17.2 The accounting method

The i^{th} operation cost c_i in a sequence is assigned an amortized cost $\hat{c}_i$.

$\hat{c}_i$ can be more or less than c_i, but

$$\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i$$

for any sequence of n operations.

stack operations

	cost	amortized cost
Push	1	2
Pop	1	0
Multipop	$\min(s,k)$	0.

The amortized cost of Push is 2 which includes the cost of Push and the future cost of popping the object - we have assigned a credit to the object.

Thus the cost of pop and Multipop is 0.

The cost of n operations is then $O(n)$.

Incrementing a binary counter

charge 2 for flipping a 0 to 1
and 0 for flipping a 1 to 0

when a 0 is flipped to 1 we have a credit for when it needs to be flipped back to 0.

				cost
0	0	0	0	
0	0	0	1	2
0	0	1	0	2
0	0	1	1	2
0	1	0	0	2
	⋮			

Only one $0 \to 1$ flip occurs in an increment so cost of n operations is $O(2n) = \underline{O(n)}$

17.3 The potential method.

A potential function is a way of obtaining amortized costs for operations.

A potential function Φ assigns a value to a data structure D.

Let D_0 denote initial data structure & D_i the data structure after i operations.

then $\Phi(D_i)$ is a real number for each i.

We think of $\Phi(D_i)$ as the 'potential' of the data structure.

Given Φ, define the amortized cost of operation i is:

$$\hat{c}_i = c_i + \underbrace{\Phi(D_i) - \Phi(D_{i-1})}_{\text{change in Potential}}$$

Recall we want $\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i \quad \forall n$

so $\sum_{i=1}^{n} (c_i + \Phi(D_i) - \Phi(D_{i-1})) \geq \sum_{i=1}^{n} c_i$

$\therefore \sum_{i=1}^{n} (\Phi(D_i) - \Phi(D_{i-1})) \geq 0$

telescoping series gives :

$$\Phi(D_n) - \Phi(D_0) \geq 0$$

$$\Phi(D_n) \geq \Phi(D_0)$$

Thus, a requirement for a potential function Φ is that $\Phi(D_n) \geq \Phi(D_0)$ for all n.

stack operations

Let D be the stack and let
$\Phi(D_i)$ be the number of objects on the stack after the i^{th} operation.

Since we start from an empty stack,
$\Phi(D_0) = 0$ and so $\Phi(D_n) \geq \Phi(D_0)$

If the i^{th} operation is a push then

$$\hat{c}_i = c_i + \underbrace{\Phi(D_i) - \Phi(D_{i-1})}$$

$$\hat{c}_i = c_i + \underbrace{\Phi(D_i) - \Phi(D_{i-1})}$$

$$= 1 + 1$$

$$= 2$$

If the i^{th} operation is a pop then

$$\hat{c}_i = 1 + (-1) = 0$$

If the i^{th} operation is a pop then

$$\hat{c}_i = 1 + (-1) = 0$$

If the i^{th} operation is a multipop then

$$\hat{c}_i = \min(s, k) + (-\min(s, k)) = 0$$

Incrementing a binary counter

Let D be the boolean array.

Let $\Phi(D_i)$ be the number of 1's in D after the i^{th} increment.

The amortized cost of the i^{th} increment is

$$\hat{c}_i = c_i + \underbrace{\Phi(D_i) - \Phi(D_{i-1})}$$

$$\leq (t_i + 1) + \underbrace{(1 - t_i)} = 2$$

$$\hat{c}_i = c_i + \underbrace{\Phi(D_i) - \Phi(D_{i-1})}$$

$$\leq (t_i + 1) + \underbrace{(1 - t_i)} = 2$$

- # of 1s flipped to 0 plus 1.

# of 1s flipped to 0

flipping 0 to 1

$$1\ 0\ 1\ 0\ 0\ \underbrace{1\ 1\ 1\ 1}_{t_i}$$