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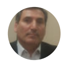

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A Numerical Method for Solution of the Heat Equation with Nonlocal Nonlinear Condition

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Abstract: This paper deals with a numerical method for the solution of the heat equation with nonlinear nonlocal boundary conditions. Here nonlinear terms are approximated by Richtmyer's linearization method. The integrals in the boundary equations are approximated by the composite Simpson rule. A difference scheme is considered for the one-dimensional heat equation. In final part, the numerical results produced by this method are compared.

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Key word: Heat equation . nonlocal boundary condition . initial condition . finite-difference scheme . nonlinear

INTRODUCTION

Considered as a fascinating element of nature, nonlinearity is regarded by many scholars as the most significant frontier for the fundamental understanding of nature. Many complex physical phenomena are frequently described and modeled by the nonlinear partial differential equations or partial differential equations with the nonlinear boundary conditions. However, in recent years, new numerical methods [1-3], analytical and semi analytical methods have been developed considerably to be used for these problems, such as the Variational Iteration Method (VIM) [4-6], Homotopy Perturbation Method (HPM) [7-9], (G'/G)-expansion method [10, 16], Exp-function method [11-16].

This paper is concerned with the numerical solution of the heat equation

$$u_t - u_{xx} = f(x,t),\ \ x \in (0,1),\ \ t \in (0,T] \qquad (1)$$

Subject to the nonlocal boundary conditions

$$\begin{cases} u(0,t) = \int_0^1 k_0(x)u^p(x,t)dx + g_0(t) \\ u(1,t) = \int_0^1 k_1(x)u^p(x,t)dx + g_1(t) \end{cases} \qquad (2)$$

and the initial condition

$$u(x,0) = g(x),\ \ x \in [0,1] \qquad (3)$$

where f, k_0, k_1, g_0, g_1 and g are known functions and p>1 is an integer constant. Over the last few years, many other physical phenomena were formulated into nonlocal mathematical models [17, 18]. Hence the numerical solution of parabolic partial differential equations with nonlocal boundary specifications is currently an active area of research. The nonlocal problems are very important in the transport of reactive and passive contaminates in aquifers, an area of active interdisciplinary research of mathematicians, engineers and life scientists [19, 20]. We refer the reader to [21, 22] for the derivation of mathematical models and for the precise hypothesis and analysis. Parabolic problems with nonlocal boundary specifications also arise in quasi-static theory of thermo elasticity [23, 24]. An interesting collection of nonlocal parabolic problems in one-space dimension is discussed in [25]. In the present paper, the difference scheme for the one-dimensional heat equation with constant coefficients and nonlocal boundary conditions is investigated. We selected the implicit difference scheme as the main object of study in the paper.

THE FINITE DIFFERENCE SCHEME

We divide the domain [0, 1]×[0,T] into M ×N mesh with spatial step size h = 1/N in x-direction and the time step size k= T/M respectively. Where M is a positive integer and N is a positive even integer. The grid points are given by

$$x_n = nh, \quad n = 0,1,...,N$$

$$t_m = mk, \quad m = 0,1,...,M$$

We define the following difference operators:

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$$u^k_{i-\frac{1}{2}} = \frac{1}{2}\left(u^k_i + u^k_{i-1}\right)$$

$$\delta_x u^k_{i-\frac{1}{2}} = \frac{1}{h}\left(u^k_i - u^k_{i-1}\right)$$

$$u_i^{k-\frac{1}{2}} = \frac{1}{2}\left(u^k_i + u^{k-1}_i\right)$$

$$\delta_t u_i^{k-\frac{1}{2}} = \frac{1}{k}\left(u^k_i - u^{k-1}_i\right)$$

$$\delta_x^2 u^k_i = \frac{1}{h^2}\left(u^k_{i+1} - 2u^k_i + u^k_{i-1}\right)$$

where

$$u^k_i = u(x_i, t_k)$$

The inner product <.,. > for (N + 1)-dimensional vectors is defined by:

$$< u,v > = \frac{h}{3}\sum_{i=0}^{\frac{N}{2}-1}\left(u_{2i}v_{2i} + 4u_{2i+1}v_{2i+1} + u_{2i+2}v_{2i+2}\right)$$

and also

$$k_0^* = \left(k_0(x_0), k_0(x_1), \ldots, k_0(x_N)\right)$$
$$k_1^* = \left(k_1(x_0), k_1(x_1), \ldots, k_1(x_N)\right)$$

Our difference scheme for (1) is as follow [26]:

$$\frac{1}{12}\left(\delta_t u_{n-1}^{m-\frac{1}{2}} + 10\delta_t u_n^{m-\frac{1}{2}} + \delta_t u_{n+1}^{m-\frac{1}{2}}\right) - \delta_x^2 u_n^{m-\frac{1}{2}} = f_n^m, \quad 0 \le m \le M,\ 0 \le n \le N \tag{4}$$

and for boundary conditions and the initial condition, we define the following difference operators:

$$\begin{cases} u_0^m = < k_0^*, u_m > + g_0(t_n) \\ u_N^m = < k_1^*, u_m > + g_1(t_n) \end{cases} \tag{5}$$
$$u_n^0 = g(x_n), \quad 0 \le n \le N$$

Let N be an even integer,

$$h = \frac{1}{N},\ x_n = nh, 0 \le n \le N$$

If $g(x) \in C^4[0,1]$, then

$$\int_0^1 g(x)dx - \frac{h}{3}\sum_{i=0}^{\frac{N}{2}-1}\left[g(x_{2i}) + 4g(x_{2i+1}) + g(x_{2i+2})\right] = -\frac{1}{180}h^4 \left.\frac{d^4 g(x)}{dx^4}\right|_{x=\zeta},\ \zeta \in (0,1)$$

By Taylor's expansion about the point (n,m)

$$\left(u_n^{m+1}\right)^p = \left(u_n^m\right)^p + k\frac{\partial\left(u_n^m\right)^p}{\partial t} + \ldots = \left(u_n^m\right)^p + k\frac{\partial\left(u_n^m\right)^p}{\partial u_n^m}\frac{\partial u_n^m}{\partial t} + \ldots = \left(u_n^m\right)^p + p\left(u_n^m\right)^{p-1}\left(u_n^{m+1} - u_n^m\right) + \ldots$$

Hence to terms of order k,

$$\left(u_n^{m+1}\right)^p = p\left(u_n^m\right)^{p-1}u_n^{m+1} + (1-p)\left(u_n^m\right)^p \tag{6}$$

The integrals in the nonlocal boundary conditions (2) are discretized by the Simpson rule. We have

$$u_0^{m+1} = \frac{h}{3}\left[\begin{array}{l} k_0(x_0)\left(u_0^{m+1}\right)^p + 4k_0(x_1)\left(u_1^{m+1}\right)^p + 2k_0(x_2)\left(u_2^{m+1}\right)^p \\ + \ldots + 4k_0(x_{N-1})\left(u_{N-1}^{m+1}\right)^p + k_0(x_N)\left(u_N^{m+1}\right)^p \end{array}\right] + g_0^{m+1} \tag{7}$$

By (6) we obtain:

$$a_0^m u_0^{m+1} + a_1^m u_1^{m+1} + a_2^m u_2^{m+1} + \ldots + a_{N-I}^m u_{N-1}^{m+1} + a_N^m u_N^{m+1} = L_N^m \quad (8)$$

where:

$$\begin{cases} a_0^m = phk_0(x_0)\left(u_0^m\right)^{p-1} - 3, \; a_N^m = phk_0(x_N)\left(u_N^m\right)^{p-1} \\ a_{2n-1}^m = 4phk_0(x_{2n+1})\left(u_{2n+1}^m\right)^{p-1}, \; n = 0,1,\ldots,\frac{N}{2}-1 \\ a_{2n}^m = 2phk_0(x_{2n})\left(u_{2n}^m\right)^{p-1}, \; n = 1,2,\ldots,\frac{N}{2}-1 \end{cases} \quad (9)$$

and

$$L_N^m = (p-1)hk_0(x_0)\left(u_0^m\right)^p + 4(p-1)hk_0(x_1)\left(u_1^m\right)^p + 2(p-1)hk_0(x_2)\left(u_2^m\right)^p + \ldots$$
$$+4(p-1)hk_0(x_{N-1})\left(u_{N-1}^m\right)^p + (p-1)hk_0(x_N)\left(u_N^m\right)^p - 3g_0^{m+1}$$

and also:

$$b_0^m u_0^{m+1} + b_1^m u_1^{m+1} + b_2^m u_2^{m+1} + \ldots + b_{N-I}^m u_{N-1}^{m+1} + b_N^m u_N^{m+1} = Q_N^m \quad (10)$$

where

$$\begin{cases} b_0^m = phk_1(x_0)\left(u_0^m\right)^{p-1}, \; b_N^m = phk_1(x_N)\left(u_N^m\right)^{p-1} - 3 \\ b_{2n-1}^m = 4phk_1(x_{2n+1})\left(u_{2n+1}^m\right)^{p-1}, \; n = 0,1,\ldots,\frac{N}{2}-1 \\ b_{2n}^m = 2phk_1(x_{2n})\left(u_{2n}^m\right)^{p-1}, \; n = 1,2,\ldots,\frac{N}{2}-1 \end{cases} \quad (11)$$

and

$$Q_N^m = (p-1)hk_1(x_0)\left(u_0^m\right)^p + 4(p-1)hk_1(x_1)\left(u_1^m\right)^p + 2(p-1)hk_1(x_2)\left(u_2^m\right)^p + \ldots$$
$$+4(p-1)hk_1(x_{N-1})\left(u_{N-1}^m\right)^p + (p-1)hk_1(x_N)\left(u_N^m\right)^p - 3g_1^{m+1}$$

By (4) we obtain:

$$(1-6r)u_{n-1}^{m+1} + (10+12r)u_n^{m+1} + (1-6r)u_{n+1}^{m+1} = (1+6r)u_{n-1}^m + (10-12r)u_n^m + (1+6r)u_{n+1}^m + 12kf_n^m \quad (12)$$

where

$$r = \frac{k}{h^2}$$

We now consider the following difference scheme (Crank-Niklson):

$$\delta_t u_n^m + \frac{1}{2}\left(\delta_x^2 u_n^{m+1} + \delta_x^2 u_n^m\right) = \frac{1}{2}\left(f_n^{m+1} + f_n^m\right)$$

We obtain:

$$u_{n+1}^{m+1} - \frac{2+2r}{r}u_n^{m+1} + u_{n-1}^{m+1} = M_n^m \quad (13)$$

where

$$M_n^m = -u_{n+1}^m + \frac{2r-2}{r}u_n^m - u_{n-1}^m - \frac{2k}{r}f\left(x_m, \frac{1}{2}\left(t_n + t_{n+1}\right)\right)$$

By the left-hand side of (12) and the right-hand side of (13), (8) and (10); we obtain the following matrix equation:

$$\begin{bmatrix} a_0^m & a_1^m & a_2^m & \cdots & a_{N-1}^m & a_N^m \\ \alpha & \beta & \alpha & & & \\ & & \ddots & \ddots & & \\ & & & & \beta & \alpha \\ b_0^m & b_1^m & b_2^m & \cdots & b_{N-1}^m & b_N^m \end{bmatrix} \begin{bmatrix} u_0^{m+1} \\ u_1^{m+1} \\ \vdots \\ u_{N-1}^{m+1} \\ u_N^{m+1} \end{bmatrix} = \begin{bmatrix} L_N^m \\ \vdots \\ M_n^m \\ \vdots \\ Q_N^m \end{bmatrix} \quad (14)$$

Table 1

		Error	
t	Exact	h = 0.05	h = 0.005
0.0000	0.0100	0.0000	0.0000
0.0100	0.0098	0.0093	0.0098
0.0200	0.0096	0.0091	0.0096
0.0300	0.0094	0.0090	0.0094
….	….	….	….
0.1000	0.0083	0.0079	0.0083

where $\alpha = 1\text{-}6r$, $\beta = 10+12r$, $n = 1,\ldots,N\text{-}1$ and a_i, $b_i (i = 0,1,\ldots,N)$ are given by (9) and (11).

NUMERICAL EXAMPLE

In this section, we test the proposed difference method on an example, whose exact solution is known to us. The right-hand side functions as well as the nonlocal boundary value conditions and initial value conditions are obtained from the exact solution. The systems of linear algebraic equations have been solved by using the Gaussian pivot method.

$$u_t - u_{xx} = \frac{-2(x^2 + t + 1)}{(t+1)^3}, \quad 0 < x \le 1,\ t \in (0,T]$$

Subject to the nonlocal boundary conditions and the initial condition

$$u(0,t) = \int_0^1 k_0(x) u^2(x,t)dx - \frac{1}{6(t+1)^4}$$

$$u(1,t) = \int_0^1 k_1(x) u^2(x,t)dx + \frac{6t^2 + 12t + 5}{6(t+1)^4}$$

$$u(x,0) = x^2 \qquad x \in (0,1]$$

which is easily seen to have exact solution

$$u(x,t) = \left(\frac{x}{t+1}\right)^2$$

The results with h = 0.05, 0.005 and r = 0.4 using the finite difference formulate discussed in section 2 are shown in Table 1. In this table, we present the error for x = 0.1 and t = 0.01, 0.02, 0.03, …, 0.1.

SUMMARY AND CONCLUDING REMARKS

In this paper, a new numerical method was applied to the one-dimensional heat equation with non-linear nonlocal boundary conditions replacing standard boundary conditions. These techniques applied well for the one-dimensional heat equation with integral conditions. One example with closed form solution is studied carefully in order to illustrate the possible practical use of this method.

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