5.2-1

Probability of hiring exactly once:
this occurs if the best candidate is interviewed first, which occurs with probability $\frac{1}{n}$.

Probability of hiring n times:
this occurs if the candidates are interviewed in ascending order, which occurs with probability $\frac{1}{n!}$.

(There are $n!$ permutations. the permutation with candidates in sorted ascending order is just 1 of them.).

5.2 - 2

Probability of hiring exactly two times:

list of candidates:

$$\underbrace{c_1, c_2, c_3, \ldots c_{k-1}}_{\substack{\text{Want exactly} \\ \text{1 hire} \\ \text{before } c_k}} \; c_k \text{ ——— }, c_n$$

↑ best candidate.

No more hires after this

$$\text{Prob} = \frac{1}{k-1}$$

Note: best could occur in any position with prob $\frac{1}{n}$.

Probability of 2 hires

$$= \sum_{k=2}^{n} P(\text{best is in position } k) \cdot P(\text{one hire before } k).$$

$$= \sum_{k=2}^{n} \frac{1}{n} \cdot \frac{1}{k-1}$$

$$= \frac{1}{n} \sum_{k=2}^{n} \frac{1}{k-1}$$

$$= \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k} \approx \frac{1}{n}\left(\ln(n-1) + O(1) \right).$$

$$\approx \frac{\ln n}{n}$$

5.2-4 Hat-check Problem

Let X_i be the indicator random variable:

$$X_i = I\{\text{customer } i \text{ gets their own hat}\}.$$

Let X be random variable for the number of customers that get their own hat back.

Then $X = X_1 + X_2 + \cdots + X_n$.

$$E[X] = E[X_1 + X_2 + \cdots + X_n]$$

$$= E[X_1] + E[X_2] + \cdots + E[X_n]$$

$$= Pr\{X_1\} + Pr\{X_2\} + \cdots + Pr\{X_n\}.$$

$$Pr\{X_1\} = \frac{1}{n}$$

$$Pr\{X_2\} = \left(\frac{n-1}{n}\right)\left(\frac{1}{n-1}\right) = \frac{1}{n}$$

$$Pr\{X_3\} = \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right)\left(\frac{1}{n-2}\right) = \frac{1}{n}$$

$$\vdots$$

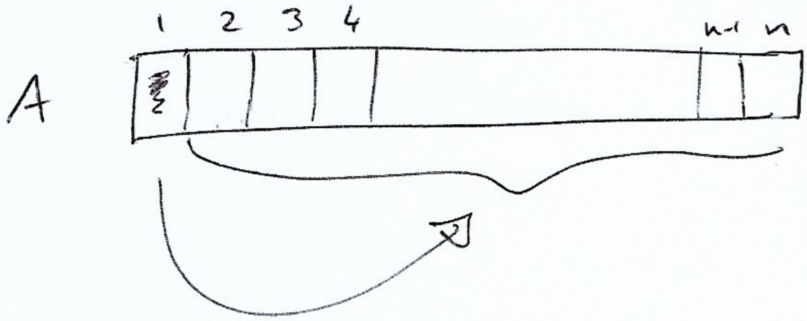
$$Pr\{X_n\} = \cdots = \frac{1}{n}$$

$$\therefore E[X] = \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}$$

$$= n\left(\frac{1}{n}\right) = \underline{\underline{1}}$$

5.3 -2

```
Permute-without-identity (A)
  for i = 1 to n-1
     swap A[i] with A[Random(i+1, n)]
```



The value in position 1 will never remain in position 1.

Same for other positions.

There are many permutations of A that cannot happen.

This algorithm fails to produce a uniform random permutation since not all permutations are equally likely to occur.

5.3 - 3

Permute-with-all (A)

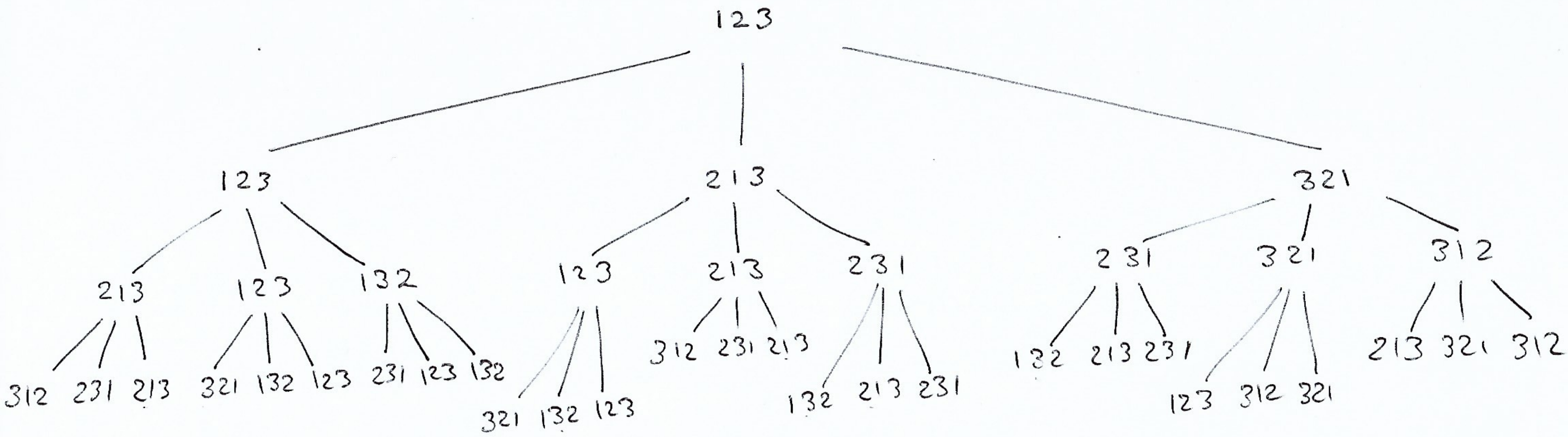
for $i = 1$ to n
    swap $A[i]$ with $A[\text{Random}(1, n)]$.

A 

Are all permutations equally likely?

Consider an array A $\boxed{1 \mid 2 \mid 3}$

Draw the tree of all possible outcomes:



$\Pr\{123\} = 4/27$

$\Pr\{132\} = 5/27$

$\Pr\{213\} = 5/27$

$\Pr\{231\} = 5/27$

$\Pr\{312\} = 4/27$

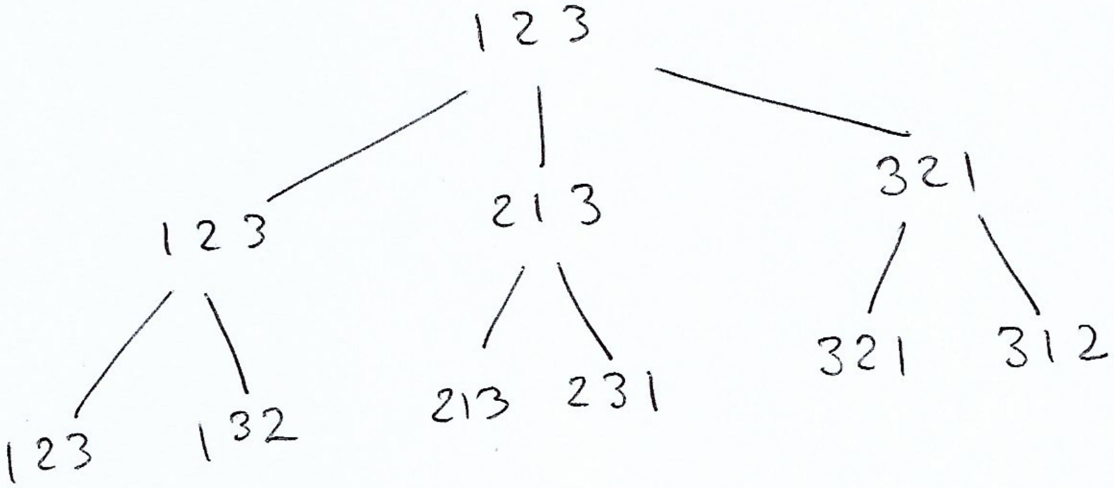
$\Pr\{321\} = 4/27$

So not all permutations are equally likely.

Randomize-in-place (A)

for $i = 1$ to $n-1$

swap $A[i]$ with $A[\text{Random}(i,n)]$



All permutations have $\frac{1}{6}$ probability of occurring.