4.3-1 Show $T(n) = T(n-1) + n$ is $O(n^2)$

Use substitution method:

Assume $T(m) \le cm^2$ for all $m < n$.

Then
$$
\begin{aligned}
T(n) &= T(n-1) + n \\
&\le c(n-1)^2 + n \quad (\text{since } n-1 < n) \\
&= cn^2 - 2cn + c + n \\
&= \underbrace{cn^2 - c(2n-1) + n \le cn^2}
\end{aligned}
$$

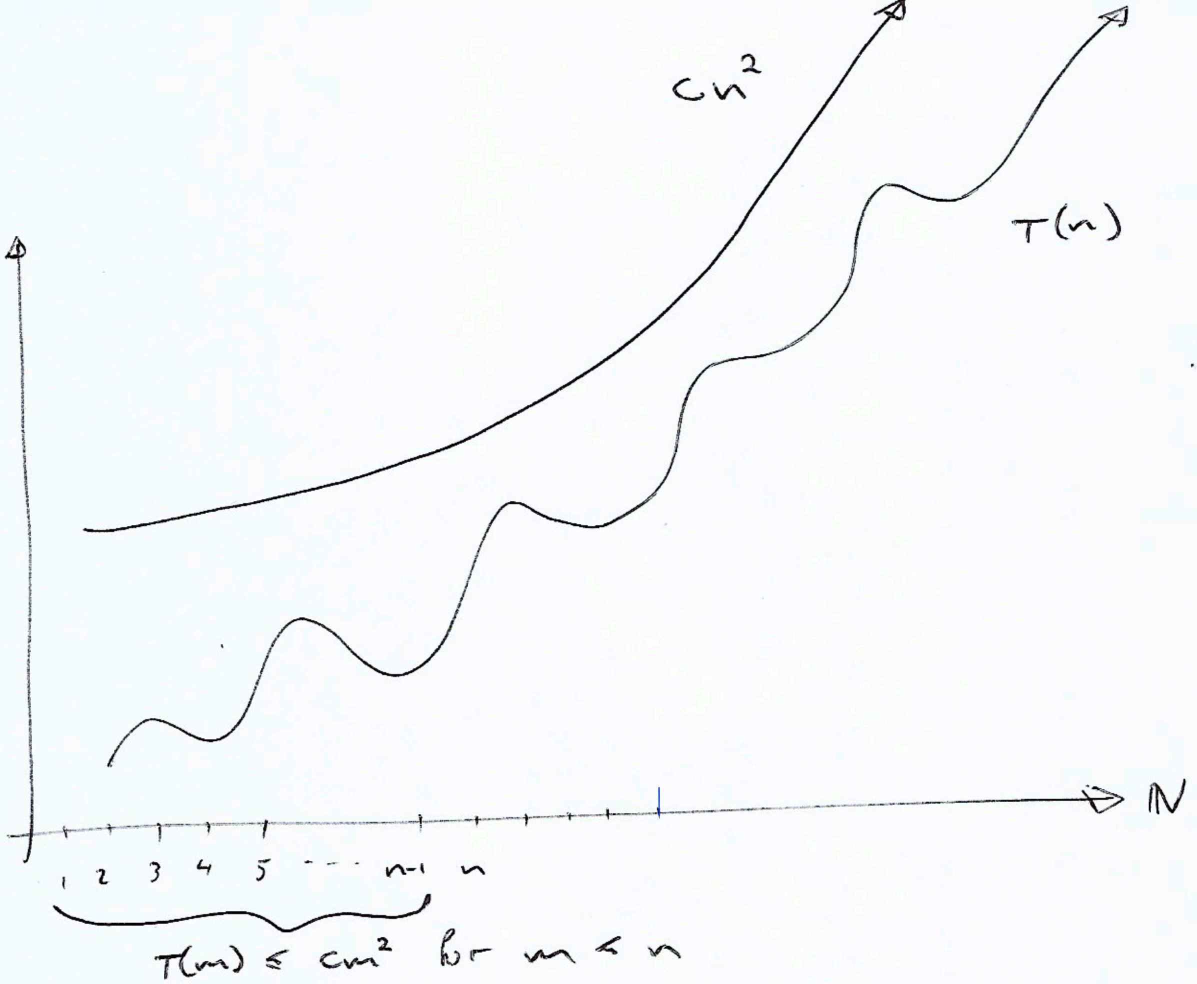
$$\iff n \le c(2n-1)$$

$$\iff \frac{n}{2n-1} \le c$$

For all $n \ge 1$, $\frac{n}{2n-1} = \frac{1}{2-\frac{1}{n}} \le \underline{1}$

Take $c = 1$.

Thus, $T(n)$ is $\underline{O(n^2)}$.



To show $T(n) \le cn^2$ for $n \ge n_0$ and for $c > 0$.

Essentially, we want to find a $c > 0$ that makes the induction step work.

4.3-2 Show $T(n) = T(\lceil \frac{n}{2} \rceil) + 1$ is $O(\log n)$.

Use substitution method:

Assume $T(m) \leq c \log m$ for all $m < n$.

Then $T(n) = T(\lceil \frac{n}{2} \rceil) + 1$

$\leq c \log(\lceil \frac{n}{2} \rceil) + 1$ (since $\lceil \frac{n}{2} \rceil < n$ Note: $n \geq 2$.)

[Note: can't drop ceiling brackets because $\lceil \frac{n}{2} \rceil \nleq \frac{n}{2}$.
Also using $\lceil \frac{n}{2} \rceil \leq \frac{n}{2} + 1$ is not useful.
Also using $\lceil \frac{n}{2} \rceil \leq n$ is not useful.]

We use: $\lceil \frac{n}{2} \rceil \leq \frac{3}{4} n$ for $n \geq 2$.

then $T(n) \leq \underbrace{c \log(\frac{3}{4} n) + 1 \leq c \log n}$

$\Leftrightarrow 1 \leq c(\log n - \log(\frac{3}{4} n))$

$\Leftrightarrow 1 \leq c \log \frac{4}{3}$

$\Leftrightarrow \frac{1}{\log \frac{4}{3}} \leq c$.

Choose $c = 3$.

Thus, $T(n) = O(\log n)$.

<u>4.3-3</u> Show $T(n) = \Omega(n \log n)$

if $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + n$.

We show that $T(n) = \Omega((n+1)\log(n+1))$.

Assume $T(m) \geq c(m+1)\log(m+1)$ for $m < n$.

Then $T(n) \geq 2c(\lfloor \frac{n}{2} \rfloor + 1)\log(\lfloor \frac{n}{2} \rfloor + 1) + n$

$\geq 2c(\frac{n}{2})\log(\frac{n}{2}) + n$ (since $\lfloor \frac{n}{2} \rfloor + 1 \geq \frac{n}{2}$)

$= cn\log n - cn + n$

We want $cn\log n - cn + n \geq c(n+1)\log(n+1)$

$\Longleftrightarrow$ $n \geq c((n+1)\log(n+1) - n\log n + n)$

$\Longleftrightarrow$ $1 \geq c\left(\left(\frac{n+1}{n}\right)\log(n+1) - \log n + 1\right)$.

Now, $\left(\frac{n+1}{n}\right)\log(n+1) - \log n \geq 0$

& $\lim_{n \to \infty}\left(\left(\frac{n+1}{n}\right)\log(n+1) - \log n\right) = 0$

(check!)

$\therefore$ For large enough $n_0 \in \mathbb{N}$ we have

$\left(\frac{n+1}{n}\right)\log(n+1) - \log n \leq \varepsilon$ for all $n \geq n_0$

where ε is a small positive value.

So we want $\quad 1 \geq c(\varepsilon + 1)$

(because then $\quad 1 \geq c(\varepsilon+1) \geq c\left(\frac{n+1}{n}\log(n+1) - \log n + 1\right)$)

So choose $\quad c \leq \dfrac{1}{\varepsilon+1}$.

e.g. $c = \frac{1}{2}$ will do.

Thus $\quad T(n) = \Omega((n+1)\log(n+1))$

but $\quad (n+1)\log(n+1) = \Omega(n\log(n))$

so $\quad T(n) = \Omega(n\log n)$.

4.3-6 $\quad T(n) = 2T(\lfloor \frac{n}{2} \rfloor + 17) + n$

Show: $\quad T(n) = O(n \log n)$

First, we show $T(n) = O((n-17)\log(n-17))$.

Assume $T(m) \leq c(n-17)\log(n-17)$ for all $m < n$.

Then $T(n) \leq 2c(\lfloor \frac{n}{2} \rfloor + 17 - 17)\log(\lfloor \frac{n}{2} \rfloor + 17 - 17) + n$

$$= 2c\lfloor \tfrac{n}{2} \rfloor \log(\lfloor \tfrac{n}{2} \rfloor) + n$$

$$\leq cn \log(\tfrac{n}{2}) + n$$

$$= cn\log n - cn + n$$

We want: $\quad cn\log n - cn + n \leq c(n-17)\log(n-17)$

$\iff \quad n \leq c((n-17)\log(n-17) - n\log n + cn)$

$\iff \quad 1 \leq c((\frac{n-17}{n})\log(n-17) - \log n + 1)$

using the fact that

$$(\tfrac{n-17}{n})\log(n-17) \leq \log n$$

$$\& \quad \lim_{n \to \infty}\left((\tfrac{n-17}{n})\log(n-17) - \log n\right) = 0$$

(check!)

there exists an $n_0 \in \mathbb{N}$ such that

for $n \geq n_0$, $(\frac{n-17}{n})\log(n-17) - \log n \geq -\varepsilon$

for some fixed small positive value ε.

So it suffices to find a value for c such that $1 \leq c(-\varepsilon + 1)$.

(because then $c\left(\left(\frac{n-17}{n}\right)\log(n-17) - \log n + 1\right)$
$\geq c(-\varepsilon + 1)$
≥ 1.)

choose $c \geq \frac{1}{-\varepsilon + 1}$.

since ε is a small value, we can choose $c = 2$, for example.

Thus, $T(n) = O((n-17)\log(n-17))$.

But $(n-17)\log(n-17) \in O(n \log n)$

so $T(n) = O(n \log n)$

4.3-3 Show $T(n) = T(\lfloor \frac{n}{2} \rfloor) + n$ is $\Omega(n \log n)$

Use substitution method:

Assume $T(m) \geq c\, m \log m$ for all $m < n$.

then
$$
\begin{aligned}
T(n) &= T(\lfloor \tfrac{n}{2} \rfloor) + n \\
&\geq c \lfloor \tfrac{n}{2} \rfloor \log \lfloor \tfrac{n}{2} \rfloor + n \quad (\text{since } \lfloor \tfrac{n}{2} \rfloor < n) \\
&\geq \\
&\vdots \\
&\geq c n \log n
\end{aligned}
$$

solve for suitable c ...

4.3-6 show $T(n) = 2T(\lfloor \frac{n}{2} \rfloor + 17) + n$ is $O(n\log n)$

Assume $T(m) \le c\, m \log m$ for all $m < n$.

then $T(n) = 2T(\lfloor \frac{n}{2} \rfloor + 17) + n$

$\le$??

(check: $\lfloor \frac{n}{2} \rfloor + 17 < n$ if $n > 34$)

$\therefore$ $T(n) \le c(\lfloor \frac{n}{2} \rfloor + 17) \log(\lfloor \frac{n}{2} \rfloor + 17) + n$

if $n \ge 35$

(use: $\lfloor \frac{n}{2} \rfloor + 17 \le \frac{3n}{4}$ if $n \ge 68$)

then $T(n) \le c\,\frac{3n}{4} \log(\frac{3n}{4}) + n$ if $n \ge 68$

want $c\,\frac{3n}{4} \log(\frac{3n}{4}) + n \le c\, n \log n$

$\iff \dfrac{1}{\log n - \frac{3}{4}(\log 3 + \log n - 2)} \le c$

$\iff \dfrac{4}{3\log n - 3\log 3 + 6} \le c$

choose $c = 1$

4.4-1 $\quad T(n) = 3T(\lfloor \frac{n}{2} \rfloor) + n$

use $\quad T(n) = 3T(\frac{n}{2}) + n$

$$T(\tfrac{n}{2}) = 3T(\tfrac{n}{4}) + \tfrac{n}{2}$$

Recursion Tree:

$T(n)$

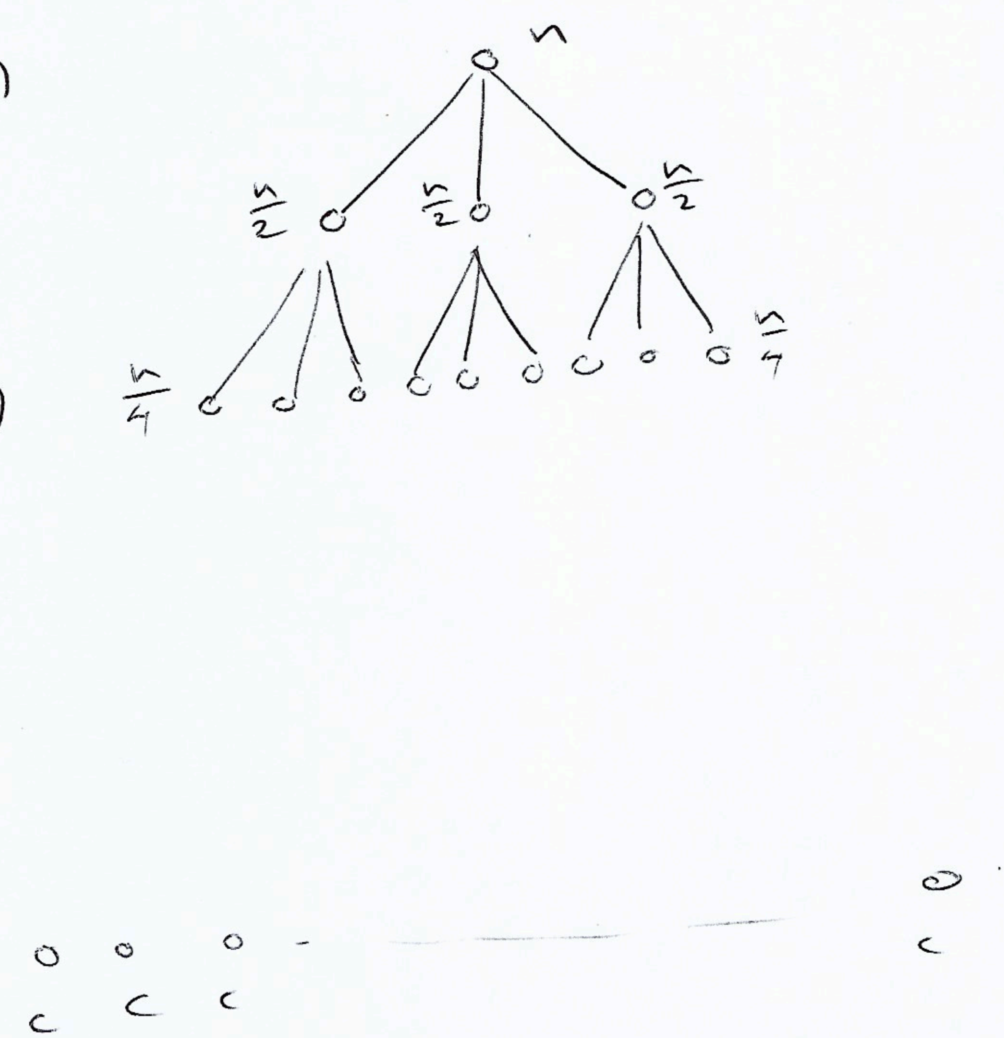
$T(\frac{n}{2})$

$T(\frac{n}{4})$

$\vdots$

$T(\frac{n}{2^i})$

$T(1)$



n

$3\frac{n}{2}$

$3^2\frac{n}{2^2}$

$\vdots$

$3^i \cdot \frac{n}{2^i}$

$\vdots$

$\frac{n}{2^k} \leq 1 \iff n \leq 2^k \iff \log n \leq k.$

height of tree is $\log n$.

$\#\text{leaves} = (\text{Branching factor})^{\text{height}} = 3^{\log n}$

$= n^{\log 3}$

Thus, $T(n) = \sum_{i=0}^{\log n - 1} \frac{3^i n}{2^i} + cn^{\log 3}$

$$= n \sum_{i=0}^{\log n - 1} \left(\tfrac{3}{2}\right)^i + cn^{\log 3}$$

$$= n \left(\frac{\left(\tfrac{3}{2}\right)^{\log n} - 1}{\tfrac{3}{2} - 1} \right) + cn^{\log 3}$$

$$= 2n \cdot n^{\log 3/2} - 2n + cn^{\log 3}$$

$$= 2n \cdot n^{\log 3 - \log 2} - 2n + cn^{\log 3}$$

$$= 2n \cdot n^{\log 3 - 1} - 2n + cn^{\log 3}$$

$$= 2n^{\log 3} - 2n + cn^{\log 3}$$

$$= O(n^{\log 3})$$

4.4-4 $T(n) = 2T(n-1) + 1$

$T(n-1) = 2T(n-2) + 1$

Recursion Tree

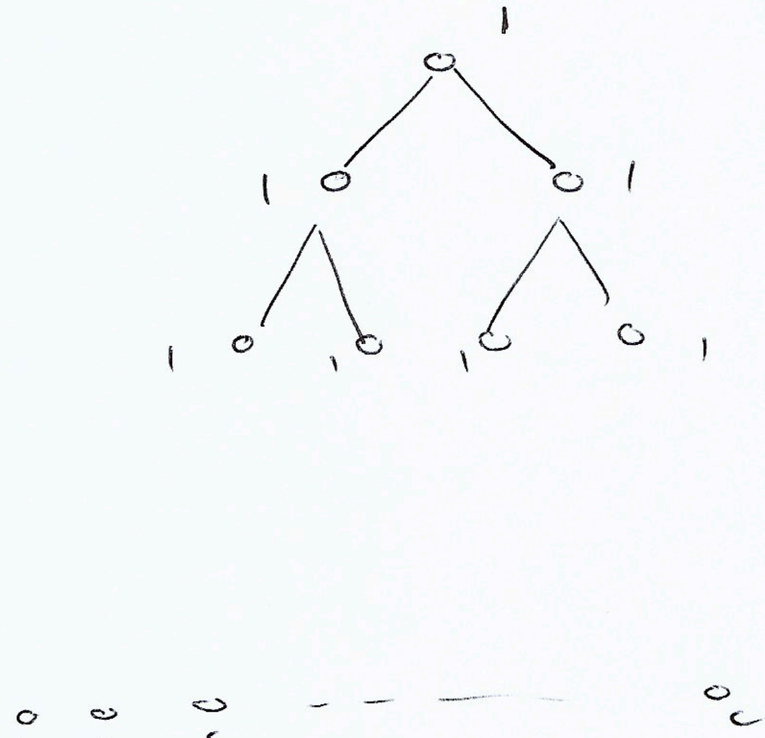
$T(n)$

$T(n-1)$

$T(n-2)$

$T(\ldots)$

$T(0)$



$1 = 2^0$

$2 = 2^1$

$4 = 2^2$

2^i

$\vdots$

At each level the argument decreases by 1

$n - k(1) \leq 0$

$n \leq k$

$\therefore$ height $= n$

$\therefore$ # leaves $= 2^n$

Thus, $T(n) = \sum_{i=0}^{n-1} 2^i + c.2^n$

$= \dfrac{2^n - 1}{2-1} + c.2^n$

$= 2^n - 1 + c.2^n$

$= \Theta(2^n)$