Applied Nonlinear Control

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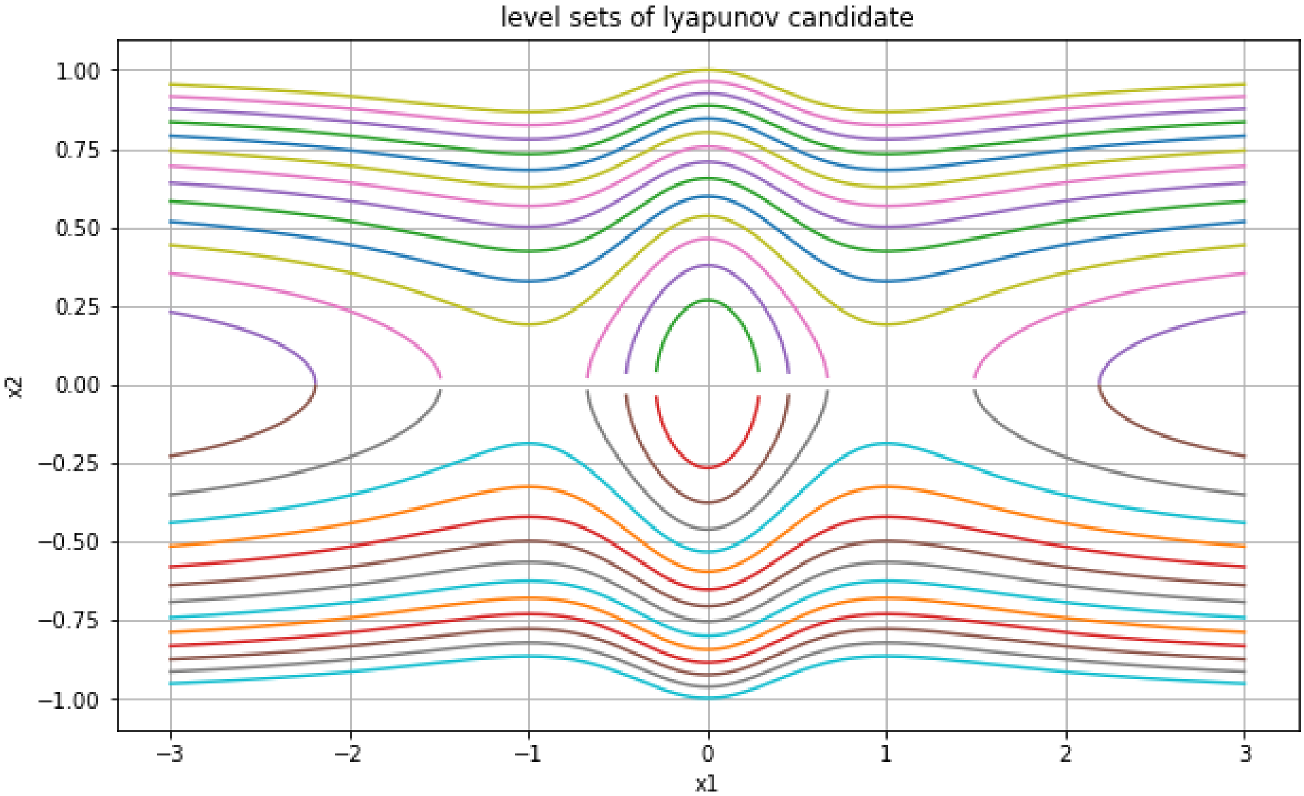
HW3_Report

Problem 1

To Plot the level sets of Lyapunov candidate, I plotted x_2 as a function of $V(x)$ and x_1, I set x_1 as an array from (-3,3) and set $V(x)$ as an array from (0,1) with 15 points, to see the behavoir of the function for 15 different level.

$$x_2 = \pm \sqrt{V(x) - \frac{x_1^2}{\left(1 + x_1^2\right)^2}}$$

and I got the following results:



- As we can see from the plot, the function $V(x)$ doesn't diverge to infinity when $x \to \infty$
- And if we find the derivative of $V(x)$:

$$\dot{V(x)} = \left(\frac{-4x_1^3}{\left(1 + x_1^2\right)^3} + \frac{2x_1}{\left(1 + x_1^2\right)^2} \right) \dot{x_1} + 2x_2 \, \dot{x_2}$$

we can see that $\dot{V(x)}$ is not negative definite.

- Then we can't use $V(x)$ to deduce the **global asymptotic stability** for some systems.

Problem 2

To find whether the given system is **locally** or **globally** stable:

$$\dot{x_1} = x_1\left(x_1^2 + x_2^2 - c\right) - 4x_1x_2^2$$

$$\dot{x_2} = 2x_1^2x_2 + x_2\left(x_1^2 + x_2^2 - c\right)$$

we can define Lyapunov candidate $V(x) = x_1^2 + x_2^2$, find it's derivative $\dot{V(x)} = 2x_1\dot{x_1} + 2x_2\dot{x_2}$ then substitute the equations of $\dot{x_1} and\ \dot{x_2}$ from the given system. We obtain:

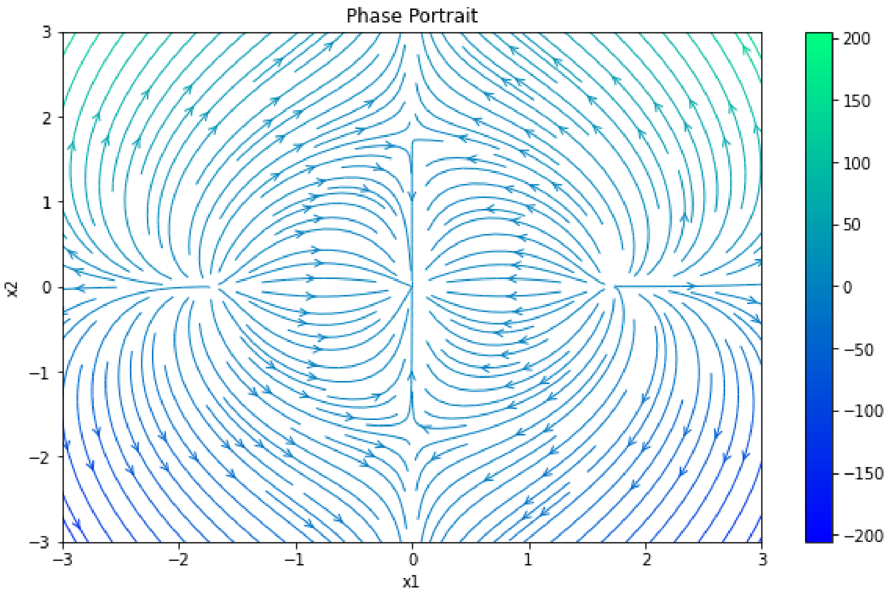
$$\dot{V(x)} = -2cx_1^2 - 2cx_2^2 + 2x_1^4 + 4x_1^2x_2^2 + 2x_2^4$$

since $\dot{V(x)}$ is not positive definite, so the system is **not globally stable.**
To find the region where the function $\dot{V(x)}$ is negative definite:

$$\dot{V(x)} < 0$$
$$-2cx_1^2 - 2cx_2^2 + 2x_1^4 + 4x_1^2x_2^2 + 2x_2^4 \quad < 0$$
$$-2c\left(x_1^2 + x_2^2\right) \ < \ -2\left(x_1^2 + x_2^2\right)^2$$
$$x_1^2 + x_2^2 \ < \ c$$

$\dot{V(x)}$ is negative definite in the ball $x_1^2 + x_2^2 \ < \ c$
Then the system is **locally stable** in ball $x_1^2 + x_2^2 \ < \ c$

The system has the following phase portrait at $c = 3$.

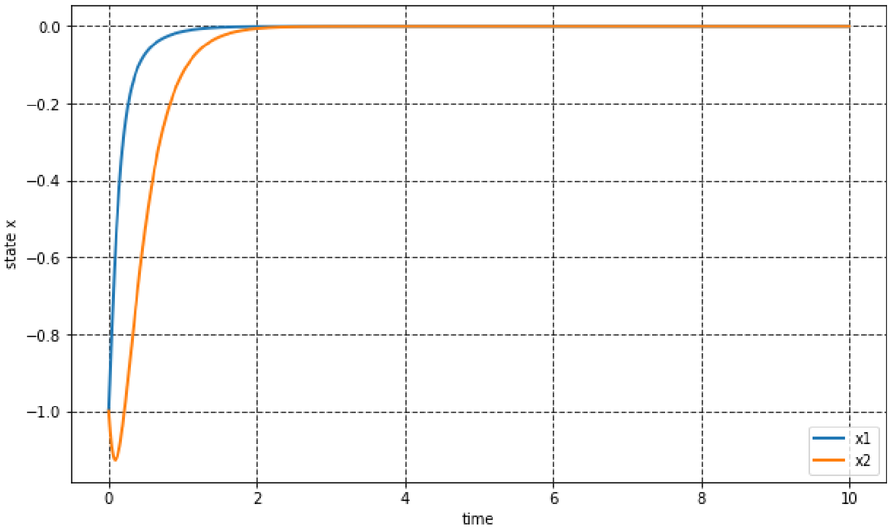


We can notice that the system is locally stable around its equilibrium points.

$$\left\{(0,\ 0),(0,\ -\sqrt{c}),(0,\ \sqrt{c}),(-\sqrt{c},\ 0),\left(-\frac{\sqrt{c}}{2},\ -\frac{i\sqrt{c}}{2}\right),\left(-\frac{\sqrt{c}}{2},\ \frac{i\sqrt{c}}{2}\right),\left(\frac{\sqrt{c}}{2},\ -\frac{i\sqrt{c}}{2}\right),\left(\frac{\sqrt{c}}{2},\ \frac{i\sqrt{c}}{2}\right),(\sqrt{c},\ 0)\right\}$$

The region of attraction:
As we can see from the previous phase portrait that the all trajectories converges to the invariant set Ω defined by all $x_1\ and\ x_2\ \in\ \Omega$ such that $x_1^2 + x_2^2 < c$ for $c = 3$. so the region of attraction is the circle $x_1^2 + x_2^2 < c$.

Plotting the system dynamics time impulse response.



we can notice that the dynamics is exponentially stable.

Problem 3

Given nonlinear pendulum:

$$\ddot{\theta} + sin\ \theta + \dot{\theta} = u$$
$$\ddot{\theta} = u - sin\ \theta - \dot{\theta} \tag{1}$$

with energy defined as:

$$H = \frac{1}{2}\dot{\theta}^2 + 1 - cos\ \theta$$

and energy error:

$$\widetilde{H} = H_d - H$$
$$\widetilde{H} = H_d - \frac{1}{2}\dot{\theta}^2 - 1 + cos\ \theta \tag{2}$$

using the lyapunov candidate:

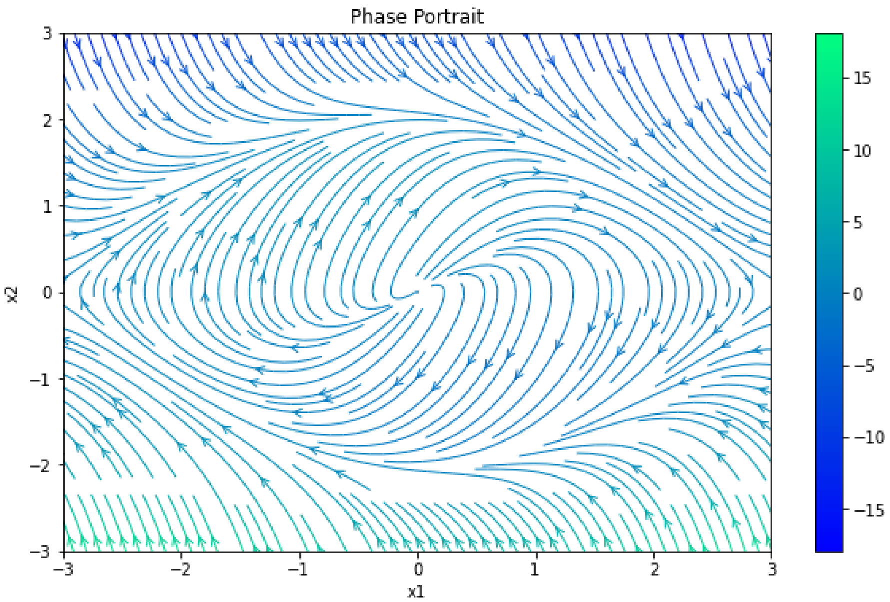
$$V = \frac{1}{2}\widetilde{H}^2$$

we need to find the drivative of V and find the controller u such that $V < 0$ as the following steps:

$$\dot{V} = \widetilde{H}\dot{\widetilde{H}} \tag{3}$$
$$finding\ \dot{\widetilde{H}}\ form\ eq\ (2)$$
$$\dot{\widetilde{H}} = \dot{H}_d - \dot{\theta}\ddot{\theta} - sin\theta$$
$$substituting\ from\ eq\ (1)\ yields$$
$$\dot{\widetilde{H}} = \dot{H}_d - \dot{\theta}(u - sin\ \theta - \dot{\theta}) - sin\theta$$
$$where\ H_d\ is\ constant\ desired\ energy,\ so\ \dot{H}_d = 0$$
$$\dot{\widetilde{H}} = -\dot{\theta}(u - \dot{\theta})$$
$$substituting\ back\ in\ eq\ (3)\ yields$$
$$\dot{V} = -\dot{\theta}\widetilde{H}(u - \dot{\theta}) < 0$$
$$-\dot{\theta}\widetilde{H}(u - \dot{\theta}) < 0$$
$$we\ need\ to\ find\ u\ to\ satisfy\ the\ previous\ inequality\ and\ make\ \dot{V} = -\dot{\theta}^2\widetilde{H}^2$$
$$-\dot{\theta}\widetilde{H}(u - \dot{\theta}) = -\dot{\theta}^2\widetilde{H}^2$$

$$u = \dot{\theta}\widetilde{H} + \dot{\theta}$$
$$u = \dot{\theta}\left(\widetilde{H} + 1\right)$$
$$u = \dot{\theta}\left(H_d - \frac{1}{2}\dot{\theta}^2 + cos\ \theta\right)$$

Drawing the phase portrait of the closed loop dynamics yields:



As we can see, the closed loop dynamics results a limit cycle which converges to the set defined by constant desired energy H_d.

Problem 4

For the given system:

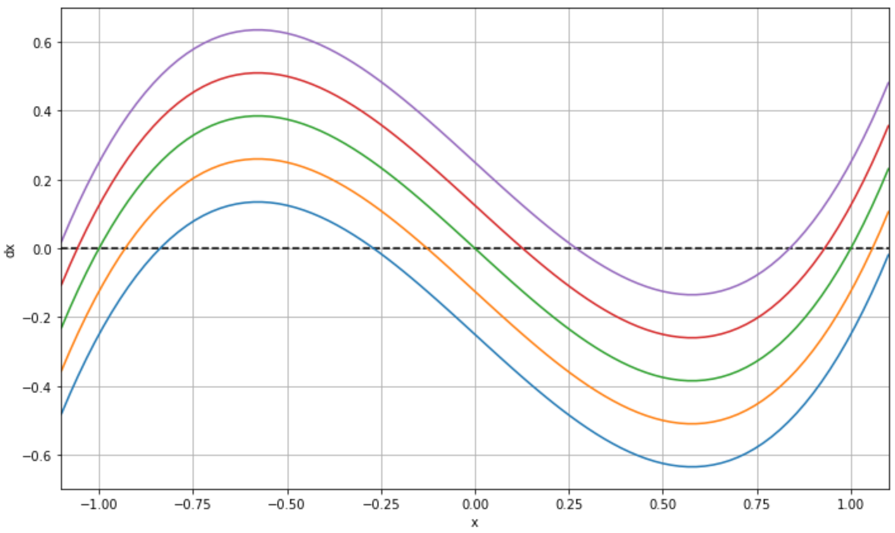
$$\dot{x} = -x + x^3 + \alpha\ ,\ where\ \frac{-1}{4} < \alpha < \frac{1}{4}$$

Taking lyapunov candidate $V(x) = x^2$ and computing its derivative then substitute with the system:

$$\dot{V}(x) = 2x\dot{x} = 2x\left(-x + x^3 + \alpha\right)$$
$$since\ \dot{V}(x) < 0$$
$$2x\left(-x + x^3 + \alpha\right) < 0$$
$$-x^2 + x^4 + \alpha x < 0$$
$$in\ order\ to\ satisfy\ the\ inequality\ above$$
$$we\ need\ to\ find\ the\ range\ of\ x$$
$$x^4 + \alpha x < 0$$
$$x^3 < \alpha$$
$$x < \sqrt[3]{\alpha}$$

Then the boundaries of the robust invariant set is $x < \sqrt[3]{\alpha}$ for any value of α in given region.

Using the analytical tool which is plotting the system dynamics for 5 values of α in the given region, we obtain the following graph:



We can notice that the boundaries for of the robust invariant set is $x < \sqrt[3]{\alpha}$ for any value of α in given region. $\left(-\sqrt[3]{\frac{1}{4}},\ \sqrt[3]{\frac{1}{4}}\right) = (-0.63,\ 0.63)$

Problem 5

For the mechanical system:

$$M(q)\,\ddot{q} + C(q,\dot{q})\dot{q} + g(q) \ = \ u \tag{4}$$

Has the following energy-like function:

$$V = \frac{1}{2}\,\dot{\tilde{q}}^T M(q)\,\dot{\tilde{q}} + \frac{1}{2}\,\tilde{q}\,K_P\,\tilde{q}$$

Where $\tilde{q} = q_d - q$ and q_d us constant desired position.
Then we have:

$$\dot{\tilde{q}} \ = \ -\dot{q} \tag{5}$$

$$\ddot{\tilde{q}} \ = \ -\ddot{q} \tag{6}$$

PD controller:

$$u = \ K_P\,\tilde{q} \ + \ K_D\,\dot{\tilde{q}} \ + g(q) \tag{7}$$

Computing the derivative of V:

$$\dot{V} = \ \dot{\tilde{q}}\,M\,\ddot{\tilde{q}} \ + \ \tilde{q}\,Kp\,\dot{\tilde{q}} \tag{8}$$

$$\textit{substituting with eq (6) \& (7) together yields}:$$
$$\ddot{q} = \ (u - g(q) - C(q,\dot{q})\dot{q})\ M^{-1}$$

$$\textit{then substituting back with eq (5) in (8) yields}:$$
$$\dot{V} = \dot{q}\ (u - g(q) - C(q,\dot{q})\dot{q}) \ - \ \ K_P\,\dot{q}\,\tilde{q}$$

$$\textit{substituting with eq (7) yields}:$$
$$\begin{aligned}\dot{V} \ &= \ \dot{q}\,\left(K_P\,\tilde{q} \ + \ K_D\,\dot{\tilde{q}} \ + g(q) - g(q) - C(q,\dot{q})\dot{q}\right) \ - \ \ K_P\,\dot{q}\,\tilde{q} \\ &= \ K_P\,\tilde{q}\,\dot{q} \ \ - \ K_D\,\dot{q} \ \ \dot{q} \ - C(q,\dot{q})\dot{q}\,\dot{q} \ \ - \ \ K_P\,\dot{q}\,\tilde{q} \\ &= -\ K_D\,\dot{q} \ \ \dot{q} \ \ - C(q,\dot{q})\dot{q}\,\dot{q}\end{aligned}$$

$$since\ \dot{V} < 0$$
$$-\dot{q}^T K_D \dot{q} - \dot{q}^T C(q, \dot{q})\dot{q} < 0$$

Then the gravity compensation PD controller is asymptotic stable.