ANC HW 3: Lyapunov Direct Method

Problem 1. Consider the following Lyapunov candidate:

$$V(\mathbf{x}) = \frac{x_1^2}{(1+x_1^2)^2} + x_2^2$$

Plot the level sets of Lyapunov candidate. Can one use this $V(\mathbf{x})$ to deduce the global asymptotic stability for some system? Motivate your answer.

Problem 2. Given the system:

$$\begin{cases} \dot{x}_1 = x_1(x_1^2 + x_2^2 - c) - 4x_1x_2^2 \\ \dot{x}_2 = 4x_1^2x_2 + x_2(x_1^2 + x_2^2 - c) \end{cases}$$

where c is positive constant.

Do the following:

1. Find is the system is globally or locally stable.
2. Determine the region of attraction.
3. Check out if dynamics is exponentially stable.

Problem 3. Given nonlinear pendulum:

$$\ddot{\theta} + \sin\theta + \dot{\theta} = u$$

With energy defined as:

$$\mathcal{H} = \frac{1}{2}\dot{\theta}^2 + 1 - \cos\theta$$

Use Lyapunov candidate $V = \frac{1}{2}\tilde{\mathcal{H}}^2$ to:

1. Find the controller that will ensure the convergence of energy error $\tilde{\mathcal{H}} = \mathcal{H}_d - \mathcal{H}$ thus making the trajectories converge to the set defined by constant desired energy $\mathcal{H}_d$.
2. Simulate the response of your controller and draw the phase portrait of closed loop dynamics. Discuss the results.

Problem 4. For many forms of uncertainty, we might not even know the location of the fixed points of the uncertain system, however one still may conclude something on the **invariant sets**. For instance consider the system:

$$\dot{x} = -x + x^3 + \alpha, \quad -\frac{1}{4} < \alpha < \frac{1}{4}$$

Use the graphical tools (phase portrait) and Lyapunov-like arguments to find the boundaries of the **robust invariant set** (invariant for any value of α in given region).

Problem 5. Consider mechanical system:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{u}$$

Where $\mathbf{M}$ is P.D. inertia matrix, $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ is contribution of coriolis and centrifugal terms, while $\mathbf{g}(\mathbf{q}) = \frac{\partial \mathcal{P}}{\partial \mathbf{q}}$ is the potential forces associated with potential energy $\mathcal{P}$.

Choose the Lyapunov candidate to be the following "energy-like" function:

$$V = \mathcal{K} + \frac{1}{2}\tilde{\mathbf{q}}^T \mathbf{K}_P \tilde{\mathbf{q}} = \frac{1}{2}\dot{\tilde{\mathbf{q}}}^T \mathbf{M}(\mathbf{q})\dot{\tilde{\mathbf{q}}} + \frac{1}{2}\tilde{\mathbf{q}}^T \mathbf{K}_P \tilde{\mathbf{q}}$$

Where $\tilde{\mathbf{q}} = \mathbf{q}_d - \mathbf{q}$ is regulation error and $\mathbf{q}_d$ is constant desired position.

Prove the **asymptotic stability** of gravity compensation PD controller:

$$\mathbf{u} = \mathbf{K}_P \tilde{\mathbf{q}} + \mathbf{K}_D \dot{\tilde{\mathbf{q}}} + \mathbf{g}(\mathbf{q})$$

Note: Recall that derivative of the energy is mechanical power: $\dot{\mathcal{H}} = \dot{\mathcal{K}} + \dot{\mathcal{P}} = \dot{\mathbf{q}}^T \mathbf{u}$