3.1-2

Show $(n+a)^b = \Theta(n^b)$, $\quad a,b \in \mathbb{R}$, $b > 0$

Proof: We must find $c_1, c_2 \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that

$$c_1 n^b \leq (n+a)^b \leq c_2 n^b$$

for all $n \geq n_0$.

Case 1. Suppose $a \geq 0$.

Then $n^b \leq (n+a)^b$ for all $n \geq 0$

So we take $c_1 = 1$.

For c_2, solve: $(n+a)^b \leq c_2 n^b$

$$\iff c_2 \geq \left(1 + \frac{a}{n}\right)^b$$

We know $\lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^b = 1$

So we can find $n_0 \in \mathbb{N}$ such that $\left(1 + \frac{a}{n}\right)^b \leq 2$ for all $n \geq n_0$.

and then we choose $c_2 = 2$.

Case 2. If $a < 0$, ... similar

3.1-4

Is $2^{n+1} = O(2^n)$?

Can we find $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ s.t.

$2^{n+1} \leq c \cdot 2^n$ for all $n \geq n_0$?

$\Leftrightarrow \quad c \geqslant 2$.

Yes, choose $c = 2$ and $n_0 = 0$.

Is $2^{2n} = O(2^n)$?

Can we find $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ s.t.

$2^{2n} \leq c \cdot 2^n$ for all $n \geq n_0$?

$\Leftrightarrow \quad c \geqslant \frac{2^{2n}}{2^n} = 2^n$.

No, there is no $c \in \mathbb{R}^+$ for which

$c \geq 2^n$ for all $n \geqslant n_0$

(for any n_0).

Thus $2^{2n} \neq O(2^n)$

3.2 - 2

Prove $a^{\log_b c} = c^{\log_b a}$

Set $\log_b c = x$

then $c = b^x$

so $c^{\log_b a} = (b^x)^{\log_b a}$

$= b^{x \log_b a}$

$= (b^{\log_b a})^x$

$= a^x$

$= a^{\log_b c}$

3.2 - 3

Prove $\log(n!) = \Theta(n \log n)$ (3.19)

Stirling's approx:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\alpha_n}$$

where $\frac{1}{12n+1} < \alpha_n < \frac{1}{12n}$.

Thus,

$$\log(n!) = \log\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\alpha_n}\right)$$

$$= \frac{1}{2}\log 2\pi + \frac{1}{2}\log n + n\log n - n\log e + \alpha_n \log e$$

$$= \Theta(n \log n)$$

Note: We can easily get $\log(n!) = O(n \log n)$:

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

$$\leq n\, n\, n \cdots n\cdot n\cdot n$$

$$= n^n$$

$\therefore \log(n!) \leq \log(n^n) = n \log n$.

But $\log(n!) = \Omega(n \log n)$ is harder.

Problem 3.4 T or F: For all asymptotically positive functions $f(n)$ and $g(n)$,

a. $f(n) = O(g(n)) \Rightarrow g(n) = O(f(n))$

False: choose $f(n) = n$ and $g(n) = n^2$.

Then $f(n) = O(g(n))$ since $n \leq n^2$ for $n \geq 0$.

But $g(n) \neq O(f(n))$ ie $n^2 \neq O(n)$

($n^2 \leq cn$ for $n \geq n_0$

$\Leftrightarrow$ $\underline{c \geq n}$ for $n \geq n_0$

not possible.

d. $f(n) = O(g(n)) \Rightarrow 2^{f(n)} = O(2^{g(n)})$

True? Suppose $f(n) = O(g(n))$

Then $f(n) \leq c\,g(n)$ for some $c \in \mathbb{R}^+$ and all $n \geq n_0$.

Then $2^{f(n)} \leq 2^{c\,g(n)} \leq d.\, 2^{g(n)}$

?

False ?

d. Can we find $f(n)$, $g(n)$ such that

$f(n) = O(g(n))$ but $2^{f(n)} \neq O(2^{g(n)})$?

Try $f(n) = 2n$ and $g(n) = n$.

$2n = O(n)$ ✓ True

$2^{2n} \neq O(2^n)$ ✓ True (see exercise 3.1-4)

So d. is False.

e. False

g. False

h. True