1. Suprema and infima

Definition 1.1. Suppose $\Omega \subset \mathbb{F}$ where $\mathbb{F}$ is an ordered field. Then $b \in \mathbb{F}$ is an upper bound of Ω if $b \geq x$ for all $x \in \Omega$.

Definition 1.2. Suppose $\Omega \subset \mathbb{F}$ where $\mathbb{F}$ is an ordered field. Then $b \in \mathbb{F}$ is a lower bound of Ω if $b \leq x$ for all $x \in \Omega$.

Definition 1.3. Suppose $\Omega \subset \mathbb{F}$ where $\mathbb{F}$ is an ordered field. Then $b \in \mathbb{F}$ is the *least upper bound*, or *supremum*, if $b \leq c$ for every upper bound c.

Definition 1.4. Suppose $\Omega \subset \mathbb{F}$ where $\mathbb{F}$ is an ordered field. Then $b \in \mathbb{F}$ is the *greatest lower bound*, or *infimum*, if $b \geq c$ for every lower bound c.

Proposition 1.1. *The supremum of a set is unique.*

Proposition 1.2. *The infimum of a set is unique.*

Proposition 1.3. *If* $\sup \Omega = \inf \Omega$ *then* Ω *has only one point.*

Proposition 1.4. *Suppose* $\Omega \subset \mathbb{R}$ *and* $\Omega \neq \emptyset$. *Then* $S = \sup \Omega$ *if and only if*

1. *For all* $x \in \Omega, s \geq X$.
2. *For all* $\epsilon > 0$, *there exists* $x \in \Omega$ *such that* $s - \epsilon < x$.

2. Elements of analysis

Definition 2.1. An ordered field $\mathbb{F}$ possess' the *least upper bound property* if and only if every non-empty bounded set in $\mathbb{F}$ has a least upper bound in $\mathbb{F}$.

Proposition 2.1. *Every ordered field with the least upper bound property is isomorphic to* $\mathbb{R}$.

Definition 2.2 (Metric space)**.** A set X is a metric space iff there is a function $d : X \times X \to \mathbb{R}$ such that for any *points* $p, q \in X$

1. $d(p, q) = d(q, p)$,
2. $d(p, q) > 0$ if $p \neq 0$,
3. $d(p, p) = 0$, and
4. $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$.

Lemma 2.2 (Triangle inequality)**.** *If* $a, b \in \mathbb{R}$, *then* $||a| - |b|| \leq |a + b| \leq |a| + |b|$.

Proposition 2.3. *The set* $\mathbb{R}$ *is a metric space.*

Definition 2.3. We say two sets Ω_1 and Ω_2 have the same *cardinality*, that is $|\Omega_1| = |\Omega_2|$, iff there is a bijection between Ω_1 and Ω_2.

Definition 2.4. A set Ω is *countable* if it has the same cardinality as $\mathbb{N}$.

Proposition 2.4. *The set* $\mathbb{Z}$ *is countable.*

Proposition 2.5. *The set* $\mathbb{Q}$ *is countable, but* $\mathbb{R}$ *is not.*

Proposition 2.6 (Axiom of choice)**.** *Suppose we have a family of sets* $S(w)$ *indexed by* $w \in W$. *Then there is a* choice function

$$f : W \to \bigcup_{w \in W} S(w)$$

such that $f(w) \in S(w)$ *for all* $w \in W$.

3. Sequential limits

Definition 3.1. A sequence is a mapping from $\mathbb{N}$ to a set.

Definition 3.2. Let $(a_n)_{n=1}^{\infty}$ be a sequence of reals. We write $\lim_{n\to\infty} a_n = a$, or $a_n \to a$ as $n \to \infty$, if and only if for every $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - a| < \epsilon$.

Definition 3.3. If $(a_n)_{n=1}^{\infty}$ has a limit then it *converges*. Otherwise, it diverges.

Proposition 3.1. *All limits are unique. Suppose the sequence* $a_n \to L$ *and* $a_n \to M$ *as* $n \to \infty$. *Then* $L = M$.

Proposition 3.2 (Squeeze theorem)**.** *Suppose the sequences* $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, *and* $(c_n)_{n=1}^{\infty}$ *satisfy*

1. $a_n \leq b_n \leq c_n$ *for all* $n \in \mathbb{N}$, *and*
2. $a_n \to L$ *and* $b_n \to L$ *as* $n \to \infty$.

Then $b_n \to L$ *as* $n \to \infty$.

Definition 3.4. The sequence $(a_n)_{n=1}^{\infty}$ is *monotone increasing* if and only if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

Definition 3.5. The sequence $(a_n)_{n=1}^{\infty}$ is *strictly monotone increasing* if and only if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.

Definition 3.6. The sequence $(a_n)_{n=1}^{\infty}$ is *monotone decreasing* if and only if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

Definition 3.7. The sequence $(a_n)_{n=1}^{\infty}$ is *strictly monotone increasing* if and only if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$.

Lemma 3.3 (Convergent sequences are bounded)**.** *If* $(a_n)_{n=1}^{\infty}$ *converges then there exists* $M > 0$ *such that* $|a_n| \leq M$ *for all* n.

Proposition 3.4. *Let* $(a_n)_{n=1}^{\infty}$ *be a monotone sequence. This sequence converges iff it is bounded.*

Proposition 3.5. *If the sequences* $a_n \to a$ *and* $b_n \to b$ *as* $n \to \infty$, *then as* $n \to \infty$,

1. $a_n + b_n \to a + b$, $a_n - b_n \to a - b$,
2. $\lambda a_n \to \lambda a$ *for constant* λ,
3. $a_n b_n \to ab$, *and*
4. $1/a_n \to 1/a$ *for* $a \neq 0$.

Lemma 3.6 (Bernoulli inequality)**.**

1. *Assuming* $x \geq -1$ *and* $n \in \mathbb{N} \cup \{0\}$ *but* $x = -1$ *and* $n = 0$ *do not hold at the same time,*
$$(1 + x)^n \geq 1 + nx.$$
2. *We have a strict inequality in the above formula if and only if* $n > 1$ *and* $x \neq 0$.

Proposition 3.7. *Define* $(a_n)_{n=1}^{\infty}$ *and* $(b_n)_{n=1}^{\infty}$ *by*

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

and

$$b_n = \left(1 + \frac{1}{n}\right)^{n+1}.$$

Then a_n *is strictly increasing,* b_n *is strictly decreasing and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Definition 3.8 (Euler's number)**.** Euler's number e is given by

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

which converges, as per the previous proposition.

4. Subsequential limits

Definition 4.1. Suppose $(x_n)_{n=1}^{\infty} \subset \mathbb{R}$. The point $x \in \mathbb{R}$ is a *cluster point* of $(x_n)_{n=1}^{\infty}$ iff for all $\epsilon > 0$

$$\left|\left\{n \in \mathbb{N} \mid |x_n - x| < \epsilon\right\}\right| = |\mathbb{N}|.$$

Proposition 4.1. *Assume* $(x_n)_{n=1}^{\infty} \subset \mathbb{R}$ *and* $x \in \mathbb{R}$. *Then* x *is a cluster point of* $(x_n)_{n=1}^{\infty}$ *iff for all* $\epsilon > 0$ *and* $N \in \mathbb{N}$ *there exists a natural* $n \geq N$ *such that* $|x_n - x| < \epsilon$.

Proposition 4.2. *Assume* $(x_n)_{n=1}^{\infty} \subset \mathbb{R}$ *and* $x \in \mathbb{R}$. *Then* x *is a cluster point iff there exists a subsequence* $(x_{n_k})_{k=1}^{\infty}$ *such that*

$$\lim_{n\to\infty} x_{n_k} = x.$$

Proposition 4.3. *Assume* $(x_n)_{n=1}^{\infty} \subset \mathbb{R}$ *and* $x \in \mathbb{R}$. *Then*

$$\lim_{n\to\infty} x_n = x$$

iff every subsequence of $(x_n)_{n=1}^{\infty}$ *converges to* x.

Proposition 4.4. *Assume* $(x_n)_{n=1}^{\infty} \subset \mathbb{R}$ *and* $x \in \mathbb{R}$. *Then* $(x_n)_{n=1}^{\infty}$ *converges iff* $(x_n)_{n=1}^{\infty}$ *is bounded and has exactly one cluster point.*

Definition 4.2. Suppose $(a_n)_{n=1}^{\infty}$ is a sequence and $n : \mathbb{N} \to \mathbb{N}$. Then $(a_{n_k})_{k=1}^{\infty}$ is a *subsequence* of $(a_n)_{n=1}^{\infty}$.

Definition 4.3 (Limit supremum and infimum)**.** For the sequence $(a_n)_{n=1}^{\infty}$,

$$\limsup_{n\to\infty} a_n = \lim_{N\to\infty} \sup_{n\geq N} a_n$$

and

$$\liminf_{n\to\infty} a_n = \lim_{N\to\infty} \inf_{n\geq N} a_n.$$

Proposition 4.5. *We have*

$$x = \limsup_{n\to\infty} a_n$$

iff x *is the greatest cluster point of the sequence. Similarly*

$$x = \liminf_{n\to\infty} a_n$$

iff x *is the least cluster point of* $(a_n)_{n=1}^{\infty}$.

Proposition 4.6. *If* $(a_n)_{n=1}^{\infty}$ *converges, then*

$$\limsup_{n\to\infty}\ (a_n)_{n=1}^{\infty}$$

and

$$\liminf_{n\to\infty}\ (a_n)_{n=1}^{\infty}$$

exist.

5. The completeness of the reals

Definition 5.1. Suppose $\Omega \neq \emptyset$ then the *diameter* of Ω is

$$\operatorname{diam} \Omega = \sup_{x,y\in\Omega} |x - y|.$$

Definition 5.2. The sequence $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$ is a *Cauchy sequence* iff for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies $|a_m - a_n| < \epsilon$.

Definition 5.3. We say $k \in \mathbb{N}$ is a *peak point* of the sequence $(a_n)_{n=1}^{\infty}$ iff for all $n > k$ we have $a_n < a_k$.

Lemma 5.1. *Any sequence* $(a_n)_{n=1}^{\infty}$ *contains a subsequence which is either non-decreasing or non-increasing.*

Proposition 5.2 (Bolzano-Weierstass Theorem)**.** *Every bounded sequence has a convergent subsequence.*

Definition 5.4. Let X be a metric space. Suppose that given any Cauchy sequence $(a_n)_{n=1}^{\infty} \subset X$, $a_n \to L$ where $L \in X$. Then we say that X is *complete.*

Lemma 5.3. *If* $(a_n)_{n=1}^{\infty}$ *is Cauchy then it converges.*

Lemma 5.4. *If* $(a_n)_{n=1}^{\infty}$ *converges then it is Cauchy.*

Proposition 5.5. *The set* $\mathbb{R}$ *is a complete metric space.*

Lemma 5.6. *For any sequence* $(a_n)_{n=1}^{\infty}$,

$$\operatorname{diam}\ (a_n)_{n=1}^{\infty} = \sup\ (a_n)_{n=1}^{\infty}\ -\ \inf\ (a_n)_{n=1}^{\infty}.$$

Proposition 5.7. *For any bounded sequence* $(a_n)_{n=1}^{\infty}$, $(a_n)_{n=1}^{\infty}$ *is Cauchy iff*

$$\lim_{N\to\infty} \operatorname{diam}\ (a_n)_{N}^{\infty} = 0.$$

6. Infinite sums

Definition 6.1. Consider the sequence

$$S_n = \sum_{k=1}^{n} a_n$$

where $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$. We define the following sum thus:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n.$$

Proposition 6.1. *If* $\sum_{n=1}^{\infty} a_n$ *converges, then*

$$\lim_{n\to\infty} a_n = 0.$$

Proposition 6.2 (Cauchy criterion). *The series $\sum_{n=1}^{\infty} a_n$ converges iff for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $q > p \geq N$ then*

$$\left| \sum_{n=p+1}^{q} a_n \right| < \epsilon.$$

Proposition 6.3 (Convergence laws for series). *Assume $\sum_{n=1}^{\infty} a_n = a$ and $\sum_{n=1}^{\infty} b_n = b$.*

(1) *Then*

$$\sum_{n=1}^{\infty} (a_n + b_n) = a + b$$

(2) *for constant $\lambda \in \mathbb{R}$*

$$\sum_{n=1}^{\infty} (\lambda a_n) = \lambda a$$

However, though, by definition of the product of these sequence,

$$\left(\sum_{n=1}^{\infty} a_n \right) \left(\sum_{n=1}^{\infty} b_n \right) = ab,$$

it could be the case that

$$\sum_{n=1}^{\infty} (a_n b_n) \neq ab.$$

Proposition 6.4 (Comparison theorem). *If $\sum_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$ then $\sum_{n=1}^{\infty} a_n$ converges.*

Lemma 6.5.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Proposition 6.6. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

Proposition 6.7. *If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.*

Proposition 6.8 (Leibniz test). *If the series a_n is a non-negative non-increasing sequence which converges to zero, then*

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges.

Definition 6.2. If $\sum_{n=1}^{\infty} |a_n|$ converges then we say $\sum_{n=1}^{\infty} a_n$ converges *absolutely*. If it converges, but $\sum_{n=1}^{\infty} |a_n|$ does not converge - we say it converges *conditionally*. Absolute convergence implies conditional convergence however the converse is not true.

Proposition 6.9 (Ratio test). *The series $\sum a_n$*

(1) *converges if*

$$\limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

(2) *diverges if*

$$\liminf_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$$

(3) *has uncertain convergence if*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$$

(could go either way.)

Proposition 6.10 (Root test). *Set*

$$a = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

(1) *If $a < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.*
(2) *If $a > 1$,then $\sum_{n=1}^{\infty} a_n$ diverges.*
(3) *If $a = 1$, test is inconclusive.*

7. Functional limits

Definition 7.1. We write

$$\lim_{x\to\infty} f(x) = \infty$$

if and only if for every $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n > M$.

Definition 7.2. We write

$$\lim_{x\to\infty} f(x) = -\infty$$

if and only if for every $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n < M$.

Definition 7.3. We write

$$\lim_{x\to\infty} f(x) = L$$

iff for every $\epsilon > 0$ there exists M such that $x > M$ implies $|f(x) - L| < \epsilon$.

Definition 7.4. We write

$$\lim_{x\to-\infty} f(x) = L$$

iff for every $\epsilon > 0$ there exists M such that $x < M$ implies $|f(x) - L| < \epsilon$.

Definition 7.5. Assume X is a subset of $\mathbb{R}$. The point $a \in \mathbb{R}$ is called a *limit point* of X if every neighbourhood of a contains another point $A \neq a$ such that $A \in X$. That is, for all $\epsilon > 0$ there exists $x \in X$ such that $x \in (a - \epsilon, a + \epsilon) \setminus \{a\}$.

Definition 7.6. The *r-neighbourhood* of a is the set of all $x \in \mathbb{R}$ such that $x \in (a - r, a + r)$.

Definition 7.7. The *deleted r-neighbourhood* of a is the set of all $x \in \mathbb{R}$ such that $x \in (a - r, a + r) \setminus a$.

Definition 7.8. Assume a is a limit point and f is a function from $\Omega \subset \mathbb{R}$ to $\mathbb{R}$. We write

(1)
$$\lim_{x\to a} f(x) = L$$
iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all x, $|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta$.

(2)
$$\lim_{x\to a^+} f(x) = L$$
iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x > a$, $|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta$.

(3)
$$\lim_{x\to a^-} f(x) = L$$
iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x < a$, $|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta$.

Lemma 7.1. *Given a function $f : \Omega \to \mathbb{R}$ and a limit point $a \in \Omega$,*
$$\lim_{x\to a} f(x) = L \tag{1}$$
implies
$$\lim_{n\to\infty} f(a_n) = L.$$
for any sequence $(a_n)_{n=1}^{\infty} \in S$ with
$$S = \left\{ (a_n)_{n=1}^{\infty} \in \Omega^{\mathbb{N}} \;\middle|\; \lim_{n\to\infty} a_n = a, a_n \neq a \right\}.$$
($\Omega^{\mathbb{N}}$ is the set of functions from $\mathbb{N}$ to Ω)

Lemma 7.2. *Given a function $f : \Omega \to \mathbb{R}$ and a limit point $a \in \Omega$,*
$$\lim_{x\to a} f(x) = L \tag{2}$$
if
$$\lim_{n\to\infty} f(a_n) = L. \tag{3}$$
for any sequence $(a_n)_{n=1}^{\infty} \in S$ with
$$S = \left\{ (a_n)_{n=1}^{\infty} \in \Omega^{\mathbb{N}} \;\middle|\; \lim_{n\to\infty} a_n = a, a_n \neq a \right\}.$$
($\Omega^{\mathbb{N}}$ is the set of functions from $\mathbb{N}$ to Ω)

Proposition 7.3 (Sequential criterion for functional convergence)**.** *Given a function $f : \Omega \to \mathbb{R}$ and a limit point $a \in \Omega$,*
$$\lim_{x\to a} f(x) = L \tag{4}$$
iff
$$\lim_{n\to\infty} f(a_n) = L. \tag{5}$$
for any sequence $(a_n)_{n=1}^{\infty} \in S$ with
$$S = \left\{ (a_n)_{n=1}^{\infty} \in \Omega^{\mathbb{N}} \;\middle|\; \lim_{n\to\infty} a_n = a, a_n \neq a \right\}.$$
($\Omega^{\mathbb{N}}$ is the set of functions from $\mathbb{N}$ to Ω)

Proposition 7.4. *Suppose*
$$\lim_{x\to a} f(x) = L$$
and
$$\lim_{x\to a} g(x) = M.$$
Then the following holds

(1)
$$\lim_{x\to a} (f+g)(x) = L + M,$$
(2)
$$\lim_{x\to a} (fg)(x) = LM,$$
(3) *if $m \neq 0$,*
$$\lim_{x\to a} \left(\frac{f}{g}\right)(x) = \frac{L}{M}.$$

Proposition 7.5 (Squeeze theorem)**.** *Define three functions: $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$, and $h : X \to \mathbb{R}$. If*
$$f(x) \leq h(x) \leq g(x)$$
for all $x \in X$ and if
$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$
then
$$\lim_{x\to a} f(x) = \lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

8. Continuity

Definition 8.1 (Continuity)**.** The function f from $\Omega \subset \mathbb{R}$ to $\mathbb{R}$ is *continuous* at $x_0 \in \Omega$ iff
$$\lim_{x\to x_0} f(x) = f(x_0).$$

Definition 8.2 (Continuity on an open interval)**.** The function f from $\Omega \subset \mathbb{R}$ is continuous on $(a, b) \subset \Omega$ iff
$$\lim_{x\to x_0} = f(x_0).$$
for all $x_0 \in (a, b)$.

Definition 8.3 (Continuity on a closed interval)**.** The function f from $\Omega \subset \mathbb{R}$ is continuous on $[a, b] \subset \Omega$ iff it is continuous on (a, b),
$$\lim_{x\to a^-} f(x) = f(a),$$
and
$$\lim_{x\to b^+} f(x) = f(b).$$

Proposition 8.1. *If $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ are continuous at $x_0 \in (a, b)$ then*

(1) *$f + g$ is continuous at x_0*
(2) *fg is continuous at x_0*
(3) *f/g is continuous at x_0 if $g(x_0) \neq 0$.*

Corollary 8.2. *Any polynomial or rational function is continuous where defined.*

Proposition 8.3 (Intermediate Value Theorem)**.** *Suppose f is a function from $\Omega \subset \mathbb{R}$ to $\mathbb{R}$ that is continuous on $[a, b] \subset \Omega$. If $f(a) < \lambda < f(b)$ then there exists $c \in [a, b]$ such that $f(c) = \lambda$.*

Definition 8.4. Suppose f is a function from $\Omega \subset \mathbb{R}$ to $\mathbb{R}$. Then f is *uniformly continuous* on Ω iff given any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \Omega$, $f(x) - f(y) < \epsilon$ if $|x - y| < \delta$.

Proposition 8.4. *Suppose f is continuous on a closed and bounded interval $[a, b]$. Then f is uniformly continuous on $[a, b]$.*

9. Sequences and series of functions

Definition 9.1. A *sequence of functions* is a function $f : \mathbb{N} \times \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}$. We say that the sequence of functions f_n converge pointwise to f on Ω iff for all $x \in \Omega$, $f_n(x) \to f(x)$.

Definition 9.2. We say f_n converges to f *uniformly* on Ω if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \epsilon$ for all $n \geq N$ and $x \in \Omega$.

Proposition 9.1. *If $f_n : [a, b] \to \mathbb{R}$ are continuous functions and f_n uniformly converges to f then f is also continuous.*

Proposition 9.2. *The functions f_n converge uniformly to f iff for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $k, l > N$ for all x, $|f_k(x) - f_l(x)| < \epsilon$.*

Proposition 9.3. *Suppose $(a_n)_{n=1}^{\infty}$ is Cauchy. Then for any uniformly continuous function f, $(f(a_n))_{n=1}^{\infty}$ is Cauchy.*

Definition 9.3. A *series of functions* is sequence $s : \mathbb{N} \times \Omega \to \mathbb{R}$ of the form

$$s_n(x) = \sum_{i=1}^{n} f_n(x)$$

where $f : \mathbb{N} \times \Omega \to \mathbb{R}$ is a function and $\Omega \subset \mathbb{R}$. We say that

$$\sum_{n=1}^{\infty} f_n(x) = f(x)$$

converges pointwise on Ω to the *uniform sum* $f(x)$ iff $s_n \to s$ pointwise. Additionally, we say that the series *converges pointwise* on Ω to the *uniform sum* $f(x)$ iff s_n converges uniformly iff $s_n \to s$ uniformly.

Proposition 9.4 (Weierstrass M-test)**.** *Suppose $(f_n(x))_n^{\infty}$ is defined for all $x \in \Omega$ where $\Omega \subset \mathbb{R}$ such that for all $n \in \mathbb{N}$ $|f_n(x)| < M_n$ where $(M_n)_{n=1}^{\infty}$ is a sequence. Then $\sum_{n=1}^{\infty}$ converges uniformly on Ω if $\sum_{n=1}^{\infty} M_n$ converges.*

10. Differentiation

Definition 10.1. Suppose $x_0 \in [a, b)$ and $f : [a, b) \to \mathbb{R}$. The *right derivative* of f at x_0 is

$$\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}.$$

If these limits exist, then f is *right differentiable* at x_0.

Definition 10.2. Suppose $x_0 \in (a, b)$ and $f : (a, b) \to \mathbb{R}$. The *left derivative* of f at x_0 is

$$\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

If these limits exist, then f is *left differentiable* at x_0.

Definition 10.3. Suppose $x_0 \in (a, b)$ and $f : (a, b) \to \mathbb{R}$. If the left derivative of f at x_0 is equal to the right derivative of f at x_0, then the common value these limits is called the *derivative* of f at x_0 and f is said to be *differentiable* at x_0.

Definition 10.4. Define differentiability on open and closed intervals, analogously to how continuity was defined for open and closed intervals.

Proposition 10.1. *Suppose $x_0 \in (ab)$ and $f : (a, b) \to \mathbb{R}$. If f is differentiable at x_0 then its derivative is given by*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Proposition 10.2. *If f is differentiable at x_0 then f is continuous at x_0.*

Lemma 10.3. *Suppose*

(1) *the intervals $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$ are open,*
(2) *the function $g : I \to J$ is differentiable at $x_0 \in I$, and*
(3) *the function $f : J \to \mathbb{R}$ is differentiable at $g(x) \in J$*

then $f' \circ g(x_0)$ is given by

$$\lim_{h \to 0} \frac{f[g(x_0) + h] - f \circ g(x_0)}{h} = \lim_{h \to 0} \frac{f \circ g(x_0 + h) - f \circ g(x_0)}{g(x_0 + h) - g(x_0)}. \tag{6}$$

Proposition 10.4 (Chain rule)**.** *Suppose*

(1) *the intervals $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$ are open,*
(2) *the function $g : I \to J$ is differentiable at $x_0 \in I$, and*
(3) *the function $f : J \to \mathbb{R}$ is differentiable at $g(x) \in J$*

Then the function $f \circ g : I \to \mathbb{R}$ is differentiable at x_0 and

$$(f \circ g)'(x_0) = f' \circ g(x_0) \cdot g'(x_0). \tag{7}$$

Proposition 10.5. *Take $f : I \to \mathbb{R}$, and $g : I \to \mathbb{R}$ where $I \subset R$ is open and assume they are differentiable at $x \in I$. Then $f + g$, fg are differentiable. Similarly f/g is differentiable if $g \neq 0$ for all values of g. Further*

(1) *$(f + g)' = f' + g'$ and*
(2) *$fg = fg' + gf'$.*

Definition 10.5. The function f from $E \subset \mathbb{R}$ to $\mathbb{R}$ has a *local maximum* at $c \in E$ iff there exists an open interval $U \subset R$ such that $c \in U$ and $f(c) \geq f(x)$ for all $x \in U \cap E$.

Definition 10.6. The function f from $E \subset \mathbb{R}$ to $\mathbb{R}$ has a *local minimum* at $c \in E$ iff there exists an open interval $U \subset R$ such that $c \in U$ and $f(c) \leq f(x)$ for all $x \in U \cap E$.

Definition 10.7. Suppose that the function f from $E \subset \mathbb{R}$ to $\mathbb{R}$ and there exists $c \in E$ such that $f(c) \geq f(x)$ for all $x \in E$. Then f is said to have an absolute maximum at c.

Definition 10.8. Suppose that the function f from $E \subset \mathbb{R}$ to $\mathbb{R}$ and there exists $c \in E$ such that $f(c) \leq f(x)$ for all $x \in E$. Then f is said to have an absolute minimum at c.

Proposition 10.6. *Suppose $f : [a, b] \to \mathbb{R}$ is a function and $c \in (a, b)$. If f has a local extremum at c and $f'(c)$ exists then $f'(c) = 0$.*

Proposition 10.7 (Rolle's theorem)**.** *Suppose f is a function from $\Omega \subset \mathbb{R}$ to $\mathbb{R}$ that is continuous on $[a, b] \subset \Omega$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists $\xi \in (a, b)$ such that $f'(\xi) = 0$.*

Proposition 10.8 (Mean value theorem)**.** *Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proposition 10.9. *Assume $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$ then $f = a$ for $a \in [a, b]$.*

Proposition 10.10. *Assume $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Also assume $|f'(x)| \leq M$ for all $x \in (a, b)$. Then $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in (a, b)$.*

Proposition 10.11. *Assume f is continuous on $[a, b]$ and differentiable on (a, b).*

1. *if $f' \geq 0$ on (a, b) then f is non-decreasing.*
2. *If $f' \leq 0$ on (a, b) then f is non-increasing.*
3. *if $f' > 0$ then f is strictly increasing*
4. *$f' < 0$ then f is strictly decreasing.*

Proposition 10.12. *Assume f is continuous on $[a, b]$ and differentiable on (a, b). Assume f is twice differentiable on (a, b). Let $x_0 \in (a, b)$. Then*

1. *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a local minimum.*
2. *If $f'(x_0) = 0$ and $f'(x_0) < 0$, then x_0 is a local maximum.*

11. Riemann integration

Definition 11.1 (Partition)**.** A *partition* P of $[a, b]$ is a set of points $\{x_0, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < \ldots < x_n = b.$$

Assume $f : [a, b] \to \mathbb{R}$ is bounded, but not necessarily continuous. Then *lower sum* of P is

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

where

$$m_i = \inf f(x)$$

and $x \in [x_i, x_{i-1}]$. Similarly, the *upper sum* of P is

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

where

$$M_i = \sup f(x)$$

and $x \in [x_i, x_{i-1}]$.

Definition 11.2. The partition P' is a *refinement* of P if $P' \supset P$.

Lemma 11.1. *If $P' \supset P$, then*

$$L(f, P) \leq L(f, P')$$

and

$$U(f, P) \geq U(f, P').$$

Definition 11.3. The number

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup\big\{L(f, P) \mid P \text{ is a partition of } [a, b]\big\}$$

is called the *lower integral* of f over $[a, b]$. Similarly, the number

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = \inf\big\{U(f, P) | P \text{ is a partition of } [a, b]\big\}$$

is called the *upper integral* of f over $[a, b]$.

Definition 11.4. If

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x$$

then f is called *integrable* and

$$\int_a^b f(x)\,\mathrm{d}x = \underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x.$$

Proposition 11.2. *The function $f : [a, b] \to \mathbb{R}$ is integrable iff for every $\epsilon > 0$ there is some P such that*

$$U(f, P) - L(f, P) < \epsilon.$$

Proposition 11.3. *If $f[a, b] \to \mathbb{R}$ is bounded and continuous at all but finitely many points, then f is integrable on $[a, b]$.*

Proposition 11.4. *If f is increasing or decreasing, then it's integrable.*

Proposition 11.5.

1. *If f is integrable on $[a, b]$ and $k \in \mathbb{R}$, then kf is integrable and*
$$\int_a^b kf\,\mathrm{d}x = k\int_a^b f\,\mathrm{d}x.$$
2. *If f, g are integrable on $[a, b]$ then so is $f + g$ and*
$$\int_a^b (f + g)\,\mathrm{d}x = \int_a^b f\,\mathrm{d}x + \int_a^b g\,\mathrm{d}x.$$

(3) *If f, g are integrable and $f \leq g$ on $[a,b]$ then*

$$\int_a^b f \,\mathrm{d}x \leq \int_a^b g \,\mathrm{d}x.$$

(4) *If f is integrable on $[a,b]$ and $[b,c]$ then it is integrable on $[a,c]$ and*

$$\int_a^c f \,\mathrm{d}x = \int_a^b f \,\mathrm{d}x + \int_b^c$$

(5) *If f is integrable on $[a,b]$ then so is $|f|$ and*

$$\left| \int_a^c f \,\mathrm{d}x \leq \int_a^b |f| \,\mathrm{d}x \right|$$

Proposition 11.6 (Mean value theorem for integrals)**.** *If f is continuous on $[a,b]$ then there exists $c \in [a,b]$ such that*

$$\int_a^b f(x) \,\mathrm{d}x = f(c)(b-a).$$

Definition 11.5. The function F is an antiderivative of f iff $F' = f$.

Proposition 11.7 (Fundamental theorem of calculus)**.** *Assume $f : [a,b] \to \mathbb{R}$ is continuous and let*

$$F(x) = \int_a^x f(x) \,\mathrm{d}x.$$

Then F is an antiderivative of f and

$$\int_a^b f(x) \,\mathrm{d}x = F(b) - F(a).$$

Proposition 11.8 (Integral test)**.** *Assume f is continuous, non-negative and non-increasing on $[1, \infty)$. Then $\int_1^\infty f(x) \,\mathrm{d}x$ converges iff $\sum_{n=1}^\infty f(n)$ converges.*

Proposition 11.9 (Integration by parts)**.** *If $u, v : [a,b] \to \mathbb{R}$ are differentiable on (a,b) and u', v' are continuous on (a,b) then*

$$\int_a^b uv' \,\mathrm{d}x = uv\bigg|_a^b - \int_a^b u'v \,\mathrm{d}x$$

Definition 11.6 (Improper integral)**.** Consider a function $f : (a,b] \to \mathbb{R}$ which is not necssarily bounded. Assume f is integrable on $[a+\epsilon, b]$ for all $\epsilon \in (0, b-a)$. Then

$$\lim_{\epsilon \to 0^+} \int_{a+\epsilon}^b f(x) \,\mathrm{d}x$$

is called the *improper integral* of the first kind of f on $[a,b]$.

Definition 11.7. Assume $f : [a, \infty) \to \mathbb{R}$ is integrable on $[a,b]$ for all $b > a$. Then

$$\int_a^\infty f(x) \,\mathrm{d}x = \lim_{b \to \infty} \int_a^b f(x) \,\mathrm{d}x$$

is called the *improper integral* of the second kind of f on $[a, \infty)$. We also define

$$\int_{-\infty}^\infty f(x) \,\mathrm{d}x = \int_{-\infty}^0 f(x) \,\mathrm{d}x + \int_0^\infty f(x) \,\mathrm{d}x,$$

provided both integrals on the right exist.

12. The trigonometric, exponential, and logarithmic functions

Definition 12.1. The well known constant π is defined thus:

$$\pi = 2 \int_{-1}^1 \sqrt{1-x^2}$$

Definition 12.2. Let

$$A(x) = (A_1 + A_2)(x)$$

where

$$A_1 = \frac{x\sqrt{1-x^2}}{2}$$

and

$$A_2 = \int_x^1 \sqrt{1-s^2} \,\mathrm{d}s.$$

Definition 12.3. If $x \in [0, \pi]$, then $\cos(x)$ is defined as the unique number such that

$$A(\cos(x)) = \frac{x}{2}.$$

Definition 12.4. Given $x \in [0, \pi]$ define

$$\sin(x) = \sqrt{1 - \cos^2 x}.$$

Proposition 12.1. *If $x \in (0, \pi)$ then*

(1) $\cos'(x) = -\sin x$
(2) $\sin'(x) = \cos x$*.*

Definition 12.5. We extend the definition of sin and cos to $[0, 2\pi]$ by setting

$$\sin x = -\sin(2\pi - x)$$

if $x \in [\pi, 2\pi]$, and

$$\cos x = \cos(2\pi - x).$$

It is also trivial to extend cos and sin periodically to $\mathbb{R}$.

Definition 12.6. Define the logarithmic function so

$$\log x = \int_1^x \frac{1}{t} \,\mathrm{d}t.$$

Proposition 12.2. *If $x, y > 0$, then*

$$\log(xy) = \log(x) + \log(y).$$

Definition 12.7. The exponenetial function exp is defined as the inverse function of log.

Proposition 12.3. *We have that $\exp'(x) = \exp(x)$ for all $x \in \mathbb{R}$.*

Proposition 12.4. *We have that*

$$\exp(x+y) = \exp(x) \cdot \exp(y).$$

Definition 12.8. Define

$$a^x = \exp(x \log(a))$$

for $a > 0$.

13. Taylor series

Definition 13.1. Suppose Suppose f is a function from $\Omega \subset \mathbb{R}$ to $\mathbb{R}$. Then f is *n-times differentiable* on $(a,b) \subset \Omega$ iff $f^{(n+1)}(t)$ exists for any $t \in (a,b)$.

Lemma 13.1 (Rolle's theorem for $(n+1)$-times differentiable functions)**.** *Suppose f is a function from $\Omega \subset \mathbb{R}$ to $\mathbb{R}$ that is continuous on $[a,b] \subset \Omega$ and $(n+1)$-times differentiable on (a,b). Assume, that*

$$f(a) = f'(a) = f''(a) = \cdots = f^{(n)}(a) = 0 = f(b).$$

Then for some $\xi \in (a,b)$, $f^{(n+1)}(\xi) = 0$.

Proposition 13.2 (The mean value theorem for $(n+1)$-times differentiable functions, otherwise known by the unenlightened as Taylor's theorem)**.** *Suppose f is a function from $\Omega \subset \mathbb{R}$ to $\mathbb{R}$ that is continuous on $[\alpha,\beta] \subset \Omega$ and $(n+1)$-times differentiable on (α,β). Then for all distinct $a,b \in [\alpha,\beta]$ and $n \in \mathbb{N}$ there exists $\xi \in (\min\{a,b\}, \max\{a,b\})$ such that*

$$f(b) = (P_n + R_n)(b)$$

where

$$P_n(b) = \sum_{k=0}^{n} \frac{f^{(n)}(a)}{k!}(b-a)^k$$

is defined to be the nth Taylor polynomial *of f centred at a and*

$$R_n(b) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}$$

is defined to be the Lagrange form *of the nth* Taylor remainder.

Proposition 13.3. *If the Taylor remainder of f, $R_n(x) \to 0$ as $n \to \infty$ for some x, then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Definition 13.2. If the Taylor remainder of f, $R_n(x) \to 0$ as $n \to \infty$ for some x, then,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the *Taylor series* of f at a. If $a = 0$, it is also called a *Maclaurin series.*

Definition 13.3. We say that the *radius of convergence* of a power series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

is r iff f converges for all $x \in (a-r, a+r)$ and diverges for all $x \in \mathbb{R}\backslash[a-r, a+r]$.

Proposition 13.4. *If $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ then the series is the Taylor series of f.*

14. Elements of multivariable analysis

Definition 14.1. Suppose $x, y \in \mathbb{R}^m$. Then

$$|x| = \sqrt{\sum_{i=1}^{m} x_i^2}$$

Proposition 14.1. *If $x, y \in \mathbb{R}^m$ then the Cauchy-Schwarz inequality holds: $|(x,y)| \leq |x||y|$*

Proposition 14.2. *If $x, y \in \mathbb{R}^m$ then the triangle inequality holds $|x+y| \leq |x| + |y|$.*

Definition 14.2. A vector function is a function $f : \Omega \to \mathbb{R}^m$ where $\Omega \subset \mathbb{R}^n$.

15. Multivariable functional limits

Definition 15.1. Suppose $\Omega \subset \mathbb{R}^n$. The point $a \in \Omega$ is called a *limit point* of Ω if for every $\epsilon > 0$ there exists $y \in \Omega$ such that $0 < |y - a| < \epsilon$.

Definition 15.2. Let $f : \Omega \to \mathbb{R}^m$ be a function where $\Omega \subset \mathbb{R}^n$. Also let $\mathbf{x}_0$ be a limit point of Ω and $\mathbf{y}_0 \in \mathbb{R}^m$. Then

$$\lim_{\mathbf{x}\to\mathbf{x_0}} \mathbf{f}(\mathbf{x}) = \mathbf{y}_0$$

iff for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |\mathbf{x} - \mathbf{x_0}| < \delta$ and $x \in \Omega$ then $|\mathbf{f}(x) - \mathbf{y}_0| < \epsilon$.

Definition 15.3. The function $f : \Omega \to \mathbb{R}^m$ be a function where $\Omega \subset \mathbb{R}^n$ is continuous at $\mathbf{x}_0$ iff

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$$

16. Multivariable implicit and inverse function theorems

Definition 16.1 (Interior points)**.** For $S \subset \mathbb{R}^n$, x is an interior point of S if there exists an open ball centered at x which is completely contained in S.

Definition 16.2 (Open sets)**.** A set S is open iff every point within it is an interior point.

Definition 16.3 (Closed sets)**.** Let X be a metric space. A set is $S \subset X$ closed iff its complement $X\backslash S$ is open.

Definition 16.4. Let $f : \Omega \to \mathbb{R}^m$ where $\Omega \subset \mathbb{R}^n$ and Ω be open. The *Jacobian matrix* of $f(x_1, \ldots, x_n)$ is

$$J_f(x_1,\ldots,x_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}. \end{bmatrix}$$

The determinant of the Jacobian matrix is called the *Jacobian* of f.

Proposition 16.1 (Inverse function theorem)**.** *Consider $f : \Omega \to \mathbb{R}^m$ where $\Omega \subset \mathbb{R}^n$ and $f^{-1} : f(\Omega) \to \Omega$ such that $f(f^{-1}(x)) = x$ for all $x \in \Omega$. Suppose the function $f(x_1, \ldots, x_m)$ from $\mathbb{R}^n$ to $\mathbb{R}^m$. If*

(1) *for all* $i, j \in \mathbb{N}$ *if* $1 \le i \le m$ *and* $1 \le j \le n$ *then* $\partial f_i / \partial x_j$ *is continuous.*
(2) *there exists* $\mathbf{x_0}$ *such that the Jacobian* J *of* f *is such that* $J(x_0) \neq 0$,

then

(1) *The Jacobian* J^* *of* f^{-1} *is such that* $J^*(f(\mathbf{x}_0))$ *is the inverse of* $J(\mathbf{x}_0)$.
(2) *there exists an open set* Ω *such that* $x_0 \in \Omega$ *and* f *has a continuous inverse.*

Proposition 16.2 (Implicit function theorem)**.** *Let* $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ *be an open set and let* $F : \Omega \to \mathbb{R}^m$ *be a function with continuous first derivatives. Assume that* $(x_0, y_0) \in \Omega$, $F(x_0, y_0) = 0$, *and* $\det J(x_0, y_0) \neq 0$. *Then there are open sets* $U \subset \mathbb{R}^n$ *and* $V \subset \mathbb{R}^m$ *such that* $x_0 \in U$, $y_0 \in V$ *and there is a unique function* $f : U \to V$ *such that* $F(x, f(x)) = 0$ *for* $x \in U$.