Math2400 notes

These notes were taken during MATH2400 in 2014 with Dr. Artem Pulemotov as the lecturer; he was away when the section on sequences and series of functions was written — Dr. Masoud Kamgarpour lectured in his stead during this time.

The LaTeX code for this document is avaiable at `https://github.com/ndurrani/analysisNotes`

Contents

1. Suprema and infima

Definition 1.1. Suppose $\Omega \subset \mathbb{F}$ where $\mathbb{F}$ is an ordered field. Then $b \in \mathbb{F}$ is an upper bound of Ω if $b \geq x$ for all $x \in \Omega$.

Definition 1.2. Suppose $\Omega \subset \mathbb{F}$ where $\mathbb{F}$ is an ordered field. Then $b \in \mathbb{F}$ is a lower bound of Ω if $b \leq x$ for all $x \in \Omega$.

Definition 1.3. Suppose $\Omega \subset \mathbb{F}$ where $\mathbb{F}$ is an ordered field. Then $b \in \mathbb{F}$ is the *least upper bound*, or *supremum*, if $b \leq c$ for every upper bound c.

Definition 1.4. Suppose $\Omega \subset \mathbb{F}$ where $\mathbb{F}$ is an ordered field. Then $b \in \mathbb{F}$ is the *greatest lower bound*, or *infimum*, if $b \geq c$ for every lower bound c.

Example 1.1. Take $\mathbb{F} = \mathbb{R}$, $\Omega = [0, 1]$, then $\sup \Omega = 1$ and $\inf \Omega = 0$.

Proposition 1.1. *The supremum of a set is unique.*

Proof. Let S_1 and S_2 be least upper bounds of Ω. Then, by definition, they are both upper bounds and $S_1 \leq S_2$ and $S_2 \leq S_1$. Therefore $S_1 = S_2$. $\square$

Proposition 1.2. *The infimum of a set is unique.*

Example 1.2. Let $\Omega = \mathbb{N}$. Then $\sup \Omega$ does not exist and $\inf \Omega = 1$.

Example 1.3. Let $\Omega = \mathbb{Q}$. Then $\sup \Omega$ and $\inf \Omega$ do not exist.

Proposition 1.3. *If* $\sup \Omega = \inf \Omega$ *then* Ω *has only one point.*

Proof. Exercise $\square$

Example 1.4. The set $\mathbb{R}$ has the least upper bound property.

Example 1.5. The set $\mathbb{Q}$ does not have the least upper bound property. Counterexample:

$$\Omega : \{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots\}.$$

Proposition 1.4. *Suppose* $\Omega \subset \mathbb{R}$ *and* $\Omega \neq \emptyset$. *Then* $S = \sup \Omega$ *if and only if*

(1) *For all* $x \in \Omega, s \geq X$.
(2) *For all* $\epsilon > 0$, *there exists* $x \in \Omega$ *such that* $s - \epsilon < x$.

Proof. We first prove the forward implication. Assume $S = \sup \Omega$. Then s is an upper bound of Ω, implying (1). Assume (2) is false. Then there exists $\epsilon > 0$ such that for all $x \in (s - \epsilon, s)$, $x \notin \Omega$. This implies that $s - \epsilon$ is an upper bound for Ω. Therefore S is not the least upper bound for Ω.

We will now prove the reverse implication. If (1) holds, the s is an upper bound of Ω. We need now to prove that it's the least upper bound. We need to prove that any other upper bound s^* is such that $s^* \geq s$. Assume $s^* < s$. Then take

$$\epsilon = \frac{|s^* - s|}{2}$$

to get that $x \in (s - \epsilon, s + \epsilon)$ implies $x > s^*$. so $(s - \epsilon, s + \epsilon) \cap \Omega = \emptyset$. But this contradicts (2). $\square$

2. Elements of analysis

Definition 2.1. An ordered field $\mathbb{F}$ possess' the *least upper bound property* if and only if every non-empty bounded set in $\mathbb{F}$ has a least upper bound in $\mathbb{F}$.

Remark. The least upper bound property can only be formalized in 2nd predicate order logic. Note that we only interface with 2nd order logic through ZFC set theory.

Proposition 2.1. *Every ordered field with the least upper bound property is isomorphic to* $\mathbb{R}$.

Remark. This can be used as a working definition of the reals.

Definition 2.2 (Metric space)**.** A set X is a metric space iff there is a function $d : X \times X \to \mathbb{R}$ such that for any *points* $p, q \in X$

(1) $d(p,q) = d(q,p)$,
(2) $d(p,q) > 0$ if $p \neq 0$,
(3) $d(p,p) = 0$, and
(4) $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$.

Lemma 2.2 (Triangle inequality)**.** *If $a, b \in \mathbb{R}$, then* $||a| - |b|| \leq |a+b| \leq |a| + |b|$.

Proof. Since

$$|a+b|^2$$

we have

$$|a+b| \leq |a| + |b|. \tag{1}$$

Furthermore,

$$|a-b+b| \leq |a-b| + |b|$$

whence we get

$$|a| - |b| \leq |a-b|. \tag{2}$$

Furthermore, using (1),

$$|a| - |b| = |a| - |a-a+b| \geq |a| - |a| - |a-b|. \tag{3}$$

Now, using (2) and (3),

$$-|a-b| \leq |a| - |b| \leq |a-b|.$$

Therefore

$$||a| - |b|| \leq |a-b|. \tag{4}$$

By (1) and (4),

$$||a| - |b|| \leq |a+b| \leq |a| + |b|$$

□

Proposition 2.3. *The set $\mathbb{R}$ is a metric space.*

Proof. The absolute value function satisfies the axioms of a metric space. □

Definition 2.3. We say two sets Ω_1 and Ω_2 have the same *cardinality*, that is $|\Omega_1| = |\Omega_2|$, iff there is a bijection between Ω_1 and Ω_2.

Definition 2.4. A set Ω is *countable* if it has the same cardinality as $\mathbb{N}$.

Remark. According to some definitions, finite sets are countable.

Proposition 2.4. *The set $\mathbb{Z}$ is countable.*

Proof. Use the function $s : \mathbb{N} \to \mathbb{Z}$ given by

$$s(n) = \begin{cases} -n-2 & \text{if } n \text{ is odd} \\ 1 & \text{if } n = 0 \\ n+2 & \text{if } n \text{ is even} \end{cases}.$$

□

Proposition 2.5. *The set $\mathbb{Q}$ is countable, but $\mathbb{R}$ is not.*

Proof. The proof is well known and important, but too intricate to describe without a figure. □

Proposition 2.6 (Axiom of choice)**.** *Suppose we have a family of sets $S(w)$ indexed by $w \in W$. Then there is a* choice function

$$f : W \to \bigcup_{w \in W} S(w)$$

such that $f(w) \in S(w)$ for all $w \in W$.

Remark.

(1) The choice function is a function which picks an element from each set $S(w)$. Note that we can't always have it be the least or greatest element of the set - since these might not exist if any $S(w)$ is an open set.
(2) The power of the axiom of choice is that the family of sets can be *infinite*. It's not needed for finite sets since we could just use induction.
(3) The axiom of choice is also formalized using 2nd order logic, just like the least upper bound property.

Example 2.1. Consider the family of sets indexed by $y \in \mathbb{Q}$

$$S(y) = \{x \in \mathbb{R} \mid x < y\}.$$

Now noting that

$$\bigcup_{y\in\mathbb{Q}} S(y) = \mathbb{R},$$

using the axiom of choice, there is a function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(y) \in S(y)$ for all $y \in \mathbb{Q}$.

3. Sequential limits

Definition 3.1. A sequence is a mapping from $\mathbb{N}$ to a set.

Definition 3.2. Let $(a_n)_{n=1}^{\infty}$ be a sequence of reals. We write $\lim_{n\to\infty} a_n = a$, or $a_n \to a$ as $n \to \infty$, if and only if for every $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - a| < \epsilon$.

Definition 3.3. If $(a_n)_{n=1}^{\infty}$ has a limit then it *converges*. Otherwise, it diverges.

Proposition 3.1. *All limits are unique. Suppose the sequence $a_n \to L$ and $a_n \to M$ as $n \to \infty$. Then $L = M$.*

Proof. Fix $\epsilon > 0$. Then there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then

$$|a_n - L| < \frac{\epsilon}{2}.$$

Also, there exists $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then

$$|a_n - M| < \frac{\epsilon}{2}.$$

Set $N = \max\{N_1, N_2\}$. Now if $n \geq N$, then

$$|L-M| = |L-a_n+a_n-M| \leq |a_n-L|+|a_n-M| < \epsilon.$$

Therefore for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that if $n \geq N$ then $|L - M| < \epsilon$, so

$$\lim_{n\to\infty} L = M.$$

Therefore $L = M$. □

Example 3.1. Prove that

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

We want to show that for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all natural $n \geq N$,

$$\left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < \epsilon,$$

or equivalently,

$$n > \frac{1}{\epsilon} - 1.$$

Setting $\epsilon > 0$,

$$\left\lceil\frac{1}{\epsilon}\right\rceil > \frac{1}{\epsilon} - 1.$$

Therefore, letting

$$N = \left\lceil\frac{1}{\epsilon}\right\rceil,$$

if $n \geq N$ then

$$n > \frac{1}{\epsilon} - 1.$$

Therefore, for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all natural $n \geq N$,

$$n > \frac{1}{\epsilon} - 1,$$

which is what we wanted to show.

Proposition 3.2 (Squeeze theorem)**.** *Suppose the sequences $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, and $(c_n)_{n=1}^{\infty}$ satisfy*

(1) *$a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, and*
(2) *$a_n \to L$ and $b_n \to L$ as $n \to \infty$.*

Then $b_n \to L$ as $n \to \infty$.

Proof. Note that

$$\begin{aligned} |b_n - L| &= |(b_n - a_n) + (a_n - L)| \\ &\leq |b_n - a_n| + |a_n - L| \\ &\leq |c_n - a_n| + |a_n - L| \\ &= |(c_n - L) + (L - a_n)| + |a_n - L| \\ &\leq |c_n - L| + 2|a_n - L| \end{aligned} \tag{5}$$

Set $\epsilon > 0$. Now since $a_n \to L$ as $n \to \infty$, there exists $N_a \in \mathbb{N}$ such that if $n \geq N_a$ then

$$|a_n - L| < \frac{\epsilon}{4},$$

and since $c_n \to L$ as $n \to \infty$, there is $N_c \in \mathbb{N}$ such that

$$|c_n - L| < \frac{\epsilon}{2}$$

when $n \geq N_c$. Take $N = \max\{N_a, N_c\}$. Now, using (5), if $n \geq N$ then

$$|b_n - L| \leq |c_n - L| + 2|a_n - L| < \frac{\epsilon}{2} + 2\frac{\epsilon}{4} = \epsilon.$$

□

Example 3.2. Find

$$\lim_{n\to\infty} \frac{|\sin\left(1+n^2\right)|}{n^2}.$$

Note that

$$0 \leq \frac{\left|\sin\left(1+n^2\right)\right|}{n^2} \leq \frac{1}{n^2}$$

But

$$\lim_{n\to\infty} \frac{1}{n^2} = 0 = \lim_{n\to\infty} 0.$$

Therefore, by the squeeze theorem,

$$\lim_{n\to\infty} \frac{|\sin(1+n^2)|}{n^2} = 0.$$

Definition 3.4. The sequence $(a_n)_{n=1}^{\infty}$ is *monotone increasing* if and only if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$.

Definition 3.5. The sequence $(a_n)_{n=1}^{\infty}$ is *strictly monotone increasing* if and only if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.

Definition 3.6. The sequence $(a_n)_{n=1}^{\infty}$ is *monotone decreasing* if and only if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

Definition 3.7. The sequence $(a_n)_{n=1}^{\infty}$ is *strictly monotone increasing* if and only if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$.

Lemma 3.3 (Convergent sequences are bounded)**.** *If $(a_n)_{n=1}^{\infty}$ converges then there exists $M > 0$ such that $|a_n| \leq M$ for all n.*

Proof. Since $(a_n)_{n=1}^{\infty}$ is convergent, for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all natural $n \geq N$, $|a_n - L| < \epsilon$ for some L. Setting $\epsilon = 1$ and L, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - L| < 1$. Therefore, if $n \geq N$, then

$$|a_n| < 1 + |L|.$$

To show that it is bounded for $n < N$, let

$$M = \max\{a, a_n, \ldots, a_{N-1}, |L| + 1\}.$$

Now, $|a_n| \leq M$ for all n. □

Proposition 3.4. *Let $(a_n)_{n=1}^{\infty}$ be a monotone sequence. This sequence converges iff it is bounded.*

Proof. The forward implication, that if $(a_n)_{n=1}^{\infty}$ converges then it is bounded, follows from the lemma. To prove the reverse implication, we want to show that if $(a_n)_{n=1}^{\infty}$ is bounded then it converges. Assume, without loss of generality, that $(a_n)_{n=1}^{\infty}$ is non-decreasing (otherwise we can consider $(-a_n)_{n=1}^{\infty}$ since $(a_n)_{n=1}^{\infty}$ converges if $(-a_n)_{n=1}^{\infty}$ does). Now we need only to show that

$$\lim_{n\to\infty} a_n = a$$

where $a = \sup\ (a_n)_{n=1}^{\infty}$ (we know a exists by the least upper bound property of the reals and the fact that $(a_n)_{n=1}^{\infty}$ is bounded).

To do this, we will now make ample use of the properties of supremum such as a. Setting $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $a_N \in (a - \epsilon, a]$. Now, since $(a_n)_{n=1}^{\infty}$ is non-decreasing, we have that for all $n \geq N$,

$$a - \epsilon \leq a_N \leq a_n \leq a,$$

so $|a - a_n| < \epsilon$ for such n.

Therefore, for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that if $n \geq N$ then $|a - a_n| < \epsilon$. Therefore,

$$\lim_{n\to\infty} a_n = a,$$

which is what we wanted to show. □

Proposition 3.5. *If the sequences $a_n \to a$ and $b_n \to b$ as $n \to \infty$, then as $n \to \infty$,*

(1) $a_n + b_n \to a + b$, $a_n - b_n \to a - b$,
(2) $\lambda a_n \to \lambda a$ *for constant* λ,
(3) $a_n b_n \to ab$, *and*
(4) $1/a_n \to 1/a$ *for* $a \neq 0$.

Proof.

(1) We will first show that

$$\lim_{n\to\infty} a_n + b_n = a + b,$$

that is, for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all natural $n \geq N$, $|a_n + b_n - (a + b)| < \epsilon$.

Set $\epsilon > 0$. Now since

$$\lim_{n\to\infty} a_n = a$$

there exists $N_a \in \mathbb{N}$ such that for all natural $n \geq N_a$

$$|a_n - a| < \frac{\epsilon}{2},$$

and since

$$\lim_{n\to\infty} b_n = b$$

there exists $N_b \in \mathbb{N}$ such that for all natural $n \geq N_b$

$$|b_n - b| < \frac{\epsilon}{2}.$$

Now, letting $N = \max\{N_a, N_b\}$, if $n \geq N$ then

$$|a_n + b_n - (a + b)| \leq |a_n - a| + |b_n - b| < \epsilon.$$

Therefore, for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all natural $n \geq N$, $|a_n + b_n - (a + b)| < \epsilon$, which is what we wanted to show.

To show that

$$\lim_{n\to\infty} a_n - b_n = a - b,$$

we can reason analogously.

(2) We want to show that for all $\epsilon > 0$ there exists some natural $N \in \mathbb{N}$ such that for all natural $n \geq N$,

$$|\lambda a_n - \lambda a| = \lambda |a_n - a| < \epsilon,$$

that is, $|a_n - a| < \epsilon/\lambda$. Set $\epsilon > 0$. Since

$$\lim_{n\to\infty} a_n = a,$$

we can set

$$\epsilon^* = \epsilon/\lambda > 0,$$

to conclude that there exists some $N \in \mathbb{N}$ such that for all natural $n \geq N$,

$$|a_n - a| < \epsilon^* = \epsilon/\lambda.$$

Therefore, for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all natural $n \geq N$, $|a_n - a| < \epsilon/\lambda$, which is what we wanted to show.

(3) Note that

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &= |a_n(b_n - b) + b(a_n - a)| \\ &\leq |a_n||b_n - b| + |b||a_n - a| \\ &\leq M(|b_n - b| + |a_n - a|), \end{aligned} \tag{6}$$

where $M = \max\{M_a, M_b\}$ and M_a and M_b are respectively upper bounds of $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ (we know these exists since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converge).

We want to show that for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all natural $n \geq N$, $|a_n b_n - ab| < \epsilon$.

Set $\epsilon > 0$. Now since

$$\lim_{n\to\infty} a_n = a,$$

there exists some $N_a \in \mathbb{N}$ such that for all natural $n \geq N_a$,

$$|a_n - a| < \frac{\epsilon}{2M}$$

and since

$$\lim_{n\to\infty} b_n = b,$$

there exists some $N_b \in \mathbb{N}$ such that for all natural $n \geq N_b$,

$$|a_n - a| < \frac{\epsilon}{2M}.$$

Therefore, recalling (6) and letting

$$N = \max\{N_a, N_b\},$$

if $n \geq N$ then $|a_n b_n - ab| < \epsilon$, which is what we wanted to show.

(4) We want to show that for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all natural $n \geq N$,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b - b_n|}{|b||b_n|} < \epsilon$$

if $b \neq 0$.

To this end, we will first show that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < M|b - b_n|$$

for some M, knowing that we can make $|b - b_n|$ as small as we want.

To make $|b_n|$ as small as possible in order to find an upper bound for $|b - b_n|/|b|/|b_n|$, we will use the reverse triangle inequality: Since

$$\lim_{n\to\infty} b_n = b$$

there exists $N_1 \in \mathbb{N}$ such that for all natural $n \geq N_1$,

$$|b| - |b_n| \leq ||b| - |b_n|| \leq |b - b_n| < \frac{|b|}{2}$$

so

$$\frac{|b|}{2} < |b_n|$$

(note: the trick we used here is similar to when we set δ to get a range for x when doing an $\epsilon - \delta$ proof for quadratic polynomials. Here $|b_n|$ is analogous to x, but b_n varies along the vertical axis, so we have to set ϵ rather than δ. We could also use $|b_n| - |b|$ as a lower bound to $|b|/2$ to find $3|b|/2$ as an upper bound.) Now

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b - b_n|}{|b||b_n|} \leq \frac{2}{b^2}|b - b_n|.$$

But, since

$$\lim_{n\to\infty} b_n = b$$

there is some $N_2 \in \mathbb{N}$ such that for all natural $n \geq N_2$,

$$|b_n - b| < \frac{b^2}{2}\epsilon.$$

Now, setting $\epsilon > 0$ and letting

$$N = \max\{N_1, N_2\},$$

if $n \geq N$ then

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b - b_n|}{|b||b_n|} \leq \frac{2}{b^2}|b - b_n| < \frac{2}{b^2}\frac{|b|^2}{2}\epsilon.$$

Therefore, for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all natural $n \geq N$,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < \epsilon.$$

□

Example 3.3. Fix $\lambda \in (0, 1)$. Set $a_n = \lambda^n$. Then $\lim_{n\to\infty} a_n = 0$

Proof. Note $\lambda > 0$. Now, setting $0 < \lambda < 1$,

$$0 < \lambda^{n+1} < \lambda^n < 1.$$

This means that λ^n is *monotone decreasing.* Such sequences always converge if bounded so $\lambda^n \to \alpha$ as $n \to \infty$ for some $\alpha \in \mathbb{R}$. To prove the theorem we need to prove that $\alpha = 0$ by showing that

(1) Assume $\alpha < 0$. Take $\epsilon = \frac{|\alpha|}{2} = \frac{-\alpha}{2}$. Then there exists $N \in \mathbb{N}$ such that $n > N$ implies $|\lambda^n - \alpha| < -\alpha/2$ so that
$$\frac{\alpha}{2} < \lambda^n - \alpha < \frac{-\alpha}{2}$$
and, adding α to the inequality,
$$\frac{3\alpha}{2} < \lambda^n < \frac{\alpha}{2}$$
whence we conclude
$$\lambda^n < 0,$$
contradicting our assumption.

(2) Assume $\alpha \geq 1$. Then a contradiction arises using a similar proof to case (1).

(3) Assume $0 < \alpha < 1$. Take
$$\epsilon = \frac{\alpha}{\lambda} - \alpha.$$
Then there is some $n \in \mathbb{N}$ such that $n > N$ implies $|\lambda^n - \alpha| < \epsilon$. This means $\alpha - \epsilon < \lambda^N < \alpha + \epsilon$ so that $2\alpha - \frac{\alpha}{\lambda} < \lambda^N < \alpha + \frac{\alpha}{\lambda}$ and $\lambda^{N+1} < \alpha$. But this is impossible because λ^n is monotone decreasing, leading to a contradiction.

Therefore $\alpha = 0$. □

Lemma 3.6 (Bernoulli inequality)**.**

(1) *Assuming $x \geq -1$ and $n \in \mathbb{N} \cup \{0\}$ but $x = -1$ and $n = 0$ do not hold at the same time,*
$$(1 + x)^n \geq 1 + nx.$$

(2) *We have a strict inequality in the above formula if and only if $n > 1$ and $x \neq 0$.*

Proof. Recalling the assumption $x \geq -1$, note that

$$(1 + x)^n \geq 1 + nx \tag{7}$$

always holds for $x = -1$. Otherwise, $x > -1$. In this case, assume (7) holds for n. We show that it must then hold for $n+1$:

$$\begin{aligned}(1+x)^{n+1} &= (1+x)(1+x)^n \\ &\geq (1+x)(1+nx) \\ &= 1+nx+x+nx^2 \\ &\geq 1+(n+1)x\end{aligned}$$

Now, since (7) holds for $n=0$, it must hold for all $n \in \mathbb{N}$.

To show part (2) of the proposition holds, repeat the argument with obvious modifications. □

Proposition 3.7. *Define $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ by*

$$a_n = \left(1+\frac{1}{n}\right)^n$$

and

$$b_n = \left(1+\frac{1}{n}\right)^{n+1}.$$

Then a_n is strictly increasing, b_n is strictly decreasing and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Proof. We first prove the sequences' monotonicity. Note that

$$a_n = \left(\frac{n+1}{n}\right)^n$$

and

$$b_{n-1} = \left(\frac{n}{n-1}\right)^n.$$

Now

$$\begin{aligned}\frac{a_n}{b_{n-1}} &= \left(\frac{(n-1)(n+1)}{n^2}\right)^n \\ &= \left(\frac{n^2-1}{n^2}\right)^n \\ &= \left[1+\left(\frac{-1}{n^2}\right)\right]^n. \qquad (8)\end{aligned}$$

But by the second part of the lemma,

$$\left[1+\left(\frac{-1}{n^2}\right)\right]^n > 1+n\left(\frac{-1}{n^2}\right) = 1-\frac{1}{n}.$$

Therefore

$$\begin{aligned}a_n &= \left[1-\frac{1}{n^2}\right]^n b_{n-1} \\ &> \left(1-\frac{1}{n}\right) b_{n-1} \\ &= \frac{n-1}{n}\left(\frac{n}{n-1}\right)^n \\ &= a_{n-1}.\end{aligned}$$

We conclude that $(a_n)_{n=1}^{\infty}$ is strictly increasing.

Similarly, using (8) and the lemma, $(b_n)_{n=1}^{\infty}$ is strictly decreasing:

$$\begin{aligned}b_{n-1} &= \left(1+\frac{1}{n^2-1}\right)^n a_n \\ &> \left(1+\frac{1}{n^2}\right)^n a_n \\ &> \left(1+\frac{n}{n^2}\right) a_n \\ &> \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n \\ &= b_n\end{aligned}$$

Now, observing that

$$2 = a_1 \leq a_n < b_n \leq b_1 = 4,$$

the sequences are monotone and bounded, so they converge. Therefore the laws of limits apply, and we can conclude that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} a_n.$$

□

Definition 3.8 (Euler's number)**.** Euler's number e is given by

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

which converges, as per the previous proposition.

4. Subsequential limits

Definition 4.1. Suppose $(x_n)_{n=1}^{\infty} \subset \mathbb{R}$. The point $x \in \mathbb{R}$ is a *cluster point* of $(x_n)_{n=1}^{\infty}$ iff for all $\epsilon > 0$

$$\left|\left\{n \in \mathbb{N} \mid |x_n - x| < \epsilon\right\}\right| = |\mathbb{N}|.$$

Example 4.1. The sequence $\{0, 1, -1, 0, 1, -1, \ldots\}$ has cluster points 0, 1, and -1.

Proposition 4.1. *Assume $(x_n)_{n=1}^{\infty} \subset \mathbb{R}$ and $x \in \mathbb{R}$. Then x is a cluster point of $(x_n)_{n=1}^{\infty}$ iff for all $\epsilon > 0$ and $N \in \mathbb{N}$ there exists a natural $n \geq N$ such that $|x_n - x| < \epsilon$.*

Proof. Assume x is a cluster point of $(x_n)_{n=1}^{\infty}$, set $\epsilon > 0$, and let

$$S = \left\{n \in \mathbb{N} \mid |x_n - x| < \epsilon\right\}.$$

To prove the forward implication, assume $|S| = |\mathbb{N}|$. But now $|S| = |\mathbb{N}|$ and $S \subset \mathbb{N}$ so $S = \mathbb{N}$. Therefore $N + 1 \in S$ so there is some natural $n \geq N$ such that $|x_n - x| < \epsilon$.

We now prove the reverse implication by induction. Suppose $|S| \geq C \geq 1$ so there is some $r \in S$ such that $|x_r - x| < \epsilon$. Now, by assumption, for all $N \in \mathbb{N}$ there exists a natural $n \geq N$ such that $|x_n - x| < \epsilon$, so

(1) if $N = r + 1$ then there is some $n \geq N$ such that $n \in S$ and $n \neq r$, and
(2) if $N = 1$ then there is some natural $n \in S$.

Now by (a) if $|S| \geq C \geq 1$ then $|S| \geq C + 1$, and by (b), $S \neq \emptyset$. Therefore $|S| = |\mathbb{N}|$. □

Proposition 4.2. *Assume $(x_n)_{n=1}^{\infty} \subset \mathbb{R}$ and $x \in \mathbb{R}$. Then x is a cluster point iff there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that*

$$\lim_{n \to \infty} x_{n_k} = x.$$

Proof. We first prove the forward implication. If x is a cluster point, then for $\epsilon = 1$ there exists infinitely many x_n such that $x_n - x < 1$. Take one, and call it $x_{n_1}^{\infty}$. Then by (i), for $\epsilon = 1/2$ and $N = n_1$ there is $n_2 > n_1$ such that

$$|x - x_{n_2}| < \frac{1}{2}.$$

For $\epsilon = 1/2$ and $N = n_2$ there is n_2 such that $n_2 > n_2$ and $x_{n_3} - x| < 1/2$. Continuing this, we obtain a subsequence $(x_n)_{n=1}^{\infty}$. It is easy to show this converges to x. (exercise)

We will now prove the reverse implication. If there is a subsequence $(x_{n_k})_{n=1}^{\infty}$ such that

$$\lim_{k \to \infty} x_{n_k} = x.$$

So fix $\epsilon > 0$. Then there is $N \in \mathbb{N}$ such that $k \geq \mathbb{N}$ implies $|x_{n_k} - x| < \epsilon$. But there are infinitely many such k. Therefore there are infinitely many x_{n_k} such that $|x_{n_k} - x| < \epsilon$. □

Proposition 4.3. *Assume $(x_n)_{n=1}^{\infty} \subset \mathbb{R}$ and $x \in \mathbb{R}$. Then*

$$\lim_{n \to \infty} x_n = x$$

iff every subsequence of $(x_n)_{n=1}^{\infty}$ converges to x.

Proof. We first prove the forward implication, that if

$$\lim_{n \to \infty} x_n = x$$

then for any subsequence $(x_{n_k})_{k=1}^{\infty}$,

$$\lim_{k \to \infty} x_{n_k} = x,$$

that is, for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all natural $k \geq N$, $|x_{n_k} - x| < \epsilon$. Assume

$$\lim_{n \to \infty} x_n = x.$$

Therefore, setting $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all natural $m \geq N$, $|x_m - x| < \epsilon$. But for the subsequence $(x_{n_k})_{k=1}^{\infty}$, $n_k \geq k$ so if $k \geq N$ then $n_k \geq k \geq N$. Now, setting $m = n_k$, if $k \geq N$, then $|x_{n_k} - x| < \epsilon$. Therefore for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such if $k \geq N$ then $|a_{n_k} - a| < \epsilon$, which is what we wanted to show.

To prove the reverse implication, note that $(x_n)_{n=1}^{\infty}$ is a subsequence of itself, so if every subsequence of $(x_n)_{n=1}^{\infty}$ converges to x, then $(x_n)_{n=1}^{\infty}$ itself must converge x. □

Proposition 4.4. *Assume $(x_n)_{n=1}^{\infty} \subset \mathbb{R}$ and $x \in \mathbb{R}$. Then $(x_n)_{n=1}^{\infty}$ converges iff $(x_n)_{n=1}^{\infty}$ is bounded and has exactly one cluster point.*

Proof. Let x be a cluster point of $(x_n)_{n=1}^{\infty}$. We want to show that

$$\lim_{n \to \infty} x_n = x.$$

Setting $\epsilon > 0$,

$$\left|\left\{n \in \mathbb{N} \mid |x_n - x| < \epsilon\right\}\right| = |\mathbb{N}|$$

since x is a cluster point of $(x_n)_{n=1}^{\infty}$. Equivalently, if $n \in \mathbb{N}$ then $|x_n - x| < \epsilon$. Therefore, letting $N = 1$, if $n \geq N$ then $|x_n - x| < \epsilon$. Therefore, for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n \geq N$,

$|x_n - x| < \epsilon$, or equivalently,

$$\lim_{n\to\infty} x_n = x,$$

which is what we wanted to show. □

Definition 4.2. Suppose $(a_n)_{n=1}^{\infty}$ is a sequence and $n : \mathbb{N} \to \mathbb{N}$. Then $(a_{n_k})_{k=1}^{\infty}$ is a *subsequence* of $(a_n)_{n=1}^{\infty}$.

Example 4.2. Let consider the sequence $(a_n)_{n=1}^{\infty}$ given by $a_n = (-1)^n$. It has subsequences $(a_{2n})_{n=1}^{\infty}$ and $(a_{3n})_{n=1}^{\infty}$ such that $a_{2n} = 1$ and $a_{3n} = -1$.

Definition 4.3 (Limit supremum and infimum)**.** For the sequence $(a_n)_{n=1}^{\infty}$,

$$\limsup_{n\to\infty} a_n = \lim_{N\to\infty} \sup_{n\geq N} a_n$$

and

$$\liminf_{n\to\infty} a_n = \lim_{N\to\infty} \inf_{n\geq N} a_n.$$

Proposition 4.5. *We have*

$$x = \limsup_{n\to\infty} a_n$$

iff x is the greatest cluster point of the sequence. Similarly

$$x = \liminf_{n\to\infty} a_n$$

iff x is the least cluster point of $(a_n)_{n=1}^{\infty}$.

Example 4.3. Define $a : \mathbb{N} \to \mathbb{Z}$ such that

$$a_n = n \bmod 3 - 1$$

so $a_1 = 0$, $a_2 = 1$, $a_3 = -1$, $a_4 = 1$, and so on. Find the limit supremum and infimum of $(a_n)_{n=1}^{\infty}$. Let $x : \mathbb{N} \to \mathbb{Z}$ be given by

$$x_n = \sup\left\{a_i \mid i \geq n\right\}.$$

Now $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, and, in general, $x_n = 1$ for all n. Therefore

$$\limsup_{n\to\infty} a_n = 1.$$

Using similar reasoning,

$$\liminf_{n\to\infty} a_n = -1.$$

Example 4.4. Informally speaking,

$$\limsup_{n\to\infty} n = \lim_{n\to\infty} \infty = \infty$$

and

$$\liminf_{n\to\infty} n = \lim_{n\to\infty} n = \infty.$$

Example 4.5.

$$\limsup_{n\to\infty} \left(1 + \frac{1}{n}\right) = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$$

Proposition 4.6. *If $(a_n)_{n=1}^{\infty}$ converges, then*

$$\limsup_{n\to\infty} (a_n)_{n=1}^{\infty}$$

and

$$\liminf_{n\to\infty} (a_n)_{n=1}^{\infty}$$

exist.

Example 4.6. Let us prove that $(a_n)_{n=1}^{\infty}$ where $a_n = (-1)^n$ diverges. Assume

$$\lim_{n\to\infty} (-1)^n = \alpha.$$

Then there are three possible cases: $\alpha = 1$, $\alpha = -1$, and $\alpha = x$ for some x satisfying $x \neq 1$ and $x \neq -1$. We will consider the last case, then the proof of the other two cases can be reasoned about similarly. In the last case, by the definition of a limit, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $|a_n - \alpha| < \epsilon$. Choose

$$\epsilon = \frac{\min\{|\alpha - 1|, |\alpha + 1|\}}{2}.$$

Now if $a_n = 1$ then

$$|\alpha - a_n| = |\alpha - 1| > \frac{|\alpha - 1|}{2} \geq \epsilon.$$

Otherwise, if $a_n = -1$ then

$$|\alpha - a_n| = |\alpha - (-1)| > \frac{|\alpha - (-1)|}{2} \geq \epsilon.$$

Thus $|\alpha - a_n| < \epsilon$ fails for all ϵ.

Example 4.7. The sequence

$$x_n = \frac{(-1)^n}{n}$$

is bounded and has a cluster point of 0. The limit suprememum and infimum are also 0.

5. The completeness of the reals

Definition 5.1. Suppose $\Omega \neq \emptyset$ then the *diameter* of Ω is

$$\operatorname{diam} \Omega = \sup_{x,y \in \Omega} |x - y|.$$

Definition 5.2. The sequence $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$ is a *Cauchy sequence* iff for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies $|a_m - a_n| < \epsilon$.

Remark. As we will prove, for any bounded sequence $(a_n)_{n=1}^{\infty}$, $(a_n)_{n=1}^{\infty}$ is Cauchy iff

$$\lim_{N \to \infty} \operatorname{diam} (a_n)_N^{\infty} = 0.$$

Definition 5.3. We say $k \in \mathbb{N}$ is a *peak point* of the sequence $(a_n)_{n=1}^{\infty}$ iff for all $n > k$ we have $a_n < a_k$.

Lemma 5.1. *Any sequence $(a_n)_{n=1}^{\infty}$ contains a subsequence which is either non-decreasing or non-increasing.*

Remark. The proof is a good example of using the axiom of choice from ZFC set theory (where does it use it?).

Proof. Consider the following cases.

1. There are infinitely many peak points. Take the sequence of peak point subsequence and we are done.
2. There are finitely many peak point $p_1, \ldots, p_k$. Then take N to be a point in a sequence after the peak points. Then for each a_n for $n > N$, there exists integral $k > 0$ such that a_{n+k_1} is bigger than a_n since there are no peak points after N, and we continue this process to choose a monotone increasing sequence.

□

Proposition 5.2 (Bolzano-Weierstass Theorem). *Every bounded sequence has a convergent subsequence.*

Remark. The converse, that a convergent subsequence is bounded is not true: It is possible for a divergent sequence to have a convergent subsequence.

Proof. Every sequence has a monotone subsequence by the lemma. This sequence must be bounded. □

Definition 5.4. Let X be a metric space. Suppose that given any Cauchy sequence $(a_n)_{n=1}^{\infty} \subset X$, $a_n \to L$ where $L \in X$. Then we say that X is *complete.*

Lemma 5.3. *If $(a_n)_{n=1}^{\infty}$ is Cauchy then it converges.*

Proof. We want to show that if $(a_n)_{n=1}^{\infty}$ is Cauchy, then it converges. Since $(a_n)_{n=1}^{\infty}$ is Cauchy, there exists N such that for integral $n, k \geq N$, $|a_k - a_n| < \epsilon$. Now setting $\epsilon = 1$, we can set k and N such that $k \geq N$ and

$$|a_k| - |a_N| \leq |a_k - a_N| < 1$$

so

$$|a_k| < 1 + |a_n| < 1 + |a_N|.$$

Therefore for all $k \geq N$ we have an upper bound for $(a_n)_{n=1}^{\infty}$. To find such an upper bound M for *all* $k \in \mathbb{N}$, set

$$M = \max \left\{ a_1, a_2, \ldots, a_{N-1}, |a_N| + 1 \right\}.$$

Now, by the Bolzano-Weierstrass theorem, $(a_n)_{n=1}^{\infty}$ has a convergent subsequence $(a_{n_m})_{m=1}^{\infty}$. Denote it by

$$\lim_{m \to \infty} a_{n_m} = a$$

We prove that

$$\lim_{m \to \infty} a_m = a.$$

Take $\epsilon > 0$ so

$$\lim_{k \to \infty} a_{n_k} = a$$

there is $N_1 \in \mathbb{N}$ such that $k \geq N_1$ implies that $|a_{n_k} - a| < \epsilon/2$. Now since $(a_n)_{n=1}^{\infty}$ is Cauchy, there is $N_2 \in \mathbb{N}$ such that $n, k \geq N_2$ implies $|a_n - a_k| < \epsilon/2$. Now consider, assuming $n \geq \max\{N_1, N_2\}$,

$$|a_n - a| = |a_n - a_{n_k} + a_{n_k} - a|$$

□

Lemma 5.4. *If $(a_n)_{n=1}^{\infty}$ converges then it is Cauchy.*

Proof. Let

$$x = \lim_{n \to \infty} x_n.$$

Fix $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|x_n - x| < \frac{\epsilon}{2}.$$

Then if $n, m \geq \mathbb{N}$ then

$$|x_n - x_m| = |x_n - x + x - x_m|$$

implying

$$|x_n - x_m| \leq |x_n - x| + |x_n - x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

□

Proposition 5.5. *The set $\mathbb{R}$ is a complete metric space.*

Proof. The proposition follows directly from the Lemmas. □

Lemma 5.6. *For any sequence* $(a_n)_{n=1}^{\infty}$,

$$\text{diam } (a_n)_{n=1}^{\infty} = \sup\ (a_n)_{n=1}^{\infty}\ -\ \inf\ (a_n)_{n=1}^{\infty}.$$

Proposition 5.7. *For any bounded sequence* $(a_n)_{n=1}^{\infty}$, $(a_n)_{n=1}^{\infty}$ *is Cauchy iff*

$$\lim_{N\to\infty} \text{diam } (a_n)_N^{\infty} = 0.$$

Proof. Note that

$$\begin{aligned}\lim_{N\to\infty} \text{diam } (a_n)_{n\geq N} &= \lim_{N\to\infty} \left(\sup\ (a_n)_N^{\infty}\ -\ \inf\ (a_n)_N^{\infty}\right)\\ &= \limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n.\end{aligned}$$

But this becomes zero iff $(a_n)_{n=1}^{\infty}$ converges, when

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

□

6. Infinite sums

Definition 6.1. Consider the sequence

$$S_n = \sum_{k=1}^{n} a_n$$

where $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$. We define the following sum thus:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n.$$

Proposition 6.1. *If* $\sum_{n=1}^{\infty} a_n$ *converges, then*

$$\lim_{n\to\infty} a_n = 0.$$

Proof. Note that

$$a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = S_n - S_{n-1}.$$

Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = a - a = 0.$$

□

Remark. The converse is *not* true.

Example 6.1. The sum

$$\sum_{n=1}^{\infty} \frac{n+1}{n+2}$$

since

$$\lim_{n\to\infty} \frac{n+1}{n+2} = 1 > 0.$$

Proposition 6.2 (Cauchy criterion)**.**
The series $\sum_{n=1}^{\infty} a_n$ *converges iff for every* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that if* $q > p \geq N$ *then*

$$\left|\sum_{n=p+1}^{q} a_n\right| < \epsilon.$$

Proof. A series converges by definition when

$$S_n = \sum_{k=1}^{n} a_k$$

does, which occurs iff S_n is Cauchy. But S_n is Cauchy iff for all $\epsilon > 0$ there exists $N \geq \mathbb{N}$ such that if $q > p \geq N$ then

$$|S_q - S_p| = \left|\sum_{n=p+1}^{q} a_n\right| < \epsilon.$$

Since this is the case, we are done. □

Example 6.2. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. Assume it converges. Then, apply the Cauchy criterion, with $\epsilon = 1/3$. Then series convergence means that there exists $N \in \mathbb{N}$ such that $q > p \geq N$ implies

$$\left| \sum_{n=p+1}^{q} \frac{1}{n} \right| < \frac{1}{3}.$$

In particular, this must hold for $p = N$ and $q = 2N$. Then

$$\left| \sum_{n=N+1}^{2N} \frac{1}{n} \right| < 1/3.$$

But

$$\left| \sum_{n=N+1}^{2N} \right| > \frac{1}{2N} \times N = \frac{1}{2}.$$

Hence we have reached a contradiction, so the series does not converge.

Proposition 6.3 (Convergence laws for series)**.** *Assume* $\sum_{n=1}^{\infty} a_n = a$ *and* $\sum_{n=1}^{\infty} b_n = b$*.*

(1) *Then*

$$\sum_{n=1}^{\infty} (a_n + b_n) = a + b$$

(2) *for constant* $\lambda \in \mathbb{R}$

$$\sum_{n=1}^{\infty} (\lambda a_n) = \lambda a$$

However, though, by definition of the product of these sequence,

$$\left(\sum_{n=1}^{\infty} a_n \right) \left(\sum_{n=1}^{\infty} b_n \right) = ab,$$

it could be the case that

$$\sum_{n=1}^{\infty} (a_n b_n) \neq ab.$$

Proposition 6.4 (Comparison theorem)**.** *If* $\sum_{n=1}^{\infty} b_n$ *converges and* $0 \leq a_n \leq b_n$ *then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof. Apply the comparison theorem for the sequence for partial sums. □

Example 6.3. Assume $p < 1$ then the series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges because $1/n^p > 1/n$.

Example 6.4. The series

$$\sum_{n=1}^{\infty} \frac{1}{n + 2^n}$$

converges since

$$\frac{1}{n + 2^n} < \frac{1}{2^n}.$$

Lemma 6.5.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Proof. Note that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

and that the partial sums S_i of the series are given by

$$S_1 = \frac{1}{1 \cdot 2} = \frac{1}{2}$$

$$S_2 = \frac{1}{2 \cdot 3} = \frac{1}{2} - \frac{1}{3},$$

$$S_3 = \frac{1}{3 \cdot 4} = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = \frac{1}{2} + \frac{1}{2} - \frac{1}{4},$$

and in general

$$S_n = 1 - \frac{1}{n+1};$$

we could verify this using induction. Therefore

$$\lim_{n \to \infty} S_n = 1 - \frac{1}{n+1} = 1.$$

□

Proposition 6.6. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

Proof. Note that

$$\frac{1}{(n+1)^2} < \frac{1}{n(n+1)}.$$

But

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} + 1.$$

Now $1/(n+1)^2$ converges by the lemma, and the comparison test. Therefore $\sum_{n=1}^{\infty} \left(1/n^2\right)$ converges. In fact it converges to $\pi^2/6$. □

Proposition 6.7. *If* $\sum_{n=1}^{\infty} |a_n|$ *converges then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof. Since

$$0 \leq a_n + |a_n| \leq 2|a_n|,$$

we have by the comparison theorem that

$$\sum_{n=1}^{\infty} (a_n + |a_n|)$$

converges if $|a_n|$ converges. Now $a_n = (a_n + |a_n|) - |a_n|$. Therefore

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|.$$

if $|a_n|$ converges. Now, since the difference between two convergent sequences is convergent, the theorem follows. $\square$

Remark.

(1) The converse does not hold.
(2) We couldn't simply apply the comparison theorem to

$$0 \leq a_n \leq 2|a_n|.$$

because it is not known that $a_n > 0$ for all n.

Proposition 6.8 (Leibniz test)**.** *If the series a_n is a non-negative non-increasing sequence which converges to zero, then*

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges.

Example 6.5. Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges.

Proof. Use the Leibniz test. $\square$

Definition 6.2. If $\sum_{n=1}^{\infty} |a_n|$ converges then we say $\sum_{n=1}^{\infty} a_n$ converges *absolutely.* If it converges, but $\sum_{n=1}^{\infty} |a_n|$ does not converge - we say it converges *conditionally.* Absolute convergence implies conditional convergence however the converse is not true.

Proposition 6.9 (Ratio test)**.** *The series $\sum a_n$*

(1) *converges if*

$$\limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

(2) *diverges if*

$$\liminf_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$$

(3) *has uncertain convergence if*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$$

(could go either way.)

Proposition 6.10 (Root test)**.** *Set*

$$a = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

(1) *If $a < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.*
(2) *If $a > 1$,then $\sum_{n=1}^{\infty} a_n$ diverges.*
(3) *If $a = 1$, test is inconclusive.*

Example 6.6. Consider the series $\sum_{n=1}^{\infty} a_n$ where

$$a_n = \begin{cases} \frac{1}{n^n} & n \text{ is odd} \\ \frac{1}{3^n} & n \text{ is even} \end{cases}$$

so

$$a_n = \frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \dots$$

We will use the ratio test to test its convergence:

$$\begin{aligned} \limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \limsup_{n\to\infty} \begin{cases} \frac{1/2^{n+1}}{1/3^n} & n \text{ is even} \\ \frac{1/3^{n+1}}{1/2^n} & n \text{ is odd} \end{cases} \\ &= \limsup_{n\to\infty} \begin{cases} \frac{1}{2}\frac{3^n}{2^n} & n \text{ is even} \\ \frac{1}{3}\frac{2^n}{3^n} & n \text{ is odd} \end{cases} \\ &= \infty \end{aligned}$$

Similarly, we can show the limit infimum goes to zero. Hence, the ratio test gives us no information. Therefore, we will use the root test instead.

$$\begin{aligned} \sqrt[n]{|a_n|} &= \begin{cases} \sqrt[n]{\frac{1}{3^n}} & n \text{ is even} \\ \sqrt[n]{\frac{1}{2^n}} & n \text{ is odd} \end{cases} \\ &= \begin{cases} \frac{1}{3} & n \text{ is even} \\ \frac{1}{2} & n \text{ is odd} \end{cases} \end{aligned}$$

Therefore

$$\limsup_{n\to\infty} \sqrt{|a_n|} = \frac{1}{2} < 1.$$

Thus, the series converges.

7. Functional limits

Definition 7.1. We write

$$\lim_{x \to \infty} f(x) = \infty$$

if and only if for every $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n > M$.

Definition 7.2. We write

$$\lim_{x \to \infty} f(x) = -\infty$$

if and only if for every $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n < M$.

Definition 7.3. We write

$$\lim_{x \to \infty} f(x) = L$$

iff for every $\epsilon > 0$ there exists M such that $x > M$ implies $|f(x) - L| < \epsilon$.

Definition 7.4. We write

$$\lim_{x \to -\infty} f(x) = L$$

iff for every $\epsilon > 0$ there exists M such that $x < M$ implies $|f(x) - L| < \epsilon$.

Example 7.1. Show that

$$\lim_{x \to \infty} \frac{x^2 + x}{x^2 + 4x + 7} = 1.$$

Take $\epsilon > 0$. We want M such that when $x > M$

$$\begin{aligned} \left| \frac{x^2 + x}{x^2 + 4x + 7} - 1 \right| &= \left| \frac{-3x - 7}{x^2 + 4x + 7} \right| \\ &< \epsilon \end{aligned}$$

Assume $x > 3$ then $3x > 7$ and

$$\left| \frac{-3x - 7}{x^2 + 4x + 7} \right| \leq \frac{3x + 7}{x^2} \leq \frac{6x}{x^2} = \frac{6}{x}.$$

Therefore if $x > \max\left\{3, 6/\epsilon\right\}$ then

$$\left| \frac{x^2 + x}{x^2 + 4x + 7} - 1 \right| < \epsilon.$$

Definition 7.5. Assume X is a subset of $\mathbb{R}$. The point $a \in \mathbb{R}$ is called a *limit point* of X if every neighbourhood of a contains another point $A \neq a$ such that $A \in X$. That is, for all $\epsilon > 0$ there exists $x \in X$ such that $x \in (a - \epsilon, a + \epsilon) \setminus \{a\}$.

Remark. The limit point need not be in X.

Example 7.2. Consider $X = (0, 1) \cup [2, 3]$. Its set of limit points is $(0, 1) \cup [2, 3]$.

Example 7.3. The natural number $\mathbb{Z}$ contains no limit points. For example,

$$\left\{n \in \mathbb{N} \mid m - 1/2 < n < m + 1/2,\ m \in \mathbb{N}\right\} = \emptyset.$$

Example 7.4. Consider $\left\{1, 1/2, 1/3, 1/4, \ldots\right\}$. This set has one limit point, namely, 0.

Definition 7.6. The *r-neighbourhood* of a is the set of all $x \in \mathbb{R}$ such that $x \in (a - r, a + r)$.

Definition 7.7. The *deleted r-neighbourhood* of a is the set of all $x \in \mathbb{R}$ such that $x \in (a - r, a + r) \setminus a$.

Definition 7.8. Assume a is a limit point and f is a function from $\Omega \subset \mathbb{R}$ to $\mathbb{R}$. We write

(1)
$$\lim_{x \to a} f(x) = L$$
iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all x, $|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta$.

(2)
$$\lim_{x \to a^+} f(x) = L$$
iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x > a$, $|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta$.

(3)
$$\lim_{x \to a^-} f(x) = L$$
iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x < a$, $|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta$.

Remark. The following remarks apply to definition (1), but could be adapted to the other cases.

1. Less formally, but perhaps more clearly, the definition can be stated thus: for all ϵ-neighbourhoods $N_\epsilon(L)$ of L there exists a deleted δ-neighbourhood $N_\delta(a) \setminus a$ of a such that $f(x) \in N_\epsilon(L)$ for all $x \in N_\delta(a)$. (this definition is less formal because you can't quantify over sets, unless using the axioms or theorems of set theory; research the difference between first and second order logic)
2. The definition involves $N_\delta(a) \setminus a$ instead of $N_\delta(a)$ to allow f to be undefined at a.

Example 7.5. Show that

$$\lim_{x \to a} x = a.$$

Take $\epsilon > 0$ Then set $\delta = \epsilon$. Now, if $0 < |x - a| < \delta$ then $|x - a| < \epsilon$.

Example 7.6. Show that

$$\lim_{x \to 2} \left(x^2 + x\right).$$

Let $\epsilon > 0$ be given. So, guess that $\delta = 1$ then $|x - 2| < \delta = 1$ (we can adjust this δ later). Then, by the triangle inequality, $|x| < 3$ so if the limit exists, $|x| < \delta = 3$ needs to imply $|x-2||x-3| < \epsilon$. So far, it already implies that Now take $\delta = \min\{\epsilon/6, 1\}$.

Example 7.7. Show

$$\lim_{x\to\infty} \frac{x^2}{x^2-4x} = 1.$$

We need to show that for all $\epsilon > 0$ there exists $M \in \mathbb{R}$ such that $x > M$ implies

$$\left|\frac{x^2}{x^2-4x} - 1\right| < \epsilon.$$

Let $\epsilon > 0$ be given and note, assuming $x > 4$, that

$$\begin{aligned}\left|\frac{x^2}{x^2-4x} - 1\right| &= \left|\frac{x^2-x^2+4x}{x^2-4x}\right| \\ &= \frac{4}{x-4}.\end{aligned}$$

Now

$$\frac{4}{x-4} < \epsilon$$

when

$$x - 4 > \frac{4}{\epsilon}$$

or

$$x > \frac{4}{\epsilon} + 4.$$

Set

$$M = \frac{4}{\epsilon} + 4.$$

Now if $x > M$, then

$$\left|\frac{x^2}{x^2-4x} - 1\right| < \epsilon.$$

Example 7.8. Show that

$$\lim_{x\to a}\left(x^2 + x\right) = a^2 + a.$$

Let $\epsilon > 0$ be given. Then we need to find some $\delta > 0$ such that

$$|x^2 + x - a^2 - a| < \epsilon$$

if

$$0 < |x-a| < \epsilon.$$

Observe that

$$\begin{aligned}|x^2 + x - a^2 - a| &= |(x-a)(x+a) + (x-a)| \\ &= |(x-a)(x+a+1)| \\ &= |x-a||x+a+1|.\end{aligned}$$

Notice that if $|x-a| < 1$ then $|x| < 1 + |a|$ by the triangle inequality. Also,

$$x + (a+1) \le |x| + |a+1| < 1 + |a| + |a+1|.$$

This means

$$\begin{aligned}|x^2 + x - a^2 - a| &= |x-a||x+a+1| \\ &< |x-a|(1 + |a| + |a+1|).\end{aligned}$$

Accordingly, set

$$\delta = \min\left\{1, \frac{\epsilon}{1+|a|+|a+1|}\right\},$$

and we're done.

Remark. The variable δ may not depend on x. However, it may depend on ϵ and a. This is easy to see if you consider a function that's steeper towards the right. For fixed ϵ, the δ towards the right is clearly going to have to be smaller.

Lemma 7.1. *Given a function $f : \Omega \to \mathbb{R}$ and a limit point $a \in \Omega$,*

$$\lim_{x\to a} f(x) = L \tag{9}$$

implies

$$\lim_{n\to\infty} f(a_n) = L.$$

for any sequence $(a_n)_{n=1}^\infty \in S$ with

$$S = \left\{(a_n)_{n=1}^\infty \in \Omega^{\mathbb{N}} \;\middle|\; \lim_{n\to\infty} a_n = a, a_n \neq a\right\}.$$

($\Omega^{\mathbb{N}}$ is the set of functions from $\mathbb{N}$ to Ω)

Remark.

(1) We require $a_n \neq a$ because f need not be defined at p. That is, if we relax the requirement, then $f(a_n)$ may not exist for some n.
(2) Suppose $f : \mathbb{R} \to \mathbb{R}$ and $f^* : \mathbb{N} \to \mathbb{R}$ satisfy $f^*(n) = f(n)$. It may be the case that $f^*(n) \to L$ as $n \to \infty$ without it being the case that $f(x) \to L$ as $x \to \infty$. On the other hand, if $f(x) \to L$ as $x \to \infty$ then it is the case that $f^*(n) \to L$ as $n \to \infty$.

Proof. Set $\epsilon > 0$. Now, assuming (9), there exists a $\delta > 0$ such that for all x, $|f(x) - L| < \epsilon$ if $0 < |x-a| < \delta$. But if $(a_n)_{n=1}^\infty \in S$, then for all natural $n \ge N$ we can set $0 < |a_n - a| < \delta$. Therefore $|f(a_n) - L| < \epsilon$ for all $n \ge N$ — that is,

$$\lim_{n\to\infty} f(a_n) = L$$

□

Lemma 7.2. *Given a function $f : \Omega \to \mathbb{R}$ and a limit point $a \in \Omega$,*

$$\lim_{x \to a} f(x) = L \tag{10}$$

if

$$\lim_{n \to \infty} f(a_n) = L. \tag{11}$$

for any sequence $(a_n)_{n=1}^{\infty} \in S$ with

$$S = \left\{ (a_n)_{n=1}^{\infty} \in \Omega^{\mathbb{N}} \;\middle|\; \lim_{n \to \infty} a_n = a, a_n \neq a \right\}.$$

($\Omega^{\mathbb{N}}$ is the set of functions from $\mathbb{N}$ to Ω)

Remark.

(1) In logical notation, the definition of a limit is

$$\forall \epsilon > 0 \exists \delta > 0 \forall x.$$
$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

Its negation is

$$\exists \epsilon > 0 \forall \delta > 0 \exists x.$$
$$0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon$$

— or equivalently, for some ϵ-neighbourhood $N_\epsilon(L)$ of L, no matter what size you set the deleted δ-neighbourhood $N_\delta(a) \setminus a$ of a, you can fit $f(x)$ in $\mathbb{R} \setminus N_\epsilon(L)$ but not in $N_\epsilon(L)$ if $x \in N_\delta(a) \setminus a$.

(2) It's easiest to make peace from the outset with the following fact, which mathematicians have chosen as part of the model of geometry formalized by the definition of a limit: we can prove a limit exists by proving that its negation leads to a contradiction. Alternately, there are two ways to pin down the definition of a limit — it's definition and the negation of the definition. This is of course true of any logical formula, but it's important to verify that this idea leads to a consistent and reasonable mental picture. If the mental picture was not consistent, contradictions could reasonably be expected to occur.

(3) A sketch of the proof follows. Assume a mental image of all possible sequences approaching a, and the corresponding sequences in the range of f approaching L. Then we show that the negation of (10) does not fit with this idea — so it is not the case that for some ϵ-neighbourhood $N_\epsilon(L)$ of L, no matter what size you set the deleted δ-neighbourhood $N_\delta(a) \setminus a$ of a, you can fit $f(x)$ only in $\mathbb{R} \setminus N_\epsilon(L)$ if $x \in N_\delta(a) \setminus a$. Now if $S = \emptyset$ then anything is true for it — otherwise, the sequence $(f(a_n))_{n=1}^{\infty}$ exists and can't exist in $\mathbb{R} \setminus N_\epsilon(L)$. Therefore the only place left for it to exist is in $N_\epsilon(L)$.

(4) The proof below is a good example of using the axiom of choice from ZFC set theory (where does it use it?).

Proof. Suppose (11) is true. Now if (10) is false then there exits an $\epsilon > 0$ such that for all $\delta > 0$ there exists a a such that $0 < |x - a| < \delta$ but $|f(a) - L| > \epsilon$. Therefore, setting $\delta = 1/n$ and using the fact that a is a limit point, we can create a sequence $(a_n)_{n=1}^{\infty}$ converging to a for which $(f(a_n))_{n=1}^{\infty}$ does not converge. But this contradicts our assumption that (11) holds. ☐

Proposition 7.3 (Sequential criterion for functional convergence)**.** *Given a function $f : \Omega \to \mathbb{R}$ and a limit point $a \in \Omega$,*

$$\lim_{x \to a} f(x) = L \tag{12}$$

iff

$$\lim_{n \to \infty} f(a_n) = L. \tag{13}$$

for any sequence $(a_n)_{n=1}^{\infty} \in S$ with

$$S = \left\{ (a_n)_{n=1}^{\infty} \in \Omega^{\mathbb{N}} \;\middle|\; \lim_{n \to \infty} a_n = a, a_n \neq a \right\}.$$

($\Omega^{\mathbb{N}}$ is the set of functions from $\mathbb{N}$ to Ω)

Proof. The proposition follows directly from the lemmas. ☐

Proposition 7.4. *Suppose*

$$\lim_{x \to a} f(x) = L$$

and

$$\lim_{x \to a} g(x) = M.$$

Then the following holds

(1)
$$\lim_{x \to a} (f + g)(x) = L + M,$$

(2)
$$\lim_{x \to a} (fg)(x) = LM,$$

(3) *if $m \neq 0$,*
$$\lim_{x \to a} \left(\frac{f}{g} \right)(x) = \frac{L}{M}.$$

Remark. The proposition follows directly from the corresponding laws for sequences and the sequential criterion for functional convergence. For pedagogical purposes we will prove it using $\epsilon - \delta$ proofs instead.

Proof.

(1) Exercise

(2) Let $\epsilon > 0$ be given. We want $\delta > 0$ such that if
$$0 < |x - a| < \delta$$
then
$$|(fg)(x) - LM| < \epsilon.$$
Note that
$$\begin{aligned} &|(fg)(x) - LM| \\ &= |(fg)(x) + Mf(x) - Mf(x) - LM| \\ &= |f(x)(g(x) - M) + M(f(x) - L)| \\ &\leq |f(x)||g(x) - M| + |M||f(x) - L| \end{aligned}$$
For $\epsilon \leq 1$ there is $\delta_1 > 0$ such that if
$$0 < |x - a| < \delta_1$$
then
$$|f(x) - L| < \epsilon.$$
Or equivalently, using the triangle inequality, if
$$0 < |x - a| < \delta_1$$
then
$$|f(x)| < |l| + 1.$$
For such x we have
$$\begin{aligned} |(fg)(x) - LM| \leq &(|L| + 1)|g(x) - M| + \\ &|m||f(x) - L|. \end{aligned}$$
For
$$\frac{\epsilon}{2(|l| + 1)}$$
there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2$ implies
$$|g(x) - m| < \frac{\epsilon}{2(|L| + 1)}.$$
Also, there is $\delta_3 > 0$ such that $0 < |x - a| < \delta_3$ implies
$$|f(x) - L| < \frac{\epsilon}{2(|M| + 1)}.$$
Take δ to be the minimum of all these δs. If $0 < |x - a| < \delta$ then
$$\begin{aligned} &|(fg)(x) - LM| \\ &< \frac{(|L| + 1)\epsilon}{2(|L| + 1)} + \frac{(|M| + 1)|M|\epsilon}{2(|M| + 1)} \\ &= \epsilon. \end{aligned}$$

(3) Exercise

□

Proposition 7.5 (Squeeze theorem)**.** *Define three functions: $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$, and $h : X \to \mathbb{R}$. If*
$$f(x) \leq h(x) \leq g(x)$$
for all $x \in X$ and if
$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$
then
$$\lim_{x \to a} f(x) = \lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

8. Continuity

Definition 8.1 (Continuity)**.** The function f from $\Omega \subset \mathbb{R}$ to $\mathbb{R}$ is *continuous* at $x_0 \in \Omega$ iff

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Remark. Alternately, f is continuous iff for every ϵ-neighbourhood $N_\epsilon[f(x_0)]$ of $f(x_0)$ there exists a δ-neighbourhood $N_\delta(x_0)$ such that for all x, $f(x) \in N_\epsilon[f(x_0)]$ whenever $x \in N_\delta[x_0]$. Compared to the definition of a limit, we use a $N_\delta(x_0)$ instead of $N_\delta(x_0)\backslash x_0$.

Definition 8.2 (Continuity on an open interval)**.** The function f from $\Omega \subset \mathbb{R}$ is continuous on $(a,b) \subset \Omega$ iff

$$\lim_{x \to x_0} = f(x_0).$$

for all $x_0 \in (a,b)$.

Definition 8.3 (Continuity on a closed interval)**.** The function f from $\Omega \subset \mathbb{R}$ is continuous on $[a,b] \subset \Omega$ iff it is continuous on (a,b),

$$\lim_{x \to a^-} f(x) = f(a),$$

and

$$\lim_{x \to b^+} f(x) = f(b).$$

Proposition 8.1. *If $f : (a,b) \to \mathbb{R}$ and $g : (a,b) \to \mathbb{R}$ are continuous at $x_0 \in (a,b)$ then*

1. *$f + g$ is continuous at x_0*
2. *fg is continuous at x_0*
3. *f/g is continuous at x_0 if $g(x_0) \neq 0$.*

Proof. Follows from the properties of limits. □

Corollary 8.2. *Any polynomial or rational function is continuous where defined.*

Example 8.1. The function

$$f(x) = \frac{1}{x}$$

is continuous on $(0, \infty)$. Let $x_0 \in (0, \infty)$ be given and let $\epsilon > 0$ be given. Then we want to show that there is some $\delta > 0$ such that if $|x - x_0| < \delta$ then

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| < \epsilon.$$

We compute

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| = \left| \frac{x_0 - x}{x x_0} \right| = \frac{|x - x_0|}{|x||x_0|}.$$

Now we can see that we need to get rid of the x dependence in the denominator. Assume $|x - x_0| \leq |x_0|/2$. Then, since $x_0 \in (0, \infty)$,

$$\frac{-x_0}{2} \leq x - x_0 \leq \frac{x_0}{2}$$

so $2x \geq x_0$. Now

$$\frac{|x - x_0|}{|x||x_0|} \leq \frac{|x - x_0|}{|x_0||x_0/2|} = \frac{2|x - x_0|}{x_0^2}$$

Therefore, if $|x - x_0| < \delta < |x_0|/2$ then

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| < \frac{2|x - x_0|}{x_0^2}.$$

Set

$$\delta = \min \left\{ \frac{\epsilon x_0^2}{2}, \frac{x_0^2}{2} \right\}.$$

Now if $0 < |x - x_0| < \delta$ then

$$\left| \frac{1}{x} - \frac{1}{x_0} \right| < \epsilon.$$

Proposition 8.3 (Intermediate Value Theorem)**.** *Suppose f is a function from $\Omega \subset \mathbb{R}$ to $\mathbb{R}$ that is continuous on $[a,b] \subset \Omega$. If $f(a) < \lambda < f(b)$ then there exists $c \in [a,b]$ such that $f(c) = \lambda$.*

Proof. Without loss of generality, assume $\lambda = 0$ (if $\lambda \neq 0$ simply consider $f(x) - \lambda$ instead of f). Set

$$\mu_1 = \frac{a+b}{2}.$$

If $f(\mu_1) > 0$ set $a_1 = a$ and $b_1 = \mu_1$ If $f(\mu_1) < 0$ set $a_1 = \mu_1$ and $b_1 = b$. Set

$$\mu_2 = \frac{a_1 + b_1}{2}.$$

If $f(\mu_1) > 0$ set $a_2 = a_1$ and $b_2 = \mu_2$ If $f(\mu_1) < 0$ set $a_1 = \mu_2$ and $b_2 = b_1$. Continue this process to get the sequences $(a_n)_{n=1}^\infty$. and $(b_n)_{n=1}^\infty$ so that

1. $(a_n)_{n=1}^\infty$ is non-decreasing,
2. $(b_n)_{n=1}^\infty$ is non-increasing,
3. $(a_n)_{n=1}^\infty$ is bounded since $a_n < b$ for all n,
4. $(a_n)_{n=1}^\infty$ is bounded since $a < b_n$ for all n, and
5. the distance $|a_n - b_n| = (b-a)/2^n$.

Therefore, since monotone bounded sequences are convergent, there exists p_a and p_b such that

$$\lim_{n \to \infty} a_n = p_a$$

and

$$\lim_{n \to \infty} b_n = p_b.$$

Then, by (5),

$$\lim_{n \to \infty} |a_n - b_n| = 0 = |p_a - p_b|.$$

Therefore $p_a = p_b$. Since f is continuous,

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(p).$$

Furthermore $f(a_n) < 0 < f(b_n)$ for all n. Using these two facts, the Squeeze theorem gives us $f(p) = 0$. $\square$

Example 8.2. Let $f : [1, 2] \to \mathbb{R}$ be continuous. Assume $f(1) = 0$ and $f(2) = 3$. Show that there is some $x \in (1, 2)$ such that $f(x) = x$. Consider the function $g(x) = f(x) - x = x$. Then $g(1) = -1$, and $g(2) = 1$. Now, by the IVT, there is $x \in (1, 2)$ such that $g(x) = 0$.

Example 8.3. Show that every polynomial of degree 3 has a real root. Let

$$p(x) = ax^3+bx^2+cx+d = ax^3\left(1 + \frac{b}{ax} + \frac{c}{ax^2} + \frac{d}{ax^3}\right)$$

where $a \neq 0$ and $x \neq 0$. Since

$$\lim_{x\to\infty}\left(\frac{b}{ax} + \frac{c}{ax^2} + \frac{d}{ax^3}\right) = 0,$$

there is some $M > 0$ such for all $x \geq M$,

$$\left|\frac{b}{ax} + \frac{c}{ax^2} + \frac{d}{ax^3}\right| < \frac{1}{2}$$

so

$$\frac{1}{2} < 1 + \frac{b}{ax} + \frac{c}{ax^2} + \frac{d}{ax^3}.$$

Therefore

$$0 < \frac{ax^3}{2} < ax^3\left(1 + \frac{b}{ax} + \frac{c}{ax^2} + \frac{d}{ax^3}\right).$$

so there exists some β such that $p(\beta) > 0$.

Similarly, since

$$\lim_{x\to-\infty}\left(\frac{b}{ax} + \frac{c}{ax^2} + \frac{d}{ax^3}\right) = 0,$$

there is some $M < 0$ such for all $x \leq M$,

$$0 < \frac{\left|ax^3\right|}{2} < \left|ax^3\right|\left(1 + \frac{b}{ax} + \frac{c}{ax^2} + \frac{d}{ax^3}\right)$$

but

$$0 > \frac{ax^3}{2} > ax^3\left(1 + \frac{b}{ax} + \frac{c}{ax^2} + \frac{d}{ax^3}\right)$$

so there exists some α such that $p(\alpha) < 0$.

Now we have that

$$p(\alpha) < 0 < p(\beta).$$

Therefore, since p is continuous, the IVT tells us that there exists some x between α and β such that $f(x) = 0$.

Definition 8.4. Suppose f is a function from $\Omega \subset \mathbb{R}$ to $\mathbb{R}$. Then f is *uniformly continuous* on Ω iff given any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \Omega$, $f(x) - f(y) < \epsilon$ if $|x - y| < \delta$.

Remark. Note that now δ cannot depend on a point x in the interval, as is possible for regular continuity. Rather, you need one δ that works for *all* points in the interval.

We could choose set $\delta = \inf_{x\in\Omega}\delta_x$ for a fixed epsilon, but only if Ω is closed (as we will prove.)

Example 8.4. The function $f(x) = x$ on $\mathbb{R}$ is uniformly continuous. Take $\epsilon > 0$ and set $\delta = \epsilon$. Now if $|x - y| < \delta$ then

$$|f(x) - f(y)| = |x - y| < \delta.$$

Since δ does not depend on x or y, f is uniformly continuous on $\mathbb{R}$ as stated.

Example 8.5. The function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. Assume it is. Then fix $\epsilon > 0$. Then there is some $\delta > 0$ such that $x, y \in (0, \infty)$ and for all x, y if $|x - y| < \delta$ then $|x^2 - y^2| < \epsilon$. Choose $y = x + \delta/2$. Then if $x \in (0, \infty)$, $y \in (0, \infty)$ and

$$|x - y| = \frac{\delta}{2} < \delta.$$

Consider $|x^2 - y^2|$. We have

$$\begin{aligned}|x^2 - y^2| &= \left|x^2 - \left(x + \frac{\delta}{2}^2\right)\right| \\ &= \left|-x\delta - \frac{\delta^2}{4}\right| \\ &= x\delta + \frac{\delta^2}{4} \\ &> x\delta.\end{aligned}$$

Choose $x = \epsilon/\delta$. Then

$$\left|x^2 - y^2\right| > x\delta = \epsilon.$$

But this contradicts the definition.

Example 8.6. Show that the function $1/x$ is uniformly continuous on the interval $(0, 1)$. (Answer: No.)

Proposition 8.4. *Suppose f is continuous on a closed and bounded interval $[a, b]$. Then f is uniformly continuous on $[a, b]$.*

Proof. Want to prove that f is uniformly continuous: For every $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Assume that the uniformly continuity fails. Then there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $x, y \in [a, b]$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$. Take $\delta = 1$. There exists $x_1, y_1 \in [a, b]$ such that $|x_1 - y_1| < \delta = 1$ but $f(x) - f(y) \geq \epsilon$. There exists $x_2, y_2 \in [a, b]$ such that $|x_2 - y_2| < \delta = 1/2$ but $f(x) - f(y) \geq \epsilon$. Continuing this, for $\delta = 1/n$ there are $x_n, y_n \in [a, b]$ such that

$$x_n - y_n < \frac{1}{n}$$

but $|f(x_n) - f(y_n)| \geq \epsilon$. The sequence $(x_n)_{n=1}^{\infty} \subset [a, b]$ is bounded and therefore has a convergent subsequence $(x_{n_k})_{n=1}^{\infty}$. Set

$$\lim_{k \to \infty} x_{n_k} = x_0.$$

Furthermore,

$$\lim_{k \to \infty} y_{n_k} = x_0.$$

since

$$\begin{aligned} |y_{n_k} - x_0| &= |(y_{n_k} - x_{n_k}) + (x_{n_k} - x_0)| \\ &\leq |y_{n_k} - x_{n_k}| + |x_{n_k} - x_0|. \end{aligned}$$

Now we are assuming that f is continuous at x_0 but then

$$|f(x_{n_k}) - f(x_0)| < \frac{\epsilon}{4}$$

and

$$|f(y_{n_k}) - f(x_0)| < \frac{\epsilon}{4}$$

for large enough k. Then

$$\begin{aligned} &|f(x_{n_k}) - f(y_{n_k})| \\ &= |f(x_{n_k} - f(x_0) + f(x_0) - f(y_{n_k})| \\ &\leq |f(x_{n_k} - f(x_0)| + |f(x_0) - f(y_{n_k})| \end{aligned}$$

Therefore

$$|f(x_{n_k}) - f(y_{n_k})| < \frac{\epsilon}{2}.$$

However, this contradicts our assumption that $|f(x_n) - f(y_n)| \geq \epsilon$ for all n. □

Remark.

(1) Uniform continuity on I implies uniform continuity on any subset of I.
(2) The converse is false in general

Example 8.7. The function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$, but it is uniformly continuous on any closed and bounded interval.

9. Sequences and series of functions

Definition 9.1. A *sequence of functions* is a function $f : \mathbb{N} \times \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}$. We say that the sequence of functions f_n converge pointwise to f on Ω iff for all $x \in \Omega$, $f_n(x) \to f(x)$.

Remark. For a particular x, say, y, we could set $f_n(y) = s_n$ and $f(y) = s$. For this particular value of x, we need to show that $s_n \to s$.

Example 9.1. Let $f : [0, \infty) \to \mathbb{R}$, $x \in [0, \infty)$, and $f_n(x) = x^n$. Then $f_n(x) \to f(x)$ where

$$f(x) = \begin{cases} 0 & \text{for } x \in [0, 1) \\ 1 & \text{if } x = 1 \end{cases}$$

Remark. Note that $f_n(x)$ is continuous on $[0, 1)$ (i.e. in its second variable) but the limit $f(x)$ is discontinuous. This motivates us to devise a stronger notion of convergence, namely, that of uniform continuity.

Definition 9.2. We say f_n converges to f *uniformly* on Ω if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \epsilon$ for all $n \geq N$ and $x \in \Omega$.

Example 9.2. The functions $f_n(x) = x^n$ *do not* uniformly converge to

$$f(x) = \begin{cases} 0 & \text{for } x \in [0, 1) \\ 1 & \text{if } x = 1 \end{cases}$$

on $\mathbb{R}$. Suppose that it does converge uniformly. That is, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \epsilon$ for all $n > N$ and $x \in [0, 1]$. Now, setting $\epsilon = 1/4$ we can find $N \in \mathbb{N}$ such that

$$|0 - f_n(x)| < \frac{1}{4}$$

for all $x \in [0, 1]$. But $f_n(x)$ are continuous functions so we have

$$|f_n(1)| \leq \frac{1}{4}.$$

But $f_n(1) = 1$, so we have reached a contradiction. Therefore f_n does not uniformly converge to f. Note, however, that $f_n(x)$ uniformly converges to $f(x)$ for $x \in [0, 1)$.

Example 9.3. The functions $f_n(x) = x^n$ uniformly converge to

$$f(x) = \begin{cases} 0 & \text{for } x \in [0, 1) \\ 1 & \text{if } x = 1 \end{cases}.$$

for $x \in [0, 1/2]$. We want to show that given $\epsilon > 0$ we can find $N \in \mathbb{N}$ such that $|f_n(x)| < \epsilon$ for all $n \geq N$

and for all $x \in [0,1/2]$. Note that a maximum of x^n on $[0,1/2]$ is achieved at $(1/2)^n$ so we choose N such that $(1/2)^N < \epsilon$, when $-N\log 2 < \log\epsilon$ or equivalently

$$N > \frac{-\log\epsilon}{\log 2}.$$

Example 9.4. Set $f : \mathbb{R} \to \mathbb{R}$ such that

$$f_n(x) = [x(1-x)]^n.$$

Note that

$$|x(1-x)| = 1$$

or equivalently

$$x(1-x) = \pm 1$$

so either

$$x^2 - x + 1 = 0$$

or

$$x^2 - x - 1 = 0.$$

But x can't be complex, so

$$x = \frac{1+\sqrt{5}}{2},$$

which is the golden ratio. As an exercise, find when $f_n(x)$ is uniformly convergent (use the continued fraction of the golden ratio).

Proposition 9.1. *If $f_n : [a,b] \to \mathbb{R}$ are continuous functions and f_n uniformly converges to f then f is also continuous.*

Remark. To say that f is continuous means

$$\lim_{t\to x} f(t) = f(x).$$

Therefore, to ask whether the limit of a sequence of continuous functions is continuous is the same as to ask whether

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t).$$

Proof. We we want to show that given $\epsilon > 0$ we can find N such that $|f(x) - f_n(x)| < \epsilon$ for $n > N$ and $x \in [a,b]$. But

$$\begin{aligned} &|f(x) - f(y)| \\ &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ &\le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|. \end{aligned} \tag{14}$$

Therefore we we want to show that given $\epsilon > 0$ we can find N such that

$$r < \frac{\epsilon}{3}$$

for $r \in a, b, c$ where $a = |f(x) - f_n(x)|$, $b = |f_n(x) - f_n(y)|$, and $c = |f_n(y) - f(y)|$. We we want to show that given $\epsilon > 0$ we can find N such that

$$|f(x) - f_n(x)| < \frac{\epsilon}{3}$$

for $n > N$ and $x \in [a,b]$. Since f_n is closed and bounded, it is uniformly continuous for each $x, y \in [a,b]$. That is, given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in [a,b]$ if $|x-y| < \delta$ then $r < \epsilon/3$ for all n and $r \in \{a, b, b\}$. Therefore, using (14) , given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in [a,b]$ if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$. □

Proposition 9.2. *The functions f_n converge uniformly to f iff for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $k, l > N$ for all x, $|f_k(x) - f_l(x)| < \epsilon$.*

Proof. We want to show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon/2$ for all $k > N$ and $x \in \Omega$. So then

$$\begin{aligned} |f_k(x) - f_l(x)| &\le |f_k(x) - f(x)| + |f(x) - f_l(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}. \end{aligned}$$

Conversely, for all $x \in \Omega$ $\{f_n(x)\}$ is Cauchy so f_n converges. Let $f(x) = \lim_{n\to\infty} f_n(x)$. Then $f(x)$ is a function from Ω to $\mathbb{R}$. We claim that f_n converges to f so given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $k, l > N$ implies

$$|f_n(x) - f_l(x)| < \frac{\epsilon}{2}$$

for all x. Let $k \to \infty$ and fix l. Therefore

$$|f(x) - f_l(x)| \le \frac{\epsilon}{2} < \epsilon.$$

□

Proposition 9.3. *Suppose $(a_n)_{n=1}^{\infty}$ is Cauchy. Then for any uniformly continuous function f, $(f(a_n))_{n=1}^{\infty}$ is Cauchy.*

Proof. Suppose $(a_n)_{n=1}^{\infty}$ is Cauchy and f is uniformly continuous on $\mathbb{R}$.

(1) The function f is uniformly continuous on $\mathbb{R}$ iff for any $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for any $(x,y) \in X_\delta$ where

$$X_\delta = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; |x-y| < \delta\right\}.$$

Therefore, for any $\epsilon > 0$ there exists $\delta > 0$ such that $|f(a_i) - f(a_j)| < \epsilon$ for any $(i,j) \in I$

where

$$I_\delta = \left\{(i,j) \in \mathbb{N}^2 \mid |a_i - a_j| < \delta\right\}$$

since $I_\delta \subset X$.

(2) The sequence $(a_n)_{n=1}^\infty$ is Cauchy iff for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_i - a_j| < \epsilon$ if $i, j \geq N$. Therefore, for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $(i,j) \in I_\epsilon$ if $i, j \geq N$.

Suppose $\epsilon > 0$. Using (1) then setting $\delta = \delta_0$, $|f(a_i) - f(a_j)| < \epsilon$ if $(i,j) \in I_{\delta_0}$. But using (2) then setting $\epsilon = \delta_0$ and $N = N_0$, $(i,j) \in I_{\delta_0}$ if $i, j \geq N_0$. Therefore, $|f(a_i) - f(a_j)| < \epsilon$ if $i, j \geq N_0$. Therefore for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f(a_i) - f(a_j)| < \epsilon$ for all $i, j \geq N$; that is, $(f(a_n))_{n=1}^\infty$ is Cauchy for all n. We can conclude that if f is uniformly continuous, then $(f(a_n))_{n=1}^\infty$ is Cauchy. □

Remark. Equivalently, if $(f(a_n))_{n=1}^\infty$ is not Cauchy, then f is not uniformly continuous. Then, since a sequence is Cauchy iff it converges, if we can find a sequence $(a_n)_{n=1}^\infty$ that converges, but for which $(f(a_n))_{n=1}^\infty$ does not, then f is not uniformly continuous.

Definition 9.3. A *series of functions* is sequence $s : \mathbb{N} \times \Omega \to \mathbb{R}$ of the form

$$s_n(x) = \sum_{i=1}^{n} f_n(x)$$

where $f : \mathbb{N} \times \Omega \to \mathbb{R}$ is a function and $\Omega \subset \mathbb{R}$. We say that

$$\sum_{n=1}^{\infty} f_n(x) = f(x)$$

converges pointwise on Ω to the *uniform sum* $f(x)$ iff $s_n \to s$ pointwise. Additionally, we say that the series *converges pointwise* on Ω to the *uniform sum* $f(x)$ iff s_n converges uniformly iff $s_n \to s$ uniformly.

Example 9.5. Let

$$f_n(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Then f_n is a series of functions.

Proposition 9.4 (Weierstrass M-test)**.** *Suppose $(f_n(x))_n^\infty$ is defined for all $x \in \Omega$ where $\Omega \subset \mathbb{R}$ such that for all $n \in \mathbb{N}$ $|f_n(x)| < M_n$ where $(M_n)_{n=1}^\infty$ is a sequence. Then $\sum_{n=1}^\infty$ converges uniformly on Ω if $\sum_{n=1}^\infty M_n$ converges.*

Remark. Note that the converse is not asserted.

Proof. If $\sum_{n=1}^\infty M_n$ converges, then for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\sum_{i=N}^{n} M_i \leq \epsilon.$$

But for all i,

$$M_i > \left|f_i(x)\right| > 0$$

by assumption. Therefore, for all $x \in \Omega$,

$$\left|\sum_{i=N}^{n} f_i(x)\right| \leq \epsilon.$$

Therefore for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in \Omega$,

$$\left|\sum_{i=N}^{n} f_i(x)\right| \leq \epsilon;$$

that is, $\sum_{n=1}^\infty$ converges uniformly. □

$\square$

Definition 10.1. Suppose $x_0 \in [a,b)$ and $f : [a,b) \to \mathbb{R}$. The *right derivative* of f at x_0 is

$$\lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}.$$

If these limits exist, then f is *right differentiable* at x_0.

Definition 10.2. Suppose $x_0 \in (a,b)$ and $f : (a,b) \to \mathbb{R}$. The *left derivative* of f at x_0 is

$$\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}.$$

If these limits exist, then f is *left differentiable* at x_0.

Definition 10.3. Suppose $x_0 \in (a,b)$ and $f : (a,b) \to \mathbb{R}$. If the left derivative of f at x_0 is equal to the right derivative of f at x_0, then the common value these limits is called the *derivative* of f at x_0 and f is said to be *differentiable* at x_0.

Definition 10.4. Define differentiability on open and closed intervals, analogously to how continuity was defined for open and closed intervals.

Proposition 10.1. *Suppose $x_0 \in (ab)$ and $f : (a,b) \to \mathbb{R}$. If f is differentiable at x_0 then its derivative is given by*

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}.$$

Example 10.1. The equation of the tangent line at x_0

$$y = f'(x_0)(x-x_0)+f(x_0)$$

Proposition 10.2. *If f is differentiable at x_0 then f is continuous at x_0.*

Proof. We want to show that if

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0},$$

$$\lim_{x\to x_0} f(x) = f(x_0).$$

We know

$$f(x) = (x-x_0)\frac{f(x)-f(x_0)}{x-x_0} + f(x_0).$$

Therefore

$$\begin{aligned}
&\lim_{x\to x_0} f(x)\\
&= \lim_{x\to x_0}(x-x_0)\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} + \lim_{x\to x_0} f(x_0)\\
&= 0\times f'(x_0)+f(x_0)\\
&= f(x_0).
\end{aligned}$$

Example 10.2. The function

$$f(x) = |x|$$

is uniformly continuous on $[-1,1]$. However, it's not differentiable.

Example 10.3. Computer the derivative of x^3 by first principles thus:

$$\lim_{h\to 0}\frac{(x+h)^3-x^3}{h} = \lim_{h\to 0}\frac{x^3+h^3+3h^2x+3hx^2-x^3}{h}$$

This simplifies to

$$\lim_{h\to 0}\left(h^2+3hx+3x^2\right) = 3x^2.$$

Lemma 10.3. *Suppose*

1. *the intervals $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$ are open,*
2. *the function $g : I \to J$ is differentiable at $x_0 \in I$, and*
3. *the function $f : J \to \mathbb{R}$ is differentiable at $g(x) \in J$*

then $f' \circ g(x_0)$ is given by

$$\begin{aligned}
&\lim_{h\to 0}\frac{f[g(x_0)+h]-f\circ g(x_0)}{h}\\
&= \lim_{h\to 0}\frac{f\circ g(x_0+h)-f\circ g(x_0)}{g(x_0+h)-g(x_0)}.
\end{aligned} \tag{15}$$

Remark. In the Cartesian plane set $y = f(x)$. We claim in the limit of $\Delta y/\Delta x$ about $x = g(x_0)$ as $\Delta x \to 0$ doesn't depend on the speed that we travel along the y axis towards $f'\circ g(x_0)$ and the speed that we travel along the x axis towards $g(x_0)$.

The intuition behind the proof is that that

1. $g(x_0+h)-g(x_0) \to 0$
2. $f[g(x_0)+h] \to f\circ g(x_0)$ and $f\circ g(x_0+h) \to f\circ g(x_0)$

as $h \to 0$. In other words, for equation (15), the denominator and the numerator are approximately the same for small h.

Proof. Set $\epsilon > 0$. By the differentiability of f, there exists a $\delta > 0$ such that for all r, $0 < |r| < \delta^*$ implies

$$\left|\frac{f[g(x_0)+r]-f\circ g(x_0)}{r} - f'\circ g(x_0)\right| < \epsilon.$$

But since g is differentiable and hence continuous, we can set

$$0 < |g(x_0+h)-g(x_0)| < \delta^*$$

assuming $0 < |h| < \delta$. Now, setting $r = g(x_0 + h) - g(x_0)$,

$$\left| \frac{f \circ g(x_0 + h) - f \circ g(x_0)}{g(x_0 + h) - g(x_0)} - f' \circ g(x_0) \right| < \epsilon$$

if $0 < |h| < \delta$. Therefore

$$f' \circ g(x_0) = \lim_{h \to 0} \frac{f \circ g(x_0 + h) - f \circ g(x_0)}{g(x_0 + h) - g(x_0)}$$

as required. □

Proposition 10.4 (Chain rule)**.** *Suppose*

(1) *the intervals $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$ are open,*
(2) *the function $g : I \to J$ is differentiable at $x_0 \in I$, and*
(3) *the function $f : J \to \mathbb{R}$ is differentiable at $g(x) \in J$*

Then the function $f \circ g : I \to \mathbb{R}$ is differentiable at x_0 and

$$(f \circ g)'(x_0) = f' \circ g(x_0) \cdot g'(x_0). \tag{16}$$

Proof. The left hand side of (16) is

$$(f \circ g)'(x_0) = \lim_{x \to x_0} \frac{f \circ g(x) - f \circ g(x_0)}{x - x_0}.$$

The right hand side of (16) is the product of

$$g'(x_0) = \frac{g(x) - g(x_0)}{x - x_0}$$

with the equation given by the lemma

$$f' \circ g(x_0) = \lim_{x \to x_0} \frac{f \circ g(x) - f \circ g(x_0)}{g(x) - g(x_0)}.$$

But this product is clearly equal to the left hand side of (16), as was required to be demonstrated. □

Proposition 10.5. *Take $f : I \to \mathbb{R}$, and $g : I \to \mathbb{R}$ where $I \subset R$ is open and assume they are differentiable at $x \in I$. Then $f + g$, fg are differentiable. Similarly f/g is differentiable if $g \neq 0$ for all values of g. Further*

(1) *$(f + g)' = f' + g'$ and*
(2) *$fg = fg' + gf'$.*

Definition 10.5. The function f from $E \subset \mathbb{R}$ to $\mathbb{R}$ has a *local maximum* at $c \in E$ iff there exists an open interval $U \subset R$ such that $c \in U$ and $f(c) \geq f(x)$ for all $x \in U \cap E$.

Definition 10.6. The function f from $E \subset \mathbb{R}$ to $\mathbb{R}$ has a *local minimum* at $c \in E$ iff there exists an open interval $U \subset R$ such that $c \in U$ and $f(c) \leq f(x)$ for all $x \in U \cap E$.

Definition 10.7. Suppose that the function f from $E \subset \mathbb{R}$ to $\mathbb{R}$ and there exists $c \in E$ such that $f(c) \geq f(x)$ for all $x \in E$. Then f is said to have an absolute maximum at c.

Definition 10.8. Suppose that the function f from $E \subset \mathbb{R}$ to $\mathbb{R}$ and there exists $c \in E$ such that $f(c) \leq f(x)$ for all $x \in E$. Then f is said to have an absolute minimum at c.

Proposition 10.6. *Suppose $f : [a, b] \to \mathbb{R}$ is a function and $c \in (a, b)$. If f has a local extremum at c and $f'(c)$ exists then $f'(c) = 0$.*

Proof. Without loss of generality, assume c is a local maximum (otherwise, consider $-f$ instead of f). There exists $\delta > 0$ such that $[c - \delta, c + \delta] \subset [a, b]$ and $f(x) \leq f(c)$ for all $x \in [c - \delta, c + \delta]$. Consider the function

$$q(x) = \begin{cases} \frac{f(x) - f(c)}{x - c} & x \neq c \\ f'(c) & x = c \end{cases}$$

The function is continuous at c. When $x \in [c - \delta, c]$, $q(x) \geq 0$. Also, when $x \in [c, c + \delta]$, $q(x) \leq 0$. By continuity this implies $q(c) = 0$. But $q(c) = f'(c)$ so $f'(c) = 0$. □

Remark.

(1) If $c = a$ at $c = b$ the $f'(c)$ is not necessarily 0. For example, if $f(x) = x^3$ then $f'(0) = 3x^2|_{x=0} = 0$. Note that 0 is not a local extremum
(2) The theorem only works when f is differentiable. The function $f(x) = |x|$ when $x = 0$ is a local minimum but $f'(0)$ does not exist.

Proposition 10.7 (Rolle's theorem)**.** *Suppose f is a function from $\Omega \subset \mathbb{R}$ to $\mathbb{R}$ that is continuous on $[a, b] \subset \Omega$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists $\xi \in (a, b)$ such that $f'(\xi) = 0$.*

Proof. If $f(x) = f(a)$ for all $x \in [a, b]$, then $f' = 0$ on (a, b) and we're done. If not, assume that there exists $x_0 \in (a, b)$ such that $f(x_0) > f(a) = f(b)$ (if no such expression exists, simply consider $-f$.) Now, f achieves its maximum on $[a, b]$. Further, it's not at a or b by our assumption, so it is in (a, b). Therefore, since f is differentiable on (a, b) and has a maximum, $f'(x_0) = 0$. □

Proposition 10.8 (Mean value theorem)**.** *Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable* *on (a,b). Then there exists $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Proof. Let

$$g(x) = \frac{f(b)-f(a)}{b-a}(x-a) + f(x) - f(a).$$

The theorem follows immediately now from Rolle's theorem. ☐

Proposition 10.9. *Assume $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). If $f'(x) = 0$ for all $x \in (a,b)$ then $f = a$ for $a \in [a,b]$.*

Proof. Take $x \in (a,b)$. Apply the MVT on $[a,x]$ then $f(x) - f(a) = f'(\zeta)(x-a)$ for some $\zeta \in (a,x)$. Thus, $f(x) = f(a)$ for all $x \in (a,b]$. ☐

Proposition 10.10. *Assume $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Also assume $|f'(x)| \le M$ for all $x \in (a,b)$. Then $|f(x) - f(y)| \le M|x-y|$ for all $x, y \in (a,b)$.*

Proof. Without loss of generality, assume $y > x$. Apply the MVT on $[x,y]$ to obtain

$$f(y) - f(x) = f'(\zeta)(y-x)$$

for some $\zeta \in (x,y)$. This implies

$$|f(y) - f(x)| = |f'(\zeta)||y-x| \le M|y-x|.$$

☐

Proposition 10.11. *Assume f is continuous on $[a,b]$ and differentiable on (a,b).*

(1) *if $f' \ge 0$ on (a,b) then f is non-decreasing.*
(2) *If $f' \le 0$ on (a,b) then f is non-increasing.*
(3) *if $f' > 0$ then f is strictly increasing*
(4) *$f' < 0$ then f is strictly decreasing.*

Proof.

(1) Take some $x, y \in [a,b]$ such that $x < y$. We need to show $f(x) \le f(g)$. Applying the MVT on $[x,y]$. Now

$$f(y) - f(x) = f'(\zeta)(y-x),$$

which is non-negative since $y > x$ and $f'(\zeta)$ assumed to be non-negative. Therefore $f(y) \ge f(x)$.
(2) Exercise.
(3) Exercise.
(4) Exercise.

☐

Remark. Note that the converse of (2), (3), and (4) are false in general. For example, for (3), consider $f(x) = x^3$ (or, for (4) consider $f(x) = -x^3$.) Clearly, f is strictly increasing on $\mathbb{R}$. However, $f'(0) = 0$, which is not positive.

Proposition 10.12. *Assume f is continuous on $[a,b]$ and differentiable on (a,b). Assume f is twice differentiable on (a,b). Let $x_0 \in (a,b)$. Then*

(1) *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a local minimum.*
(2) *If $f'(x_0) = 0$ and $f'(x_0) < 0$, then x_0 is a local maximum.*

Proof. We will only prove part (1) — part (2) can be proved similarly.

By definition,

$$f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f'(x)}{x - x_0},$$

and

$$\lim_{x \to x_0} \frac{f'(x)}{x - x_0} > 0.$$

Because f' is differentiable at x_0, it is continuous. Therefore,

$$\frac{f'(x)}{x - x_0} > 0$$

for $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$ for some $\delta > 0$. This means that $f'(x)$ is negative on $(x_0 - \delta, x_0)$. Similarly, $f'(x) > 0$ for $x \in (x_0, x_0 + \delta)$. Therefore, f is decreasing on $(x_0 - \delta, x_0)$ and increasing on $(x_0, x_0 + d)$. This means f attains its minimum at x_0 on $(x_0 - \delta, x_0 + \delta)A$ by the previous theorem. Thus, f has a local minimum at x_0. ☐

11. Riemann integration

Definition 11.1 (Partition)**.** A *partition* P of $[a,b]$ is a set of points $\{x_0,\dots,x_n\}$ such that

$$a = x_0 < x_1 < \ldots < x_n = b.$$

Assume $f:[a,b]\to\mathbb{R}$ is bounded, but not necessarily continuous. Then *lower sum* of P is

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

where

$$m_i = \inf f(x)$$

and $x \in [x_i, x_{i-1}]$. Similarly, the *upper sum* of P is

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

where

$$M_i = \sup f(x)$$

and $x \in [x_i, x_{i-1}]$.

Remark. If $|f(x)| \le M$ for some rectangle,

$$-M(b-a) \le L(f,P) \le U(f,P) \le M(b-a).$$

Example 11.1. Consider the function $f:[0,4]\to[0,5]$ such that

$$f(x) = \begin{cases} x & x\in[0,1) \\ 2 & x\in[1,3) \\ 5 & x\in[3,4] \end{cases}.$$

Take $P_1 = \{0,2,4\}$. To find $L(f,P_1)$ note that

$$L(f,P_1) = \sum_{i=1}^{2} \inf_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}).$$

and

$$U(f,P_1) = \sum_{i=1}^{2} \sup_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}).$$

Therefore

$$L(f,P_1) = 0\times(2-0) + 2\times(4-2) = 4$$

and

$$U(f,P_1) = 2\times(2-0) + 5\times(4-2) = 14.$$

Now take $P_2 = \{0,1,2,3,4\}$. Then

$$L(f,P_2) = 0\times(1-0)+2\times(2-1)+2\times(3-2)+5\times(4-3) = 9$$

and

$$U(f,P_2) = 2\times1 + 2\times1 + 5\times1 + 5\times1 = 14.$$

Definition 11.2. The partition P' is a *refinement* of P if $P' \supset P$.

Lemma 11.1. *If $P' \supset P$, then*

$$L(f,P) \le L(f,P')$$

and

$$U(f,P) \ge U(f,P').$$

Proof. Let

$$P = \{x_0, x_1, \dots, x_n\}$$

and

$$P' = \left\{x_0, x_1, \dots, x_{j-1}, \gamma, x_j, \dots, x_n\right\}.$$

Then

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}) \\ &= \sum_{i=1, i\neq j}^{n} \inf_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}) + \\ &\quad \sum_{j=1}^{n} \inf_{x\in[x_{j-1},x_j]} f(x)(x_j - x_{j-1}) \\ &\le \sum_{i=1, i\neq j}^{n} \inf_{x\in[x_{i-1},x_i]} f(x)(x_i - x_{i-1}) + \\ &\quad \inf_{x\in[x_{j-1},\gamma]} f(x)(\gamma - x_{j-1}) + \\ &\quad \inf_{x\in[\gamma,x_j]} f(x)(x_j - \gamma). \end{aligned}$$

Now if P' differs from P by m points, just repeat this procedure m times. The proof for the upper sums in analogously proved. ☐

Definition 11.3. The number

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup\left\{L(f,P) \mid P \text{ is a partition of } [a,b]\right\}$$

is called the *lower integral* of f over $[a,b]$. Similarly, the number

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = \inf\left\{U(f,P) | P \text{ is a partition of } [a,b]\right\}$$

is called the *upper integral* of f over $[a,b]$.

Definition 11.4. If

$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x$$

then f is called *integrable* and

$$\int_a^b f(x)\,\mathrm{d}x = \underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x.$$

Proposition 11.2. *The function $f : [a, b] \to \mathbb{R}$ is integrable iff for every $\epsilon > 0$ there is some P such that*

$$U(f, P) - L(f, P) < \epsilon.$$

Proof. The forward implication is left as an exercise. Then

$$\overline{\int_a^b} f(x)\,\mathrm{d}x - \underline{\int_a^b} f(x)\,\mathrm{d}x \le U(f, P) - L(f, P) < \epsilon.$$

Thus

$$0 \le \overline{\int_a^b} f(x)\,\mathrm{d}x - \underline{\int_a^b} f(x)\,\mathrm{d}x < \epsilon$$

□

Example 11.2. Take any partition

$$P = \{x_0, \ldots, x_n\}$$

of $[0, 1]$. Then

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} \inf_{[x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} 1(x_i - x_{i-1}) \\ &= x_n - x_0 \\ &= 1. \end{aligned}$$

This means that f is integrable and

$$\int_0^1 f(x)\,\mathrm{d}x = 1.$$

Example 11.3. Consider

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}.$$

We have

$$L(f, P) = \sum_{i=1}^{n} \inf_{[x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) = 0$$

and

$$U(f, P) = \sum_{i=1}^{n} \sup_{[x_{i-1}, x_i]} f(x)(x_i - x_{i-1}) = 1.$$

Proposition 11.3. *If $f[a, b] \to \mathbb{R}$ is bounded and continuous at all but finitely many points, then f is integrable on $[a, b]$.*

Proof. Suppose f is continuous on $[a, b]$. Then f is uniformly continuous on $[a, b]$. Fix $\epsilon > 0$. Then there exists $\delta > 0$ such that if $|x - y| < \delta$ then

$$f(x) - f(y) < \frac{\epsilon}{b - a}.$$

Take a partition P of $[a, b]$ such that

$$P = \{x_0, \ldots, x_n\}$$

and $|x_i - x_{i-1}| < \delta$ for all $i \in \{1, \ldots, n\}$. Then

$$\begin{aligned} &U(f, P) - L(f, P) \\ &= \sum_{i=1}^{n} \sup_{[x_{i-1}, x_i]} f(x_i - x_{i-1}) - \sum_{i=1}^{n} \inf_{[x_{i-1}, x_i]} f(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} [\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x)](x_i - x_{i-1}) \\ &\le \sum_{i=1}^{n} \frac{\epsilon(x_i - x_{i-1})}{b - a} \\ &= \epsilon \end{aligned}$$

This means f is integrable. □

Example 11.4. case B: f has exactly one discontinuity. case C:

Proposition 11.4. *If f is increasing or decreasing, then it's integrable.*

Proof. Without loss of generality, assume f is decreasing (otherwise, consider $-f$). Fix $\epsilon > 0$. We want to find a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. Let P be the partition of $[a, b] = [x_0, x_n]$ into n equal parts. Thus,

$$x_k = a + k\frac{b - a}{n}.$$

Then

$$\begin{aligned} &U(f, P) - L(f, P) \\ &= \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \right) \frac{b - a}{n} \\ &= \sum_{i=0}^{n} \frac{b - a}{n} \left[f(x_i) - f(x_{i+1}) \right] \\ &= \frac{b - a}{n} \left[f(a) - f(b) \right] \end{aligned}$$

Choose n such that

$$n > \frac{(b - a)(f(a) - f(b))}{\epsilon}.$$

Then $U(f, P) - L(f, P) < \epsilon$. □

Remark. If f is increasing or decreasing, then f is bounded on $[a, b]$ to ensure the infima exists.

Proposition 11.5.

(1) *If f is integrable on $[a,b]$ and $k \in \mathbb{R}$, then kf is integrable and*

$$\int_a^b kf \,\mathrm{d}x = k \int_a^b f \,\mathrm{d}x.$$

(2) *If f, g are integrable on $[a,b]$ then so is $f+g$ and*

$$\int_a^b (f+g) \,\mathrm{d}x = \int_a^b f \,\mathrm{d}x + \int_a^b g \,\mathrm{d}x.$$

(3) *If f, g are integrable and $f \leq g$ on $[a,b]$ then*

$$\int_a^b f \,\mathrm{d}x \leq \int_a^b g \,\mathrm{d}x.$$

(4) *If f is integrable on $[a,b]$ and $[b,c]$ then it is integrable on $[a,c]$ and*

$$\int_a^c f \,\mathrm{d}x = \int_a^b f \,\mathrm{d}x + \int_b^c$$

(5) *If f is integrable on $[a,b]$ then so is $|f|$ and*

$$\left| \int_a^c f \,\mathrm{d}x \leq \int_a^b |f| \,\mathrm{d}x \right|$$

Proposition 11.6 (Mean value theorem for integrals)**.** *If f is continuous on $[a,b]$ then there exists $c \in [a,b]$ such that*

$$\int_a^b f(x) \,\mathrm{d}x = f(c)(b-a).$$

Proof. If f is constant on $[a,b]$ then any point will suffice as c. Otherwise, denote

$$m = \int_{x \in [a,b]} f(x)$$

and

$$M = \sup_{x \in [a,b]} f(x).$$

Because f is continuous, there are x_m, $x_M \in [a,b]$ such that $f(x_m) = m$ and $f(x_M) = M$. We may assume that $x_m < X_M$. Then apply the IVT on $[x_m, x_M]$: Observe that

$$m(b-a) \leq \int_a^b f(x) \,\mathrm{d}x \leq M(b-a).$$

Then

$$\frac{\int_a^b f(x) \,\mathrm{d}x}{b-a} \in [m, M].$$

Now, $f(x_m) = m$ and $f(x_M) = M$ so the IVT implies the existence of $c \in [x_m, x_M]$ such that

$$(b-a)f(c) = \int_a^b f(x) \,\mathrm{d}x.$$

□

Definition 11.5. The function F is an antiderivative of f iff $F' = f$.

Proposition 11.7 (Fundamental theorem of calculus)**.** *Assume $f : [a,b] \to \mathbb{R}$ is continuous and let*

$$F(x) = \int_a^x f(x) \,\mathrm{d}x.$$

Then F is an antiderivative of f and

$$\int_a^b f(x) \,\mathrm{d}x = F(b) - F(a).$$

Remark. Note that F is well defined because f is continuous, and thus integrable, on $[a,x]$ for each $x \in [a,b]$.

Proof. We claim that $F' = f$ on (a,b). Take $x \in (a,b)$ and $h \in (0, b-a)$. The motivation for this choice of h is the following:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

Consider

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t) \,\mathrm{d}t - \int_a^t f(t) \,\mathrm{d}t \\ &= \int_x^{x+h} f(t) \,\mathrm{d}t \\ &= f(x)((x+h) - x) \\ &= f(x)h \end{aligned}$$

for some h by the integral MVT. Now f is uniformly continuous on $[a,b]$: For each $\epsilon > 0$ there exists $\delta > 0$ such that if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$; note that here, $x, y \in (x, x+h)$. If $|h| < \delta$ then $|c - x| < \delta$ and $|f(c) - f(x)| < \epsilon$. Therefore,

$$\begin{aligned} \left| \frac{F(x+h) - F(x)}{h} - f(x) \right| &= \left| \frac{f(c)h}{h} - f(x) \right| \\ &= |f(c) - f(x)| < \epsilon. \end{aligned}$$

Therefore $F'(x) = f(x)$. □

Remark.

(1) If $G'(x) = f(x)$ on (a,b) then

$$(G-F)' = f - f = 0,$$

so for $c \in \mathbb{R}$, we ahve $G - F = c$ whence we get that $G = F + c$. Also

$$\int_a^b f(x) \,\mathrm{d}x = G(b) - G(a)$$

provided that G is continuous on $[a,b]$

(2) If f is discontinuous, then

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

might not be differentiable.

Example 11.5. Take $f = \operatorname{sgn} x$ on $[-1, 1]$. Then

$$\int_{-1}^x f(t)\,\mathrm{d}t = |x|$$

on $[-1, 1]$.

Proposition 11.8 (Integral test)**.** *Assume f is continuous, non-negative and non-increasing on $[1, \infty)$. Then $\int_1^\infty f(x)\,\mathrm{d}x$ converges iff $\sum_{n=1}^\infty f(n)$ converges.*

Proof. Suppose $a = n + 1$ where $n \in \mathbb{N}$ then consider the partition

$$P = \{1, 2, \ldots, n + 1\}.$$

Then, because f is non-increasing,

$$\inf_{x \in [x_{i-1}, x_i]} f(x) = f(x_i)$$

and

$$\sup_{x \in [x_{i-1}, x_i]} f(x) = f(x_{i-1}).$$

Therefore,

$$U(f, P) = \sum_{i=1}^n f(i).$$

Similarly,

$$L(f, P) = \sum_{i=2}^{n+1} f(i) = \sum_{i=1}^n f(i + 1).$$

Then

$$L(f, P) \le \int_1^{n+1} f(x)\,\mathrm{d}x \le U(f, P).$$

Now, if the integral converges, then by the comparison theorem,

$$L(f, P) = \sum_{i=2}^n f(i)$$

converges as $n \to \infty$ so

$$\sum_{n=2}^\infty f(n)$$

converges. On the other hand, if the integral diverges, then

$$U(f, P) = \sum_{i=1}^n f(i)$$

diverges as $n \to \infty$ so

$$\sum_{n=1}^\infty f(n)$$

diverges. □

Remark. Recall that

$$\int_1^\infty f(x)\,\mathrm{d}x = \lim_{a \to \infty} \int_1^a f(x)\,\mathrm{d}x.$$

The assumption that f is continuous allows us to conclude that the integral is defined for any a.

Proposition 11.9 (Integration by parts)**.** *If $u, v : [a, b] \to \mathbb{R}$ are differentiable on (a, b) and u', v' are continuous on (a, b) then*

$$\int_a^b uv'\,\mathrm{d}x = uv\Big|_a^b - \int_a^b u'v\,\mathrm{d}x$$

Proof. The product rule yields

$$(uv)' = u'v + uv'.$$

Integrating this, we get

$$uv\Big|_a^b = \int_a^b u'v\,\mathrm{d}t + \int_a^b uv'\,\mathrm{d}t.$$

Rearranging,

$$\int_a^b u'v\,\mathrm{d}t = uv\Big|_a^b - \int_a^b uv'\,\mathrm{d}t.$$

□

Definition 11.6 (Improper integral)**.** Consider a function $f : (a, b] \to \mathbb{R}$ which is not necssarily bounded. Assume f is integrable on $[a + \epsilon, b]$ for all $\epsilon \in (0, b - a)$. Then

$$\lim_{\epsilon \to 0^+} \int_{a+\epsilon}^b f(x)\,\mathrm{d}x$$

is called the *improper integral* of the first kind of f on $[a, b]$.

Remark. This is usually denoted

$$\int_a^b f(x)\,\mathrm{d}x$$

Definition 11.7. Assume $f : [a, \infty) \to \mathbb{R}$ is integrable on $[a, b]$ for all $b > a$. Then

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b \to \infty} \int_a^b f(x)\,\mathrm{d}x$$

is called the *improper integral* of the second kind of f on $[a, \infty)$. We also define

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^0 f(x)\,\mathrm{d}x + \int_0^\infty f(x)\,\mathrm{d}x,$$

provided both integrals on the right exist.

Example 11.6. Consider
$$f(x) = \frac{1}{x}.$$
Now
$$\int_1^\infty f(x)\,\mathrm{d}x = \ln x$$
is an improper integral.

Remark. Note that
$$\int_{-\infty}^{\infty} f(x) \neq \lim_{a\to\infty} \int_{-a}^{a} f(x)\,\mathrm{d}x$$

Example 11.7. For the function $y = x$
$$\int_{-\infty}^{\infty} x\,\mathrm{d}x$$
does not exists.

Example 11.8. Compute
$$\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x.$$
By the definition of an improper integral of the first kind, we have
$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x &= \lim_{\epsilon\to 0^+} \int_\epsilon^1 \frac{dx}{\sqrt{x}} \\ &= 2\lim_{\epsilon\to 0^+} \sqrt{x}\Big|_\epsilon^1 \\ &= 2\lim_{\epsilon\to 0^+}(1-\sqrt{\epsilon}) \\ &= 2.\end{aligned}$$

Example 11.9. Compute $\int_0^1 x^p\,\mathrm{d}x$ where $p \in \mathbb{R}$.

(1) If $p \geq 0$ and x^p is continuous on $(0,1]$ so it is integrable, so the computation is trivial.

(2) If $p \in (-1,0)$ then
$$\begin{aligned}\int_0^1 x^p\,\mathrm{d}x &= \lim_{\epsilon\to 0^+} \int_\epsilon^1 x^p\,\mathrm{d}x \\ &= \frac{1}{p+1} \lim \epsilon \to 0^+ x^p \Big|_\epsilon^1 \\ &= \frac{1}{p+1} \lim_{\epsilon\to 0^+} \left(1-\epsilon^{p+1}\right) \\ &= \frac{1}{p+1}.\end{aligned}$$

(3) If $p < -1$ then the same argument implies that the integral diverges.

(4) If $p = -1$ then the improper integral
$$\int_1^\infty \frac{1}{x}\,\mathrm{d}x$$
diverges by the integral test.

12. The trigonometric, exponential, and logarithmic functions

Definition 12.1. The well known constant π is defined thus:
$$\pi = 2\int_{-1}^{1} \sqrt{1-x^2}$$

Definition 12.2. Let
$$A(x) = (A_1 + A_2)(x)$$
where
$$A_1 = \frac{x\sqrt{1-x^2}}{2}$$
and
$$A_2 = \int_x^1 \sqrt{1-s^2}\,\mathrm{d}s.$$

Remark. Note that
$$A(x) = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-s^2}\,\mathrm{d}s$$
whence we get
$$A'(x) = \frac{-1}{2\sqrt{1-x^2}}.$$
Further note that $A(x)$ is decreasing on $[-1,1]$ from $A(x) = \pi/2$ downwards to $A(x) = 0$. Thus $A(-1) = \pi/2$ and $A(1) = 0$.

Definition 12.3. If $x \in [0,\pi]$, then $\cos(x)$ is defined as the unique number such that
$$A(\cos(x)) = \frac{x}{2}.$$

Remark. This is well-defined because $\cos(0)$ is the unique number such that $A[\cos(0)] = 0$, so $\cos(0) = 1$. Further, $\cos(\pi)$ is the unique number such that $A[\cos(\pi)] = \pi/2$, so $\cos(\pi) = -1$. Now, by the intermediate value theorem, we have that for each $x \in (0,\pi)$ there exists c such that
$$A(c) = \frac{x}{2}.$$
Furthermore, c is unique because A is strictly decreasing.

Definition 12.4. Given $x \in [0,\pi]$ define
$$\sin(x) = \sqrt{1-\cos^2 x}.$$

Proposition 12.1. *If $x \in (0,\pi)$ then*

(1) $\cos'(x) = -\sin x$

(2) $\sin'(x) = \cos x$.

Proof.

(1) Let $B(x) = 2A(x)$. Then $B[\cos(x)] = x$ for $x \in (0, \pi)$. Then $B(x) = \cos^{-1}(x)$ and

$$B'(x) = 2A'(x) = \frac{-1}{\sqrt{1-x^2}}.$$

Now note that

$$\begin{aligned}\cos'(x) &= \frac{1}{B'(B^{-1}(x)} \\ &= \frac{1}{B'[\cos(x)]} \\ &= \left(\frac{-1}{\sqrt{1-\cos^2 x}}\right)^{-1} \\ &= -\sin(x)\end{aligned}$$

(2) Exercise.

□

Definition 12.5. We extend the definition of sin and cos to $[0, 2\pi]$ by setting

$$\sin x = -\sin(2\pi - x)$$

if $x \in [\pi, 2\pi]$, and

$$\cos x = \cos(2\pi - x).$$

It is also trivial to extend cos and sin periodically to $\mathbb{R}$.

Definition 12.6. Define the logarithmic function so

$$\log x = \int_1^x \frac{1}{t}\,\mathrm{d}t.$$

Remark. Suppose $a > 0$. We know how to define a^r for $r \in \mathbb{Q}$. We also know that for rational exponents n and m we have that

$$a^n \cdot a^m = a^{n+m}$$

where $m, n \in \mathbb{Q}$. In other words, if $f(x) = a^x$ then we have

$$f(x) \cdot f(y) = f(x+y) \tag{17}$$

for $x, y \in \mathbb{Q}$. Now, to extend the definition of f to the domain $\mathbb{R}$ such that (17) holds. We also want f to be differentiable. We could take $f = 0$ or $f = 1$. So we also want f such that $f(1) = a$ where $a \in \mathbb{R}$. We have

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x)\cdot f(h) - f(x)}{h} \\ &= f(x) \lim_{h\to 0} \frac{f(x)\cdot(f(h)-1)}{h}.\end{aligned}$$

Now if this limit exists and equals α then $f'(x) = \alpha f(x)$. This implies that f^{-1} satisfies

$$f^{-1}(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{\alpha f\left(f^{-1}(x)\right)} = \frac{1}{\alpha x}$$

Then we may have

$$f^{-1}(x) = \int_1^x \frac{1}{\alpha t}\,\mathrm{d}t.$$

Proposition 12.2. *If $x, y > 0$, then*

$$\log(xy) = \log(x) + \log(y).$$

Proof. We know

$$\frac{\mathrm{d}\log(x)}{\mathrm{d}x} = \frac{1}{x}.$$

Fix $y > 0$ and set $g(x) = \log(xy)$. Then

$$g'(x) = y \cdot \frac{1}{xy} = \frac{1}{x}.$$

This means

$$g(x) = \log(x) + C$$

for some constant C. Take $x = 1$. Then $g(1) = \log(1) + C$, implying that $g(1) = C$. But $g(1) = \log(y \times 1) = \log(y)$. Thus $C = \log y$ and

$$\log(xy) = g(x) = \log(x) + \log(y)$$

□

Definition 12.7. The exponenetial function exp is defined as the inverse function of log.

Remark.

(1) We have $\log : (0, \infty) \to \mathbb{R}$ is continuous and strictly increasing.
(2) We have that $\exp : \mathbb{R} \to (0, \infty)$ is continuous and strictly increasing.
(3) If $n \in \mathbb{N}$ then $\log x^n = n \log x$ and

$$\log \frac{x}{y} = \log x - \log y.$$

Proposition 12.3. *We have that $\exp'(x) = \exp(x)$ for all $x \in \mathbb{R}$.*

Proof. Note that

$$\begin{aligned}\exp'(x) &= \log^{-1'}(x) \\ &= \frac{1}{\log'(\log^{-1}(x))} \\ &= \frac{1}{1/\log^{-1} x} \\ &= \exp(x).\end{aligned}$$

□

Proposition 12.4. *We have that*

$$\exp(x+y) = \exp(x)\cdot\exp(y).$$

Proof. Set $X = \exp(x)$ and $Y = \exp(y)$. Then

$$x = \log(X)$$

and

$$y = \log(Y).$$

Now

$$x + y = \log(X) + \log(Y) = \log(XY)$$

so

$$\exp(x+y) = XY = \exp(x)\cdot\exp(y).$$

□

Definition 12.8. Define

$$a^x = \exp(x\log(a))$$

for $a > 0$.

Remark.

(1) From this definition it folllows that $\exp(x) = (\exp(1))^x$.
(2) We note that $\exp(1) = e$.

Example 12.1. Derive the following

(1) $(a^b)^c = a^{bc}$
(2) $a^{b+c} = a^b a^c$
(3) $a^1 = a$

Example 12.2. Find $\exp 1$. We have that

$$\lim_{y\to 0^+} \frac{1+y}{y} = \frac{1}{y} + 1.$$

With $y = 1/x$ for some $x > 0$ we have that

$$1 = \lim_{x\to\infty} x \log\left(1 + \frac{1}{x}.\right)$$

Therefore

$$\exp\left[\lim_{x\to\infty} x + \log\left(1+\frac{1}{x}\right)\right] = 1$$

whence we get that

$$\lim_{x\to\infty}\left[(\exp(1))^{x\log(1+1/x)}\right] = \exp(1)$$

and hence

$$\lim_{x\to\infty}\left[(\exp(1))^{x\log(1+1/x)}\right]^x = \exp(1).$$

Now we behold

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right) = e.$$

13. Taylor series

Definition 13.1. Suppose Suppose f is a function from $\Omega\subset\mathbb{R}$ to $\mathbb{R}$. Then f is *n-times differentiable* on $(a,b)\subset\Omega$ iff $f^{(n+1)}(t)$ exists for any $t\in(a,b)$.

Lemma 13.1 (Rolle's theorem for $(n+1)$-times differentiable functions)**.** *Suppose f is a function from $\Omega\subset\mathbb{R}$ to $\mathbb{R}$ that is continuous on $[a,b]\subset\Omega$ and $(n+1)$-times differentiable on (a,b). Assume, that*

$$f(a) = f'(a) = f''(a) = \cdots = f^{(n)}(a) = 0 = f(b).$$

Then for some $\xi\in(a,b)$, $f^{(n+1)}(\xi) = 0$.

Proof. Using the regular Rolle's theorem, there is some $\xi_1\in(a,b)$ such that $f'(b_1) = 0$ which by assumption equals $f'(a)$. In this manner we can repeatedly apply Rolle's theorem so

$$0 = f'(\xi_1) = f''(\xi_2) = \cdots = f^{(n+1)}(\xi_{n+1}).$$

□

Remark. The assumptions for the theorem could be relaxed so that $f(a) = f(b)$ is not necessarily zero (in fact, the proof above still works in this case). This generality is not needed for proving Taylor's theorem, so we favor symmetry.

Proposition 13.2 (The mean value theorem for $(n+1)$-times differentiable functions, otherwise known by the unenlightened as Taylor's theorem)**.** *Suppose f is a function from $\Omega\subset\mathbb{R}$ to $\mathbb{R}$ that is continuous on $[\alpha,\beta]\subset\Omega$ and $(n+1)$-times differentiable on (α,β). Then for all distinct $a,b\in[\alpha,\beta]$ and $n\in\mathbb{N}$ there exists $\xi\in(\min\{a,b\},\max\{a,b\})$ such that*

$$f(b) = (P_n + R_n)(b)$$

where

$$P_n(b) = \sum_{k=0}^{n}\frac{f^{(n)}(a)}{k!}(b-a)^k$$

is defined to be the nth Taylor polynomial *of f centred at a and*

$$R_n(b) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}$$

is defined to be the Lagrange form *of the nth* Taylor remainder*.*

Remark.

(1) For $n = 1$ this is just the mean value theorem.
(2) In the proof for the regular mean value theorem, we considered an error function then

applied the regular Rolle's theorem to it. This proof is similar.

(3) Clearly, there always exists some $\lambda \in \mathbb{R}$ such that $f(b) = P_n(b)+\lambda$ (compare this expression to $((P_n + R_N)(b)$ from the proposition). However, we prefer to use R_n in place of λ because it is structured enough to prove that goes to 0 for the right f; see the next proposition to see why we would want to do this.

Proof. Note that $P_0(t) = f(a)$, and

$$P_1(t) = P_0(t) + \frac{f^{(1)}(a)}{1!}(t-a)^1 = f(a) + f'(a)(t-a),$$

so $P_1'(a) = f'(a)$. Similarly

$$P_2(t) = P_0(t) + P_1(t) + \frac{f^{(2)}(a)}{2!}(t-a)^2$$

so $P_2^{(2)}(a) = f^{(2)}(a)$. In general, we can approximate f at a to n derivatives: $P_n^{(n)}(a) = f^{(n)}(a)$. Now

$$(P_n - f)^{(n)}(a) = P_n^{(n)}(a) - f^{(n)}(a) = 0 \tag{18}$$

so $(P_n - f)$ satisfies one of the requirements of a function suitable to apply the lemma to. To satisfy the other requirement, set $\lambda \in \mathbb{R}$ and

$$g : [\min\{a,b\}, \max\{a,b\}] \to \mathbb{R}$$

such that

$$g(t) = (P_n - f)(t) + \lambda(t-a)^{n+1},$$

and $g(b) = 0$, choosing the coefficient of λ to preserve the property of (18). Now, applying the lemma to g, there exists some ξ in the domain of g such that

$$\begin{aligned} 0 &= g^{(n+1)}(\xi) \\ &= (P_n - f)^{(n+1)}(t) + \left.\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n}\lambda(t-a)^{n+1}\right]\right|_{t=\xi} \\ &= \lambda(n+1)! - f^{(n+1)}(\xi) \end{aligned}$$

so we can solve for λ and find that

$$g(t) = (P_n - f)(t) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(t-a)^{n+1}.$$

Now since $g(b) = 0$,

$$f(b) = P_n(t) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(t-a)^{n+1}.$$

as was required to be demonstrated. □

Proposition 13.3. *If the Taylor remainder of f, $R_n(x) \to 0$ as $n \to \infty$ for some x, then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Proof. Let P_n and R_n respectively be the nth Taylor polynomial and remainder respectively.

Now we want to show that

$$f(x) = \lim_{n\to\infty} P_n(x)$$

By Taylor's theorem,

$$f(x) = P_n(x) + R_n(x)$$

so if

$$\lim_{n\to\infty} R_n(x) = 0,$$

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} P_n(x) + \lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} P_n(x).$$

□

Definition 13.2. If the Taylor remainder of f, $R_n(x) \to 0$ as $n \to \infty$ for some x, then,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the *Taylor series* of f at a. If $a = 0$, it is also called a *Maclaurin series.*

Definition 13.3. We say that the *radius of convergence* of a power series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

is r iff f converges for all $x \in (a-r, a+r)$ and diverges for all $x \in \mathbb{R}\backslash[a-r, a+r]$.

Remark.

(1) We can determine the radius of converges of a power series by using the ratio or root test. Just as the ratio test is inconclusive if

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$$

and the root test is inconclusive if

$$\limsup_{n\to\infty}\left|\sqrt[n]{a_n}\right| = 1,$$

so too is the convergence of a series if $x = a \pm r$.

(2) We now have two methods of determining if a Taylor series exists: showing that the limit of the Taylor remainder vanishes and the root or ratio tests. The prior allows us to use series tests while the latter allows us to use limit convergence tests. However, the logic

behind ξ may complicate finding the radius of convergence using the Taylor remainder; further, the ratio test usually yields the radius of convergence easily.

Proposition 13.4. *If $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ then the series is the Taylor series of f.*

Example 13.1. Find the Maclauring series of

$$y(x) = e^{x^3} + e^{2x^3}.$$

Note that

$$\frac{\mathrm{d}e^{x^3}}{\mathrm{d}x} = 3x^2 e^{x^3},$$

$$\frac{\mathrm{d}^2 e^{x^3}}{\mathrm{d}x^2} = 3x^2\left(3x^2 e^{x^3}\right) + e^{x^3}(6x) = \left(9x^4 + 6x\right)e^{x^3},$$

and

$$\frac{\mathrm{d}^3 e^{x^3}}{\mathrm{d}x^3} = \left(9x^4+6x\right)\left(3x^2e^{x^3}\right) + e^{x^3}\left(36x^3+6\right)$$

so

$$\left.\frac{\mathrm{d}e^{x^3}}{\mathrm{d}x}\right|_{x=0} = 0 = \left.\frac{\mathrm{d}^2e^{x^3}}{\mathrm{d}x^2}\right|_{x=0}$$

but

$$\left.\frac{\mathrm{d}^3e^{x^3}}{\mathrm{d}x^3}\right|_{x=0} = 6.$$

It can similarly be shown that

$$\left.\frac{\mathrm{d}^3e^{2x^3}}{\mathrm{d}x^3}\right|_{x=0} = 12,$$

and the lower derivatives of e^{2x^3} at $x=0$ vanish.

Now we know for sure that that in the expansion of the Maclaurin series

$$\sum_{k=0}^{\infty} \frac{y^{(k)}(0)}{k!}x^k,$$

the third term is

$$\frac{y^{(3)}(0)}{3!}x^3 = \frac{6+12}{3!}x^3 = 3x^3.$$

We could conjecture that the terms of the Maclaurin series vanish, save for the zero*th* term and every term with the factor x^{3n} where $n \in \mathbb{N}$. Fortunately, instead, we can verify this without much work: Since

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$$e^{x^3} + e^{2x^3} = \sum_{n=0}^{\infty}\frac{x^{3n}}{n!} + \sum_{n=0}^{\infty}\frac{2^n x^{3n}}{n!} = \sum_{n=0}^{\infty}\frac{2^n+1}{n!}x^{3n}$$

if the sum exists. This is a Taylor series by Proposition 13.4 and agrees with what we know so far, namely, that $y(0) = 2$ and the third term of the Taylor series, the first term of the power series above, is $3x^3$.

We will determine, using the ratio test, if the sum does indeed exist. Letting

$$a_n = \frac{2^n+1}{n!}x^{3n},$$

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{2^{n+1}+1}{(n+1)!}\frac{n!}{2^n+1}\frac{x^{3n+3}}{x^{3n}} \\ &= \frac{2^{n+1}+1}{2^n+1}\frac{x^3}{n+1} \\ &< \frac{2^{n+1}+1}{2^n}\frac{x^3}{n} \\ &= \left(2+\frac{1}{2^n}\right)\frac{x^3}{n}\end{aligned}$$

so

$$\lim_{n\to\infty}\left|\frac{a^{n+1}}{a^n}\right| = \lim_{n\to\infty}\frac{2x^3}{n} + \lim_{n\to\infty}\frac{x^3}{2^n\cdot n} = 0$$

for all x. Therefore, by the ratio test, the Maclaurin series converges for all x.

Example 13.2. Compute the Taylor series centred at $a=1$ for $\ln x$. Also compute the corresponding radius of convergence and the value of $\ln 1.3$ to 2 decimal places if it falls within the radius of convergence.

Note that

$$\ln'(x) = \frac{1}{x},$$

$$\ln''(x) = \frac{-1}{x^2},$$

and in general,

$$\ln^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

for $n > 0$. Therefore by Taylor's theorem,

$$\begin{aligned}&\ln x \\ &= \sum_{n=0}^{N}\frac{\ln^{(n)}(1)}{n!}(x-1)^n + \frac{\ln^{(n+1)}(\xi)}{(n+1)!}(x-1)^{n+1} \\ &= \sum_{n=1}^{N}\frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n + \\ &\frac{(-1)^N N!}{\xi^{N+1}(N+1)!}(x-1)^{N+1} \\ &= \sum_{n=1}^{N}\frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n + \frac{(-1)^N(x-1)^{N+1}}{\xi^{N+1}(N+1)} \\ &= \sum_{n=1}^{N}\frac{(-1)^{n-1}}{n}(x-1)^n + \frac{(-1)^N(x-1)^{N+1}}{\xi^{N+1}(N+1)}\end{aligned}$$

where $\xi \in (\min\{1,x\}, \max\{1,x\})$. Letting

$$a_n = \frac{(-1)^{n-1}}{n}(x-1)^n,$$

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(-1)^n}{n+1}\frac{n}{(-1)^{n-1}}\frac{(x-1)^{n+1}}{(x-1)^n} \\ &= \frac{-(x-1)}{1+1/n}\end{aligned}$$

so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-1|.$$

Therefore, by the ratio test, a_n converges if

$$|x|-1 \leq |x-1| < 1,$$

when $|x| < 2$. Therefore the radius of convergence of the Taylor series is 1.

Note that $|1.3| < 2$ so we can use the Taylor series of ln centred at 1 to calculuate $\ln 1.3$ to 3 decimal places, as follows. Set $x = 1.3$ and the Taylor remainder R_n such that $|R_n| < 0.005$ so $\xi \in (1, 1.3)$. Now we want

$$\frac{|1.3-1|^{N+1}}{\xi^{N+1}(N+1)} = |R_N| < \frac{0.3^{N+1}}{N+1} < 0.0005.$$

But if $N = 1$,

$$\frac{0.3^{N+1}}{N+1} = \frac{0.09}{2} = 0.045,$$

if $N = 2$

$$\frac{0.3^{N+1}}{N+1} = \frac{0.09 \times 0.3}{3} = \frac{0.0027}{3} = 0.009,$$

and if $N = 3$

$$\frac{0.3^{N+1}}{N+1} = \frac{0.0027 \times 0.3}{4} = \frac{0.00081}{4} = 0.002025$$

so we need to compute 3 terms of the Taylor polynomial thus:

$$\begin{aligned}&\sum_{n=1}^{3}\frac{(-1)^{n-1}}{n}(1.3-1)^n \\ &= 0.3 - \frac{0.3^2}{2} + \frac{0.3^3}{3} \\ &= 0.3 - 0.045 + 0.009 \\ &= 0.264\end{aligned}$$

Therefore $\ln 1.3 = 0.264 \pm 0.002025$. Therefore $\ln 1.3 \approx 0.26$.

Example 13.3. Compute e to 5 decimal places. By Taylor's theorem, when $x = 1$

$$e^x = \sum_{n=0}^{N}\frac{x^n}{n!} + \frac{e^\xi}{(N+1)!}x^{N+1} = \sum_{n=0}^{N}\frac{1}{n!} + \frac{e^\xi}{(N+1)!}$$

for some $\xi \in (0,1)$ when $e^\xi \in (1,e)$. Now, since $1 < e < 3$, $1 < e^\xi < 3$ so we want

$$\frac{e^\xi}{(n+1)!} < \frac{3}{(n+1)!} < 0.000005.$$

Therefore by inspection, we must set $n = 9$ in order to achieve the desired precision of 0.000005. Doing this, we get that $e \in (P, P + 0.000005)$ where $P = 1 + 1 + \ldots + 1/9!$.

Example 13.4. Let

$$f(x) = \begin{cases} 1/e^{1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Clearly f is differentiable at $x \neq 0$, and in fact, f is infinitely differentiable at 0. Therefore we can find the Maclaurin series of f:

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n.$$

But $f^{(n)}(0) = 0$ for all n. But the function itself is not 0 for all x.

14. Elements of multivariable analysis

Definition 14.1. Suppose $x, y \in \mathbb{R}^m$. Then

$$|x| = \sqrt{\sum_{i=1}^{m} x_i^2}$$

Proposition 14.1. *If $x, y \in \mathbb{R}^m$ then the Cauchy-Schwarz inequality holds: $|(x,y)| \leq |x||y|$*

Proof. Assume $x \neq 0$ and $y \neq 0$ and note that If $a, b \in \mathbb{R}$ then

$$(|a| - |b|)^2 = a^2 - 2|a||b| + b^2 \geq 0$$

implies

$$|ab| \leq \frac{a^2 + b^2}{|ab|}.$$

Now take $a = |x_i|/|\mathbf{x}|$ and $b = |y_i|/|\mathbf{y}|$ for some $i = 1, \ldots, m$. Then

$$\begin{aligned} \frac{|x_i y_i|}{|x||y|} &\leq \frac{1}{2}\left(\frac{x_i^2}{|\mathbf{x}|^2} + \frac{y_i^2}{|\mathbf{y}|^2}\right) \\ &= \frac{1}{2}\left(\frac{1}{|\mathbf{x}|^2}\sum_{i=1}^{n} x_i^2 + \frac{1}{|\mathbf{y}|^2}\sum_{i=1}^{n} y_i^2\right) \\ &= \frac{1}{2} + \frac{1}{2}. \end{aligned}$$

Therefore $\sum_{i=1}^{m} |x_i y_i| \leq |x||y|$. Also, by the triangle inequality for $\mathbb{R}$ we have that

$$\left|\sum_{i=1}^{m} x_i y_i\right| \leq \sum_{i=1}^{m} |x_i y_i|.$$

Therefore $|x||y| \geq \left|\sum_{i=1}^{m} x_i y_i\right| = |(x,y)|$. ☐

Proposition 14.2. *If $x, y \in \mathbb{R}^m$ then the triangle inequality holds $|x + y| \leq |x| + |y|$.*

Proof. Note that

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y}) \\ &= (\mathbf{x}, \mathbf{x}) + (\mathbf{x}, \mathbf{y}) + (\mathbf{y}, \mathbf{x}) + (\mathbf{y}, \mathbf{y}) \\ &\leq |\mathbf{x}|^2 + 2|(\mathbf{x}, \mathbf{y})| + |\mathbf{y}|^2 \\ &\leq |\mathbf{x}|^2 + 2|\mathbf{x}||\mathbf{y}| + |\mathbf{y}|^2 \\ &= (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

Therefore, $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$. ☐

Remark.

(1) $|\mathbf{x}| = \mathbf{0}$ iff $\mathbf{x} = \mathbf{0}$.
(2) $(\mathbf{x}, \mathbf{0}) = \mathbf{0}$ for all $\mathbf{x}$
(3) $(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ does not necessarily imply that $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$.

Definition 14.2. A vector function is a function $f : \Omega \to \mathbb{R}^m$ where $\Omega \subset \mathbb{R}^n$.

Example 14.1. Consider the function

$$f(x, y) = (\sqrt{1 - x^2 - y^2}, x + y).$$

Its domain is

$$\Omega = \left\{(x, y) \in \mathbb{R} \mid x^2 + y^2 \leq 1\right\}$$

and its range is $\mathbb{R}^2$.

Definition 15.1. Suppose $\Omega \subset \mathbb{R}^n$. The point $a \in \Omega$ is called a *limit point* of Ω if for every $\epsilon > 0$ there exists $y \in \Omega$ such that $0 < |y - a| < \epsilon$.

Example 15.1. If

$$\Omega = B_1(0) = \left\{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| < 1\right\},$$

then the set of limit points of Ω is $\left\{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| \leq 1\right\}$ where $B_1(0)$ is a ball of radius 1 centred at 0.

Example 15.2. Suppose $\Omega = B_1(0) \backslash \{0, 0\}$. Then the set of limit points is $\left\{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| \leq 1\right\}$.

Definition 15.2. Let $f : \Omega \to \mathbb{R}^m$ be a function where $\Omega \subset \mathbb{R}^n$. Also let $\mathbf{x}_0$ be a limit point of Ω and $\mathbf{y}_0 \in \mathbb{R}^m$. Then

$$\lim_{\mathbf{x} \to \mathbf{x_0}} \mathbf{f}(\mathbf{x}) = \mathbf{y}_0$$

iff for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |\mathbf{x} - \mathbf{x_0}| < \delta$ and $x \in \Omega$ then $|\mathbf{f}(x) - \mathbf{y}_0| < \epsilon$.

Definition 15.3. The function $f : \Omega \to \mathbb{R}^m$ be a function where $\Omega \subset \mathbb{R}^n$ is continuous at $\mathbf{x}_0$ iff

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$$

Remark. Every $f : \mathbb{R}^n \to \mathbb{R}^m$ can be written as $(f_1, f_2, f_3, \ldots, f_m)$ where for each i, f_i is from $\mathbb{R}^n \to \mathbb{R}$ for all $i = 1, \ldots, n$. The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous iff for each i, each of these f_i are continuous.

Example 15.3. Consider $f : \mathbb{R}^2 \backslash \left\{(0, 0)\right\} \to \mathbb{R}$ where

$$f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}.$$

Show that

$$\lim_{(x,y) \to (0,0)} f(x, y)$$

exists. Fix $\epsilon > 0$. We want to find $\delta > 0$ such that if $0 < |(x, y) - (0, 0)| < \delta$ then

$$\left|\frac{xy}{x^2 + y^2} - L\right| < \epsilon$$

for some L. Assume that $|(x, y) - (0, 0)| = \sqrt{x^2 + y^2} \leq \delta$ and

$$(x - y)^2 \geq 0,$$

that is,

$$x^2 + y^2 \geq 2|xy|.$$

Therefore

$$\frac{|xy|}{\sqrt{x^2 + y^2}} \leq \frac{1}{2} \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} = \frac{\sqrt{x^2 + y^2}}{2} < \frac{\delta}{2}.$$

Therefore, setting $\delta = 2\epsilon$, $|(x, y) - (0, 0)| < \delta$ will imply

$$|f(x, y) - 0| < \epsilon,$$

and we're done.

Example 15.4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that

$$f(x, y) = \frac{xy^2}{x^2 + y^2}.$$

Show that

$$\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^2} = 0.$$

Note that

$$|f(x, y)| = \frac{|x|y^2}{x^2 + y^2} \leq |x|$$

for $(x, y) \neq (0, 0)$. Now suppose that $0 < |(x, y) - (0, 0)| < \delta$. Then

$$|f(x, y) - 0| \leq |x| = \sqrt{x^2} < \sqrt{x^2 + y^2} < \delta,$$

and we're done by setting $\delta = \epsilon$.

Example 15.5. Show that

$$\lim_{(x,y) \to (0,0)} \frac{x^2 + xy}{x^2 + y^2}$$

does not exist. Suppose it does. Then for all (x, y) such that $0 < |(x, y) - (0, 0)| < \delta$ we would have

$$\left|\frac{x^2 + xy}{x^2 + y^2} - L\right| < \epsilon$$

for some L. To find a counterexample to this, take $\epsilon = 1/2$, $x = 0$ and $0 < |(x, y)| < \delta$. Now

$$\left|\frac{x^2 + xy}{x^2 + y^2} - L\right| = |L| < \frac{1}{2}$$

However, taking x, y such that $x = y$, we have that no matter how close (x, y) is to $(0, 0)$,

$$|1 - |L|| \leq \left|\frac{x^2 + xy}{x^2 + y^2} - L\right| = |1 - L| < \frac{1}{2}$$

so $|L| \in (1 - 1/2, 1 + 1/2)$. But this means $|L| > 1/2$ — a contradiction.

Example 15.6. Consider the limit

$$\lim_{(x,y,z) \to (0,0,0)} \frac{xy + y^2 + zx}{x^2 + y^2 + z^2}.$$

Approaching the origin on the z-axis by setting $x = 0$ and $y = 0$ so

$$\frac{xy + y^2 + zx}{x^2 + y^2 + z^2} = 0$$

on the z-axis. Hence the function approaches 0. But, the origin along $x = y = z$. Then, along this line,

$$\frac{xy + y^2 + zx}{x^2 + y^2 + z^2} = \frac{x^2 + x^2 + x^2}{x^2 + x^2 + x^2} = 1.$$

so the function approaches 1. Therefore the function approaches the origin and reaches two different values, so the limit does not exist.

Example 15.7. Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$$

does not exist. Approaching the original along $y = 0$,

$$\frac{x^2 y}{x^4 + y^2} = 0.$$

However, approaching along $y = x^2$

$$\frac{x^2 y}{x^4 + y^2} = \frac{x^4}{2x^4} = \frac{1}{2}.$$

Therefore the function takes different values at the origin as it approaches it along different curves. Therefore the limit does not exist.

16. Multivariable implicit and inverse function theorems

Definition 16.1 (Interior points)**.** For $S \subset \mathbb{R}^n$, x is an interior point of S if there exists an open ball centered at x which is completely contained in S.

Definition 16.2 (Open sets)**.** A set S is open iff every point within it is an interior point.

Definition 16.3 (Closed sets)**.** Let X be a metric space. A set is $S \subset X$ closed iff its complement $X \backslash S$ is open.

Definition 16.4. Let $f : \Omega \to \mathbb{R}^m$ where $\Omega \subset \mathbb{R}^n$ and Ω be open. The *Jacobian matrix* of $f(x_1, \ldots, x_n)$ is

$$J_f(x_1, \ldots, x_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}. \end{bmatrix}$$

The determinant of the Jacobian matrix is called the *Jacobian* of f.

Proposition 16.1 (Inverse function theorem)**.** *Consider $f : \Omega \to \mathbb{R}^m$ where $\Omega \subset \mathbb{R}^n$ and $f^{-1} : f(\Omega) \to \Omega$ such that $f(f^{-1}(x)) = x$ for all $x \in \Omega$. Suppose the function $f(x_1, \ldots, x_m)$ from $\mathbb{R}^n$ to $\mathbb{R}^m$. If*

1. *for all $i, j \in \mathbb{N}$ if $1 \le i \le m$ and $1 \le j \le n$ then $\partial f_i / \partial x_j$ is continuous.*
2. *there exists $\mathbf{x_0}$ such that the Jacobian J of f is such that $J(x_0) \neq 0$,*

then

1. *The Jacobian J^* of f^{-1} is such that $J^*(f(\mathbf{x}_0))$ is the inverse of $J(\mathbf{x}_0)$.*
2. *there exists an open set Ω such that $x_0 \in \Omega$ and f has a continuous inverse.*

Example 16.1. Consider $f(x, y) = (x^3 - 2xy^2, x + y)$. Take $(x_0, y_0) = (1, -1)$. We can show that f is invertible near (x_0, y_0). That is, we can show that there exists an open set Ω such that $(x_0, y_0) \in \Omega$ and f is invertible in Ω. Note that the Jacobian J of f is

$$J(x, y) = \begin{vmatrix} 3x^2 - 2y^2 & -4xy \\ 1 & 1 \end{vmatrix}.$$

Therefore $J(-1, -1) = -3$. Therefore f is invertible near $(1, -1)$.

Example 16.2. Show that $f(x, y) = (x^{26} - y^{26}, xy)$ is (locally) invertible near every $(x, y) \neq (0, 0)$. Using the inverse function theorem, the Jacobian of f is

$$J(x,y) = \begin{vmatrix} 26x^{25} & -26y^{25} \\ y & x \end{vmatrix}.$$

Since $J(x,y) \neq 0$ unless $(x,y) \neq (0,0)$, f is locally invertible. However, it is not *globally invertible,* so it does not have an inverse of *all* $\mathbb{R}^2$. In fact,

$$f(1,1) = f(-1,-1) = (0,1)$$

Proposition 16.2 (Implicit function theorem)**.** *Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open set and let $F : \Omega \to \mathbb{R}^m$ be a function with continuous first derivatives. Assume that $(x_0, y_0) \in \Omega$, $F(x_0, y_0) = 0$, and $\det J(x_0, y_0) \neq 0$. Then there are open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ such that $x_0 \in U$, $y_0 \in V$ and there is a unique function $f : U \to V$ such that $F(x, f(x)) = 0$ for $x \in U$.*

Example 16.3. We can solve the system of

$$xu + yv^2 = 0$$

and

$$xv^3 + y^2 + u^6 = 0$$

for u and v near $x = 1$, $y = -1$, $u = 1$, and $v = -1$. Define $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $F = (F_1, F_2)$ where $F_1(x,y,u,v) = xu + yv^2$, $F_2(x,y,u,v) = xv^3 + y^2 + u^6$. Note that $F(1,-1,1,-1) = 0$ and the Jacobian of F is

$$J(x,y,u,v) = \begin{vmatrix} x & 2vy \\ 6u^5y^2 & 3xv^2 \end{vmatrix}$$

so $J(1,-1,1,-1) \neq 0$. Therefore we can apply the implicit function theorem: We can solve for u and v where $u = f_1(x,y)$ and $v = f_2(x,y)$ in some neighbourhood of $x = 1$ and $y = 1$.