1 Introduction

The aim of this work is to estimate the parameters of the SIR model. In particular, I will focus on the first wave of contagion of the Coronavirus pandemic in Italy.

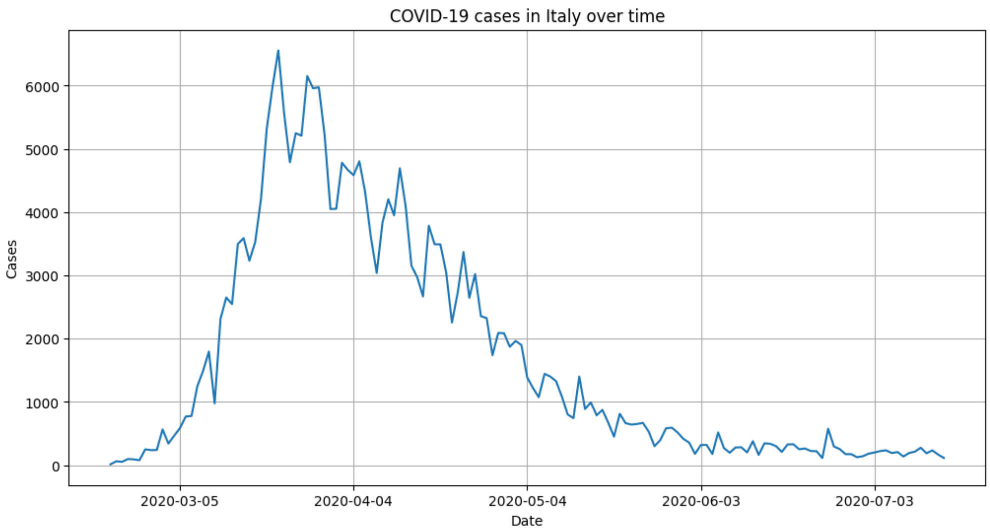


Figure 1: New infections per day in Italy

2 Model

2.1 Standard SIR model

My first try was with the following differential equation for the total infected population:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad N(0) = N_0$$

Which admits the following solution:

$$N(t) = \frac{K}{1 + \left(\frac{K - N_0}{N_0}\right) e^{-rt}}$$

That has a flex (i.e. the max of the new infections curve) in:

$$t^* = \frac{1}{r} \ln\left(\frac{K - N_0}{N_0}\right)$$

To use a linear regression, the differential equation can be discretized into:

$$N_{t+1} = \alpha_1 N_t + \alpha_2 N_t^2$$

So the parameter can be found using:

$$r = \alpha_1 - 1, \quad K = \frac{1 - \alpha_1}{\alpha_2}$$

2.2 Bernoulli equation model

However, the solution of the SIR model yields a symmetric curve, therefore a more appropriate model that can deliver the right-hand skewness of the contagion curve is:

$$N_{t+1} = \alpha_1 N_t + \alpha_2 N_t^{1+\varepsilon}, \tag{1}$$

with $\varepsilon \in (0, 1)$
Using Bernoulli transformation:

$$V = N^{-\varepsilon}, \qquad \frac{\dot{V}}{V} = -r\varepsilon\frac{\dot{N}}{N}, \quad \Rightarrow \quad \dot{V} = -r\varepsilon V + \frac{r\varepsilon}{K}.$$

Which has close form solution:

$$V(t) = K^{-1} + \left(V_0 - K^{-1}\right) e^{-r\varepsilon t} \quad \Rightarrow \quad N(t) = \frac{1}{\left(K^{-1} + \left(N_0^{-\varepsilon} - K^{-1}\right) e^{-r\varepsilon t}\right)^{\varepsilon}}.$$

The peak of the new cases is determined by solving:

$$\ddot{N} = -\frac{V^{-\frac{1+\varepsilon}{\varepsilon}}}{\varepsilon}\left(\ddot{V} - \frac{1+\varepsilon}{\varepsilon}\frac{\dot{V}^2}{V}\right) \quad \rightarrow \quad t^* = \frac{1}{\varepsilon r}\ln\left(\varepsilon K N_0^{-\varepsilon} - \varepsilon\right)$$

Where:

$$\dot{V} = -r\varepsilon\left(V_0 - K^{-1}\right) e^{-r\varepsilon t}, \qquad \ddot{V} = (r\varepsilon)^2\left(V_0 - K^{-1}\right) e^{-r\varepsilon t}.$$

In terms of finite differences:

$$V_{t+1} = \beta_0 + \beta_1 V_t,$$

where:

$$\beta_0 = -\alpha_2\varepsilon = \frac{r\varepsilon}{K}, \quad \beta_1 = \varepsilon(1-\alpha_1) = -r\varepsilon.$$

3 Estimation

3.1 Preliminary Regression

Let's regress the following model with OLS:

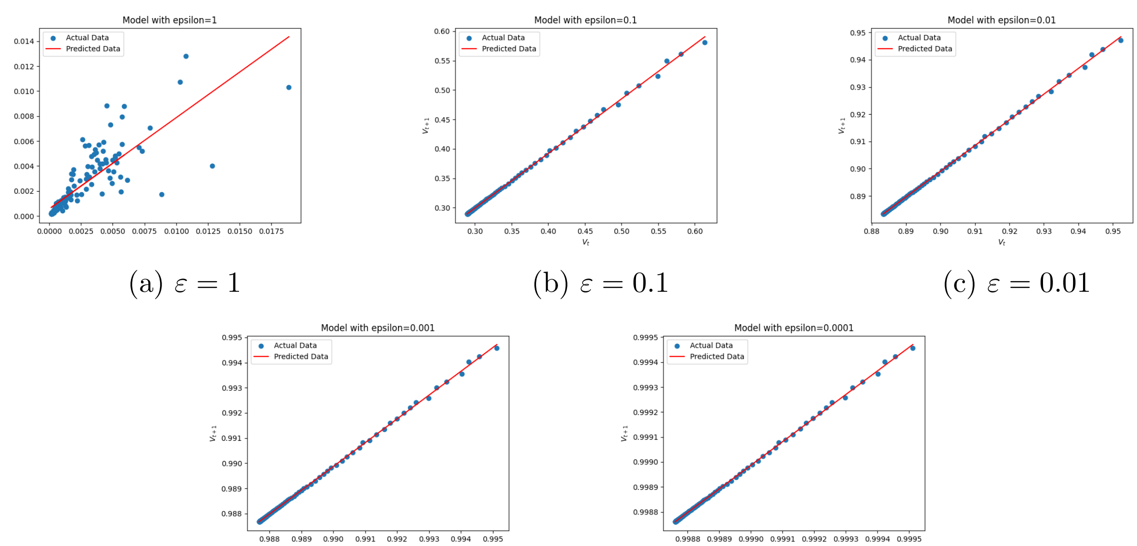
$$V_{t+1} = \beta_0 + \beta_1 V_t + U_t, \tag{2}$$

where $V_t = N_t^{-\varepsilon}$, for given values of ε. I tried with the following values: $\varepsilon \in \{1, 0.1, 0.01, 0.001, 0.0001\}$. The OLS regression yielded the following results:

	ε =: 1.0	ε =: 0.1	ε =: 0.01	ε =: 0.001	ε =: 0.0001
β_0	0.0000*	0.0211***	0.0501***	0.0547***	0.0552***
	(0.0000)	(0.0006)	(0.0013)	(0.0014)	(0.0014)
β_1	0.6348***	0.9278***	0.9433***	0.9446***	0.9447***
	(0.0071)	(0.0019)	(0.0014)	(0.0014)	(0.0014)
R-squared	0.9827	0.9994	0.9997	0.9997	0.9997
R-squared Adj.	0.9826	0.9994	0.9997	0.9997	0.9997
N	142	142	142	142	142

Table 1: Standard errors in parentheses.
$*p < 0.1$, $**p < 0.05$, $***p < 0.01$

Overall, the assumption of a linear relationship between V_t and V_{t-1} seems to hold, as long as ε is small enough.



(a) $\varepsilon = 1$ (b) $\varepsilon = 0.1$ (c) $\varepsilon = 0.01$

(d) $\varepsilon = 0.001$ (e) $\varepsilon = 0.0001$

Figure 2: Regression Plot V_t over V_{t-1}

3.2 Ramsey Test

Assuming $\varepsilon \simeq 0$, we can expand $N_t^{1+\varepsilon}$ as:

$$N_t^{1+\varepsilon} \simeq N_t \left(1 + \varepsilon \ln N_t + \frac{\varepsilon^2 (\ln N_t)^2}{2} + \frac{\varepsilon^3 (\ln N_t)^3}{6}\right). \tag{3}$$

So:

$$N_{t+1} = (\alpha_1 + \alpha_2) N_t + \varepsilon \alpha_2 N_t \ln N_t + \varepsilon^2 \alpha_2 N_t \frac{(\ln N_t)^2}{2} + \varepsilon^3 \alpha_2 N_t \frac{(\ln N_t)^3}{6}. \tag{4}$$

So by regressing on the nonlinear terms:

N_{t-1}	-3.953
	(-1.00)
$\hat{Y} \ln \hat{Y}$	1.606
	(1.62)
$\hat{Y} (\ln \hat{Y})^2$	-0.158
	(-1.90)
$\hat{Y} (\ln \hat{Y})^3$	0.00493*
	(2.12)
Constant	-13649.1
	(-1.39)
Observations	151

Table 2: t statistics in parentheses, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

I can then reasonably assume there is some nonlinearity at play in the relationship between N_{t-1} and N_t.

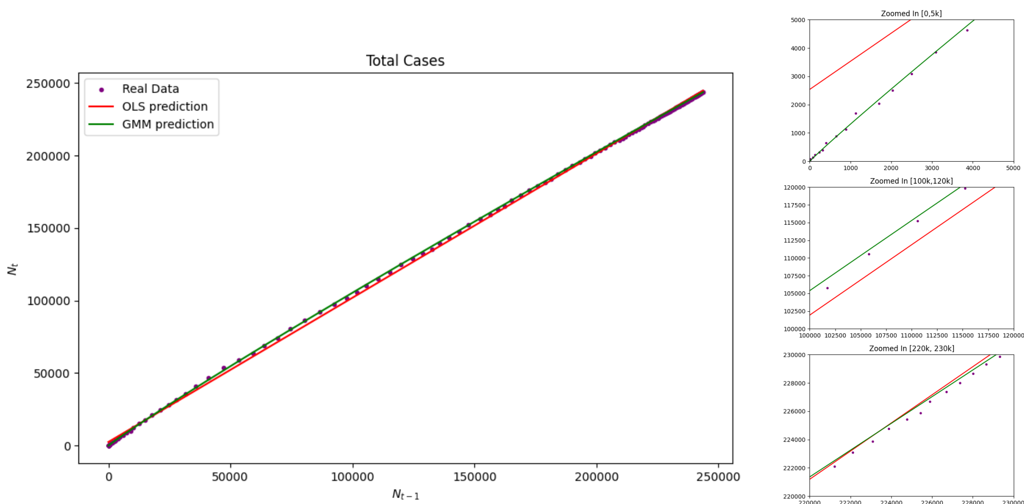


Figure 3: N_t vs N_{t-1} comparison between OLS and G2 model

3.3 GMM estimation with Stata

Let's consider the model:

$$N_t = \alpha_1 N_{t-1} + \alpha_2 N_{t-1}^{1+\varepsilon} + U_t \tag{5}$$

I assume that U_t is mean independent of all the N_s for $s \leq t-1$. For 3 parameters, at least 3 instruments are needed. The following set of instruments were tried:

Model	Instruments
S2	N_{t-1}, N_{t_2}
S3	$N_{t-1}, N_{t-2}, N_{t-3}$
S4	$N_{t-1}, N_{t-2}, N_{t-3}, N_{t-4}$
Sln	$N_{t-1}, N_{t-2}, N_{t-3}, N_{t-1} \ln N_{t-1}$

The results are as follows:

Table 3: GMM results from Stata

Model	S2	S3	S4	Sln
α_1	4.280	7.049	7.457	7.503
	(6.681)	(.)	(.)	(.)
α_2	-2.528	-4.981***	-5.466***	-5.474***
	(6.514)	(0.0550)	(0.0453)	(0.0493)
ε	0.0210	0.0157***	0.0135***	0.0140***
	(0.0436)	(0.000896)	(0.000672)	(0.000730)
J		252496718.1	161236752.4	209017433.8
J_df	0	2	3	3
rank	3	2	2	2

Standard errors in parentheses

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

All the model failed to converge in Stata (tried with different criteria, but without luck).

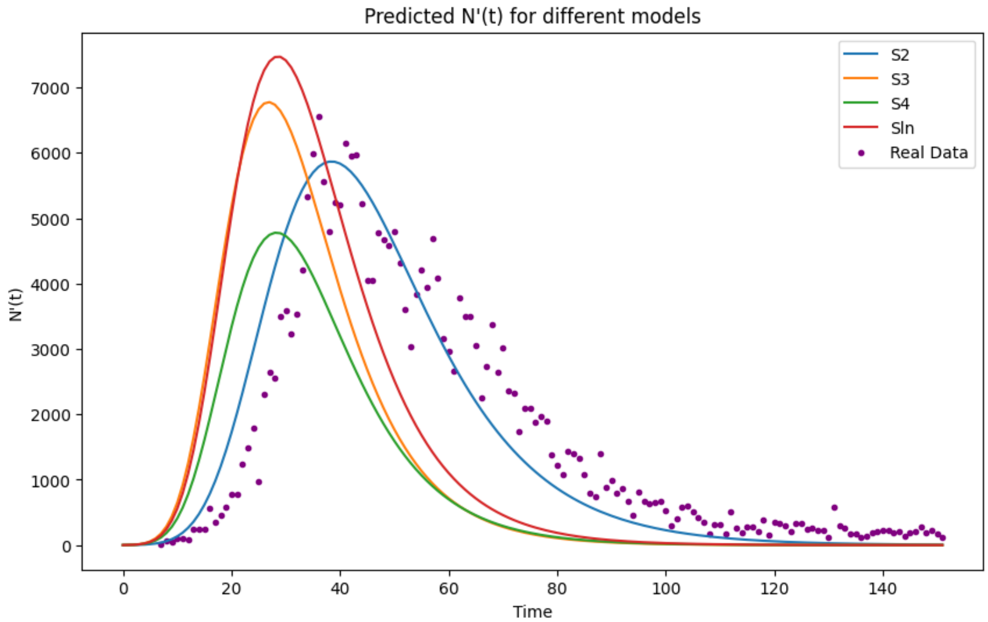


Figure 4: Comparison between real data and the predicted new cases using the parameters from Stata.

3.4 GMM estimation with grid search

I therefore implemented the GMM in python, with the minimization done by grid search on the parameters. The following set of instruments were tried:

Model	Instruments
G2	N_{t-1}, N_{t_2}
G3	$N_{t-1}, N_{t-2}, N_{t-3}$
G4	$N_{t-1}, N_{t-2}, N_{t-3}, N_{t-4}$
G5	$N_{t-1}, N_{t-2}, N_{t-3}, N_{t-4}, N_{t-5}$

The results are as follows:

Table 4: GMM results using grid search

Model	G2	G3	G4	G5
α_1	3.409	3.273	3.545	3.545
	(4.149)	(3.919)	(5.548)	(5.547)
α_2	-1.758	-1.636	-1.924	-1.924
	(3.997)	(3.765)	(5.381)	(5.381)
ε	0.0255	0.0265	0.0226	0.0226
	(0.0446)	(0.0465)	(0.0498)	(0.0498)
J-stat		6.7297	14.5866	14.7678
OverId	0	1	2	3
Samples	151	151	151	151

Standard errors in parentheses

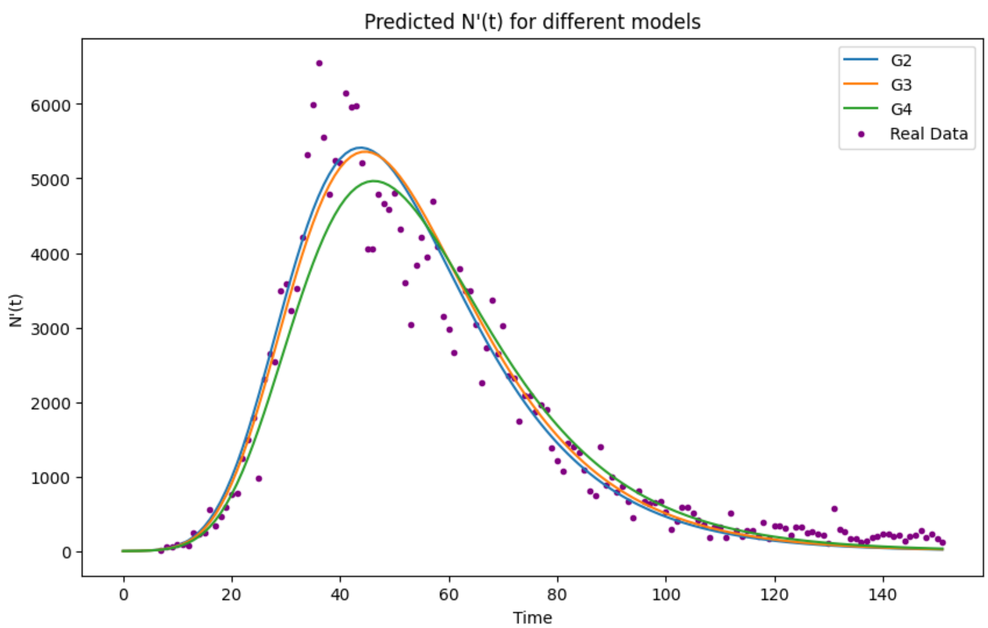


Figure 5: Comparison between real data and the predicted new cases using the parameters found with the grid search minimization.

3.5 Specification Test

I tested the specifications of model G4 and the validity of instruments N_{t-4}, N_{t-5}:

- Hansen test for G4 model specification:

$$\hat{J}_4 = 0.0966 < 3.841 = Q_{\chi^2_2}(0.95) \tag{6}$$

  The test fail to reject hypothesis of valid specifications, suggesting the model is well-defined.

- EHS test for exogeneity of N_{t-4}, N_{t-5}:

$$\hat{J}_5 - \hat{J}_3 = 0.005 < 3.841 = Q_{\chi^2_2}(0.95) \tag{7}$$

Once again, the test does not reject the validity of instruments N_{t-4}, N_{t-5}.