, \quad \Rightarrow \quad \dot{V} = -r\varepsilon V + \frac{r\varepsilon}{K}.$$

Which has close form solution:

$$V(t) = K^{-1} + (V_0 - K^{-1}) e^{-r\varepsilon t} \quad \Rightarrow \quad N(t) = \frac{1}{\left(K^{-1} + (N_0^{-\varepsilon} - K^{-1}) e^{-r\varepsilon t} \right)^{\frac{1}{\varepsilon}}}.$$

The maximum of its derivative is determined by solving:

$$\ddot{N} = -\frac{V^{-\frac{1+\varepsilon}{\varepsilon}}}{\varepsilon} \left(\ddot{V} - \frac{1+\varepsilon}{\varepsilon} \frac{\dot{V}^2}{V} \right)$$

Where:

$$\dot{V} = -r\varepsilon (V_0 - K^{-1}) e^{-r\varepsilon t}, \quad \ddot{V} = (r\varepsilon)^2 (V_0 - K^{-1}) e^{-r\varepsilon t}.$$

In terms of finite differences:

$$V_{t+1} = \beta_0 + \beta_1 V_t,$$

where:

$$\beta_0 = \frac{r\varepsilon}{K}, \quad \beta_1 = r\varepsilon.$$

2 Estimation

2.1 Preliminary Regression

Let's regress the following model with OLS:

$$V_{t+1} = \beta_0 + \beta_1 V_t \tag{2}$$

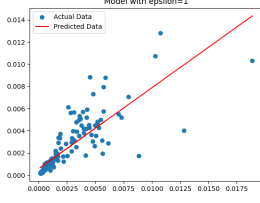
where $V_t = N_t^{-\varepsilon}$, for different values of ε .

Table 1

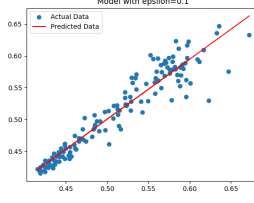
	$\varepsilon =: 1$	$\varepsilon =: 0.1$	$\varepsilon =: 0.01$	$\varepsilon =: 0.001$	$\varepsilon =: 0.0001$
const	0.0006*** (0.0002)	0.0293** (0.0131)	0.0454** (0.0221)	0.0474** (0.0233)	0.0476** (0.0234)
x1	0.7310*** (0.0427)	0.9422*** (0.0253)	0.9514*** (0.0236)	0.9522*** (0.0234)	0.9523*** (0.0234)
R-squared	0.6771	0.9080	0.9206	0.9218	0.9219
R-squared Adj.	0.6748	0.9073	0.9200	0.9212	0.9213
N	142	142	142	142	142

Standard errors in parentheses.

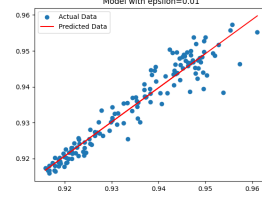
* p_i.1, ** p_i.05, ***p_i.01



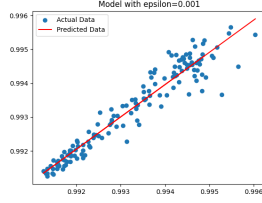
(a) Caption for plot 1



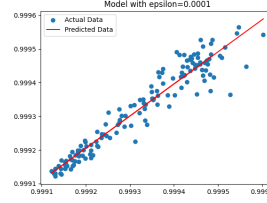
(b) Caption for plot 2



(c) Caption for plot 3



(d) Caption for plot 4



(e) Caption for plot 5

Figure 1: Regression Plot

2.2 GMM Regression

Let's consider the model:

$$N_t = \alpha_1 N_{t-1} + \alpha_2 N_{t-1}^{1+\varepsilon} + U_t \quad (3)$$

For a dataset of size T . We use GMM estimation, and since we have 3 parameters, we need to use 3 moment conditions. We consider different UCMR:

- Assuming U_t is orthogonal to N_{t-1} and N_{t-2} every period.
- Assuming U_t is orthogonal to N_{t-1} and N_{t-1}^ε every period.

2.3 Case 1

Let's assume that U_t is exogenous to N_{t-1} and N_{t-2} every period:

$$E[U_t | N_{t-2}, N_{t-1}] = 0 \quad (4)$$

Which implies the following unconditional moment restriction:

$$E \begin{bmatrix} 1 \\ N_{t-1} \\ N_{t-2} \end{bmatrix} U_t = 0 \quad (5)$$

In canonical form:

$$E g(N_{t-2}, N_{t-1}, N_t, \theta) = 0, \quad (6)$$

Where:

$$g(N_{t-2}, N_{t-1}, N_t, \theta) = \begin{bmatrix} 1 \\ N_{t-1} \\ N_{t-2} \end{bmatrix} (N_t - \alpha_1 N_{t-1} - \alpha_2 N_{t-1}^{1+\varepsilon}), \quad \theta = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \varepsilon \end{bmatrix} \quad (7)$$

Bookkeeping:

- The estimator is just identified:

$$\dim \theta = \dim g = 3 \quad (8)$$

- Function g is non-linear in the parameters.
- Its Jacobian is:

$$D(\theta) = \frac{\partial g}{\partial \theta} = \begin{bmatrix} \frac{\partial g}{\partial \alpha_1} & \frac{\partial g}{\partial \alpha_2} & \frac{\partial g}{\partial \varepsilon} \end{bmatrix} = - \begin{bmatrix} N_{t-1} & N_{t-1}^{1+\varepsilon} & \alpha_2 N_{t-1}^{1+\varepsilon} \ln N_{t-1} \\ N_{t-1}^2 & N_{t-1}^{2+\varepsilon} & \alpha_2 N_{t-1}^{2+\varepsilon} \ln N_{t-1} \\ N_{t-1} N_{t-2} & N_{t-1}^{1+\varepsilon} N_{t-2} & \alpha_2 N_{t-1}^{1+\varepsilon} N_{t-2} \ln N_{t-1} \end{bmatrix} \quad (9)$$

The estimator $\hat{\theta}$ is the minimizer:

$$\hat{\theta} = \arg \min_{\theta} E g^T(N_{t-2}, N_{t-1}, N_t, \theta) S E g(N_{t-2}, N_{t-1}, N_t, \theta) \quad (10)$$

for some weighting matrix S . Result:

$$\theta = [8.00 \quad -6.95 \quad 8.69 e - 04] \quad (11)$$

2.4 Case 2

Let's assume that U_t is exogenous to N_{t-1} every period:

$$E[U_t | N_{t-1}] = 0 \quad (12)$$

Which implies the following unconditional moment restriction:

$$E \begin{bmatrix} 1 \\ N_{t-1} \\ N_{t-1}^\varepsilon \end{bmatrix} U_t = 0 \quad (13)$$

In canonical form:

$$E g(N_{t-1}, N_{t-1}^\varepsilon, N_t, \theta) = 0, \quad (14)$$

This time we find:

$$\theta = [8.00 \quad -6.95 \quad 8.69 e - 04] \quad (15)$$

2.5 Numerical procedure

2.5.1 Try 1

1. Start by generating a random weighting matrix S_0 .
2. Make a 3×3 grid of values for the parameters $(\alpha_1, \alpha_2, \varepsilon)$.
3. Compute the sample equivalent of $m(\theta) = Eg(N_t, N_{t+1}, \theta)$ for all points in the grid.
4. Take $\hat{\theta}^{(1)}$ for which the target function $Eg(N_t, N_{t+1}, \theta) S Eg^T(N_t, N_{t+1}, \theta)$ is minimum.
5. Estimate the variance as the sample equivalent of $Eg(N_t, N_{t+1}, \theta) g^T(N_t, N_{t+1}, \theta)$ for $\theta = \hat{\theta}^{(1)}$:

$$\hat{V}^{(1)} = \frac{1}{T-1} \sum_{t=1}^{T-1} g(N_t, N_{t+1}, \hat{\theta}^{(1)}) g^T(N_t, N_{t+1}, \hat{\theta}^{(1)}) \quad (16)$$

6. Define a new weighting matrix $S^{(1)} = \left(\hat{V}^{(1)}\right)^{-1}$.
7. Iterate until convergence, i.e., until $\left|\hat{\theta}^{(t+1)} - \hat{\theta}^{(t)}\right|_i < \Delta$ for each component $i = 1, 2, 3$.
8. Try with different starting matrices S_0 and different grids $(\alpha_1, \alpha_2, \varepsilon)$.

2.5.2 Try 2

1. Start by generating a random weighting matrix S_0 .
2. Numerically solve the FOCs for θ :

$$\hat{D}(\theta) S_0 \hat{m}(\theta) = 0 \quad (17)$$

the resulting $\hat{\theta}^{(1)}$ minimize the target function $Em\theta) S m(\theta)$.

3. Estimate the variance as the sample equivalent of $Eg(N_t, N_{t+1}, \theta) g^T(N_t, N_{t+1}, \theta)$ for $\theta = \hat{\theta}^{(1)}$:

$$\hat{V}^{(1)} = \frac{1}{T-1} \sum_{t=1}^{T-1} g(N_t, N_{t+1}, \hat{\theta}^{(1)}) g^T(N_t, N_{t+1}, \hat{\theta}^{(1)}) \quad (18)$$

4. Define a new weighting matrix $S^{(1)} = \left(\hat{V}^{(1)}\right)^{-1}$.
5. Iterate until convergence, i.e., until $\left|\hat{\theta}^{(t+1)} - \hat{\theta}^{(t)}\right|_i < \Delta$ for each component $i = 1, 2, 3$.
6. Try with different starting matrices S_0 and different grids $(\alpha_1, \alpha_2, \varepsilon)$.

For P-0 consider the stationary distribution as $t \rightarrow \infty$ and estimate t^* as an average with that distribution