Ex6

March 8, 2025

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
```

1 Convergence to Expectation (v1)

In this version, the generation in rerun for each max size

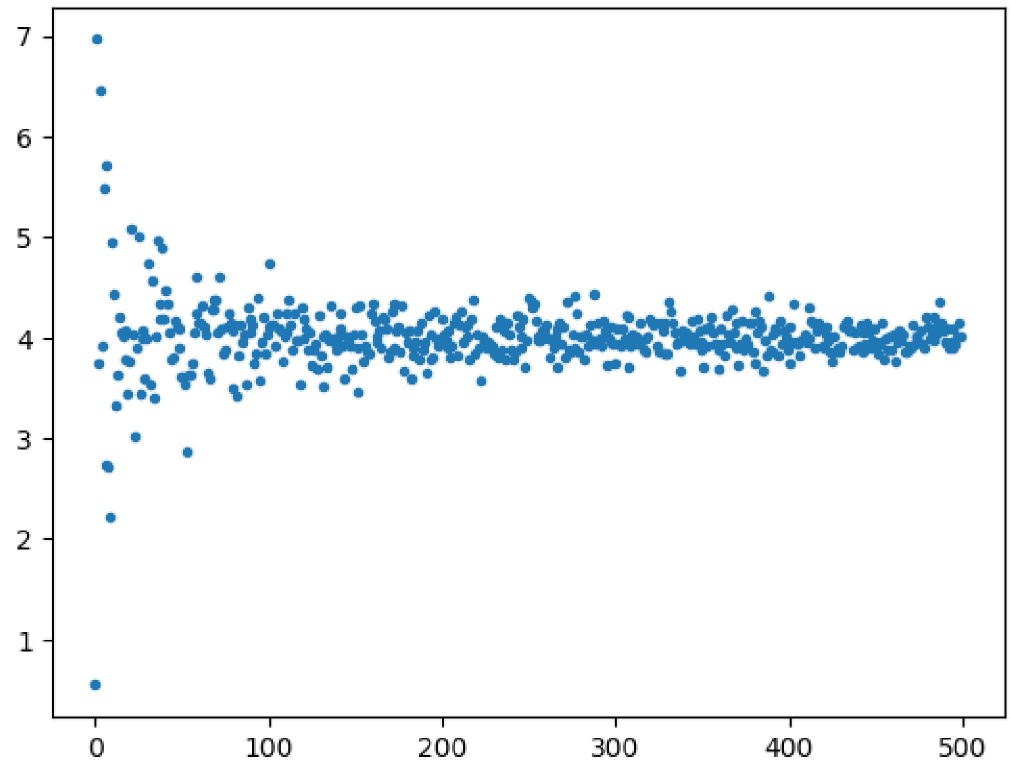
[2]:
```
np.random.seed(100)

max_size = 500

size_array= np.arange(max_size)
avg_array = np.zeros(max_size)

for size in size_array:
    np.random.seed(100+37*size)
    chi_2 = np.random.chisquare(df=4, size=size+1)
    avg_array[size] = chi_2.mean()
```

[3]:
```
plt.scatter(size_array,avg_array, marker='.')
plt.show()
```



2 Convergence to Expectation (v2)

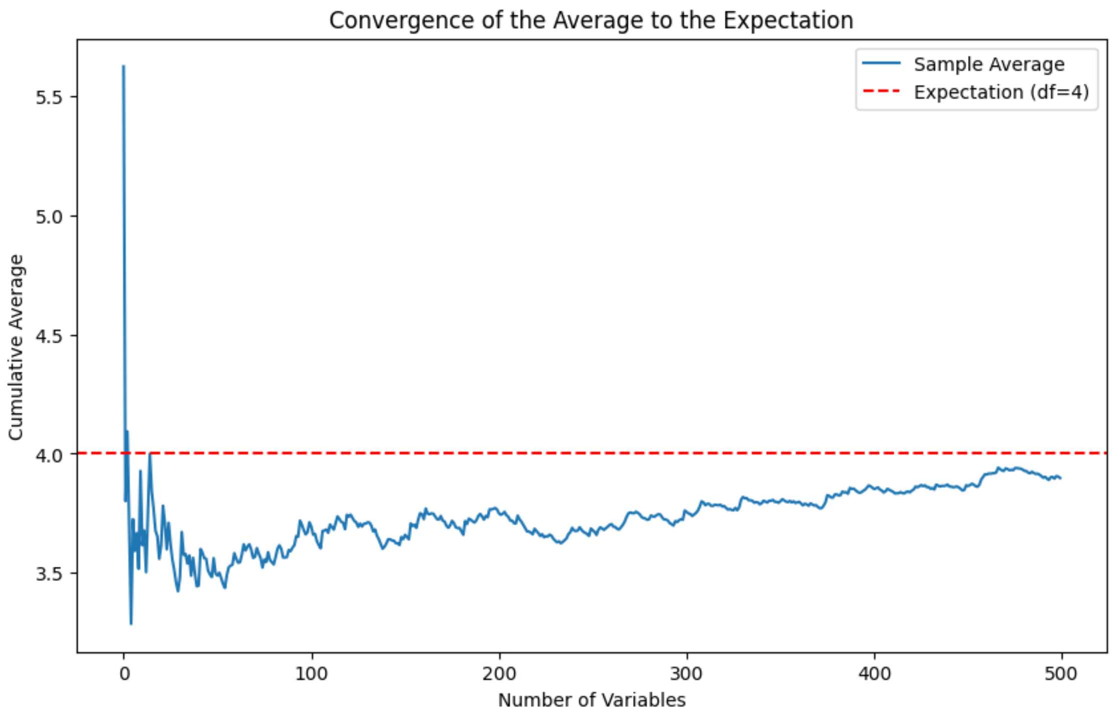
a. In this version, the cumulative average is used (i.e. a single random sequence is used)

[4]:
```
# Set seed for reproducibility
np.random.seed(1212)
size = 500              # sample size
df  = 4                 # chi2 dof

x = np.random.chisquare(df=df, size=size)
s = np.cumsum(x)                    # cumulative sum
r = s / np.arange(1, size + 1)        # elementwise division for running averages
```

[5]:
```
# Plot the simulation
plt.figure(figsize=(10, 6))
plt.plot(r, label="Sample Average")
plt.axhline(df, color="red", linestyle="--", label="Expectation (df=4)")
plt.title("Convergence of the Average to the Expectation")
plt.xlabel("Number of Variables")
plt.ylabel("Cumulative Average")
plt.legend()
plt.show()
```

```
print("Final average:", r[-1])
```




```
Final average: 3.8980893945033963
```

3 Monte Carlo distribution of final averages

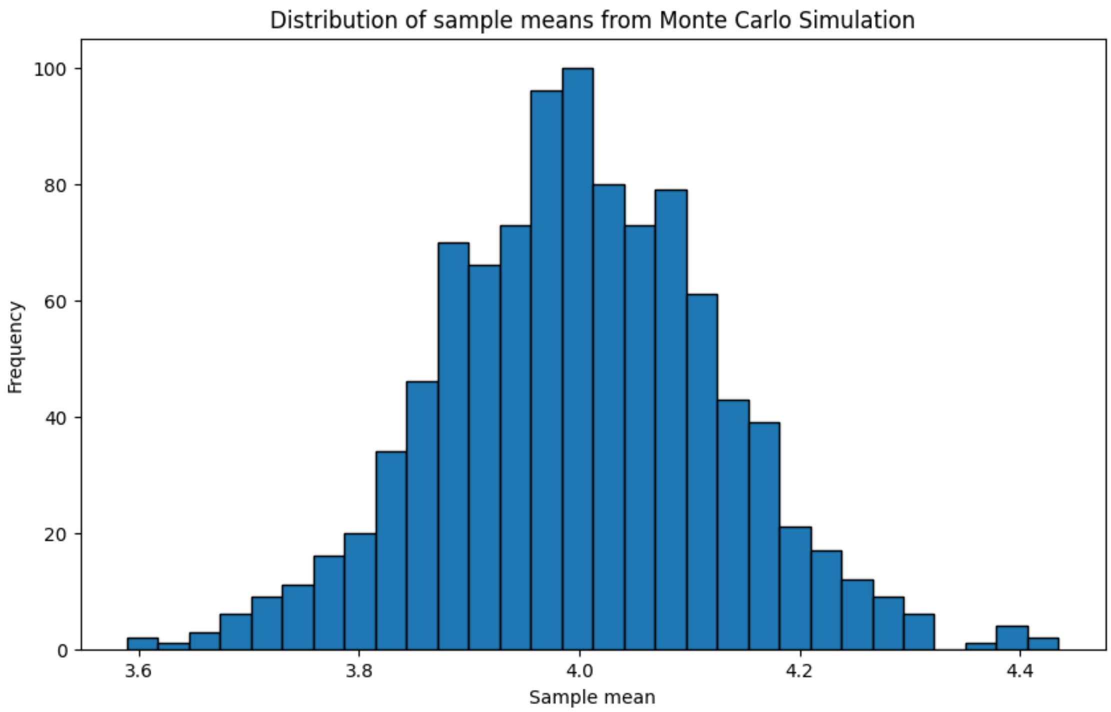
b. Here we use 1000 MC iterations for a sample of 500 random variables

[6]:
```
# Monte Carlo: Distribution of final averages
m = 1000                                # number of simulation iterations
p_last = np.empty(m)                    # initialize vector to store final averages

for i in range(m):
    x = np.random.chisquare(df=df, size=size)
    s = np.cumsum(x)                    # cumulative sum
    r = s / np.arange(1, size + 1)          # elementwise division for running
 ↪averages
    p_last[i] = r[-1]
```

[7]:
```
plt.figure(figsize=(10, 6))
plt.hist(p_last, bins=30, edgecolor="black")
plt.title("Distribution of sample means from Monte Carlo Simulation")
plt.xlabel("Sample mean")
```

```
plt.ylabel("Frequency")
plt.show()
```



c. Here we use 1000 MC iterations for a sample of 10, 100, 1000 and 100000

[8]:
```
def mc_simulation(sample_size, iterations):
    mean_array = np.empty(iterations)            # initialize vector to store␣
 ↪final averages

    for i in range(iterations):
        generated_values = np.random.chisquare(df=df, size=sample_size)
        cumulative_sum = np.cumsum(generated_values)                      #␣
 ↪cumulative sum
        running_mean = cumulative_sum / np.arange(1, sample_size + 1)       #␣
 ↪elementwise division for running averages
        mean_array[i] = running_mean[-1]

    return mean_array
```

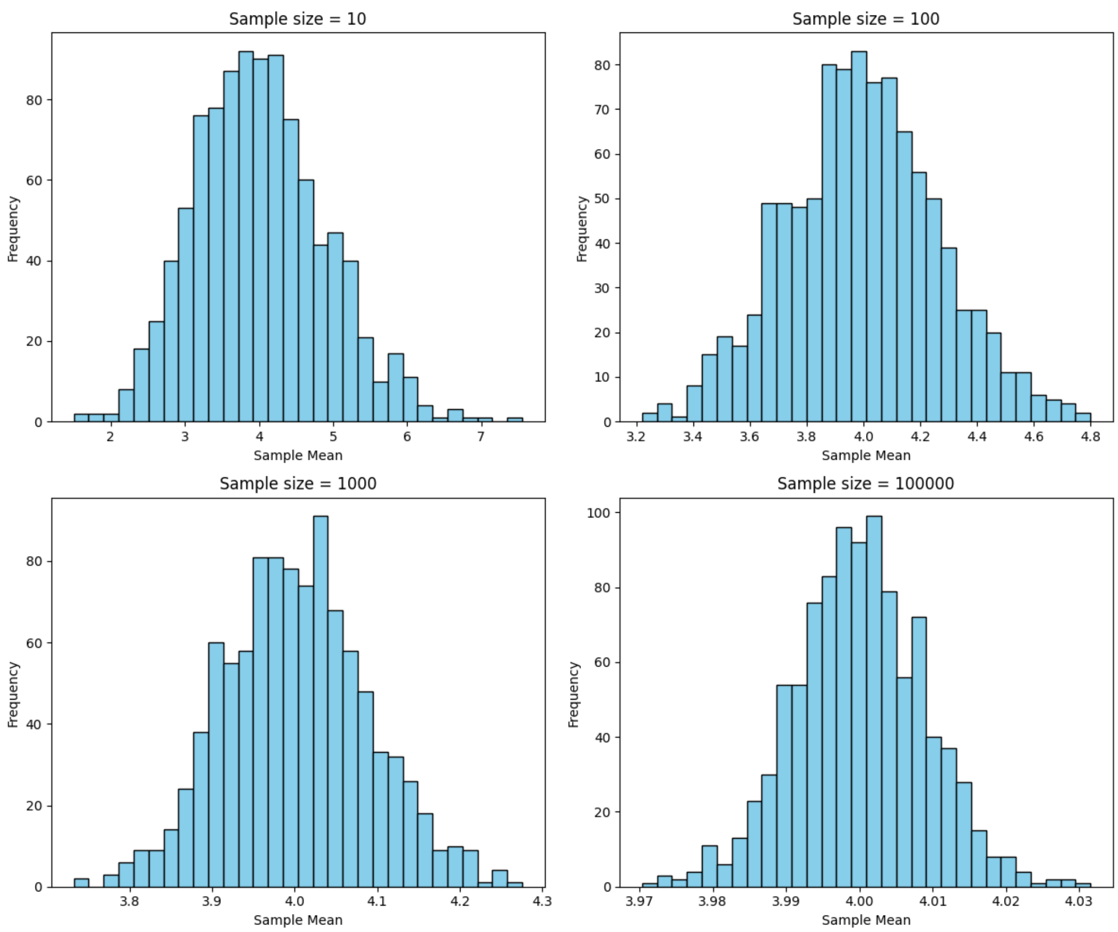
[9]:
```
# Define the simulation parameters.
sample_sizes = [10, 100, 1000, 100000]
iterations = 1000
```

```
# Run the simulations for each sample size.
results = {size: mc_simulation(size, iterations) for size in sample_sizes}

# Create a figure with 4 subplots.
fig, axs = plt.subplots(2, 2, figsize=(12, 10))
axs = axs.flatten()  # Flatten the 2D array of axes for easy iteration.

# Plot a histogram for each sample size.
for i, size in enumerate(sample_sizes):
    axs[i].hist(results[size], bins=30, color='skyblue', edgecolor='black')
    axs[i].set_title(f'Sample size = {size}')
    axs[i].set_xlabel('Sample Mean')
    axs[i].set_ylabel('Frequency')

plt.tight_layout()
plt.show()
```

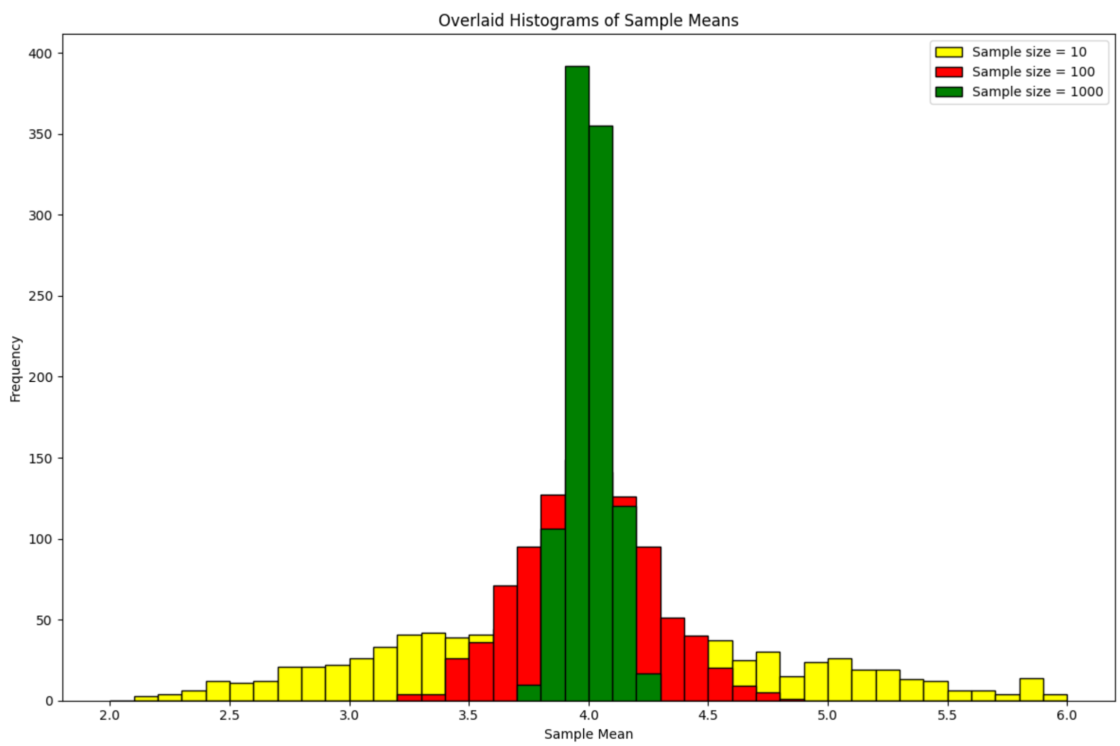


[10]:
```
# Create a single figure overlaying the 4 histograms.
plt.figure(figsize=(12, 8))
colors = ['yellow', 'red', 'green', 'skyblue']

bin_edges = np.arange(2, 6 + 0.1, 0.1)

for size, color in zip(sample_sizes[:-1], colors[:-1]):
    plt.hist(results[size], bins =bin_edges, color=color, alpha=1,
             label=f'Sample size = {size}', edgecolor='black')

plt.xlabel('Sample Mean')
plt.ylabel('Frequency')
plt.title('Overlaid Histograms of Sample Means')
plt.legend()
plt.tight_layout()
plt.show()
```



We can observe that the variance of the distribution of averages decreases as the sample size increases: the histograms get slimmer around the population mean of 4. This suggests that the CLT holds for the Chi-square distribution.

4 Cauchy Distribution

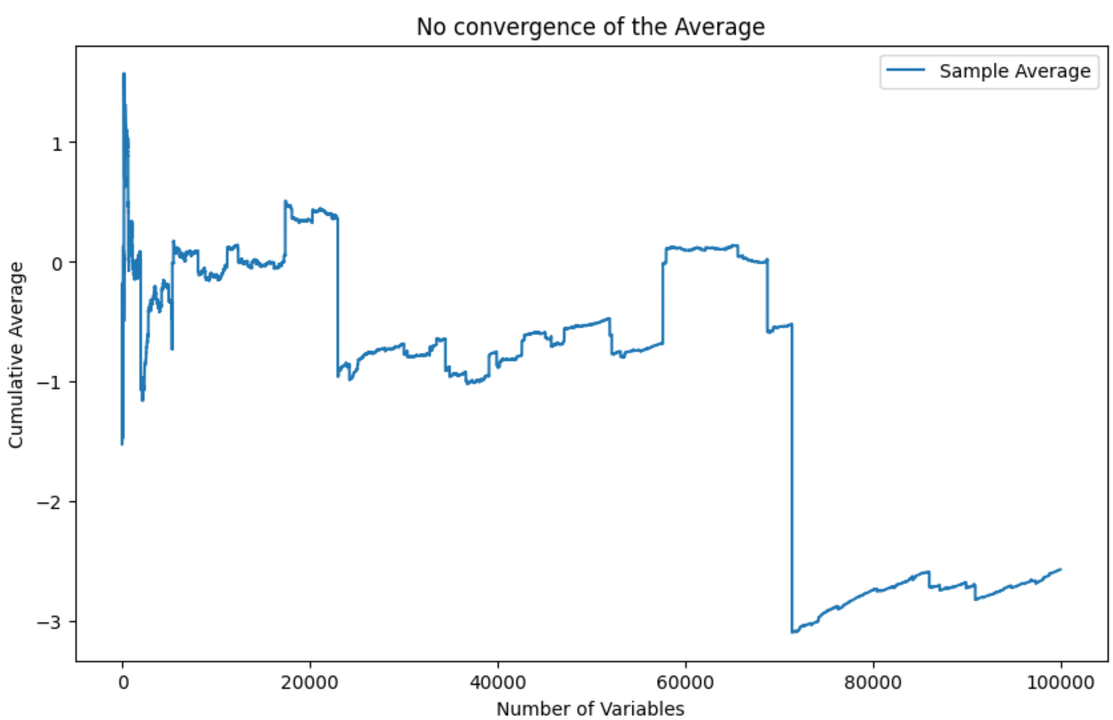
d. Because the population mean of the Cauchy distribution is not defined, the sample mean does not converge and we can observe the characteristic jumps.

[11]:
```
np.random.seed(1212)
size = 100000              # sample size

x = np.random.standard_cauchy(size=size)
s = np.cumsum(x)                # cumulative sum
r = s / np.arange(1, size + 1)
```

[12]:
```
# Plot the simulation
plt.figure(figsize=(10, 6))
plt.plot(r, label="Sample Average")
plt.title("No convergence of the Average")
plt.xlabel("Number of Variables")
plt.ylabel("Cumulative Average")
plt.legend()
plt.show()

print("Final average:", r[-1])
```




```
Final average: -2.5673072399462145
```

[13]:
```
def mc_simulation_cauchy(sample_size, iterations):
    mean_array = np.empty(iterations)            # initialize vector to store␣
 ↪final averages

    for i in range(iterations):
        generated_values = np.random.standard_cauchy(size=sample_size)
        cumulative_sum = np.cumsum(generated_values)                 #␣
 ↪cumulative sum
        running_mean = cumulative_sum / np.arange(1, sample_size + 1)       #␣
 ↪elementwise division for running averages
        mean_array[i] = running_mean[-1]

    return mean_array
```

[ ]:
```
# Define the simulation parameters.
sample_sizes = [10, 100, 1000, 100000]
iterations = 1000

# Run the simulations for each sample size.
results = {size: mc_simulation_cauchy(size, iterations) for size in␣
 ↪sample_sizes}
```

For Cauchy distribution, the CLT does not hold, i.e. more iterations does not lead to lower variance on the distribution of averages, because the population mean is not defined.

[ ]:
```
radius =25
# Create a figure with 4 subplots.
fig, axs = plt.subplots(2, 2, figsize=(12, 10))
axs = axs.flatten()  # Flatten the 2D array of axes for easy iteration.

# Plot a histogram for each sample size.
for i, size in enumerate(sample_sizes):
    axs[i].hist(np.clip(results[size],-radius,radius), bins=30,␣
 ↪color='skyblue', edgecolor='black')
    axs[i].set_title(f'Sample size = {size}')
    axs[i].set_xlabel('Mean')
    axs[i].set_ylabel('Frequency')
    axs[i].set_xlim([-radius,radius])

plt.tight_layout()
plt.show()
```

[ ]:
```
```