

# Introduction to Electromagnetism

## 1 Coulomb's Law



$$\vec{F}_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1 Q}{r_1^2} \hat{r}_1$$

If there was another charge  $q_2$  at  $\vec{r}_2$ :



The force on  $Q$  due to  $q_2$  is given by:

$$\vec{F}_2 = \frac{1}{4\pi\epsilon_0} \frac{q_2 Q}{r_2^2} \hat{r}_2$$

And the net force on  $Q$  is given by:

$$\vec{F} = \vec{F}_1 + \vec{F}_2$$

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 Q}{r_1^2} \hat{r}_1 + \frac{1}{4\pi\epsilon_0} \frac{q_2 Q}{r_2^2} \hat{r}_2$$

If there were  $n$  charges, the net force on  $Q$  would be:

$$\vec{F} = \sum_{i=1}^n \frac{1}{4\pi\epsilon_0} \frac{q_i Q}{r_i^2} \hat{r}_i$$

$$\vec{F} = Q \sum_{i=1}^n \frac{1}{4\pi\epsilon_0} \frac{q_i}{r_i^2} \hat{r}_i$$

where  $\sum_{i=1}^n \frac{1}{4\pi\epsilon_0} \frac{q_i}{r_i^2} \hat{r}_i$  is the electric field  $\vec{E}$  at  $\vec{R}$  due to the  $n$  charges.

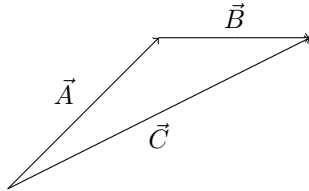
**Field** A value, vector or tensor, that is defined for every point in space and time.

## 2 Vector Calculus

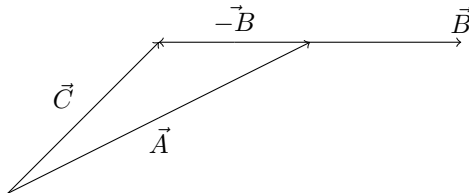
**Vectors** A vector is a quantity that is said to transform like a displacement.

### 2.1 Operations on Vectors

- **Addition:**  $\vec{A} + \vec{B} = \vec{C}$



- **Subtraction:**  $\vec{A} - \vec{B} = \vec{C}$



- **Multiplication by a scalar:**  $c\vec{A} = \vec{C}$
- **Dot Product:**  $\vec{A} \cdot \vec{B} = AB \cos(\theta)$ . Dot product is a scalar. It is commutative and distributive.
- **Cross Product:**  $\vec{A} \times \vec{B} = AB \sin(\theta) \hat{n}$  where  $\hat{n}$  is the unit vector perpendicular to the plane containing  $\vec{A}$  and  $\vec{B}$ .

Cross product is a vector. It is anti-commutative and distributive.

## 2.2 Component form of a Vector:

A vector  $\vec{A}$  can be written as:

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

where  $\hat{i}, \hat{j}, \hat{k}$  are unit vectors along the  $x, y, z$  axes respectively.

Given that  $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$  and  $\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$ , we can perform the following operations:

- **Addition (and Subtraction):**  $\vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$
- **Multiplication by a scalar:**  $c\vec{A} = cA_x\hat{i} + cA_y\hat{j} + cA_z\hat{k}$ , where  $c$  is a scalar.
- **Dot Product:**  $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$
- **Modulus:**  $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$
- **Cross Product:**  $\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$  This can also be written in a determinant form:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

## 2.3 Triple Product:

**Scalar Triple Product:**  $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$  Geometrically the scalar triple product is the volume of the parallelepiped formed by the three vectors.

**Vector Triple Product:**  $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

# 3 Differential Calculus

## 3.1 'Ordinary' Derivative:

$\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$  Geometrically, the derivative is the slope of the tangent to the curve at a point.

### 3.2 Gradient:

Consider a scalar,  $T$ , which exists at every point in space.

$$dT = \left( \frac{\partial T}{\partial x} \right) dx + \left( \frac{\partial T}{\partial y} \right) dy + \left( \frac{\partial T}{\partial z} \right) dz$$

This can be written in the dot product form as:

$$dT = \left( \frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} \right) \cdot (dx \hat{x} + dy \hat{y} + dz \hat{z})$$

this is denoted as:

$$dT = \nabla T \cdot d\vec{r}$$

Here, we treat  $\nabla$  as an operator. It takes a scalar and returns a vector.

**The  $\nabla$  operator:**

$$\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$$

### 3.3 Divergence:

Consider a vector,  $\vec{A}$ , which exists at every point in space. The divergence of  $\vec{A}$  is defined as:

$$\begin{aligned} \nabla \cdot \vec{A} &= \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \\ \nabla \cdot \vec{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \end{aligned}$$

It behaves like a dot product, and is a scalar.

### 3.4 Curl:

Consider a vector,  $\vec{A}$ , which exists at every point in space. The curl of  $\vec{A}$  is defined as:

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

It behaves like a cross product, and is a vector.

### 3.5 Second Derivatives:

- **Divergence of a Gradient:**  $\nabla \cdot (\nabla T)$

$$\begin{aligned} \nabla \cdot (\nabla T) &= \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot \left( \frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} \right) \\ \nabla \cdot (\nabla T) &= \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \end{aligned}$$

It is denoted by  $\nabla^2 T$  and is called the Laplacian of  $T$ .

**Note:** Laplacian of a vector is theoretically not defined. But when  $\nabla^2 \vec{A}$  is written, it means:

$$\nabla^2 \vec{A} = (\nabla^2 A_x) \hat{x} + (\nabla^2 A_y) \hat{y} + (\nabla^2 A_z) \hat{z}$$

- **Curl of a Gradient:**  $\nabla \times (\nabla T)$

$$\nabla \times (\nabla T) = 0$$

Curl of a gradient is always zero.

- **Gradient of Divergence:**  $\nabla (\nabla \cdot \vec{A})$

- **Divergence of Curl:**  $\nabla \cdot (\nabla \times \vec{A})$

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

Divergence of a Curl is always zero.

- **Curl of Curl:**  $\nabla \times (\nabla \times \vec{A})$

$$\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

## 4 Integral Calculus

### 4.1 Line Integral:

Consider a vector field  $\vec{V}$  and a curve  $C$  joining points  $a$  and  $b$ . The line integral of  $\vec{V}$  along  $C$  is defined as:

$$\int_{aP}^b \vec{V} \cdot d\vec{l}$$

where  $d\vec{l}$  is the differential displacement along the curve  $C$ .

when  $C$  is a closed figure, it is written as:

$$\oint \vec{V} \cdot d\vec{l}$$

### 4.2 Surface Integral:

Consider a vector field  $\vec{V}$  and a surface  $S$ . The surface integral of  $\vec{V}$  over  $S$  is defined as:

$$\int_S \vec{V} \cdot d\vec{a}$$

where  $d\vec{a}$  is the differential area vector of the surface  $S$ .

When  $S$  is a closed surface:

$$\oint \vec{V} \cdot d\vec{a}$$

### 4.3 Volume Integral:

Consider a scalar field  $T$  and a volume  $V$ . The volume integral of  $T$  over  $V$  is defined as:

$$\int_V T dV$$

If we consider a vector field  $\vec{A}$ , then:

$$\int \vec{A} dV$$
$$\hat{x} \int A_x dV + \hat{y} \int A_y dV + \hat{z} \int A_z dV$$

In Cartesian coordinates:

$$dV = dx dy dz$$

In Spherical Polar coordinates:

$$dV = r^2 \sin(\theta) dr d\theta d\phi$$

### 4.4 Fundamental Theorem of Calculus:

If  $f(x)$  is a continuous function and  $F(x)$  is the anti-derivative of  $f(x)$ , then:

$$\int_a^b f(x) = F(b) - F(a)$$

- **Fundamental Theorem for Gradients:**

Consider a scalar field  $T$  and a curve  $C$ , from  $\vec{a}$  to  $\vec{b}$ , then:

$$\int_{\vec{a}}^{\vec{b}} \nabla T \cdot d\vec{l} = T(\vec{b}) - T(\vec{a})$$

If the curve is a closed curve, then:

$$\int_{\vec{a}}^{\vec{b}} \nabla T \cdot d\vec{l} = 0$$

- **Fundamental Theorem for Divergence:** a.k.a Gauss' Theorem, Green's Theorem or divergence theorem. Consider a vector field  $\vec{A}$  and a volume  $V$ , then:

$$\int_V (\vec{\nabla} \cdot \vec{v}) = \oint \vec{v} \cdot d\vec{s}$$

where  $ds$  is the differential area vector of the surface  $S$ .

- **Fundamental Theorem for Curl:** a.k.a Stokes' Theorem. Consider a vector field  $\vec{A}$  and a surface  $S$ , then:

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot \vec{da} = \oint_C \vec{v} \cdot d\vec{l}$$

where  $d\vec{l}$  is the differential displacement vector along the boundary curve  $C$  of the surface  $S$ .

## 5 Dirac Delta Function:

### 5.1 One-Dimensional Dirac Delta Function:

It is a function that is zero everywhere except at  $x = 0$  and has an integral of 1. It is defined as:

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

Mathematically, it is not a function, i.e., don't tell your maths profs that its a function.

### 5.2 Three-Dimensional Dirac Delta Function:

It is a function that is zero everywhere except at  $\vec{r} = \vec{r}_0$  and has a volume integral of 1. It is defined as:

$$\delta^3(\vec{r}) = \begin{cases} \infty & (x, y, z) = (0, 0, 0) \\ 0 & \text{everywhere else} \end{cases}$$

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

and

$$\int_{\text{all space}} \delta^3(\vec{r}) d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z) dx dy dz$$

Generalizing the function:

$$\int_{\text{all space}} f(r) \delta^3(r - a) d\tau = f(a)$$

## 6 Electrostatics

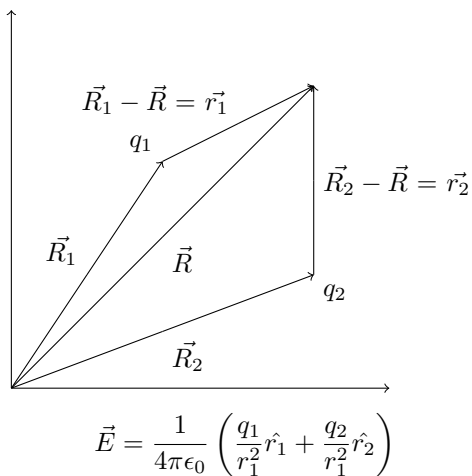
### 6.1 Electric Field



The Electric Field at  $\vec{R}$  due to  $q_1$  is given by:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_1^2} \hat{r}_1$$

If there is another charge present:



$$\vec{E} = \frac{1}{4\pi\epsilon_0} \left( \frac{q_1}{r_1^2} \hat{r}_1 + \frac{q_2}{r_2^2} \hat{r}_2 \right)$$

**Principle of Superposition:**

The net electric field at  $\vec{R}$  due to  $n$  charges is given by:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \left( \sum_{i=1}^n \frac{q_i}{r_i^2} \hat{r}_i \right)$$



**Question:**



Consider two charges,  $q$ , placed at  $r = \pm \frac{d}{2}$  respectively. Find the electric field at  $z$ .

**Answer**

$$\begin{aligned}\vec{E} &= \frac{1}{4\pi\epsilon_0} \left( \frac{q}{\left(\frac{d}{2}\right)^2 + z^2} \hat{r}_1 + \frac{q}{\left(\frac{d}{2}\right)^2 + z^2} \hat{r}_2 \right) \\ \vec{E} &= \frac{1}{4\pi\epsilon_0} \left( \frac{2q}{z^2 + \frac{d^2}{4}} \right) \left( \frac{z}{\sqrt{z^2 + \frac{d^2}{4}}} \hat{z} \right) \\ \vec{E} &= \frac{1}{4\pi\epsilon_0} \left( \frac{2qz}{\left(z^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}} \right) \hat{z}\end{aligned}$$

Now, if  $z$  is very large (but not infinity):

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \left( \frac{2q}{z^2} \right) \left( 1 + \frac{d^2}{4z^2} \right)^{-\frac{3}{2}} \hat{z}$$

this can be approximated to:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \left( \frac{2q}{z^2} \right) \left( 1 - \frac{3d^2}{8z^2} \right) \hat{z}$$

Similar methods can be used to find the electric field for any discrete charge distribution.

## 6.2 Continuous Charge Distribution

### 6.2.1 Linear Charge Distribution

Consider a linear Charge distribution,  $\lambda$  of length  $l$ , along the  $x$  axis. The electric field at point  $P$  is given by:



The Electric Field at point  $P$  due to  $dx$  is given by:

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\lambda dx}{r^2} \hat{r}$$

Hence, the electric field at point  $P$  due to the entire linear charge distribution is given by:

$$\vec{E} = \int_{\text{length } l} \frac{1}{4\pi\epsilon_0} \frac{\lambda dx}{r^2} \hat{r}$$

### 6.2.2 Surface Charge Distribution

Consider a surface charge distribution,  $\sigma$  of area  $A$ , on the  $xy$  plane. The electric field at point  $P$  is given by:



The Electric Field at point  $P$  due to  $dA$  is given by:

$$d\vec{E} = \frac{1}{4\pi\epsilon} \frac{\sigma dA}{r^2} \hat{r}$$

Hence, the electric field at point  $P$  due to the entire surface charge distribution is given by:

$$\vec{E} = \int_{\text{area } A} \frac{1}{4\pi\epsilon} \frac{\sigma dA}{r^2} \hat{r}$$

### **6.2.3 Volume Charge Distribution**