4.3 Substitution Method

Solve: $T(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2T(\lfloor \frac{n}{3} \rfloor) + n^2 & \text{if } n > 0 \end{cases}$

We guess that $T(n) = O(n^2)$
and try to prove it.

We want $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that
$T(n) \leq cn^2$ for all $n \geq n_0$

Assume that $T(m) \leq cm^2$ for all $m \leq n$

SHOW: $T(n) \leq cn^2$

$$T(n) = 2\,T(\lfloor \tfrac{n}{3} \rfloor) + n^2$$
$$\leq 2c\left(\lfloor \tfrac{n}{3} \rfloor\right)^2 + n^2 \qquad \text{(because } \lfloor \tfrac{n}{3} \rfloor < n\text{)}$$
$$\leq 2c\left(\tfrac{n}{3}\right)^2 + n^2$$
$$= \left(\tfrac{2}{9}c + 1\right)n^2$$

We want $c \in \mathbb{R}^+$ such that

$$\left(\frac{2}{9}c + 1\right)n^2 \leq cn^2$$

$$\Longleftrightarrow \quad \frac{2}{9}c + 1 \quad\quad \leq \quad c$$

$$\Longleftrightarrow \quad 1 \quad\quad \leq \quad \frac{7}{9}c$$

$$\Longleftrightarrow \quad \frac{9}{7} \quad \leq \quad c$$

For the base case, consider $n = 0$:

$$T(0) = 1 \leq c(0)^2$$

$$?!$$

$$\therefore \quad 1 \leq 0$$

Doesn't work for any $c > 0$.

We can choose a different base case

$$T(1) = 2T(\lfloor \tfrac{1}{3} \rfloor) + 1^2$$
$$= 2T(0) + 1^2$$
$$= 2(1) + 1^2$$
$$= 3$$

We want $T(1) \leq c(1)^2$

$$\therefore \quad 3 \leq c$$

Note: $T(2) = 2T(\lfloor \frac{2}{3} \rfloor) + 2^2$

$= 2T(0) + 2^2$

$= 2(1) + 2^2$

$= 6$

We want $T(2) \le c(2)^2$

$6 \le 4c$

$\therefore c = 3$ is okay

So the base cases are $n=1$ and $n=2$

& $T(n) \leq 3n^2$ for all $n \geq 1$

$\therefore \quad T(n) = O(n^2)$