Ex 1

$$p(\mu|D) = \frac{p(D|\mu)\cdot p(\mu)}{p(D)} = \frac{\prod_{i=1}^{n} p(x^{(i)}|\mu,\delta^2)\cdot p(\mu|\mu_0,\delta_0^2)}{p(D)}$$

$$\Rightarrow \log p(\mu|D) = \log \prod_{i=1}^{n} p(x^{(i)}|\mu,\delta^2) + \log p(\mu|\mu_0,\delta_0^2) - \underbrace{\log p(D)}_{\text{constant}}$$

$$= \sum_{i=1}^{n} \log\left(\frac{1}{\sqrt{2\pi\delta^2}}\exp\left(-\frac{1}{2\delta^2}(x^{(i)}-\mu)^2\right)\right) + \log\frac{1}{\sqrt{2\pi\delta_0^2}}\exp\left(-\frac{1}{2\delta_0^2}(\mu-\mu_0)^2\right) + (\text{constant})$$

$$= \underbrace{n\log\frac{1}{\sqrt{2\pi\delta^2}}}_{\text{constant}} + \left(-\frac{1}{2\delta^2}\sum_{i=1}^{n}(x^{(i)}-\mu)^2\right) + \underbrace{\log\frac{1}{\sqrt{2\pi\delta_0^2}}}_{\text{constant}} - \frac{1}{2\delta_0^2}(\mu-\mu_0)^2 + (\text{constant})$$

$$= \underbrace{-\frac{1}{2\delta^2}\sum_{i=1}^{n}x^{(i)2}}_{\text{constant}} + \frac{\mu}{\delta^2}\underbrace{\sum_{i=1}^{n}x^{(i)}}_{\bar{x}} - \frac{n}{2\delta^2}\mu^2 - \frac{1}{2\delta_0^2}(\mu^2 - 2\mu_0\mu + \underbrace{\mu_0^2}_{\text{constant}}) + (\text{constant})$$

$$= -\frac{1}{2}\left(\frac{n}{\delta^2}+\frac{1}{\delta_0^2}\right)\mu^2 + \left(\frac{\bar{x}}{\delta^2}+\frac{\mu_0}{\delta_0^2}\right)\mu + (\text{constant}) \quad \text{—— } ①$$

$$\log p(\mu|\mu_{post},\delta_{post}^2) = \log\left(\frac{1}{\sqrt{2\pi\delta_{post}^2}}\exp\left(-\frac{1}{2\delta_{post}^2}(\mu-\mu_{post})^2\right)\right)$$

$$= -\frac{1}{2\delta_{post}^2}\mu^2 + \frac{\mu_{post}}{\delta_{post}^2}\mu + (\text{constant}) \quad \text{—— } ②$$

We want ① = ② :
$$\begin{cases} -\dfrac{1}{2\delta_{post}^2} = -\dfrac{1}{2}\left(\dfrac{n}{\delta^2}+\dfrac{1}{\delta_0^2}\right) \\ \dfrac{\mu_{post}}{\delta_{post}^2} = \dfrac{\bar{x}}{\delta^2}+\dfrac{\mu_0}{\delta_0^2} \end{cases}$$

$$\Rightarrow \delta_{post}^2 = \left(\frac{1}{\delta_0^2}+\frac{n}{\delta^2}\right)^{-1}, \quad \mu_{post} = \delta_{post}^2\left(\frac{\bar{x}}{\delta^2}+\frac{\mu_0}{\delta_0^2}\right) = \frac{\cancel{\delta^2\delta_0^2}}{\delta^2+n\delta_0^2}\cdot\frac{\bar{x}\cdot\delta_0^2+\mu_0\delta^2}{\cancel{\delta^2\delta_0^2}}$$

$$= \frac{1}{n}\left(\frac{\delta_0^2}{\frac{\delta^2}{n}+\delta_0^2}\bar{x} + \frac{\delta^2}{\frac{\delta^2}{n}+\delta_0^2}\mu_0\right)$$

Therefore $p(\mu|D,\mu_0,\delta_0^2) \sim \mathcal{N}\left(\frac{1}{n}\left(\frac{\delta_0^2}{\frac{\delta^2}{n}+\delta_0^2}\bar{x} + \frac{\delta^2}{\frac{\delta^2}{n}+\delta_0^2}\mu_0\right), \left(\frac{1}{\delta_0^2}+\frac{n}{\delta^2}\right)^{-1}\right)$ $\square$