Applied Nonlinear Control

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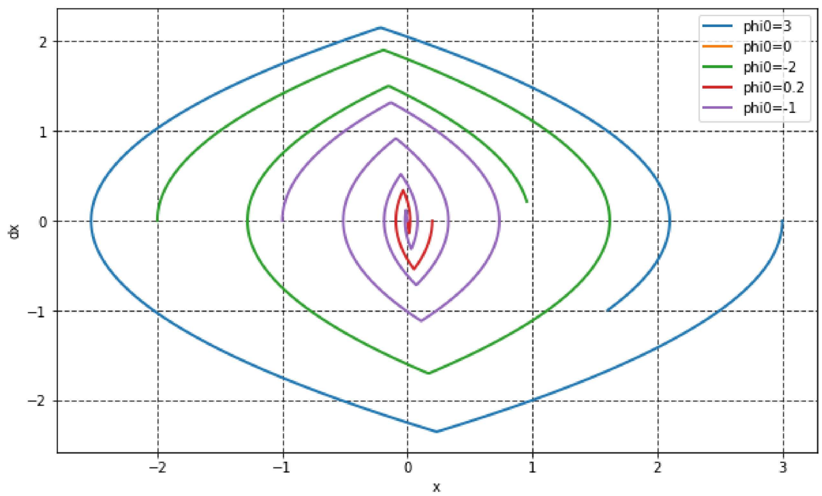
B19-RO-01

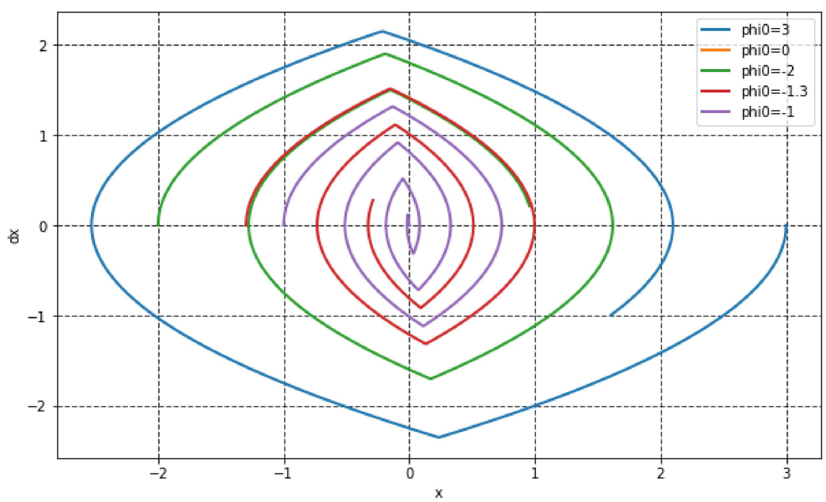
HW2_Report

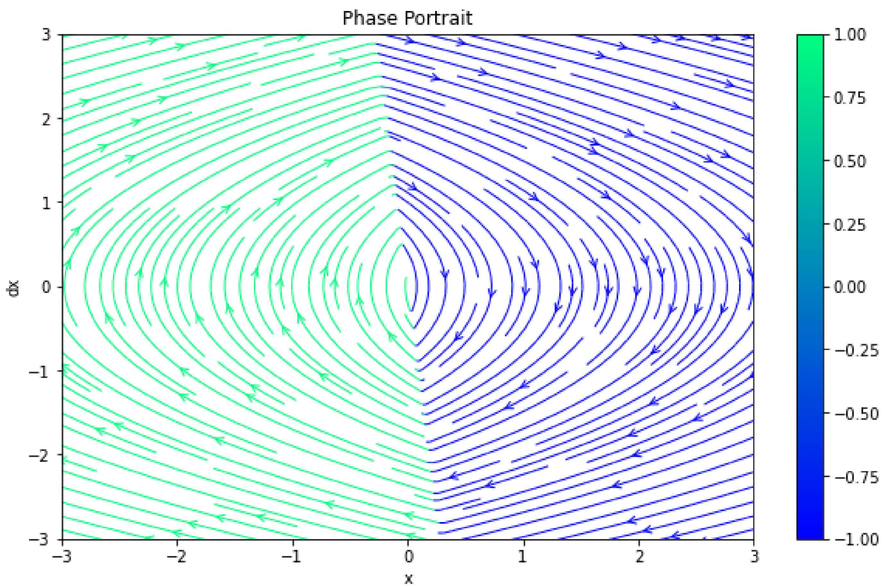
Problem 1

For the system $\ddot{\varphi} = u = -sign(k\varphi + \dot{\varphi})$:

Phase Portrait using odeint for various initial values of φ:







We can notice that from the initial values of φ_0 that:

- At $\varphi_0 = 0$, the system stays the same because it is an equilibrium point.
- For $\varphi_0 > 0\ and\ \varphi_0 < 0$, the trajectories oscillate then converge to zero angle.

so we can conclude that the system is asymptotically stable.

Problem 2

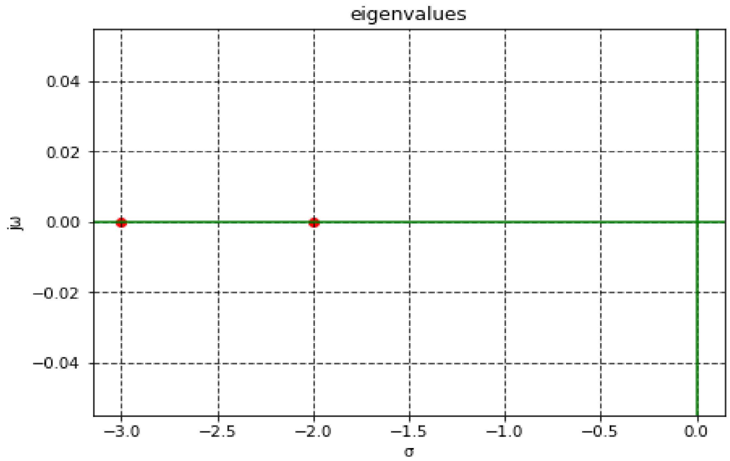
The eigenvalues of A in linear system $\dot{x} = Ax$ affects the phase portrait depending on their values. We'll explain each possible condition of the eigenvalues and how the phase portrait changes according and what means.

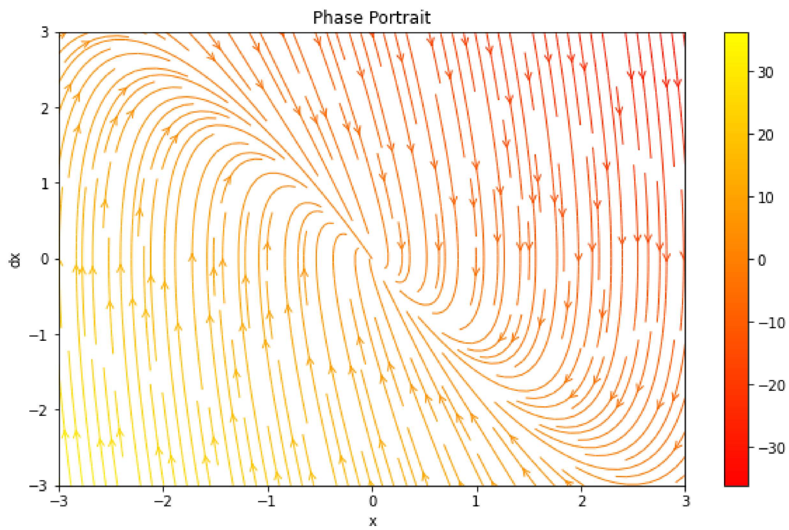
1) Stable Node:

- It is the case when $\lambda_1\ and\ \lambda_2$ are both real and have the same negative sign.
- The system trajectories converge to zero exponentially.
- The system is asymptotically stable.
- **For example:**

$$\ddot{x} = -5\dot{x} - 6x$$
$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$
$$\lambda_1 = -2,\ \lambda_2 = -3$$

This system has the following plots for its eigenvalues and phase portrait.





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2) Unstable Node:

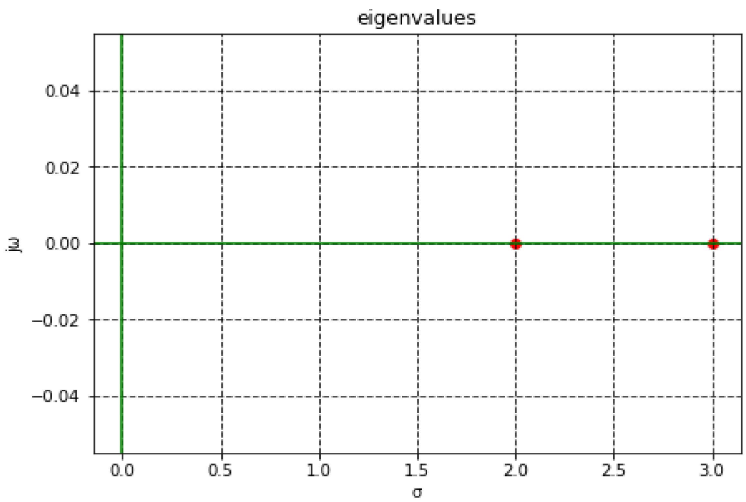
- It is the case when λ_1 *and* λ_2 are both real and have the same positive sign.
- The system trajectories diverge from zero exponentially.
- The system is unstable.
- **For example:**

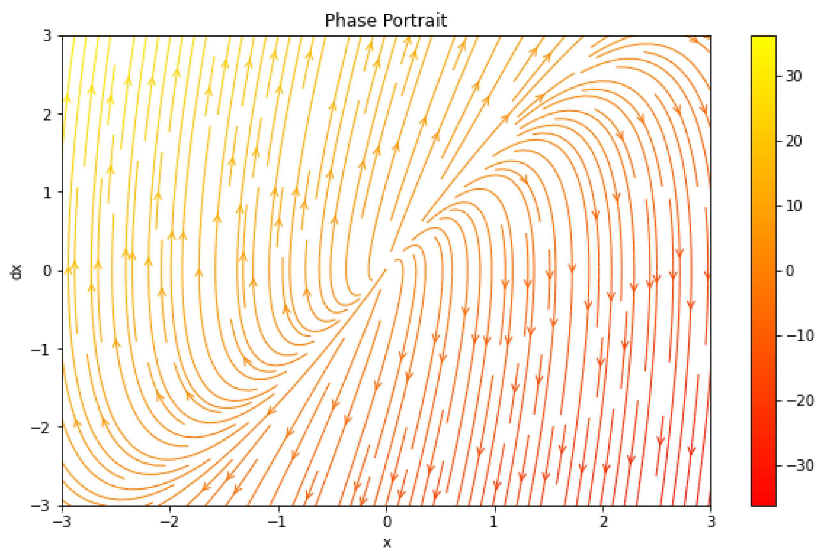
$$\ddot{x} = 5\dot{x} - 6x$$

$$A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$$

$$\lambda_1 = 2, \ \lambda_2 = 3$$

This system has the following plots for its eigenvalues and phase portrait.





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3) Saddle Point:

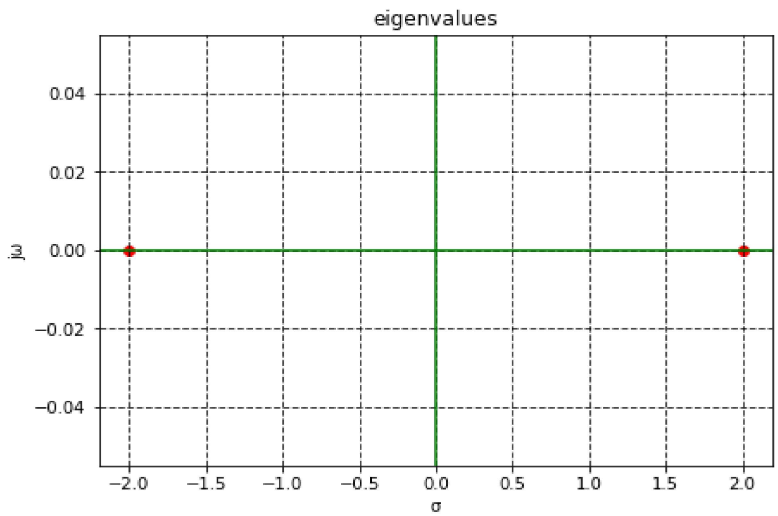
- It is the case when $\lambda_1 > 0 \; and \; \lambda_2 < 0$
- Almost all of the system trajectories diverge to infinity because of the unstable pole $\lambda_1 > 0$, and there are two straight lines passing through the origin.
- **For example:**

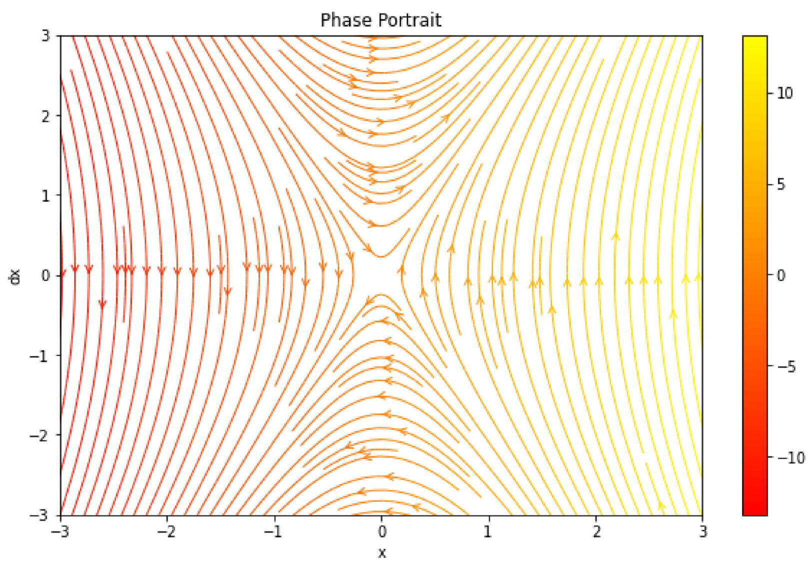
$$\ddot{x} = 4x$$

$$A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$$

$$\lambda_1 = 2, \; \lambda_2 = -2$$

This system has the following plots for its eigenvalues and phase portrait.





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4) Stable Focus:

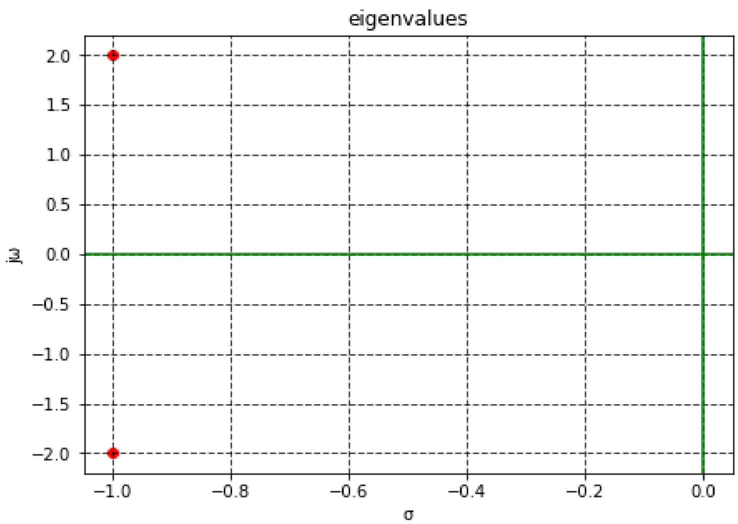
- It is the case when λ_1 *and* λ_2 are complex conjugate with negative real parts.
- The system trajectories oscillate around zero before converging to it.
- **For example:**

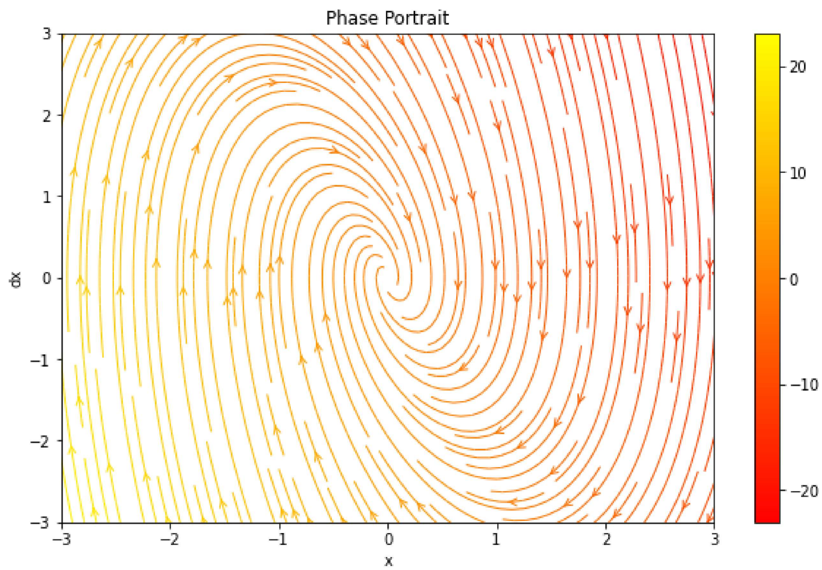
$$\ddot{x} = -5x - 2\dot{x}$$

$$A = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}$$

$$\lambda_1 = -1 + 2i,\ \lambda_2 = -1 - 2i$$

This system has the following plots for its eigenvalues and phase portrait.





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5) Unstable Focus:

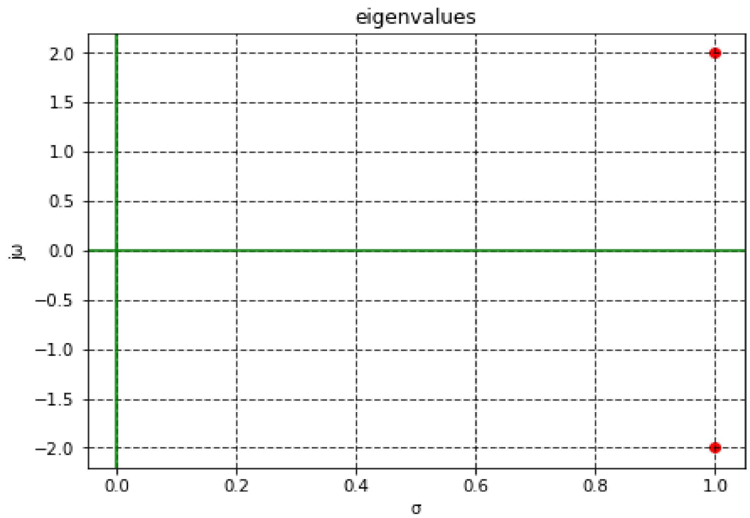
- It is the case when λ_1 *and* λ_2 are complex conjugate with positive real parts.
- The system trajectories diverge from zero to infinity.
- **For example:**

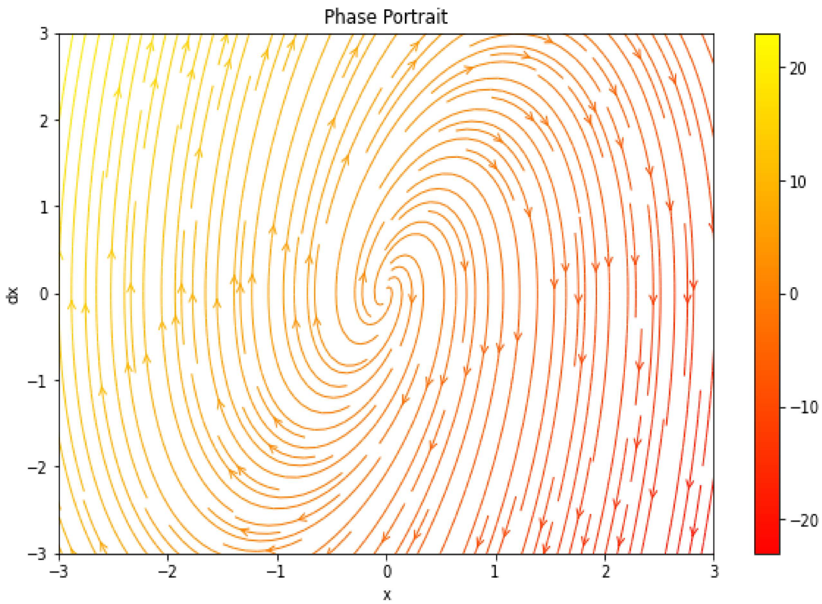
$$\ddot{x} = -5x + 2\dot{x}$$

$$A = \begin{bmatrix} 0 & 1 \\ -5 & 2 \end{bmatrix}$$

$$\lambda_1 = 1 + 2i,\ \lambda_2 = 1 - 2i$$

This system has the following plots for its eigenvalues and phase portrait.





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6) Center Point:

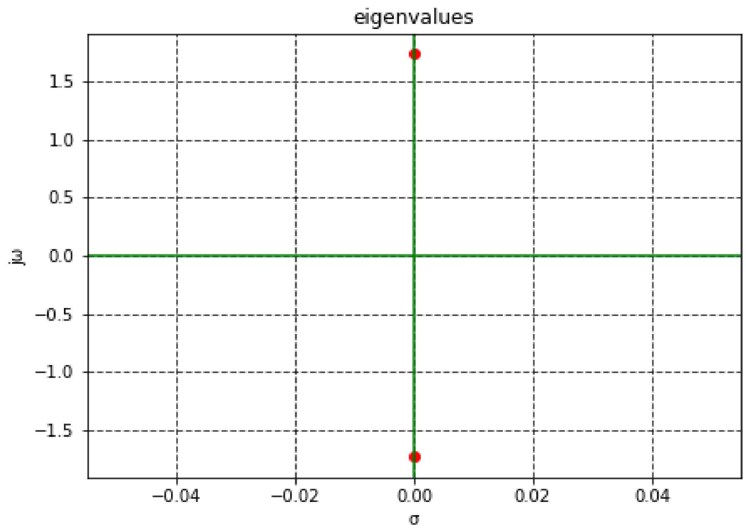
- It is the case when λ_1 *and* λ_2 are complex conjugate with zero real parts.
- The system trajectories oscillate around the equilibrium point.
- The system is marginally stable.
- **For example:**

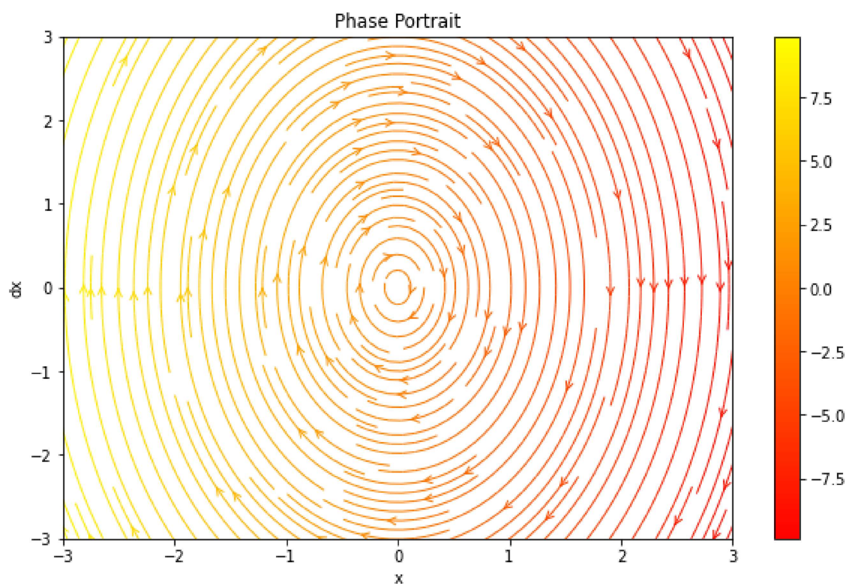
$$\ddot{x} = 3x$$

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}$$

$$\lambda_1 = \sqrt{3}i,\ \lambda_2 = -\sqrt{3}i$$

This system has the following plots for its eigenvalues and phase portrait.



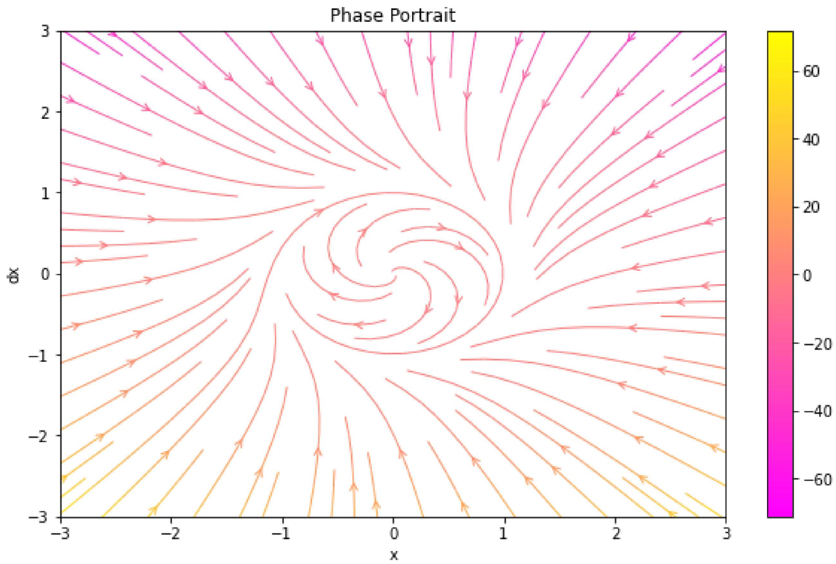


Problem 3

For the system a)

$$\begin{cases} \dot{x_1} = x_2 - x_1\left(x_1^2 + x_2^2 - 1\right) \\ \dot{x_2} = -x_1 - x_2\left(x_1^2 + x_2^2 - 1\right) \end{cases}$$

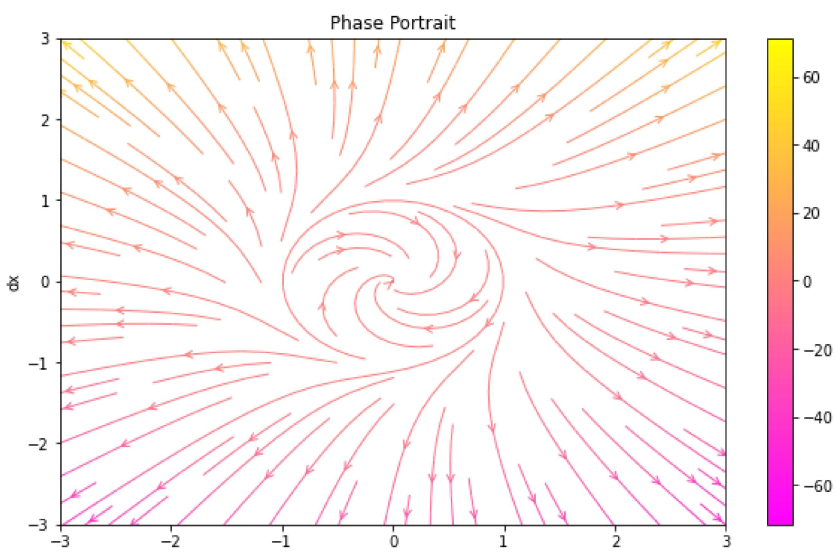
- It has a behavior of **stable limit cycles**, because all trajectories in the vicinity of the limit cycle converge to it as time goes to infinity.
- It has the following Phase Portrait.



For the system b)

$$\begin{cases} \dot{x_1} = x_2 \; + x_1\left(x_1^2 + x_2^2 - 1\right) \\ \dot{x_2} = -x_1 \; + x_2\left(x_1^2 + x_2^2 - 1\right) \end{cases}$$

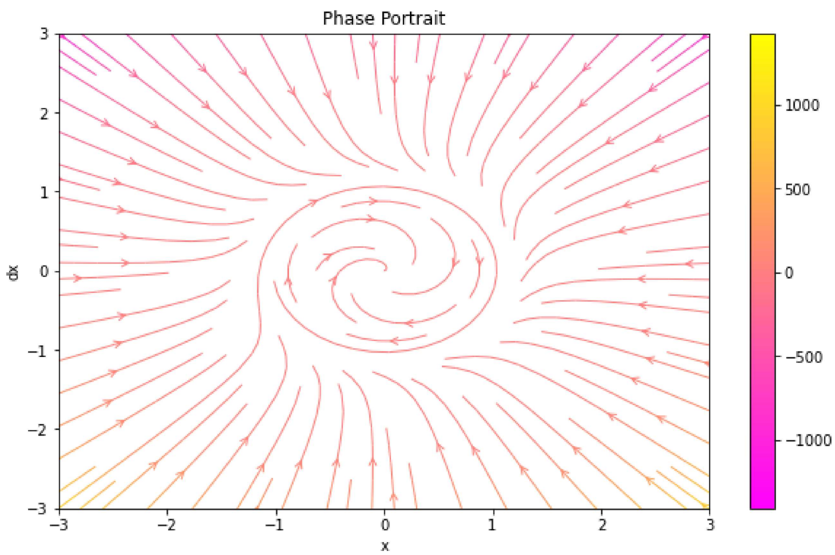
- It has a behavior of **unstable limit cycles** because all trajectories in the vicinity of the limit cycle diverge from it as time goes to infinity.
- It has the following Phase Portrait.



For the system c)

$$\begin{cases} \dot{x_1} = x_2 - x_1\left(x_1^2 + x_2^2 - 1\right)^2 \\ \dot{x_2} = -x_1 - x_2\left(x_1^2 + x_2^2 - 1\right)^2 \end{cases}$$

- It has a behavior of **semi-stable limit cycles** because some of the trajectories in the vicinity converge to it, while the others diverge from it as time goes to infinity.
- It has the following Phase Portrait.



Problem 4

For the damped nonlinear pendulum system:

$$\ddot{x} = -\dot{x} - sin(x)$$

we can follow Lyapunov linearization method which has the following steps:

1) Solving $f(x_e) = 0$ to find the equilibrium points.

- letting $x = y_1, \dot{x} = y_2$ the corresponding state space equation is:

$$\dot{y_1} = y_2$$
$$\dot{y_2} = -y_2 - sin(y_1)$$
$$\begin{bmatrix} \dot{y_1} \\ \dot{y_2} \end{bmatrix} = \begin{bmatrix} y_2 \\ -y_2 - sin(y_1) \end{bmatrix}$$

- The equilibrium points are given by

$$y_2 = 0,\ sin(y_1) = 0$$
$$eq\ points\ (y_1, y_2) = (0,0), (\pi, 0)$$
$$\text{"}or\ (n*\pi, 0)\ where\ n\ belongs\ to\ Z$$

2) Calculation of the system jacobian with respect to state x.

$$Jacobian = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix}$$

$$Jacobian \; = \begin{bmatrix} 0 & 1 \\ -cos(y_1) & -1 \end{bmatrix}$$

3) Evaluation of resulting jacobian in the equilibrium points x_e.

- Substitute with the 1st equilibrium point $(0, 0)$:

$$A_1 \; = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

- Find the eigenvalues of A_1:

$$A_1 \; = \begin{bmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix}$$

$$\lambda^2 + \lambda + 1 = 0$$

$$\lambda_1 \; = \; -\frac{1}{2} + 0.866i$$

$$\lambda_2 \; = \; -\frac{1}{2} - 0.866i$$

- Since λ_1 *and* λ_2 has negative real part, The system is asymptotically stable at $(0, 0)$.

- Substitute with the 2nd equilibrium point $(\pi, 0)$:

$$A_2 \; = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

- Find the eigenvalues of A_2:

$$A_2 \; = \begin{bmatrix} -\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix}$$

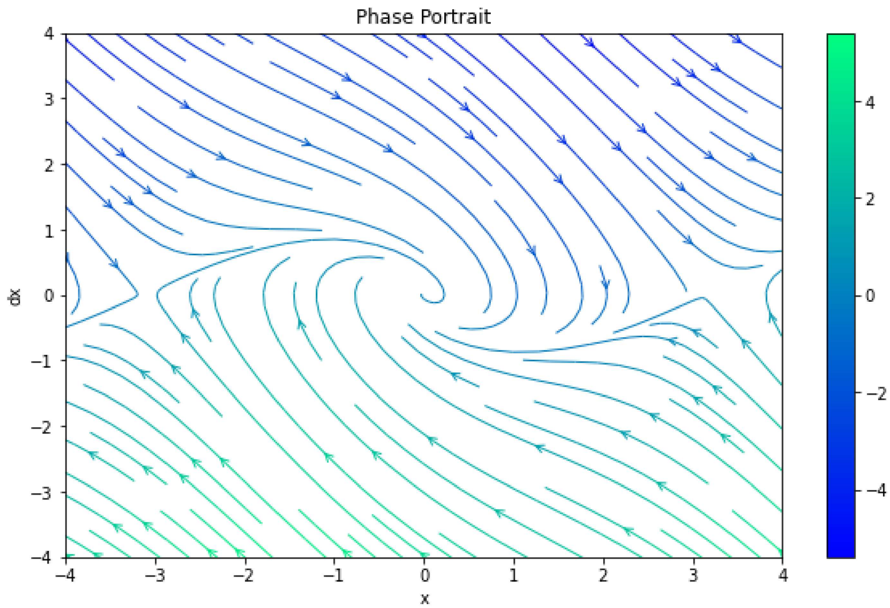
$$\lambda^2 + \lambda - 1 = 0$$

$$\lambda_1 \; = 0.618$$

$$\lambda_2 \; = \; -1.618$$

- Since $\lambda_1 \; > 0 \; and \; \lambda_2 < 0$ The system is unstable (diverges) at $(\pi, 0)$. And we can notice saddle point $(\pi, 0)$, $(-\pi, 0)$

I checked my solution by plotting the phase portrait and obtained this one.



Problem 5

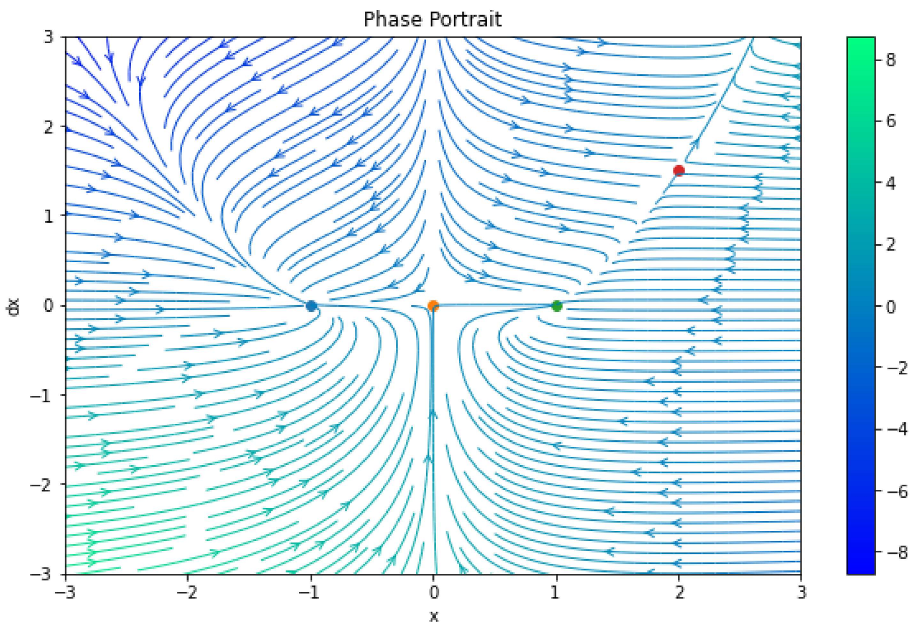
I implemented the algorithm for Lyapunov linearization method in the Attached .ipynb file. Then tested it with the following system:

$$\begin{cases} \dot{x_1} = x_1 - x_1^3 + 2x_1x_2 \\ \dot{x_2} = -x_2 + \dfrac{1}{2}x_1x_2 \end{cases}$$

- obtaining the equilibrium points:

$$(-1.0,\ 0),\ (0,\ 0),\ (1.0,\ 0),\ (2.0,\ 1.5)$$

- The jacobian symbolically and numerically then analyzing the eigenvalues.
- The system is Asymptotically Stable at the equilibrium point $(-1, 0), (1, 0)$
- The system is unstable Stable at the equilibrium point $(0, 0), (2, 1.5)$
- The phase portrait:



Phase Portrait
x
dx