Applications of Algorithms

TEST

25 August 2023

Question 1. (5 *marks*)

Let $f(n)$, $g(n)$ and $h(n)$ be asymptotically nonnegative functions.
Use the definitions of $\mathcal{O}$ and o to prove that the following are true:

(a) (2 *marks*) If $f(n) = \mathcal{O}(g(n))$, then $f(n)^3 = \mathcal{O}\left(g(n)^3\right)$.

(b) (3 *marks*) If $f(n) = \mathcal{O}(h(n))$ and $g(n) = o(h(n))$, then $f(n) + g(n) = O(h(n))$.

Question 2. (5 *marks*)

Use the **substitution method** to show that the following recurrence has a solution in $\mathcal{O}(n \log n)$:

$$T(n) = T(\tfrac{n}{3}) + T(\tfrac{2n}{3}) + n.$$

Question 3. (5 *marks*)

Use a **recursion tree** to solve the following recurrence (you may assume that n is a power of 3):

$$T(n) = 3\,T(\tfrac{n}{3}) + n \log n.$$

Question 4. (5 *marks*)

Consider the following recursive algorithm:

$\textsc{DivideAndConquer}(A, p, q)$

input: an array A with $A.length = n$ and two indices p and q
output: some value

$\ell = q - p + 1$
if $\ell \leq 2$ **return** 0
$r = \lfloor \frac{\ell}{3} \rfloor$
$c_1 \leftarrow \textsc{DivideAndConquer}(A, p, p + r)$
$c_2 \leftarrow \textsc{DivideAndConquer}(A, q - r, q)$
$c_3 \leftarrow \textsc{Combine}(A, \ell, r, c_1, c_2)$
return c_3

(a) (2 *marks*) Given that $\textsc{Combine}$ has running time $\Theta(n)$, give a recurrence for the running time of $\textsc{DivideAndConquer}$.

(b) (3 *marks*) Use the **substitution method** to show that the recurrence in (a) has a solution in $\mathcal{O}(n)$.

Question 5. (6 *marks*)

Consider the following problem:

Given an array of integers, find the maximum contiguous subarray containing only positive values.

(a) (3 *marks*) Give pseudocode for a **recursive algorithm** for solving this problem.
(Your algorithm must divide the given array section approximately in half at each call.)

(b) (3 *marks*) Give a recurrence for the running time of the algorithm in (a) and solve it using the **Master method**. (You can ignore the floor and ceiling brackets when solving the recurrence.)

Question 6. (4 *marks*)

Consider the following pseudocode for the Hire-Assistant problem.

```
HIRE-ASSISTANT(n)
best = 0
for i = 1 to n
    interview candidate i
    if candidate i is better than candidate best
        best = i
        hire candidate i
```

Let X_i be the indicator random variable for the event that candidate i is hired, and X the indicator random variable for the total number of hires.

Show that $E[X] = \ln n + \mathcal{O}(1)$. Explain all your steps.

Total marks: 30