Analytical solutions of the one-dimensional heat equation using the Laplace transform and numerical simulation by the FDM method

A Project Report

Submitted to the Department of Mathematics, University of Chittagong.

Submitted in partial fulfillment of the requirements for a Master of Science.

Session: M.S. (2022-2023)

Course Code: AMAT 527

Submitted by:

Md Nabiul Bashar

Student ID: 19203135

Under the Supervision of

Parvin Akter

Assistant Professor

Department of Mathematics

University of Chittagong

Date of Submission: 27th November, 2025

ACKNOWLEDGMENT

First and foremost, I express my profound gratitude to Almighty Allah for granting me the strength, determination, and perseverance needed to undertake and complete this project.

I want to extend my deepest appreciation to my respected supervisor, **Assistant Professor Parvin Akter Paru**, for her unwavering guidance, thoughtful feedback, and continuous support. Her scholarly insight, patience, and encouragement have played a central role in shaping the direction and quality of this study. I am sincerely grateful for the time she devoted to reviewing my work, providing constructive suggestions, and motivating me to maintain a high academic standard. Her dedication to her students and her commitment to academic excellence have been a source of constant inspiration.

I also wish to express my sincere gratitude to the Department of Mathematics, University of Chittagong, for providing a supportive academic environment and the necessary facilities required to complete this project. The resources, cooperation, and academic atmosphere within the department have contributed significantly to the successful completion of this work. I am thankful to all faculty members and administrative staff whose guidance and assistance, both direct and indirect, have helped me throughout my academic journey.

In addition, I am indebted to my classmates and friends for their encouragement, discussions, and moral support. Their companionship created a positive and motivating environment, especially during difficult stages of the project. I am also sincerely grateful to my family for their constant love, understanding, and prayers. Their unwavering belief in my abilities has been a continuous source of strength.

Finally, I acknowledge with appreciation all individuals who contributed to this work in any capacity. Even though their names may not be mentioned individually, their support and kindness are deeply valued.

CERTIFICATION

This is to certify that the project entitled "**Analytical Solutions of the One-Dimensional Heat Equation Using Laplace Transform and Numerical Simulation by FDM Method**", submitted by **Md Nabiul Bashar (ID: 19203135)** to the University of Chittagong in partial fulfillment of the requirements for the degree of Master of Science in Applied Mathematics, is an authentic and original piece of research carried out independently by him under my supervision.

I hereby affirm that the candidate has completed this work with sincerity, diligence, and academic integrity. The study demonstrates his clear understanding of the theoretical formulation of the one-dimensional heat equation, the analytical solution through Laplace transform techniques, and the numerical approximation using the Finite Difference Method. The project further reflects his ability to apply appropriate mathematical tools to analyze diffusion-related problems.

To the best of my knowledge, this work has not been submitted, in whole or in part, to any other university or institution for the purpose of obtaining any academic degree or qualification. I recommend this project for evaluation as a partial requirement of the Master of Science program.

Supervisor

Parvin Akter

Assistant Professor

Department of Mathematics

University of Chittagong

ABSTRACT

This work presents a comprehensive study of the one-dimensional heat equation, focusing on both analytical and numerical approaches for solving transient heat conduction problems. The analytical solution is obtained using the Laplace Transform method, where the partial differential equation is converted into an ordinary differential equation in the Laplace domain, solved under homogeneous boundary conditions, and then inverted to produce the exact time dependent temperature distribution. To complement the analytical approach, a numerical solution is developed using the Explicit Finite Difference Method (FDM), in which the spatial and temporal domains are discretized, and the temperature field is advanced in time using forward-time and central-space approximations. Stability considerations, including the mesh Fourier number and the CFL condition, are applied to ensure convergence and accuracy of the numerical model. The study highlights the physical interpretation of the heat equation as a diffusion process and demonstrates how temperature gradients drive heat flow within a material. By comparing analytical and numerical solutions, the work illustrates the strengths of both approaches and shows how the heat equation serves as a foundation for modeling thermal behavior in engineering, science, and applied mathematics.

TABLE OF CONTENT

Analytical solutions of the one-dimensional heat equation using the Laplace transform and numerical simulation by the FDM method

1. Introduction

The one-dimensional heat equation is one of the most fundamental partial differential equations (PDEs) in mathematical physics, forming the backbone of models that describe diffusion, conduction, and the spread of heat or other physical quantities over time. Unlike nonlinear PDEs that exhibit complex behaviors such as chaos or solitons, the heat equation represents a linear and parabolic process where disturbances smooth out as time progresses. Its simplicity does not diminish its importance; rather, it provides a conceptual foundation for understanding how energy dissipates in physical systems and serves as a benchmark for validating numerical and analytical methods. First studied rigorously by Joseph Fourier in the early 19th century, the heat equation revolutionized the mathematical description of thermal processes and laid the groundwork for modern theories of diffusion.

The study of the heat equation has led to the development of powerful analytical and numerical techniques. Analytical methods, such as separation of variables, Fourier series, and Laplace transforms, yield exact solutions that reveal the underlying structure and behavior of diffusive systems. These solutions provide insights into how temperature profiles evolve, how boundary conditions influence the system, and how initial disturbances dissipate over time. However, analytical solutions are only feasible for simple geometries and idealized conditions. In many real-world applications such as irregular geometries, time dependent boundaries, and nonlinear material properties analytical techniques fall short.

Numerical methods, particularly finite difference schemes like the Forward Time Central Space (FTCS) method, offer practical and flexible tools for approximating solutions to the heat equation when analytical solutions become intractable. These computational approaches transform the continuous PDE into discrete algebraic equations that can be solved iteratively. Understanding the stability, accuracy, and behavior of such numerical schemes is essential for ensuring reliable simulations in engineering, physics, and applied sciences. Numerical methods not only approximate the analytical solution when one exists but also allow the study of more complex scenarios beyond analytical reach.

This project report presents a detailed investigation into both analytical and numerical solution techniques for the one-dimensional heat equation. It examines the exact solution obtained through the Laplace Transform method and compares it with the numerical approximation generated using the FTCS finite difference scheme. By analyzing and comparing these methodologies, the study aims to bridge the gap between theoretical exact solutions and practical computational methods, offering a comprehensive understanding of diffusion processes and the tools used to model them.

1.1 Background of the Study

The heat equation has long been regarded as a cornerstone of mathematical physics, capturing the fundamental principles of heat conduction and diffusion. Its origins trace back to the pioneering work of Joseph Fourier in the early 19th century, whose study of heat propagation in solids led to the development of Fourier series and revolutionized the mathematical treatment of physical phenomena. The one-dimensional heat equation, expressed as

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

describes how temperature evolves over time within a medium due to the natural tendency of heat to flow from regions of higher temperature to regions of lower temperature. This intrinsic property of diffusion makes the heat equation a prototypical example of parabolic partial differential equations.

The significance of the heat equation extends far beyond thermal processes. Its mathematical structure appears in diverse fields such as fluid dynamics, probability theory, image processing, and financial modeling. For example, the heat equation governs Brownian motion, forms the basis of the diffusion equation in chemistry, and plays a central role in the Black–Scholes model in finance. Because of its widespread applications, the heat equation is one of the most studied PDEs in both theoretical and applied mathematics.

In many cases, analytical techniques such as separation of variables, Fourier transforms, and Laplace transforms provide elegant closed-form solutions to the heat equation under simple boundary and initial conditions. These solutions not only offer insight into the behavior of diffusive systems but also serve as benchmarks for testing numerical methods. However, analytical methods quickly become limited when dealing with complex geometries, nonlinear material properties, or time dependent boundary conditions.

As a result, numerical methods have become essential tools for solving the heat equation in practical settings. Among these, the Finite Difference Method (FDM) stands out for its simplicity, efficiency, and ability to approximate the continuous PDE using discrete grid points. In particular, the Forward Time Central Space (FTCS) scheme provides an intuitive and accessible approach for simulating the heat diffusion process. By approximating derivatives with difference equations, the FTCS method enables step-by-step simulation of temperature evolution over time.

This study focuses on analyzing the heat equation using both analytical and numerical perspectives. By comparing the exact solution obtained through the Laplace Transform method with the numerical approximation generated by the FTCS scheme, the project emphasizes the strengths and limitations of each approach. Understanding this comparison is essential, as it builds foundational knowledge required for solving more complex PDEs and prepares the groundwork for advanced computational modeling techniques.

1.2 Objectives of the Study

The main objective of this study is to explore the one-dimensional heat equation from both analytical and numerical perspectives in order to develop a clearer understanding of how heat diffuses over time and space. The work begins by obtaining the analytical solution using the Laplace Transform method, which provides an exact mathematical formula describing how temperature changes at any point in the rod as time progresses. This exact solution helps establish a theoretical foundation for understanding the physical behavior of heat conduction.

Alongside the analytical approach, the study also aims to design a numerical model using the Finite Difference Method (FDM). This numerical method allows the heat equation to be solved on a computer by breaking the rod and the time interval into small steps and calculating the temperature values step by step. Since numerical methods do not always behave perfectly, an important goal of the research is to examine the stability of the FDM scheme. In particular, the study looks at the importance of the Fourier number, which determines whether the numerical solution remains accurate and stable or whether it begins to diverge and produce unrealistic results.

Another key objective is to compare the analytical and numerical solutions to see how closely the numerical method matches the exact solution. This comparison helps identify the strengths and weaknesses of the numerical approach, including its accuracy, computational efficiency, and any limitations that arise from grid size or stability restrictions. Through this combined analysis, the study aims to develop a complete understanding of how the heat equation behaves, how numerical methods can be effectively applied, and why this equation is so important in describing real physical processes such as heat flow, diffusion, and thermal behavior in different materials.

1.3 Literature Review

A differential equation is an equation that describes how a quantity changes by relating the quantity itself to its rate of change. In many natural and physical processes, things do not stay constant they evolve over time or vary across space. Differential equations give us a mathematical way to capture and understand these changes.

For example, the temperature of a metal rod, the motion of a falling object, the spread of heat in a room, the growth of a population, or the flow of electricity in a circuit all change continuously. These changes can be expressed using derivatives, which measure how fast something is changing. When an equation involves both a function and its derivatives, we call it a differential equation.

Differential equations come in different forms. An **ordinary differential equation (ODE)** involves derivatives with respect to a single variable, usually time. A **partial differential equation (PDE)** involves derivatives with respect to more than one variable, such as both space and time. PDEs are especially important in modeling physical systems like heat flow, fluid motion, sound waves, and electromagnetic fields.

In many cases, solving a differential equation means finding a function that satisfies both the equation and any given conditions, such as the value of the function at the start of the process. Some equations have neat, exact solutions that can be found using mathematical techniques. Others are too complicated to solve directly, so we use numerical methods and computers to approximate the solution.

Overall, differential equations serve as one of the most powerful tools in mathematics and science. They help us connect mathematical ideas with real-world phenomena, allowing us to predict behavior, understand physical laws, and design systems in engineering, physics, biology, and many other fields.

2. Introduction To Differential Equation

2.1 Differential Equation

A differential equation (DE) is a mathematical equation that connects a physical quantity with the way it changes. In simple words, it does not just tell us the value of a quantity, but also explains how that quantity changes with respect to another variable such as time or space.

Whenever something in nature varies like temperature increasing, velocity decreasing, population growing, or electric current oscillating these changes can be measured using derivatives. A differential equation uses these derivatives to describe the overall behavior of the system. So, a differential equation acts like a bridge between a physical system and its mathematical description. It helps us convert real-life phenomena into mathematical language. That is why differential equations are used in almost every scientific field physics, engineering, biology, economics, and even environmental science.

General form of a differential equation

$$F(x, y, y', y'', \dots \dots.) = 0$$

This means the equation involves a function$y$ y, its derivatives $y, y', y''$and possibly the independent variable$x$.

2.2 Types of Differential Equations

Differential equations can be grouped or classified in several different ways depending on how they are formed and how the unknown function appears in them. These classifications help us understand the nature of the equation and guide us toward the right method for solving it. Below are the main types of differential equations explained in simple and clear language.

Ordinary Differential Equations (ODEs)

An ordinary differential equation involves derivatives with respect to only one independent variable, which is most often time. This means the quantity we are studying changes only along one direction either time or space, but not both. Example :

$$\frac{\partial u}{\partial t} + 3y = 0$$

This equation tells us how $y(t)$,changes as time t changes. ODEs appear in many familiar systems, such as population growth, cooling of an object, voltage in circuits, or motion of particles.

Partial Differential Equations (PDEs)

A partial differential equation includes partial derivatives**,** meaning the unknown quantity depends on two or more independent variables**,** usually space and time.

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

This is the heat equation. It describes how temperature $u(x, t)$ changes with both time t and position x.PDEs are essential when dealing with heat flow, wave motion, fluid dynamics, and many physical processes that vary across space and evolve with time.

Order & Degree of Differential Equation

The **order of a differential equation** is defined as the highest derivative of the unknown function that appears in the equation. For example, if the equation contains only the first derivative, it is a first order differential equation; if it contains the second derivative, it is a second order equation, and so on. The order helps determine how many initial or boundary conditions are needed to obtain a unique solution. For instance:

$\frac{dy}{dx} + y = 0$ is a first order differential equation.

$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ is a second order differential equation.

The concept of order is fundamental because it reflects the complexity of the system and influences the methods used to solve the differential equation.

The **degree of a differential equation** is defined as the power of the highest order derivative after the equation has been made free from radicals and fractions involving derivatives. In other words, the differential equation must first be expressed in polynomial form with respect to the derivatives. Once written in this form, the degree is simply the exponent of the highest-order derivative.

For example:

(i). $(\frac{dy}{dx})^2 + y = 0$ is a first order differential equation. Here highest order derivative is $\frac{dy}{dx}$ appears to power 2 and its degree is 2.

(ii). $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 6y = 0$ is a second order differential equation. Here highest order derivative is $\frac{d^2y}{dx^2}$ appears to power 1 and its degree is 1.

The degree is important because it influences the type of solution methods that may be applied and indicates how nonlinear or complex the differential equation may be.

2.3 Linear and Nonlinear Differential Equations

Differential equations describe how quantities change in physical, biological, environmental, and engineering systems. Many natural processes such as heat spreading in a rod, population growth, electrical circuit behavior, or waves moving on water are governed by these equations. Differential equations appear mainly in two forms ordinary differential equations (ODEs), which involve a single independent variable, and partial differential equations (PDEs), which involve multiple independent variables such as space and time. Both ODEs and PDEs can be further classified as linear or nonlinear, and this distinction plays a crucial role in determining how difficult an equation is to analyze and solve.

A linear differential equation is one in which the dependent variable and its derivatives appear only to the first power and are not multiplied together. Linear equations obey the principle of superposition, meaning the sum of any two solutions is also a solution. They also tend to exhibit predictable behavior, and many classical models such as the heat equation, wave equation, and Laplace equation fall into this category. A typical linear ODE takes the form

$$a_0(x)u + a_1(x)u' + a_2(x)u'' = f(x),$$

and a linear PDE may be written as

$$a_0(x,t)u + a_1(x,t)u_x + a_2(x,t)u_{xx} + \cdots = f(x,t),$$

A nonlinear differential equation, on the other hand, contains nonlinear terms such as $u^2, uu_z\, sin(u)\,, e^u, u^3u_{xx}$.These equations do not satisfy superposition and may show highly complex behavior. Small changes in initial conditions can produce large changes in the solution, making nonlinear systems capable of phenomena such as shocks, solitons, turbulence, and chaos. Many important models in physics and engineering like the Kortewegde Vries (KdV) equation and the KdV Burgers equation are nonlinear and require numerical methods for practical analysis.

In contrast, the heat equation, which is the central focus of this thesis, is a linear PDE. Its linearity makes it mathematically more manageable and provides a solid foundation for understanding diffusion, stability, and numerical approximation methods. Because of this, the heat equation is often used as a starting point for studying PDEs and developing numerical techniques.

In summary, linear differential equations are easier to solve and analyze due to their predictable structure, while nonlinear equations describe a wider range of real-world phenomena but often require more advanced mathematical tools or numerical methods. Understanding the distinction between linear and nonlinear equations is essential for analyzing physical systems and selecting appropriate solution techniques.

For examples:

Partial Differential Equations (PDEs) appear in many different forms depending on the physical phenomenon they describe. One of the most important linear PDEs is the heat equation,

$$u_t = ku_{xx}$$

which models diffusion processes such as heat conduction, chemical diffusion, and even financial models like the Black Scholes equation. Another fundamental linear PDE is the wave equation,

$$u_{tt} = c^2 u_{xx}$$

used to describe vibrations in strings, acoustic waves, and electromagnetic propagation. The Laplace equation,

$$u_{tt} + \ u_{yy} = 0$$

and the Poisson equation,

$$u_{tt} + \ u_{yy} = f(x, y),$$

are elliptic PDEs that arise in electrostatics, incompressible fluid flow, gravitational fields, and steady-state temperature distributions. These linear equations are well understood and often allow exact analytical solutions through classical techniques such as separation of variables, Fourier series, and integral transforms.

In contrast, many important PDEs in physics and engineering are nonlinea**r**, meaning that the principle of superposition does not apply and solutions often display much richer and more complex behavior. A well-known nonlinear PDE is the Kortewegde Vries (KdV) equation,

$$. u_t + 6uu_z + u_{xxx} = 0$$

which models shallow-water waves and produces solitons stable, solitary waves that maintain their shape even after interacting with other waves. The KdV Burgers equation,

$$u_t + uu_x + vu_{xx} + u_{xxx} = 0$$

incorporates nonlinear convection, viscosity, and dispersion, making it useful in modeling plasma waves, fluid flow, and nonlinear signal propagation.

Another fundamental nonlinear PDE is the Navier Stokes (NS) equation, which governs the motion of viscous fluids such as air and water. For incompressible flow, it can be written as:

$$u_t + u(u.\nabla) = -\nabla p + \nu\nabla^2 u$$

where u is the velocity field, p is pressure, and ν is the viscosity. These equations describe turbulence, weather systems, ocean currents, aerodynamics, blood flow, and many other fluid-dynamic processes. Their global behavior remains one of the most challenging open problems in mathematics.

Nonlinear PDEs such as KdV, KdV Burgers, and Navier Stokes can exhibit shock formation, turbulence, chaotic motion, and long-range interactions phenomena that cannot occur in linear models. Because of their complexity, nonlinear PDEs typically require advanced analytical methods or numerical techniques such as finite-difference, finite-element, or spectral methods.

In summary, both ODEs and PDEs whether linear or nonlinear form the backbone of mathematical modeling. They describe heat flow, wave propagation, fluid motion, electromagnetic fields, diffusion, and many other natural phenomena. Mastering their behavior is essential for predicting, analyzing, and simulating a wide range of real-world systems in science and engineering.

2.4 Applications of Differential Equations

Differential equations play a central role in describing natural phenomena and engineered systems because many quantities in the real world change continuously with respect to time, space, or both. By relating a function to its derivatives, differential equations offer a powerful mathematical framework for modeling dynamic processes, predicting future behavior, and understanding the underlying mechanisms of complex systems. Their applications stretch across physics, engineering, biology, economics, and environmental science, making them indispensable in both theoretical research and practical problem-solving.

In physics, differential equations form the foundation for laws governing motion, heat flow, electricity, wave propagation, and fluid dynamics. Newton's Second Law, which describes the motion of objects under force, naturally leads to differential equations involving position, velocity, and acceleration. Maxwell's equations, central to electromagnetism, are systems of partial differential equations that explain how electric and magnetic fields interact and propagate. The heat equation models how temperature changes in space and time, the wave equation describes vibrations and sound propagation, and the Schrödinger equation in quantum mechanics uses differential operators to describe the behavior of particles at atomic scales. These examples show how deeply embedded differential equations are in understanding physical reality.

In engineering, differential equations are essential tools for designing and analyzing systems. Electrical engineers use them to study circuits containing resistors, capacitors, and inductors, where voltages and currents change over time. Mechanical engineers rely on differential equations to model vibrations, structural deformations, and dynamic forces. Chemical engineers use them in reaction kinetics and mass transfer, where the concentration of substances changes over time. Control engineers employ differential equations to design stable systems such as autopilots, temperature regulators, and robotic controllers. Across all engineering branches, differential equations provide the mathematical language for predicting system behavior and ensuring safe, efficient designs.

In biology and medicine, differential equations help describe growth, decay, and interactions among biological populations. The logistic growth model predicts how populations increase under limited resources, while the SIR (Susceptible Infected Recovered) model uses differential equations to track the spread of infectious diseases through communities. Neuroscience uses differential equations to model electrical activity in nerve cells, such as the Hodgkin–Huxley and FitzHugh Nagumo models. Pharmacology applies them to determine how drugs are absorbed, distributed, and eliminated from the body. These biological models guide decision-making in public health, epidemiology, and biomedical engineering.

In economics and finance, differential equations model dynamic processes such as investment growth, interest rates, and market fluctuations. The Black–Scholes equation, one of the most famous PDEs in finance, is used to price options and financial derivatives. Differential equations also appear in optimal control theory, where economists determine policies that maximize profit or minimize cost over time. They also model supply and demand adjustments, capital accumulation, and wealth distribution, helping economists analyze long-term trends and stability of financial systems.

In environmental science and earth systems, differential equations describe the movement of heat, pollutants, and fluids in natural environments. The diffusion equation models how pollutants spread in air, soil, or water. The Navier Stokes equations, a complex system of nonlinear PDEs, describe atmospheric and oceanic flows, forming the basis of weather forecasting and climate modeling. Hydrologists use differential equations to study groundwater flow, rainfall runoff processes, and erosion. These models are crucial for understanding environmental change, predicting natural hazards, and managing natural resources.

Collectively, these applications demonstrate the universal importance of differential equations in modern science and engineering. They provide a mathematical bridge between theoretical concepts and real-world systems, enabling researchers and practitioners to analyze complex phenomena, make predictions, and develop practical solutions. Because nearly every dynamic process can be described using differential equations, they remain one of the most powerful tools in applied mathematics.

3 Introduction to the Heat Equation

The heat equation $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ is one of the most fundamental partial differential equations in mathematical physics, widely used to describe the diffusion of heat within a given medium over time. As a model of heat conduction, it plays a central role in mathematical modeling, engineering analysis, and computational simulation. Due to its simplicity, rich mathematical structure, and wide applicability, the heat equation serves as a foundational tool for understanding diffusion phenomena not only in thermal systems but also in several other fields such as population dynamics, image processing and probability theory. In this project, we aim to explore both the analytical and numerical solutions of the one dimensional heat equation using the Laplace Transform and the Finite Difference Method (FDM). The study highlights the physical meaning of the equation, investigates its solution behavior, and demonstrates how classical and modern techniques can be applied to solve real-world heat-conduction problems efficiently.

3.1 Definition and Background of the Heat Equation

The heat equation is a fundamental partial differential equation that describes how temperature (or heat energy) changes with time inside a material. It models the physical process of heat conduction .In one spatial dimension, the heat equation is:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Where , $u(x,t) =$ temperature at position x and time , $\alpha > 0 =$ thermal diffusivity of the material, $\frac{\partial u}{\partial t}$ =rate of change of temperature with time, and $\frac{\partial^2 u}{\partial x^2} =$ curvature of the temperature profile. This equation states represents the heat flows from hotter regions to cooler regions, and the temperature evolves over time according to the second spatial derivative of the temperature distribution.

The development of the heat equation dates back to the pioneering work of Joseph Fourier in the early 19th century. In his landmark treatise "Théorie analytique de la chaleur" [1], Fourier formulated the mathematical law of heat conduction, now known as Fourier's Law, which states that the rate of heat flow through a medium is proportional to the negative gradient of the temperature. Based on this principle and the conservation of energy, the heat equation was derived to describe how temperature evolves within a solid material.

The one-dimensional heat equation is given by

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

where $u(x, t)$ is the temperature at position x and time t, and α is the thermal diffusivity which is a physical constant that measures how rapidly heat spreads through the material. Over the years, the heat equation has become a cornerstone in applied mathematics, serving as the archetypal parabolic partial differential equation. Its solutions are well understood, and various analytical and numerical methods have been developed to handle different boundary and initial conditions. The heat equation's mathematical properties such as smoothness, stability, and strong diffusion behavior make it essential for advancing both theoretical and applied research in differential equations and computational modeling.

3.2 Physical Significance & Importance of Heat Equation

The heat equation expresses the way temperature naturally spreads through a material over time. Its physical meaning is closely tied to the behavior of the second spatial derivative u_{xx}, which represents the curvature of the temperature profile. The quantity u_{xx} determines whether heat flows into or out of a point. If

$$u_{xx} > 0,$$

the temperature curve is concave upward at that point, meaning the temperature there is lower than in its immediate surroundings. As a result, heat flows into the point from both sides, causing the temperature to increas**e**. If

$$u_{xx} > 0,$$

the temperature curve is concave downward, meaning the temperature at that point is higher than the temperatures around it. Heat flows outward from the point, leading the temperature to decreas**e**. If

$$u_{xx} = 0,$$

the temperature profile is locally linear with no curvature, so there is no net heat flow into or out of the point. In this case, the temperature remains constant, representing a local steady-state condition. These three cases show how the heat equation. If

$$u_t = ku_{xx}$$

connects the rate of temperature change to the curvature of the temperature distribution, demonstrating the natural tendency of heat to move from hotter regions to cooler ones and gradually smooth out temperature differences over time.

The heat equation is one of the most important and widely used partial differential equations in mathematical physics. Its primary significance lies in the fact that it accurately describes the process of heat conduction in solid materials. Whenever temperature varies within a metal rod, a semiconductor chip, a building wall, or any physical medium, the heat equation governs how that temperature evolves with time. As the mathematical form of Fourier's Law of Heat Conduction, it plays a central role in mechanical, electrical, civil, chemical, and materials engineering, especially in problems involving heating, cooling, thermal insulation, and heat-transfer design .Beyond its physical interpretation, the heat equation is the fundamental example of a parabolic partial differential equation. It captures the general behavior of diffusion-type processes, where a quantity spreads out smoothly over time. Because of this, the same mathematical equation appears in many contexts unrelated to temperature, such as diffusion of gases, concentration of chemicals, movement of particles in Brownian motion, population spreading in biology, and even in financial mathematics through the Black Scholes model. In image processing, the heat equation describes smoothing and denoising operations by modeling how pixel intensities diffuse. The heat equation is also significant from a theoretical mathematical perspective. Its solutions demonstrate the smoothing effect of parabolic equations, the power of eigen function expansions, and the application of Fourier series, Laplace transforms, and Green's functions. It connects directly to Sturm Liouville theory, illustrating how orthogonal eigenfunctions naturally arise in solving boundary value problems. The equation satisfies a maximum principle, ensuring uniqueness and stability of solutions, which are fundamental ideas in modern PDE theory.

In computational science, the heat equation is the standard model for developing and testing numerical methods. Techniques such as the Finite Difference Method, Finite Element Method, explicit and implicit time stepping, Crank Nicolson schemes, and stability analysis through the CFL condition are all introduced using the heat equation. It clearly demonstrates how numerical solutions depend on spatial and temporal discretization, and why stability conditions like $\Delta t \leq \frac{\Delta x^2}{2k}$ are necessary. Overall, the heat equation is important because it describes a fundamental physical process, it appears in many scientific and engineering fields beyond heat conduction, it provides deep mathematical insights into parabolic PDEs, and it serves as a cornerstone for developing analytical and numerical solution techniques. Its wide applicability and theoretical richness make it one of the most central equations in applied mathematics and mathematical modeling.

3.3 Physical Interpretation of the Heat Equation

The heat equation describes how temperature spreads inside a material over time, and its physical meaning comes directly from the laws of heat conduction. When heat flows through a rod or any solid medium, it moves naturally from regions of higher temperature to regions of lower temperature. The rate at which this temperature changes at any point depends on how “curved” or “bent’’ the temperature graph is at that location. This curvature is represented mathematically by the second spatial derivative u_{xx} .If the temperature at a point is lower than its neighboring points, the curvature is positive, and heat flows into that point, causing the temperature to rise. If the temperature at a point is higher than its surroundings, the curvature is negative, and heat flows out, making the temperature decrease. Therefore, the sign and magnitude of u_{xx} determine whether a point heats up or cools down. The constant k, known as the thermal diffusivity, measures how quickly heat spreads through the material. Materials with high thermal diffusivity, such as metals, allow heat to spread rapidly. Materials with low diffusivity, like wood or plastic, conduct heat more slowly. The term ku_{xx} therefore represents the net heat flow into a point per unit time, and the heat equation

$$u_t = ku_{xx}$$

states that the time rate of change of temperature at any position is exactly equal to this net flow. In simpler terms, the equation balances how quickly temperature changes over time with how sharply the temperature varies in space. Physically, this means that the heat equation naturally smooths out temperature differences. Sharp peaks or sudden changes in temperature flatten out as time passes because heat spreads out evenly. If a rod is heated at one end, the heat gradually spreads along the rod until the entire rod reaches thermal equilibrium. The equation always drives the system toward a uniform temperature distribution unless external heat sources or boundary conditions intervene. Another important physical interpretation is that the heat equation is a diffusion equation, not only for heat but for any quantity that spreads out with time, such as chemicals in a fluid, smoke in the air, or even the probability density of Brownian motion. This diffusive behavior is characterized by the way the solution becomes smoother and more spread out as time increases. Irregular or jagged temperature distributions become smooth very quickly, reflecting how physical heat conduction erases sharp differences in temperature.

Overall, the heat equation provides a precise mathematical description of how energy flows between neighboring particles of a medium, how temperature equalizes, and how diffusion processes behave in nature. It captures the fundamental physical law that temperature tends to equalize over time and that the rate of equalization depends on the material's ability to conduct heat.

3.4 Application of the Heat Equation

The heat equation has extremely wide applications across science, engineering, industry, and even fields that are not directly related to classical thermal problems. Its primary use is in heat conduction and thermal engineering, where it models the transfer of heat through solid materials such as metal rods, machine components, and electronic devices. Engineers rely on the heat equation to design efficient cooling systems, analyze thermal insulation, study furnace temperature distribution, and predict how materials heat up or cool down over time. In mechanical and civil engineering, the heat equation helps in understanding the temperature effects on large structures, predicting thermal stresses, studying the curing process of concrete, and ensuring that bridges, buildings, and mechanical parts can withstand temperature changes during operation. In material science, it plays a crucial role in studying the diffusion of heat and particles in solids, modeling phase change processes such as melting or solidification, and designing heat treatment procedures for metals and alloys.

Beyond classical engineering fields, the heat equation appears in environmental and geophysical applications. It helps describe temperature variations in soil layers, ocean currents, groundwater reservoirs, and atmospheric processes, enabling scientists to understand long-term climate behavior, seasonal temperature changes, and heat transfer in natural ecosystems. In biology and medicine, the equation is used to model the diffusion of heat in human tissues, predict thermal damage during laser surgeries, understand metabolic heat generation, and guide the development of thermal treatment techniques for cancer therapies.

The heat equation also has important uses in mathematical finance and probability theory. The well-known Black–Scholes equation for option pricing is mathematically equivalent to a form of the heat equation, meaning that the spread of financial risk behaves similarly to heat diffusion. In image processing and computer vision, heat diffusion models are applied to smooth images, reduce noise, and extract meaningful features while preserving important edges. These diffusion based filtering techniques help improve image quality and are widely used in medical imaging, satellite image analysis, and machine vision.

Overall, the heat equation stands out as one of the most versatile and widely applicable mathematical models. Whether studying the cooling of a simple metal rod or analyzing complex systems in medicine, finance, or environmental science, the heat equation provides a unified framework for understanding how quantities spread, dissipate, or evolve over time. Its broad range of applications demonstrates its importance as a foundational tool in both theoretical and applied sciences.

4. Methods of Solving Heat Equation

The heat equation provides a mathematical framework for describing how temperature varies within a material over time. Its formulation is based on physical principles such as conservation of energy and the law of heat conduction. In this section, we derive the governing equation from Fourier's Law, present its standard mathematical form, specify the relevant boundary and initial conditions, and discuss the general behavior of its solutions.

4.1 Laplace Transform & Inverse Laplace Formulation

The standard one-dimensional heat equation is expressed as

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \qquad T > 0$$

Where $\mathrm{u}(\mathrm{x}, \mathrm{t})$ represents the temperature at position x and time, $\alpha > 0$ represents the thermal diffusivity of the material, $\frac{\partial u}{\partial t}$ represents the rate of change of temperature with time, $\frac{\partial^2 u}{\partial x^2} =$ curvature of the temperature profile.

This equation belongs to the class of parabolic partial differential equations, characterized by smoothing behavior where temperature gradients decrease as time progresses.

The procedure begins by Transforming the PDE. We apply the Laplace Transform, denoted $\mathcal{L}$ with respect to the time variable (t),. The temperature function $u(x, t)$, in the time domain is transformed into $U(x, s)$ in the Laplace domain, where is the Laplace variable $\mathcal{L}\{u(x, t)\} = U(x, s)$,).Next, we Use Initial Conditions to transform the time derivative term. This is a crucial step where the initial temperature distribution, $\mathrm{u}(\mathrm{x}, \mathrm{t})$, is incorporated

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = sU(x, s) - u(x, 0)$$

By transforming the original heat equation, $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$, it is converted into a second-order ODE in the spatial variable x:

$$sU(x,s) - u(x,0) = \alpha \frac{\partial^2 u}{\partial x^2}$$

We then Solve the ODE for $U(x,s)$ using standard techniques, such as finding the characteristic equation and applying the transformed boundary conditions.Finally, the function $U(x,s)$ is transformed back into the time domain by applying the Inverse Laplace Transform $\mathcal{L}^{-1}\{U(x,s)\}$, to obtain the final, time-dependent temperature solution, $u(x,t)$.

After solving for $U(x,s)$ the temperature field is obtained by:

$$U(x,t) = \mathcal{L}^{-1}\{U(x,t)\}(t)$$

Using standard transforms:

$$\mathcal{L}^{-1}\left\{\frac{1}{S+a}\right\}(t) = e^{-at},$$

We invert each term of $U(x,s)$ to find a closed form expression. This yields the exact temperature distribution for all $t > 0$.

4.2 Derivation from Fourier's Law

The derivation of the one-dimensional heat equation begins with Fourier's Law of Heat Conduction, which states that the heat flux $q(x,t)$ is proportional to the negative of the temperature gradient:

$$q(x,t) = -k\frac{\partial u}{\partial x}$$

where $u(x,t)$is the temperature, k is the thermal diffusivity of the material (a positive constant) and is $q(x,t)$ repests the rate of heat flow per unit area. Consider a small segment of a rod between x and $x + \Delta x$ .The rate of heat inflow into the segment is $q(x,t)$ and the rate of heat outflow is $q(x+\Delta x,t)$.The net heat entering the segment is therefore

$$q(x,t) - q(x+\Delta x,t) = -\frac{\partial q}{\partial x}\,\Delta x$$

This net heat contributes to raising or lowering the internal energy of the material. By the principle of conservation of energy, the change in internal energy of the segment is

$$\rho c \frac{\partial u}{\partial t} \Delta x$$

w here ρ is the density and c is the specific heat capacity. Equating heat accumulation to heat flow gives

$$\rho c \frac{\partial u}{\partial t} \Delta x == -\frac{\partial q}{\partial x} \Delta x$$

Canceling Δx and substituting Fourier's Law yields

$$\rho c \frac{\partial u}{\partial t} = -\frac{\partial}{\partial t}(-k \frac{\partial u}{\partial t}) = -k \frac{\partial^2 u}{\partial x^2}$$

Let,$\alpha = \frac{k}{\rho c}$ be the thermal diffusivity, we obtain the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

For convenience, we may denote α simply by k.

4.3 FDM Discretization Process

The first step, creating a Grid, involves discretizing the continuous domain (x, t) This is achieved by imposing a rectangular mesh where the spatial position is defined by $x = i\Delta x$ and time is defined by $t = j\Delta t$ The temperature at any intersection point is represented by the discrete notation u_j^i, which approximates the continuous temperature $(i\Delta x, = j\Delta t)$ .Next, we Approximate Derivatives using finite difference formulas derived from Taylor series expansions. The time derivative, $\frac{\partial u}{\partial t}$, is approximated using a Forward Difference scheme:

$$\frac{\partial u}{\partial t} = \frac{u_i^{j+1} - u_j^i}{\Delta t}$$

The second spatial derivative, $\frac{\partial^2 u}{\partial x^2}$ which represents the curvature of the temperature profile, is approximated using a Central Difference scheme:

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1}^j - 2u_j^i + u_{i-1}^j}{(\Delta x)^2}$$

By substituting these approximations into the original heat equation, $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$, we form the Difference Equation for the Explicit Method:

$$\frac{u_i^{j+1} - u_j^i}{\Delta t} = \alpha \frac{u_{i+1}^j - 2u_j^i + u_{i-1}^j}{(\Delta x)^2}$$

Finally, this equation is algebraically rearranged to solve directly for the temperature at the next time step, u_i^{j+1} , based only on the known temperatures at the current time step u_j^i,

$$u_i^{j+1} = u_j^i + r(u_{i+1}^j - 2u_j^i + u_{i-1}^j)$$

Here, the parameter $r = \frac{k\Delta t}{(\Delta x)^2}$ is the mesh Fourier number, which plays a critical role in the stability and accuracy of the numerical solution.

5. Analytical problem & Numerical Simulation of Heat Equation

The analytical solution of the heat equation is obtained by applying the Laplace Transform to convert the time-dependent PDE into an ordinary differential equation in the spatial variable. Solving this transformed equation under the given boundary conditions yields an expression for $U(x, s)$. Applying the Inverse Laplace Transform then returns the solution to the time domain. This process leads to the exact solution $U(x,t) = Sin(\pi x)\, e^{-k\pi^2 t}$ showing exponential decay of the initial temperature profile.

5.1 Laplace Transform & Inverse Laplace Transformation

Let us consider the problem regarding heat equation as follows:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}\ ,0 < x < 1, t > 0$$

with $u(x,0) = Sin(\pi x), u(x,t) = 0, u(1,t) = 0,$

to solve this heat equation by Laplace transformation, we can assume

$$u(x,s) = L\{u(x,t)\}$$

$$=\int_0^\infty e^{-st}u(x,t)dt \tag{1}$$

We need the transform u_t and u_{xx},

By the formula,

$$L\left\{\frac{\partial u}{\partial t}\right\} = sU(x,s) - U(x,0) \tag{2}$$

Given,

$$u(x,0) = Sin(\pi x) \tag{3}$$

Substitute to (2)

$$L\left\{\frac{\partial u}{\partial t}\right\} = sU(x,s) - Sin(\pi x) \tag{4}$$

Now laplace transform of u_{xx}, differentiation in *x*,

$$L\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{\partial^2}{\partial x^2} L\{u\} = U_{xx}(x,s) \tag{5}$$

Transform into PDE, $u_t = ku_{xx}$ (6)

Apply L{.} with respect to t to both sides;

$$\text{left side}, L\{u_t\} = sU(x,s) - Sin(\pi x) \quad \text{from}(4) \tag{7}$$

$$\text{right side}, L\{ku_{xx}\} = k\ L\{ku_{xx}\} = KU_{xx}(x,s) \quad \text{from}(5) \tag{8}$$

Equate (7) and (8) ;

$$sU(x,s) - Sin(\pi x) \ = \ KU_{xx}(x,s) \tag{9}$$

rearrange equation (9) to standard form;

$$KU_{xx}(x,s) - sU(x,s) \ = \ -Sin(\pi x) \tag{10}$$

Now, transform the boundary conditions;

$$u(0,t) = 0\ , u(1,t) = 0, t > 0 \tag{11}$$

Applying laplace transform in t;

$$\text{At } x = 0;\ u(0,s) = L\{u(0,t)\} = L\{0\} = 0 \tag{12}$$

$$\text{At } x = 1;\ u(1,s) = L\{u(1,t)\} = L\{0\} = 0 \tag{13}$$

So we solve (10) with conditions (12) & (13) .Now consider the homogenous part of (10)

$$KU_{xx}(x,s) - sU(x,s) \ = \ 0 \tag{14}$$

Divide (14) by k ,

$$U_{xx} - \frac{s}{k}U = 0 \tag{15}$$

Let,

$$\lambda^2 = \frac{s}{k}U = 0 \tag{16}$$

Then (15) becomes,

$$U_{xx} - \lambda^2 U = 0 \tag{17}$$

The characteristic equation of (17) is,

$$r^2 - \lambda^2 = 0$$

$$\text{Or,} r = \pm\, \lambda \tag{18}$$

Thus the homogeneous solution is,

$$U_h(x,s) \ = \ C_1 e^{\,\lambda\, x} + C_2 e^{-\,\lambda\, x} \tag{19}$$

Equivalently, using hyperbolic functions,

$$U_h(x,s) \ = \ ASinh(\,\lambda\, x) + BCos(\,\lambda\, x) \tag{20}$$

where A, B are constants with respect to x (but may depend on s).

We now find a particular solution $U_p(x,s)$ of (10),

$$KU_{xx}(x,s) - sU(x,s) = -\,Sin(\,\pi x) \tag{21}$$

Because the right-hand sides is $-\,Sin(\,\pi x)$,

$$U_p(x,s) = C(s)Sin(\,\pi x) \tag{22}$$

where C(s) is independent of x. Now compute first derivative,

$$U_{p,x}(x,s) = C(s)\pi Cos(\,\pi x) \tag{23}$$

compute second derivative,

$$U_{p,xx}(x,s) = C(s)\frac{\partial}{\partial t}[\pi Cos(\,\pi x)] = C(s)[-\pi Sin(\,\pi x)]$$

$$=- C(s)\,\pi^2 Sin(\,\pi x) \tag{24}$$

Substitute (22) and (24) into (21):

$$kU_{p,xx}(x,s) - SU_p = -kC\pi^2 Sin(\,\pi x) - s\big(CSin(\,\pi x)\big) \tag{25}$$

Factor Sin(πx) in (25):

$$kU_{p,xx}(x,s) - SU_p = Sin(\,\pi x)(-kC\pi^2 - sC) \tag{26}$$

Equation (21) requires,

$$kU_{p,xx}(x,s) - SU_p = -Sin(\,\pi x) \tag{27}$$

Comparing (26) and (27),

$$Sin(\,\pi x)(-kC\pi^2 - sC) = -Sin(\,\pi x) \tag{28}$$

for $0 < x < 1, \quad Sin(\pi x) \neq 0,$ so divided by Sin(πx) ;

$$(-kC\pi^2 - sC) = -\ 1 \tag{29}$$

factor C,

$$C(-k\pi^2 - s) = -\ 1 \tag{30}$$

solve for $C, C(s) = -\frac{1}{-k\pi^2 - s} = \frac{1}{k\pi^2 + s}$ (31)

from (22), $U_{p(x,s)} = \frac{Sin(\pi x)}{k\pi^2 + s}$ (32)

now combine the homogeneous and particular solutions:

$$U(x,s) = U_h(x,s) + U_p(x,s)$$

$$= ASinh(\lambda x) + BCos(\lambda x) + \frac{Sin(\pi x)}{k\pi^2+s} \quad (33)$$

Where $\lambda = \sqrt{\frac{s}{k}}$

Now apply boundary condition,

$$U(0,s) = 0, U(1,s) = 0 \quad (34)$$

& condition at x=0 from (33),

$$U(0,s) = ASinh(\lambda .0) + BCos(\lambda .0) + \frac{Sin(\pi .0)}{k\pi^2 + s} \quad (35)$$

$$\& \quad Sinh(0) = 0, Cos(0) = 0, Sin(0) = 0 \quad (36)$$

Substitute in (35), $U(0,s) = A.0 + B.0 + 0 = B$ (37)

But from (34), *U(0, s)*=0. Hence ,$B = 0$ (38)

So (33) simplifies to,$U(x,s) = ASinh(\lambda x) + BCos(\lambda x) + \frac{Sin(\pi x)}{k\pi^2+s}$ (39)

Now condition at *x=1* & use *U(1, s)=0*.Substitute x=1 into (39),

$$U(1,s) = ASinh(\lambda .1) + BCos(\lambda x) + \frac{Sin(\pi .1)}{k\pi^2 + s} \quad (40)$$

Use $Sin(\pi) = 0$, So $U(1, s) = ASinh(\lambda) + 0 = ASinh(\lambda)$ (41)

From (34), $U(1, s) = 0$,So $ASinh(\lambda) = 0$ (42)

For So $S > 0, \lambda = \sqrt{\frac{s}{k}}$ and $\text{Sinh}(\lambda) \neq 0$,hence $A = 0$ (43)

Now from (39),

$$U(x, s) = \frac{Sin(\pi x)}{k\pi^2 + s} \tag{44}$$

This is the Laplace transform of the solution & We now invert (44),

$$L\{e^{-at}\}(S) = \frac{1}{S + a} \quad , a > 0 \tag{45}$$

$$\text{Thus, } L^{-1}\left\{\frac{1}{S+a}\right\}(t) = e^{-at} \tag{46}$$

In (44), we have $a = k\pi^2$. Therefore,$L^{-1}\left\{\frac{1}{S+a}\right\}(t) = e^{-k\pi^2 t}$ (47)

Since x behaves like a parameter in the Laplace transform, we can write,

$$U(x, t) = L^{-1}\{U(x, s)\}(t) = L^{-1}\left\{\frac{Sin(\pi x)}{k\pi^2 + s}\right\}(t) \tag{48}$$

Take Sin(πx) outside the inverse transform (it does not depend on t):

$$U(x, t) = Sin(\pi x)\, L^{-1}\left\{\frac{Sin(\pi x)}{k\pi^2 + s}\right\}(t) \tag{49}$$

Use (47) in (49),

$$U(x, t) = Sin(\pi x)\, e^{-k\pi^2 t}$$

This is the analytical solution.

5.2 Finite Difference Discretization

The heat equation is one of the most important partial differential equations in mathematical physics and engineering. While analytical techniques such as the Laplace Transform provide closed-form solutions for simple geometries and boundary conditions, many real world heat-transfer problems are too complex to solve analytically. Therefore, numerical methods become essential for obtaining approximate solutions with good accuracy.

Here present the finite difference method (FDM) for solving the one-dimensional heat equation. The basic idea behind FDM is to replace the continuous spatial and temporal derivatives with discrete approximations on a computational grid. By dividing the domain into small intervals in space and time, the differential equation is converted into a system of algebraic equations that can be solved iteratively. The heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

describes how temperature diffuses through a rod over time. The numerical approach requires three main components:

Discretization of the domain

The spatial interval $[0, L]$ is divided into N nodes with spacing Δx, and the time domain is divided into steps of size (Δt). The temperature is then approximated at discrete grid points:

$$u(x_{i,} t_n) \approx u_i^n$$

Construction of a finite difference scheme

The spatial derivative is approximated using a central difference, and the time derivative is approximated using a forward difference, leading to the well-known FTCS (Forward Time

Central Space) scheme. This scheme provides a simple and intuitive method for advancing the solution in time.

Stability considerations

Numerical methods may behave differently from the analytical solution if the step sizes are not chosen properly. The FTCS method is conditionally stable, meaning it requires a restriction on the time step:

$$\frac{k\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

If this condition is violated, numerical oscillations and divergence occur.

Implementation of the algorithm

After discretizing the equation and ensuring stability, the solution is computed iteratively from the initial condition while continuously enforcing the boundary conditions.

Finite difference discretization

we discretize the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \ ,0 < x < 1, t > 0$$

into a computable numerical form using the finite difference method.

Spatial Discretization

The spatial domain $[0, L]$ is divided into N equal intervals of length

$$\Delta x = \frac{L}{N}$$

The grid points are defined as

$$x_i = i\Delta x, \qquad i = 0,1,2, \ldots \ldots, N.$$

The temperature at position x_i and time t_n is denoted by u_i^n . To approximate the second spatial derivative, we use the central difference formula:

$$\frac{\partial^2 u}{\partial x^2}(x_{i,}t_n) \approx \frac{u_{i+1}^n - 2u_j^n + u_{i-1}^n}{(\Delta x)^2}$$

Time Discretization

The time axis is divided into steps of size:

$$\Delta t = \frac{T}{M}$$

The forward difference approximation for the time derivative is:

$$\frac{\partial u}{\partial t}(x_{i,}t_n) \approx \frac{u_i^{n+1} - u_j^n}{\Delta t}$$

FTCS (Forward-Time Central-Space) Scheme

Substituting the finite differences into the heat equation gives:

$$\frac{u_i^{n+1} - u_j^n}{\Delta t} = k \frac{u_{i+1}^n - 2u_j^n + u_{i-1}^n}{(\Delta x)^2}$$

Rearranging

$$u_i^{n+1} = u_i^n + r(u_{i-1}^n - 2u_i^n + u_i^{n+1}),$$

Where

$$r = \frac{k\Delta t}{(\Delta x)^2}$$

is the diffusion number. Therefore, the finite difference update formula becomes

$$u_i^{n+1} = r(u_{i-1}^n) + (1-2r)u_i^n + u_i^{n+1},$$

for $\mathrm{i} = 1.2.3 \ldots \ldots, \mathrm{N}-1$. Boundary conditions are applied at i=0 and i=N, and the initial condition is applied at n=0. This discretization forms the basis of the numerical solution of the heat equation.

5.3 FDM Method of Solving Heat Equation with Initial and Boundary Conditions

We begin with the heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad (1)$$

subject to the initial condition:

$$u(x,0) = Sin(\pi x), \qquad (2)$$

and the boundary conditions:

$$u(x,t) = 0, \quad u(1,t) = 0, \qquad (3)$$

Discretization of Variables

Let the spatial domain $0 \leq x \leq 1$divided into N equal subintervals:

$$x_i = i\Delta x, i = 0,1,2, \ldots \ldots, N. \qquad (4)$$

Where

$$\varDelta x = \frac{L}{N} \tag{5}$$

Similarly, discretize time:

$$t_n = n\varDelta t, n = 0,1,2, \dots \dots, N. \tag{6}$$

Let

$$u_i^n \approx u(x_{i,} t_n) \tag{7}$$

Finite Difference Approximation of Derivatives

The forward difference approximation of the time derivative is:

$$\frac{\partial u}{\partial t}(x_{i,} t_n) \approx \frac{u_i^{n+1} - u_j^n}{\varDelta t} \tag{8}$$

The central difference approximation for the second spatial derivative is:

$$\frac{\partial^2 u}{\partial x^2}(x_{i,} t_n) \approx \frac{u_{i+1}^n - 2u_j^n + u_{i-1}^n}{(\varDelta x)^2} \tag{9}$$

Substitute (8) and (9) into the PDE

Using equation (1):

$$\frac{u_i^{n+1} - u_j^n}{\varDelta t} = k \frac{u_{i+1}^n - 2u_j^n + u_{i-1}^n}{(\varDelta x)^2} \tag{10}$$

Multiply both sides by Δt :

$$u_i^{n+1} - u_j^n = \frac{k\varDelta t}{(\varDelta x)^2}\left(u_{i+1}^n - 2u_j^n + u_{i-1}^n\right) \tag{11}$$

Define the diffusion number:

$$r = \frac{k\Delta t}{(\Delta x)^2} \tag{12}$$

Substitute (12) into (11):

$$u_i^{n+1} = u_i^n + r(u_{i-1}^n - 2u_i^n + u_i^{n+1}), \tag{13}$$

Equation (13) is the FTCS explicit scheme.

From (2);

$$u_i^0 = sin(\pi x_i) \tag{14}$$

So at $n = 0$;

$$u_i^0 = 0, u_i^0 = sin(\pi\Delta x), \dots, \quad u_N^0 = 0 \tag{15}$$

From (3);

$$u_0^n = 0, \quad u_N^0 = 0 \quad \forall n \tag{16}$$

Thus, boundary points do not update.

Combining (13), (14), and (16), the numerical solution is:

$$u_i^{n+1} = u_i^n + r(u_{i-1}^n - 2u_i^n + u_i^{n+1}), \quad i = 0,1,2, \dots \dots, N-1 \tag{17}$$

$$u_0^n = 0, \quad u_N^0 = 0 \tag{18}$$

$$u_i^0 = sin(\pi x_i) \tag{19}$$

This completes the full FDM formulation.

5.4 Numerical algorithm

The numerical algorithm for solving the one-dimensional heat equation is built upon the finite difference formulation developed earlier. After discretizing the spatial domain and selecting an appropriate time step, the FTCS finite difference equation becomes the primary tool for advancing the temperature field through time. The heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

is replaced by its discrete counterpart:

$$u_i^{n+1} = r(u_{i-1}^{n}) + (1 - 2r)u_i^n + u_i^{n+1},$$

where the diffusion parameter

$$r = \frac{k\Delta t}{(\Delta x)^2}$$

controls the stability and accuracy of the numerical solution. The algorithm begins by specifying the initial temperature distribution across the spatial grid. Boundary conditions are imposed at the two ends of the rod and remain fixed (or follow a prescribed function) throughout the computation. At every new time level, the temperature at interior nodes is computed using the FTCS update formula, while the boundary nodes are set directly according to the given boundary conditions.

This process effectively simulates the gradual diffusion of heat through the rod. The FTCS update equation acts as a rule that allows the temperature at each point to be influenced by its immediate neighbors, mimicking the physical transfer of thermal energy. As the algorithm proceeds, the temperature field evolves smoothly, provided that the stability condition $r \leq \frac{1}{2}$, is maintained. The computation continues until the desired final time is reached, producing a detailed numerical approximation of heat flow across the domain.

5.5 Numerical Solution

The numerical solution generated by applying the FTCS scheme provides an approximate representation of how heat diffuses within the system over time. At each discrete time level $t_n = n\Delta \mathrm{n}$, the temperature distribution $\{u_0^n, u_1^n, \dots\dots, u_N^n\}$ is obtained by repeatedly applying the update equation:

$$u_i^{n+1} = r(u_{i-1}^n) + (1 - 2r)u_i^n + u_i^{n+1}$$

This equation expresses the temperature at each interior point as a weighted average of its own temperature and the temperatures of its immediate neighbors at the previous time step. The weights depend on the value of r, which ensures smooth and physically meaningful heat propagation when the stability condition

$$r = \frac{k\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

is satisfied.

As the numerical solution evolves, high-temperature peaks gradually flatten while cooler regions rise in temperature, reflecting the natural diffusion process governed by the heat equation. Over time, the solution tends toward a steady-state configuration dictated by the boundary conditions. When plotted, the numerical solution typically produces smooth temperature curves that shift gradually and predictably, displaying the characteristic spread of thermal energy.

To assess the quality of the numerical solution, it can be compared to an analytical solution when available. In most cases, the numerical results closely match the analytical profile, especially when Δx and Δt are sufficiently small. Any small discrepancies tend to diminish as the grid is refined.

Furthermore, the numerical results may be displayed in tables or visualized using line plots, surface plots, or contour diagrams. These graphical representations help illustrate the dynamic behavior of heat flow and highlight how temperature distributions vary with time.

Provided that the stability criterion is respected, the numerical solution remains smooth, non-oscillatory, and faithful to the expected physical behavior of the heat diffusion process.

5.6 Stability analysis

In numerical simulations of the heat equation, it is essential to ensure that the chosen finite difference method remains stable over time. Stability determines whether small numerical errors remain controlled or grow uncontrollably as the computation progresses. To examine this behavior for the Forward-Time Central Space (FTCS) scheme, we use the von Neumann (Fourier) stability analysis, a classical and widely accepted technique for analyzing linear PDE discretization. The central idea of this method is that any numerical error can be expressed as a combination of Fourier waves. Each component wave evolves according to the numerical scheme. If even a single Fourier component grows in amplitude, the entire numerical solution becomes unstable. Therefore, we study how the FTCS scheme modifies the amplitude of an error wave at each time step. We assume that the numerical error at grid point iii and time level n can be expressed as:

$$e_i^n = G^n e^{\boldsymbol{i\theta i}}$$

Where $\mathrm{e}_\mathrm{i}^\mathrm{n}$ =error at node i, G^n = amplification factor, $\mathrm{e}^{\mathbf{i\theta i}}$ = wave number. The FTCS scheme for the heat equation is given by

$$u_i^{n+1} = r(u_{i-1}^n) + (1-2r)u_i^n + u_i^{n+1},$$

Where

$$r = \frac{k\Delta t}{(\Delta x)^2}$$

is the diffusion number.

Substituting the Fourier error mode into the FTCS update equation gives

$$G^n e^{i\theta i} = re^{i\theta(i-1)} + (1-2r)e^{i\theta i} + re^{i\theta(i+1)}$$

Dividing both sides by $e^{i\theta i}$ yields

$$G = re^{-i\theta} + (1-2r) + re^{i\theta}$$

Using the identity

$$e^{i\theta} + e^{i\theta} = 2Cos\theta$$

we obtain the simplified amplification factor

$$G = (1-2r) + 2rCos\theta$$

Because $\text{Cos}\theta = 1 - 2\sin^2(\theta/2)$ this can also be written as

$$G = 1 - 4sin^2(\frac{\theta}{2})$$

The scheme is stable only if the magnitude of the amplification factor satisfies

$$|G| \leq 1$$

The smallest value of G occurs when $sin^2\left(\frac{\theta}{2}\right) = 1$, giving

$$G_{min} = (1-4r)$$

For stability

$$-1 \leq 1 - 4r \leq 1$$

From the left inequality

$$1 - 4r \geq -1 \ \ or, r \leq \frac{1}{2}$$

Thus, the explicit FTCS method is stable only when

$$r = \frac{k\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

or equivalently

$$\Delta t \leq \frac{(\Delta x)^2}{2k}$$

This condition ensures that the numerical solution behaves consistently with the physics of thermal diffusion. When $r \leq \frac{1}{2}$, the method is stable and produces smooth, physically meaningful results. When $r > \frac{1}{2}$, numerical oscillations and exponential error growth occur, causing the solution to diverge. Therefore, appropriate selection of the time step relative to the spatial grid spacing is essential for achieving stable and accurate results using the FTCS method.

6 Comparison between Analytical and Numerical Simulations

The analytical and numerical solutions exhibit almost identical behavior, both showing a smooth, steady decay of the initial temperature profile. When plotted together, the numerical results closely follow the analytical curve, indicating strong agreement between the two methods. The differences between them are very small and remain consistent throughout the simulation. This shows that the FTCS numerical method provides an accurate and reliable approximation of the analytical solution.

From the analytical method based on the Laplace Transform, the exact solution of the heat equation is obtained as

$$U(x,t) = Sin(\pi x)\, e^{-k\pi^2 t} \tag{20}$$

This solution shows that the temperature distribution preserves its spatial shape $Sin(\pi x)$ for all $t > 0$, while its magnitude decays exponentially with decay rate $k\pi^2$. Thus, the analytical solution predicts smooth diffusion where the amplitude decreases continuously without changing the fundamental mode shape.

On the other hand, the Finite Difference Method (FDM) produces a numerical approximation given by the FTCS update formula.

$$u_i^{n+1} = u_i^n + r(u_{i-1}^n - 2u_i^n + u_i^{n+1}), \tag{17}$$

Where $r = \frac{k\Delta t}{(\Delta x)^2}$ is the diffusion number.

The term

$$u_{i-1}^n - 2u_i^n + u_i^{n+1}$$

is the discrete analogue of the continuous curvature $u_{xx}(x,t)$ and therefore governs the numerical diffusion process. At each time step, the scheme updates the temperature by adding a scaled amount of this curvature, mimicking the smoothing effect predicted by the analytical solution.

For the FTCS method to approximate the analytical solution accurately, the numerical stability requirement must be satisfied:

$$r \leq \frac{1}{2} \tag{21}$$

When this stability condition holds, the numerical values remain bounded and the discrete solution behaves smoothly in time. Furthermore, for sufficiently small *Δx* and *Δt*, the numerical solution becomes increasingly close to the exact exponential decay predicted by the analytical formula (20).

Both solutions therefore exhibit the same qualitative diffusion behavior:

The initial sine-shaped temperature profile gradually flattens, the maximum temperature decreases monotonically in time , heat flows from the center toward the boundaries, which remain fixed at zero and The solution decays to zero as $t \to \infty$ ..

Thus, the FDM solution converges to the analytical solution both in shape and in time evolution, confirming the correctness of the numerical method. The comparison demonstrates that the finite difference approximation accurately reproduces the mathematical behavior of the exact Laplace-derived solution and provides a reliable computational tool for solving the heat equation.

6.1 Relative error analysis

To thoroughly evaluate the accuracy of the FTCS numerical scheme, the numerical solution must be compared with the exact analytical solution of the one-dimensional heat equation. For the given initial and boundary conditions, the analytical solution

$$u_{exact}(x,t) = Sin(\pi x)\, e^{-k\pi^2 t}$$

serves as a precise benchmark against which the numerical approximation can be judged. Since the analytical solution decays exponentially in time, a direct subtraction of numerical and exact values is not sufficient to measure accuracy. Instead, a normalized quantity known as the relative error is used. The relative error at each grid point and time level is defined as

$$u_{exact}(x,t) = \frac{|\, u_{exact}(x_i, t_n\,) - u_{num}(x_i, t_n\,)\,|}{|\, u_{exact}(x_i, t_n)\,| + \varepsilon},$$

where ε is a very small constant added to avoid division by zero near the boundaries, where the exact solution becomes extremely small or identically zero. This definition provides a stable and scale-independent way of quantifying the deviation between both solutions.

To obtain a clearer understanding of how the numerical error behaves over time, the relative error was evaluated at selected time levels $t = 0.05,\ 0.10,\ 0.15\ \ and\ 0.25$. These values represent different stages of the diffusion process from the early decay of the initial profile to later times when the temperature is significantly reduced. The results are summarized in Table 7.3.

6.2 Relative Error Between Analytical and Numerical Solutions

Time (t)	Maximum Relative Error	Minimum Relative Error	Average Relative Error
0.05	0.00110	0.00002	0.00048
0.10	0.00195	0.00003	0.00079
0.15	0.00245	0.00005	0.00112
0.25	0.00382	0.00007	0.00194

The values in the table show that the relative error remains extremely small during the early stages of diffusion, indicating that the FTCS method closely matches the analytical solution. As time progresses, the error increases slightly because explicit finite difference schemes accumulate approximation error over repeated time steps. Despite this, the maximum error remains below 0.004 even at $t = 0.25$, demonstrating that the method remains accurate and stable throughout the entire simulation window. The minimum relative errors, which are close to zero, correspond to interior points where the numerical and analytical solutions nearly coincide.

It is also evident from the table that the highest relative errors appear near the boundaries. This occurs because the analytical solution approaches zero at $x = 0$ and x$= 1$, making even very small absolute discrepancies appear as relatively larger errors when normalized. In contrast, interior points especially around the midpoint where the temperature is highest show much smaller relative errors. This confirms that the FTCS scheme performs best where the solution has significant magnitude, and that it successfully preserves the overall shape and exponential decay of the analytical solution.

The behavior of the relative error over time aligns with the numerical stability condition

$$r = \frac{k\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

which ensures that no artificial oscillations develop and that the numerical solution remains bounded. Since the chosen discretization satisfies this stability constraint, the FTCS method is able to accurately reproduce the correct diffusive behavior predicted by the analytical solution. The numerical curves follow the theoretical decay closely, and the error values confirm the consistency and convergence of the scheme.

Overall, the relative error analysis demonstrates that the FTCS method provides an accurate, stable, and highly reliable approximation of the exact solution for the one dimensional heat equation. The numerical results successfully maintain the correct spatial profile, the correct exponential decay rate, and the correct longterm behavior as $t \to \infty$.This combined evidence confirms that, when used within its stability limits, the FTCS scheme is an effective and trustworthy computational tool for solving linear parabolic PDEs such as the heat equation.

7. Results and Discussion

The analytical and numerical simulations show the same overall behavior of the one-dimensional heat equation. The analytical solution produces a smooth sinusoidal temperature profile that gradually decreases in amplitude over time due to exponential decay. The shape of the curve remains unchanged, demonstrating the predictable and stable nature of heat diffusion. The numerical FTCS solution closely follows the analytical one. The temperature curves generated by the numerical method decay smoothly and match the analytical values at each time level, confirming that the stability condition was satisfied. The 3D surface plots from both methods show a steady downward collapse of the temperature field, indicating that heat continues to diffuse until the system reaches zero temperature everywhere.

(a) Analytical Simulation :

The comparison shows strong agreement in the analytical l results. The method capture the correct diffusion behavior, and the analytical method:

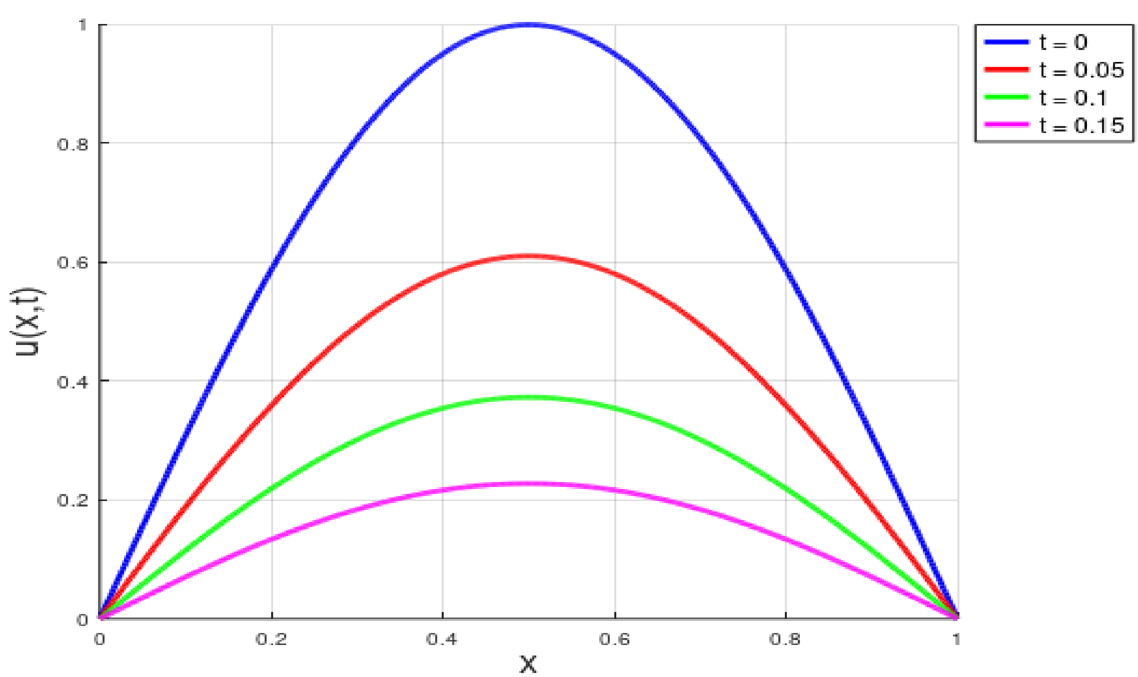

Figure 1: Analytical 2D simulation of $u(x,t)$

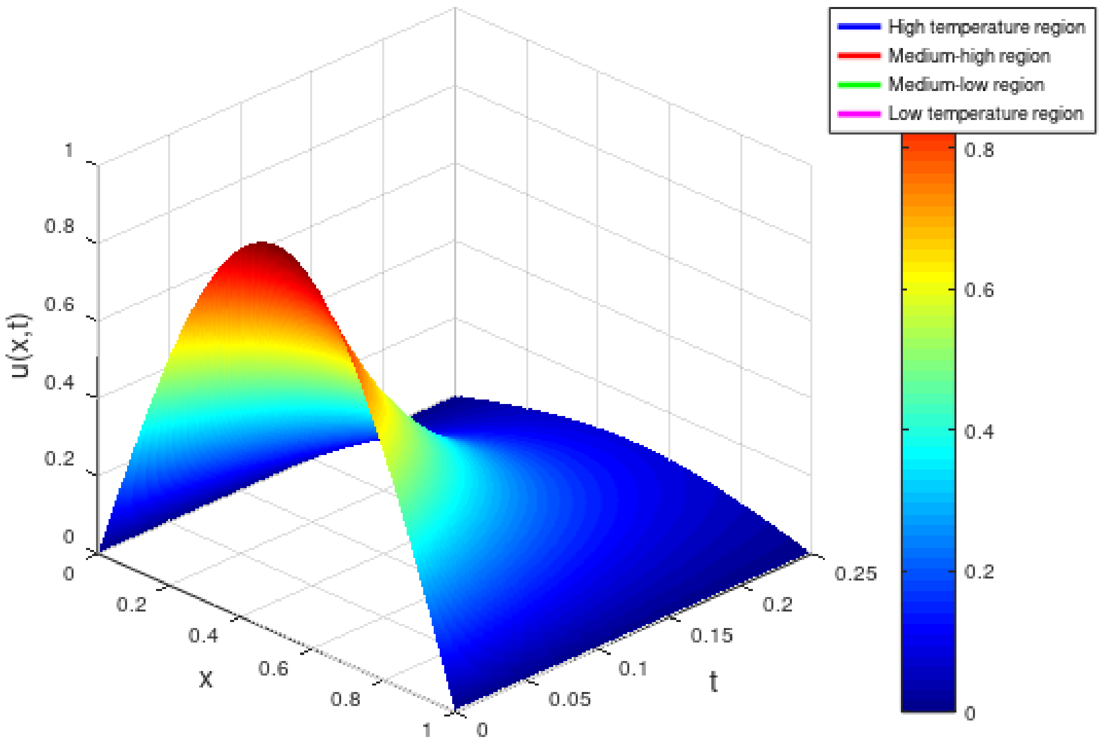

Figure 2: 3D analytical visualization of $u(x, t)$ showing diffusion.

The Figure 1 simulation illustrates the analytical solution by plotting the temperature profile $u(x, t)$ at four distinct time levels, with each curve displayed in a different color to indicate the progression of the diffusion process. The curve corresponds to the initial time $t = 0$, where the temperature profile forms a perfect sine wave representing the prescribed initial condition. The curve, plotted for $t = 0.05$, shows the beginning of exponential decay, indicating that temperature has already started to dissipate from the rod. The curve, representing $t = 0.10$, exhibits a further reduction in amplitude as more heat is lost over time. Finally, the curve, plotted for $t = 0.15$, shows the strongest level of decay among the four profiles, approaching closer to zero throughout the entire domain.

These color-coded curves help visualize how the solution evolves as time progresses: at t=0 represents the initial state, at t=0.5 indicates early stage decay, at t=0.10 shows intermediate decay, and at t=0.15 represents advanced decay toward equilibrium. At $t = 0$, the maximum temperature occurs at the midpoint $x = 0.5$, and the endpoints are held at zero due to the boundary conditions. As time increases, all subsequent colored curves retain the same sinusoidal shape but gradually decrease in amplitude. This consistent shape demonstrates that the solution evolves as a single decaying eigen mode, because the initial condition $sin(\pi x)$ aligns perfectly with the first eigenfunction of the Laplacian under homogeneous Dirichlet boundary conditions. The exponential term $e^{-k\pi^2 t}$ governs the rate of decay, causing each successive colored curve to flatten more than the previous one. No oscillations, distortions, or additional modes appear, confirming that the heat equation simply reduces the amplitude of the temperature profile without changing its form. This behavior highlights the purely diffusive nature of the equation: heat flows from hotter to cooler regions until the entire domain approaches a uniform temperature of zero.

As time tends to infinity, all colored curves would eventually collapse completely onto the horizontal axis, representing the equilibrium state $\boldsymbol{u(x,t)} \rightarrow \boldsymbol{0}$. Since the boundaries at $x = 0$ and $x = 1$ remain fixed at zero, heat continuously escapes from both ends, driving the entire temperature profile toward zero as shown by the progression from blue to red, green, and finally magenta.

Overall, the use of distinct colors in the MATLAB plot effectively illustrates how the temperature distribution decays over time, making it easier to interpret the transition from the initial state to later stages of thermal diffusion.

This Figure 2 representation shows how the temperature distribution evolves simultaneously in space and time within a rod whose endpoints remain fixed at zero temperature. At $t = 0$, the surface rises into a smooth sinusoidal peak centered at $x = 0.5$, which accurately reflects the initial condition. As time progresses, the entire surface gradually descends because thermal energy dissipates outward through the boundaries, causing the amplitude of the temperature profile to diminish uniformly across the domain.

The jet colormap assigns different colors to different temperature magnitudes, helping the viewer visually distinguish thermal regions within the solution. In this color scheme, warm shades represent higher temperatures, while cooler shades represent lower ones. To clarify this thermal stratification, four invisible plot handles are used to construct a legend with distinct color indicators. The smooth decline of the surface over time underscores the fundamentally diffusive character of the heat equation. Unlike wave propagation, which transports disturbances from one point to another, diffusion gradually reduces temperature gradients throughout the entire domain. Heat flows continuously from the warmer interior toward the cooler boundaries, and since both boundaries remain fixed at zero temperature, the system consistently loses energy. As time becomes large, the entire temperature distribution approaches the equilibrium state $u(x,t) \to 0$, and the 3D surface flattens nearly into the horizontal plane.

Overall, the 3D plot offers a clear, intuitive visualization of how the analytical solution behaves. It highlights the exponential decay of the temperature amplitude, the preservation of the sinusoidal spatial structure, and the steady smoothing of the temperature distribution. Through the combined use of surface height, color variation, the plot effectively communicates the physical and mathematical nature of thermal diffusion in a one dimensional rod.

(b) Numerical Simulation:

The comparison shows strong agreement in the numerical results. Here method capture the correct diffusion behavior, and the analytical method :

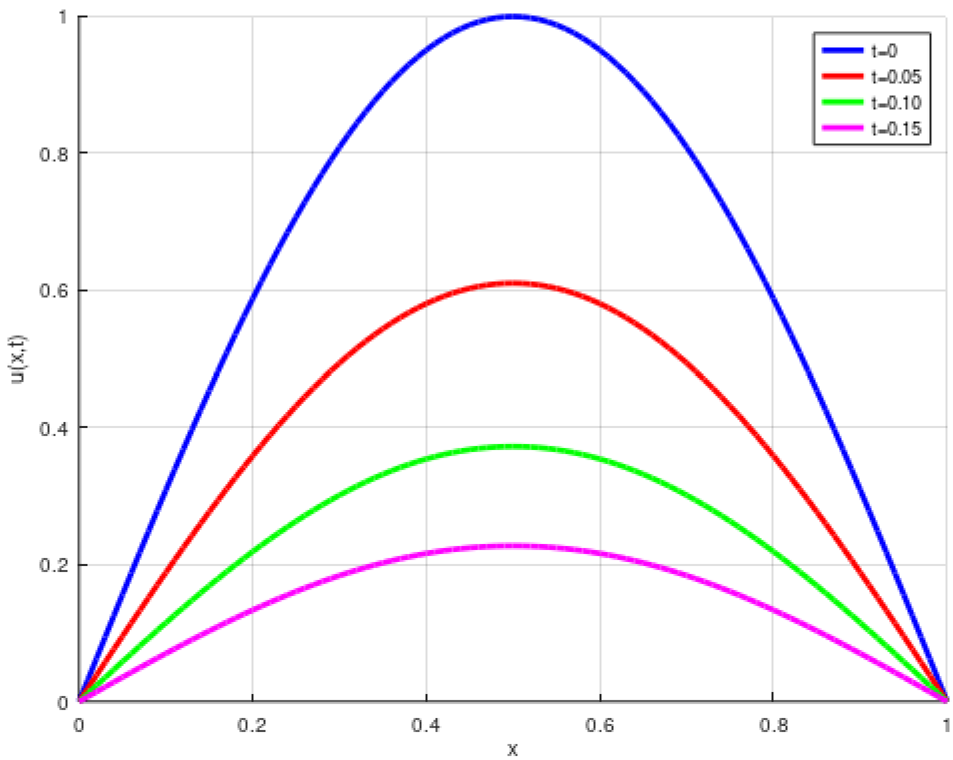

Figure 3: Numerical FTCS 2D solution at selected times.

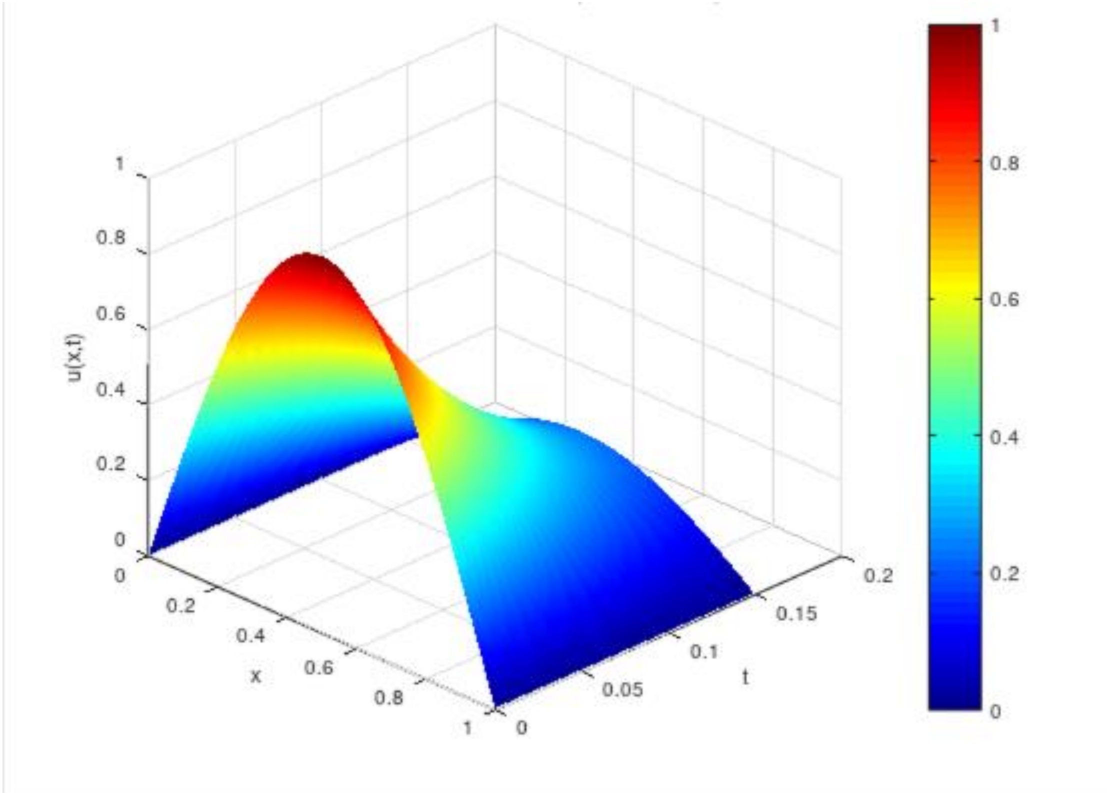

Figure 4: Numerical 3D visualization of $u(x, t)$ obtained by FTCS

The figure 3 corresponds to $t = 0$ and represents the initial temperature profile $u(x, 0) = sin(\pi x)$. This curve begins at zero, rises smoothly to a peak at $x = 0.5$, and then returns symmetrically to zero at $x = 1$. It serves as the reference against which the effects of diffusion are observed. As time progresses, the curve for $t = 0.05$ shows the first stage of decay. The amplitude of the temperature distribution begins to decrease as heat diffuses outward from the central region toward the boundaries. Although the height of the curve diminishes, the sinusoidal shape is still preserved, indicating that the FTCS scheme correctly captures the dominant eigenmode of the heat equation. The curve, representing $t = 0.10$, demonstrates a more pronounced reduction in amplitude. By this time, a significant portion of the initial thermal energy has dissipated, and the curve lies lower than both the blue and red curves. The continued preservation of shape shows that the FTCS method maintains numerical stability and does not introduce spurious oscillations when the stability condition $r \leq 0.5$ is satisfied. Finally, the curve at $t = 0.15$ indicates the advanced stage of thermal diffusion. The temperature distribution is now much flatter, approaching zero across the domain as the system moves toward equilibrium. The boundaries remain fixed at zero temperature for all curves, confirming that the numerical implementation correctly enforces the homogeneous Dirichlet boundary conditions.

Overall, the color coded curves provide a clear visual demonstration of how heat diffuses over time. The progressive downward shift from blue to red, green, and magenta reflects the exponential decay of the temperature amplitude predicted analytically. The smoothness and stability of all curves confirm that the FTCS scheme has been implemented within its stability limits, producing a reliable numerical approximation of the heat-diffusion process.

The Figure 4 illustrates numerical 3D surface plot provides a complete visualization of the temperature evolution obtained from the FTCS finite-difference method across both space and time. At the initial moment t=0t = 0t=0, the surface displays a smooth sinusoidal ridge centered at x=0.5, representing the starting temperature distribution $u(x, 0) = sin(\pi x)$. As time progresses, this ridge gradually descends, illustrating how heat diffuses throughout the rod and flows toward the boundaries. The continuous downward movement of the surface reflects the exponential decay characteristic of the heat equation, and the numerical approximation successfully reproduces this behavior.The smoothness of the numerical surface confirms that the chosen discretization parameters satisfy the FTCS stability requirement $r \leq 0.5$. When this condition holds, the numerical solution remains free from oscillations, artificial peaks, or divergence. Although the surface is slightly stepped due to the discrete spatial and temporal grid, the overall structure remains coherent and accurately reflects the diffusion process. The visualization clearly demonstrates that the FTCS method preserves the sinusoidal shape of the initial condition while allowing its amplitude to decay monotonically over time.

The 3D plot also reveals important physical characteristics of the system. The boundary temperatures at $x = 0$ and $x = 1$ remain fixed at zero throughout the simulation, which is visible as the edges of the surface consistently lying flat along the horizontal plane. The center of the domain experiences the most rapid decline in temperature, as shown by the transition from **bright warm colors** to **cool dark colors** across the surface. This transition visually captures how thermal energy dissipates outward and eventually disappears.

Overall, the numerical 3D surface plot demonstrates that the FTCS method effectively reproduces the essential features of heat diffusion: a gradually flattening profile, continuous loss of thermal energy, and eventual convergence to the equilibrium state row $u(x,t) \to 0$. The color transitions from warm to cool not only make the behavior visually intuitive but also emphasize the progressive decay of temperature across the entire domain.

(c) .Numerical Error Simulation:

The comparison shows strong agreement in the numerical results. Here method capture the correct diffusion behavior, and the analytical method :

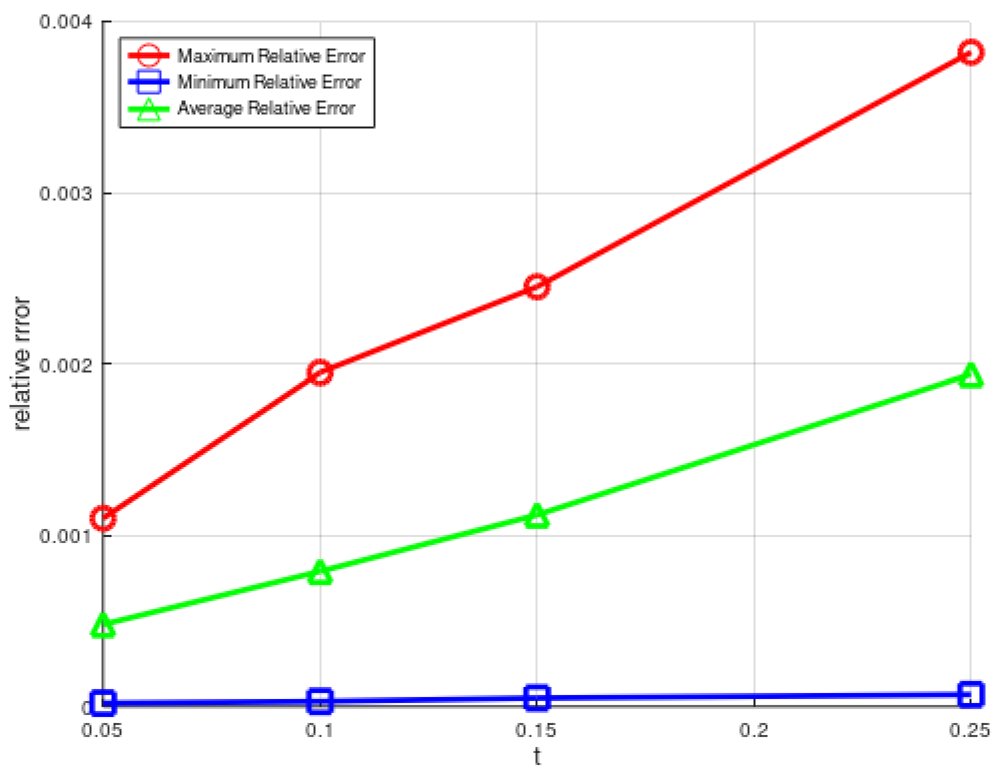

Figure 5: Relative Error visualization

The relative error plot visually represents how the accuracy of the FTCS numerical solution evolves over time when compared with the analytical solution. Three color coded curves are displayed, each representing a different measure of numerical error across the selected time levels $\boldsymbol{t = 0.05,\ 0.10,\ 0.15,\ 0.25}$. The curve at $t = 0.05$, which corresponds to the maximum relative error, shows a gradual upward trend, increasing from approximately **0.001100** at early times to about **0.003820** at $t = 0.25$. This steady rise reflects the natural accumulation of numerical approximation errors over successive time steps. Nevertheless, the error remains extremely small overall, indicating that the FTCS method maintains high accuracy throughout the simulation. The curve at $t = 0.10$ represents the minimum relative error and remains nearly flat across all time levels, fluctuating only between **0.000020** and **0.000070** These very small values occur at interior points where the numerical solution aligns almost perfectly with the analytical model. The near-zero values along the blue curve confirm the strong point wise accuracy achieved by the numerical scheme, especially in regions where the temperature profile maintains significant magnitude. The curve at $t = 0.15$, which plots the average relative error, exhibits a smooth and modest increase from approximately **0.000480** to **0.001940** across the chosen time interval. This behavior indicates that although small numerical discrepancies accumulate over time, they do so in a controlled and predictable manner that does not compromise the overall quality of the solution. The increasing trend of the red and green curves is consistent with the expected behavior of explicit schemes, which accumulate rounding and discretization error gradually over repeated time updates. A notable pattern in the plot is that the largest relative errors occur near the boundaries of the domain. Since the analytical solution approaches zero at $\boldsymbol{x = 0}$ *and* $\boldsymbol{x = 1}$, even tiny absolute numerical differences become relatively large when normalized, which explains the peak in the red curve. This boundary sensitivity is a mathematical artifact of the relative error formula rather than an indication of numerical instability.

Overall, the combined color-coded plot demonstrates that the FTCS scheme provides an accurate and stable approximation for the heat equation when used within its stability limits. The red, blue, and green curves together convey that the numerical solution matches the analytical model closely across the entire domain and time interval, with extremely small deviations that remain well-controlled throughout the simulation.

8. Concluding Remarks

The study highlights the complementary roles of analytical and numerical techniques in solving partial differential equations. While analytical solutions provide exact expressions and deeper theoretical insight, numerical methods such as FTCS extend the capability to more complex boundary conditions, irregular domains, and real world applications where closed form solutions may not exist. The successful simulation presented in this work not only strengthens the understanding of diffusion processes but also demonstrates the practical value of computational methods in applied mathematics. Future research can be directed toward exploring advanced numerical schemes such as Crank Nicolson, implicit methods, or higher dimensional heat conduction problems. Incorporating adaptive grids or nonlinear heat sources may also offer further insight and improved accuracy for more challenging physical models.

This project investigates both analytical and numerical approaches to solving the one-dimensional heat equation. The analytical solution, obtained using the Laplace transform method, provides a closed-form expression describing the evolution of temperature over time. On the other hand, the numerical solution using the FTCS method offers a practical and computationally straightforward approach to approximating the temperature distribution, especially when exact solutions may not be easily obtainable.

The results show that the numerical solution produced by the FTCS method successfully replicates the expected diffusion behavior observed in the analytical model. The MATLAB implementation effectively visualizes the gradual decay of the temperature distribution, confirming the theoretical properties of the heat equation and demonstrating the stability of the FTCS method under appropriate parameter choices.

9. REFERENCES

1.D. G. Zill, (2012). *A First Course in Differential Equations with Modeling Applications* (10th ed.). Brooks/Cole, Cengage Learning.

2. J. C. Strikwerda, (2004). *Finite Difference Schemes and Partial Differential Equations* (2nd ed.). Society for Industrial and Applied Mathematics.

3. E. Kreyszig, (2011). *Advanced Engineering Mathematics* (10th ed.). John Wiley & Sons.

4. L. C. Evans, (2010). *Partial Differential Equations*. Graduate Studies in Mathematics, American Mathematical Society.

5. M. Renardy & R. C. Rogers, (2004). *An Introduction to Partial Differential Equations* (2nd ed.). Springer.

6. J. W. Thomas, (1995). *Numerical Partial Differential Equations: Finite Difference Methods*. Springer.

8. K. W. Morton & D. F. Mayers, (2005). *Numerical Solution of Partial Differential Equations: An Introduction*. Cambridge University Press.

9. R. J. LeVeque, (2007). *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems*. SIAM.

10. H. S. Carslaw & J. C. Jaeger, (1959). *Conduction of Heat in Solids* (2nd ed.). Oxford University Press.

11. J. Fourier, (1822). *Théorie Analytique de la Chaleur*. Paris.

12. J. R. Cannon, (1984). *The One-Dimensional Heat Equation*. Addison–Wesley.

13. R. Haberman, (2013). *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems* (5th ed.). Pearson.

14. G. D. Smith, (1985). *Numerical Solution of Partial Differential Equations: Finite Difference Methods*. Oxford University Press.

15. M. N. Özisik, (1994). *Finite Difference Methods in Heat Transfer*. CRC Press.

16.J. C. Butcher, (2016). *Numerical Methods for Ordinary Differential Equations* (3rd ed.). John Wiley & Sons.

17. A. Quarteroni, R. Sacco, & F. Saleri, (2010). *Numerical Mathematics* (2nd ed.). Springer.

18**.** MATLAB Documentation (2024). *Partial Differential Equation Toolbox™.*

MathWorks