17.4 Dynamic Tables

A dynamic table is a data-structure such as an array, stack, heap, etc. that can expand when full and contract when under-utilised.

We will think of a table as an array.

For a table T, the load factor of T

is $\alpha(T) = \dfrac{\#\text{ items stored in } T}{\text{capacity of } T} = \dfrac{T.num}{T.size}$

so $0 \le \alpha(T) \le 1$

First, let's consider Table - Expansion

If the table is full and we want to insert a new item, then we declare a new table, double the size of the current table, copy existing items across and insert new item.

Table-Insert(T, x)

1   if $T.size = 0$
    allocate $T.table$ with 1 slot
    $T.size = 1$

if $T.num = T.size$
    allocate new-table with $2 \cdot T.size$ slots
    insert all items in $T.table$ in new-table
    free $T.table$
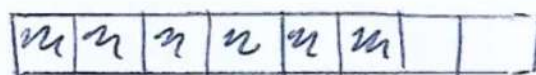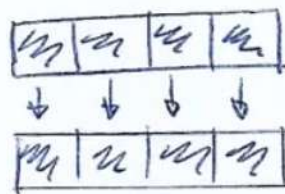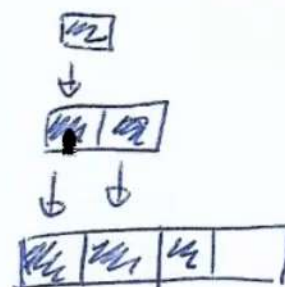 $T.table = $ new-table
    $T.size = 2 \cdot T.size$

insert x into $T.table$

insert x into T.table

T.num = T.num + 1

	Table	Cost
1st insert	$[m_1]$	1
2nd insert	$[m_1 \mid m_2]$	2
3rd insert	$[m_1 \mid m_2 \mid m_3 \mid \ ]$	3
4th insert	$[m_1 \mid m_2 \mid m_3 \mid m_4]$	1
5th insert	$[m_1 \mid m_2 \mid m_3 \mid m_4 \mid m_5 \mid \ \mid \ \mid \ ]$	5
6th insert	$[m_1 \mid m_2 \mid m_3 \mid m_4 \mid m_5 \mid m_6 \mid \ \mid \ ]$	1

etc.

what is the cost of n Table-Inserts?

$$c_i = \begin{cases} i & \text{is } i-1 \text{ is a power of 2} \\ 1 & \text{otherwise} \end{cases}$$

Thus, $\displaystyle\sum_{i=1}^{n} c_i = \underbrace{\sum_{k=0}^{\lfloor \log n \rfloor} (2^k + 1)}_{\substack{\text{when } i-1 = 2^k \\ \text{so } i = 2^k + 1}} + \underbrace{n - \lfloor \log n \rfloor - 1}_{\substack{\text{when } i-1 \text{ is} \\ \text{not a power of 2}}}$

Thus,
$$\sum_{i=1}^{n} c_i = \underbrace{\sum_{k=0}^{\lfloor \log n \rfloor} (2^k + 1)}_{\text{when } i-1 = 2^k} + \underbrace{n - \lfloor \log n \rfloor - 1}_{\text{when } i-1 \text{ is not a power of } 2}$$

when $i-1 = 2^k$
so $i = 2^k + 1$

when $i-1$ is not a power of 2

$$= \sum_{k=0}^{\lfloor \log n \rfloor} 2^k + (\lfloor \log n \rfloor + 1) + n - \lfloor \log n \rfloor - 1$$

$$= \sum_{k=0}^{\lfloor \log n \rfloor} 2^k + n$$

$$= \frac{2^{\lfloor \log n \rfloor + 1} - 1}{2 - 1} + n$$
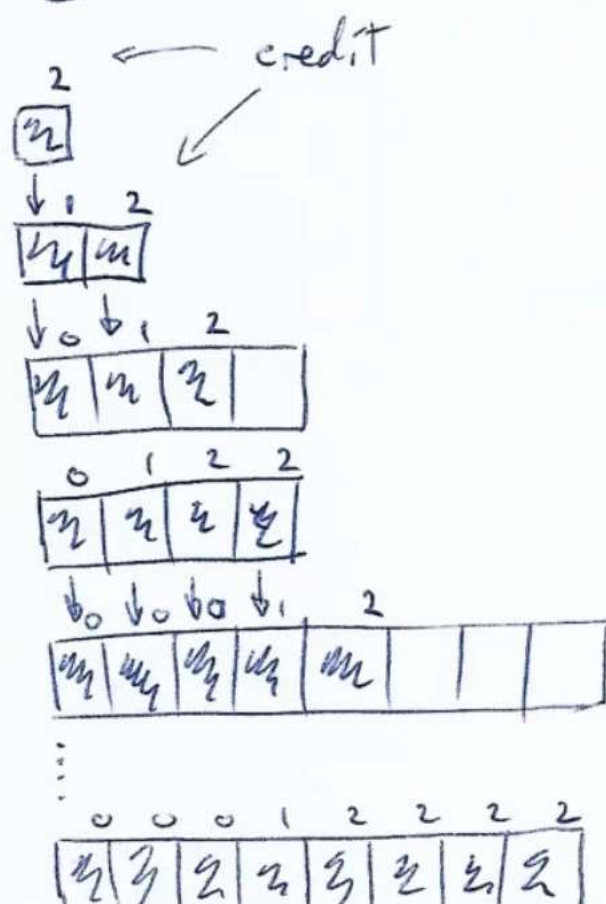
$$= 2 \cdot 2^{\lfloor \log n \rfloor} + n - 1$$

$$\leq 2 \cdot 2^{\log n} + n$$

$$= 2n + n$$

Thus, the amortized cost of each

Table-Insert is $\leq \dfrac{3n}{n} = 3$.

Thinking in terms of the accounting metho...

etc.

In terms of the potential method :

Set $\Phi(T) = 2 \cdot T.num - T.size$

(Note : $\Phi(T) \geq 0$, $\Phi(T_0) = 0$.)

Then the amortized cost of Table-Insert is

$$\hat{c}_i = c_i + \Phi(T_i) - \Phi(T_{i-1})$$

Case 1 :   No expansion

$$\hat{c}_i = \underline{1} + \underline{(2\,num_i - size_i)} - \underline{(2\,num_{i-1} - size_{i-1})}$$

$$= 1 + 2\underline{(num_{i-1} + 1)} - size_{i-1} - 2num_{i-1} + size_{i-1}$$

$$= 3$$

Case 2:   Table expansion   ($i-1$ is power of 2)

$$\hat{c}_i = i + (2num_i - size_i) - (2num_{i-1} - size_{i-1})$$

$$= i + 2(num_{i-1} + 1) - 2size_{i-1} - 2num_{i-1} + size_{i-1}$$

$$= i + 2 - size_{i-1}$$

$$= i + 2 - (i-1)$$

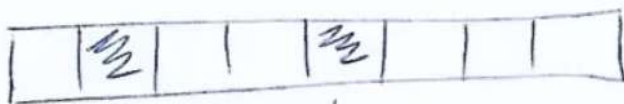$$= 3$$

Table expansion and contraction

Suppose we want to support a Table-Delete
operation and to contract the .
table when its load factor goes
below a certain value.

Contracting when the table is half-full,
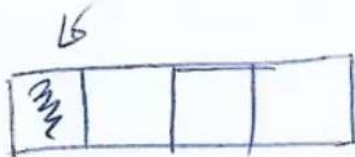
ie. $\alpha(T) = \frac{1}{2}$

is a bad idea   -   see the textbook -

We will contract the table by half
when the load factor goes below $\frac{1}{4}$,
i.e, when $\qquad \alpha(T) = \dfrac{T.num}{T.size} < \dfrac{1}{4}$.

e.g.

Delete

$\alpha(T) = \dfrac{T.num}{T.size} = \dfrac{2}{8} = \dfrac{1}{4}$

$\alpha(T) = \dfrac{T.num}{T.size} = \dfrac{1}{4}$

If $\alpha(T) = 0$ we discard Table ___

Consider a sequence of n Insert and/or Delete operations.

In order to calculate the amortized cost per operations, we use the potential functions and amortized cost analysis of earlier sections.

We use the following potential function:

$$\Phi(T) = \begin{cases} 2\,T.num - T.size & \text{if } \alpha(T) \geq \frac{1}{2} \\ \dfrac{T.size}{2} - T.num & \text{if } \frac{1}{4} \leq \alpha(T) < \frac{1}{2} \end{cases}$$

check that $\Phi(T) \geq 0$ and $\Phi(T_0) = 0$.

Recall: $\hat{c}_i = c_i + \Phi(T_i) - \Phi(T_{i-1})$

We show that $\hat{c}_i \leq 3$ regardless

of whether c_i is Insert or Delete.

For Insert, if $\alpha(T) \geqslant \frac{1}{2}$ then we show as before that $\hat{c}_i = 3$.

Consider the case that c_i is Insert and that $\frac{1}{4} \leqslant \alpha(T) < \frac{1}{2}$

After the Insert it may be that $\alpha(T) \geqslant \frac{1}{2}$ or that $\alpha(T) < \frac{1}{2}$. There are 2 cases:

$$\alpha(T_{i-1}) < \frac{1}{2}$$

(i) c_i is insert $\alpha(T_i) \geqslant \frac{1}{2}$. Then

$$\hat{c}_i = c_i + \bar{\Phi}(T_i) - \bar{\Phi}(T_{i-1})$$

$$= \underline{1} + (2\,num_i - size_i) - \left(\frac{size_{i-1}}{2} - num_{i-1}\right)$$

$$= 1 + (2(num_{i-1} + 1) - size_{i-1}) - \left(\frac{size_{i-1}}{2} - num_{i-1}\right)$$

$$= 3 + 3\,num_{i-1} - \frac{3}{2}\,size_{i-1}$$

$$= 3 + 3\left(\underbrace{num_{i-1} - \tfrac{1}{2} size_{i-1}}\right)$$

$$\text{since} \quad \alpha(T_{i-1}) = \frac{num_{i-1}}{size_{i-1}} < \tfrac{1}{2}$$

$$\leq 3$$

$$\therefore \quad num_{i-1} < \tfrac{1}{2} size_{i-1}$$

$$\therefore \quad num_{i-1} - \tfrac{1}{2} size_{i-1} < 0$$

For Delete, there are 4 cases to consider:

(i) $\alpha(T_{i-1}) \geq \frac{1}{2}$ and $\alpha(T_i) \geq \frac{1}{2}$

(ii) $\alpha(T_{i-1}) \geq \frac{1}{2}$ and $\alpha(T_i) < \frac{1}{2}$

(iii) $\frac{1}{4} < \alpha(T_{i-1}) < \frac{1}{2}$ and $\frac{1}{4} \leq \alpha(T_i) < \frac{1}{2}$

— & no contraction occurred.

(iv) $\frac{1}{4} = \alpha(T_{i-1}) < \frac{1}{2}$ and contraction occurred

& then $\frac{1}{4} \leq \alpha(T_i) < \frac{1}{2}$

(iv) $\quad \hat{c}_i = \underline{c_i} + \overline{\Phi}(T_i) - \tilde{\Phi}(T_{i-1})$

$\quad = \underline{num_{i-1}} + \left(\frac{size_i}{2} - \underline{num_i} \right) - \left(\frac{size_{i-1}}{2} - \underline{num_{i-1}} \right)$

$\quad = num_{i-1} + \frac{size_{i-1}}{4} - \frac{size_{i-1}}{2} + 1$

$\quad = num_{i-1} - \frac{1}{4} size_{i-1} + 1$

$\quad = 0 + 1 \quad$ since $\quad \dfrac{\underline{num_{i-1}}}{size_{i-1}} = \alpha(T_{i-1}) = \underline{\frac{1}{4}}$