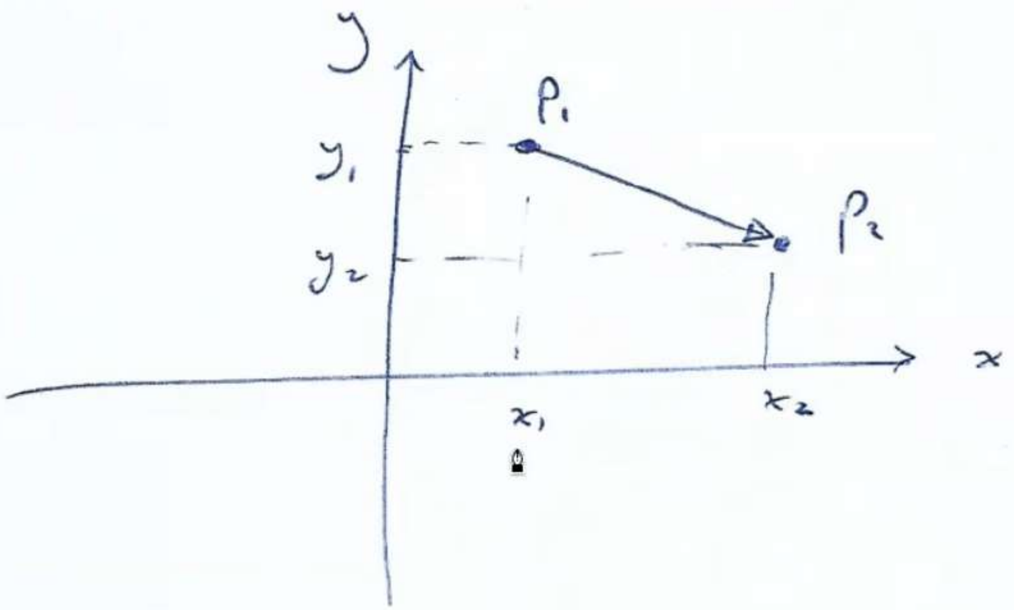
ch. 33 Computational Geometry

33.1 Consider two points in $\mathbb{R}^2$:

$$p_1 = (x_1, y_1)$$

$$p_2 = (x_2, y_2)$$

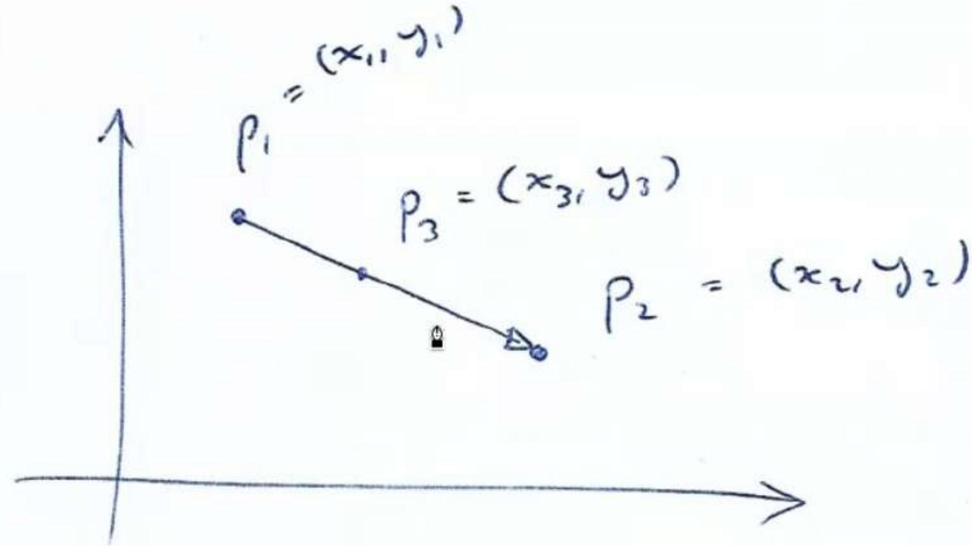


The line segment $\overline{P_1P_2}$ consists of all points on the line passing through

p_1 and p_2.

Sometimes we say directed segment $\overrightarrow{p_1 p_2}$

& the direction matters.



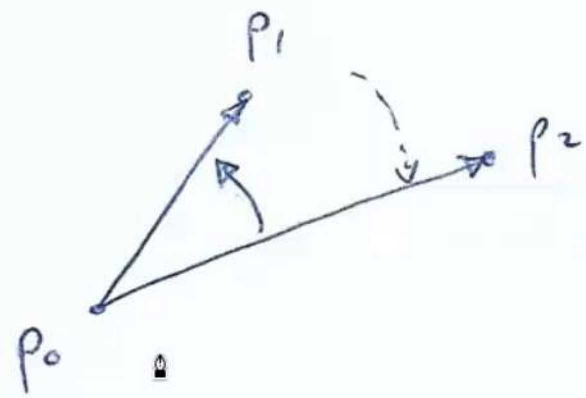
If p_3 is on the line segment $\overline{p_1 p_2}$

then $p_3 = \alpha p_1 + (1-\alpha) p_2$ for some $\alpha \in [0,1]$.

so $$\begin{cases} x_3 = \alpha x_1 + (1-\alpha) x_2 \\ y_3 = \alpha y_1 + (1-\alpha) y_2 \end{cases}$$

Questions :

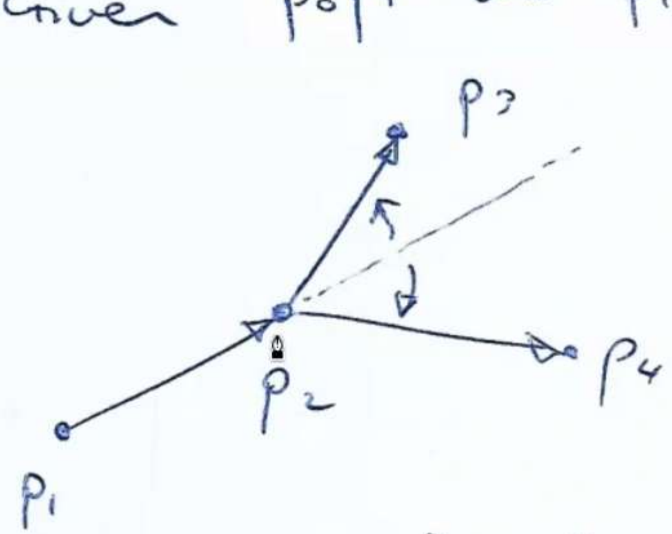
1. Given $\overrightarrow{p_0p_1}$ and $\overrightarrow{p_0p_2}$, is $\overrightarrow{p_0p_1}$ clockwise from $\overrightarrow{p_0p_2}$ or counterclockwise?



$\overrightarrow{p_0p_1}$ is counterclockwise from $\overrightarrow{p_0p_2}$

& $\overrightarrow{p_0p_2}$ is clockwise from $\overrightarrow{p_0p_1}$

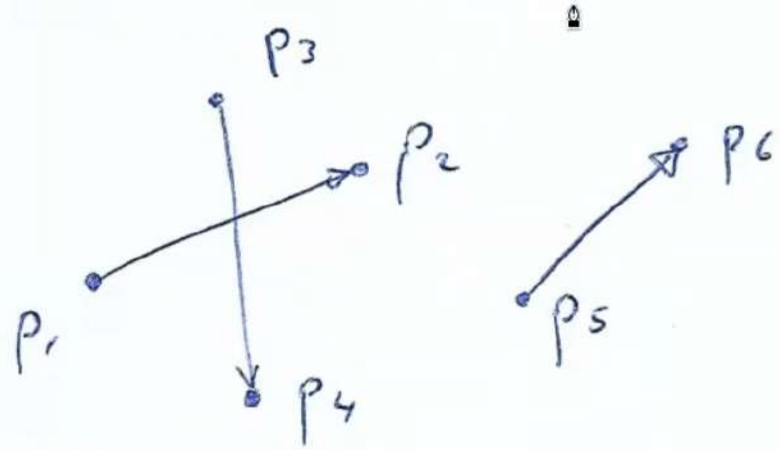
2. Given $\overrightarrow{p_0p_1}$ and $\overrightarrow{p_1p_2}$ do we turn left or right at p_1 ?



Traversing $\overrightarrow{p_1p_2}$ & $\overrightarrow{p_2p_3}$ we turn left

Traversing $\overrightarrow{p_1p_2}$ & $\overrightarrow{p_2p_4}$ we turn right

3. Do $\overrightarrow{p_1p_2}$ and $\overrightarrow{p_3p_4}$ intersect ?



Doing this using a 'High-school' method
we find straight line equations

through p_1 & p_2 and p_3 & p_4, say

$$y = m_1 x + c_1$$
$$y = m_2 x + c_2 .$$

Solve for intersection: $m_1 x + c_1 = m_2 x + c_2$

$$\Rightarrow \quad x = \frac{c_2 - c_1}{m_1 - m_2} \qquad (\text{if } m_1 \neq m_2)$$

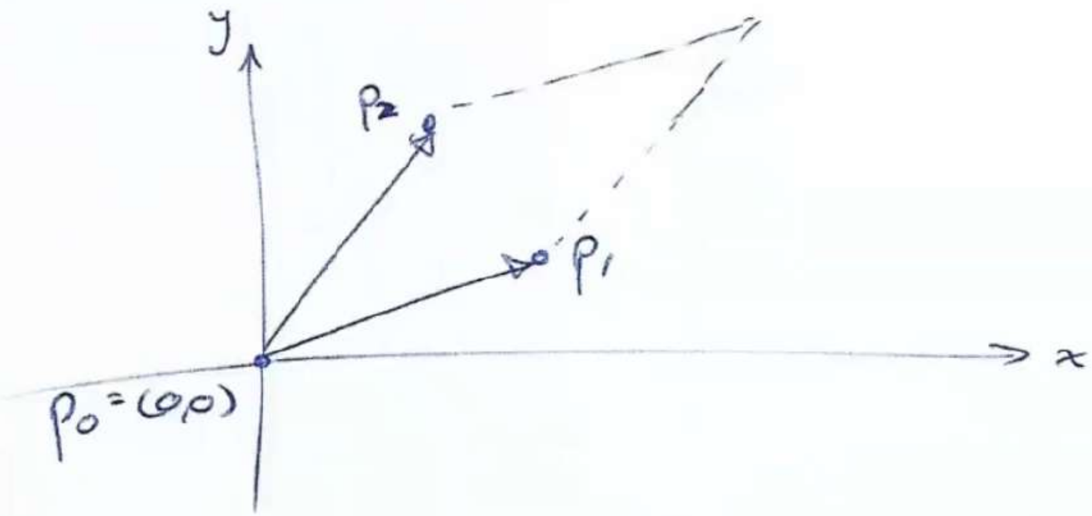
$$\Rightarrow \quad y = m_1 \left(\frac{c_2 - c_1}{m_1 - m_2} \right) + c_1$$

Then we check if the point of intersection lies on the line segment $\overrightarrow{p_1 p_2}$

(OR on $\overrightarrow{p_3 p_4}$).

Note that this method requires a 'division', which is computationally expensive.

Cross Products



$$p_1 \times p_2 = x_1 y_2 - x_2 y_1 = \det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

(gives (signed) area of parallelogram).

- If $p_1 \times p_2 > 0$ then $\overrightarrow{p_0 p_1}$ is clockwise from $\overrightarrow{p_0 p_2}$

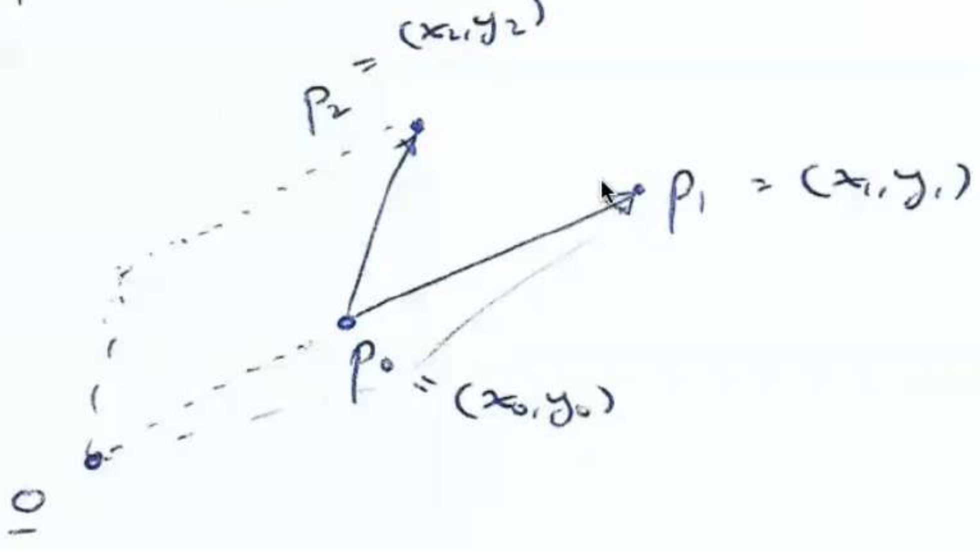
- If $p_1 \times p_2 < 0$ then $\overrightarrow{p_0 p_1}$ is counter clockwise from $\overrightarrow{p_0 p_2}$

- If $p_1 \times p_2 = 0$ then $\overrightarrow{p_0 p_1}$ and $\overrightarrow{p_0 p_2}$ are colinear

Going back to Question 1:

Given $\overrightarrow{p_0p_1}$ and $\overrightarrow{p_0p_2}$ is

$\overrightarrow{p_0p_1}$ clockwise or counterclockwise from $\overrightarrow{p_0p_2}$?



$$(p_1 - p_0) \times (p_2 - p_0)$$

$$= (x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)$$

$$= (x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)$$

if > 0 then $\overrightarrow{p_0p_1}$ clockwise from $\overrightarrow{p_0p_2}$

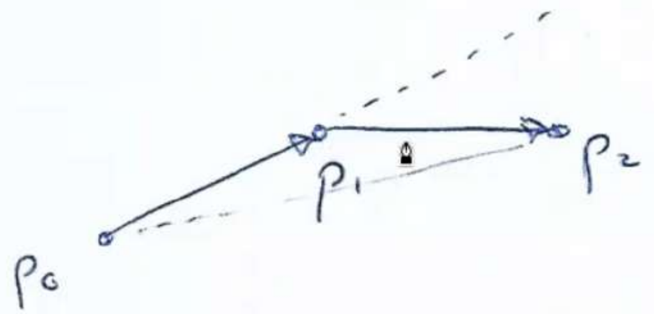
if < 0 then . . counterclockwise —

if $= 0$ then collinear -

Note: Method uses $+, -, \cdot$ only —

Returning to Question 2 :

Given $\overrightarrow{p_0p_1}$ and $\overrightarrow{p_1p_2}$ do we turn left or right ?

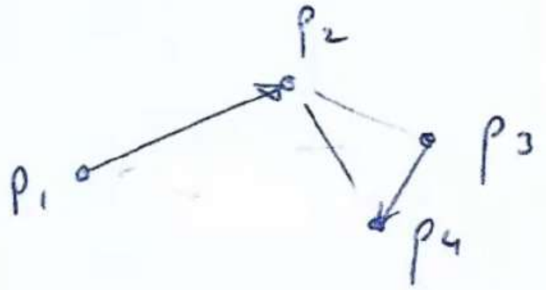


We ask: does $\overrightarrow{p_0p_2}$ lie clockwise or counterclockwise from $\overrightarrow{p_0p_1}$ -

- then use Question 1's solution -

Returning to Question 3,

Do $\overrightarrow{p_1p_2}$ and $\overrightarrow{p_3p_4}$ intersect?



If $\overrightarrow{p_1p_2}$ & $\overrightarrow{p_2p_3}$ is clockwise
& $\overrightarrow{p_1p_2}$ & $\overrightarrow{p_2p_4}$ is clockwise
$\Big\}$ No intersection.

Similarly, if both are counterclockwise then

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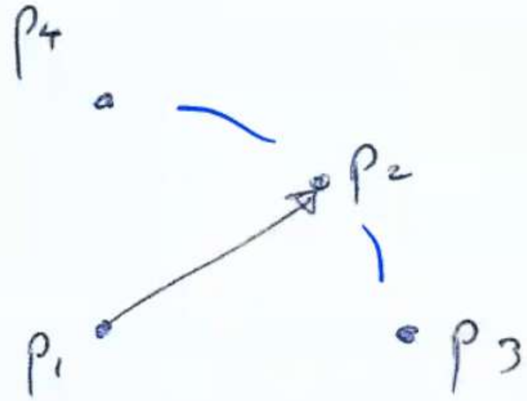
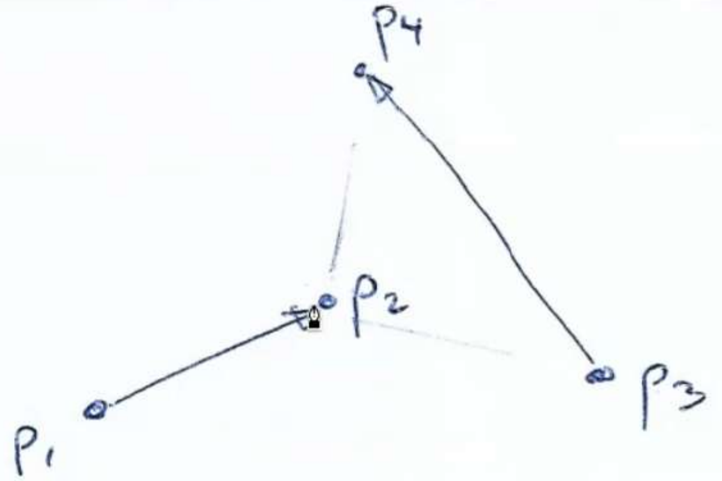
No intersection.



so one of the changes in direction must be clockwise and the other counterclockwise.

But:



So we do the same with $\overrightarrow{p_3p_4}$ &
$\overrightarrow{p_4p_1}$ and $\overrightarrow{p_4p_2}$.

In the above case both direction changes are counterclockwise.

Again one of the changes in direction

$\overrightarrow{p_3p_4}$ & $\overrightarrow{p_4p_1}$

and $\overrightarrow{p_3p_4}$ & $\overrightarrow{p_4p_2}$

must be clockwise and the