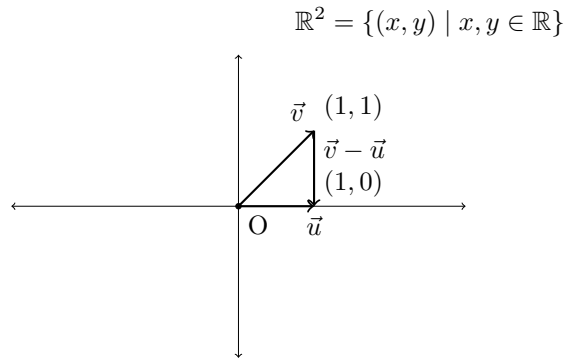


Calculus II

1 Vector Calculus

\mathbb{R} represents the set of real numbers.

\mathbb{R}^2 represents a 2 dimensional real plane.



Normally elements of \mathbb{R} are known as scalars.

We can

- add (or subtract) two vectors
- if $c \in \mathbb{R}$ and $v \in \mathbb{R}^2$, $c\vec{v}$

Dot Product

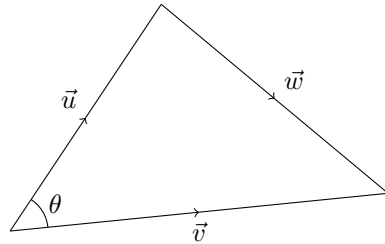
$u = (u_1, u_2)$ and $v = (v_1, v_2)$

$$u.v = u_1.v_1 + u_2.v_2$$

Theorem

$$u.v = |u||v| \cos(\theta)$$

where θ is the angle between \vec{u} and \vec{v} and $|u|$ is the length of vector \vec{u}



$$w^2 = u^2 + v^2 - 2|u||v| \cos(\theta) \dots (1)$$

$$\vec{w} = \vec{v} - \vec{u}$$

$$\begin{aligned}
w &= (v_1 - u_1, v_2 - u_2) \\
w^2 &= (v_1 - u_1)^2 + (v_2 - u_2)^2 \\
w^2 &= v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2 \\
w^2 &= v^2 + u^2 - 2v_1u_1 - 2v_2u_2 \dots (2)
\end{aligned}$$

now, as (1) = (2)

$$\begin{aligned}
u^2 + v^2 - 2|u||v|\cos(\theta) &= v^2 + u^2 - 2v_1u_1 - 2v_2u_2 \\
|u||v|\cos(\theta) &= v_1u_1 + v_2u_2 \\
|u||v|\cos(\theta) &= \vec{u} \cdot \vec{v}
\end{aligned}$$

Hence proved.

Extending the above theorem to \mathbb{R}^n : Consider $\vec{u} = (u_1, u_2, \dots, u_n), \vec{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n$$

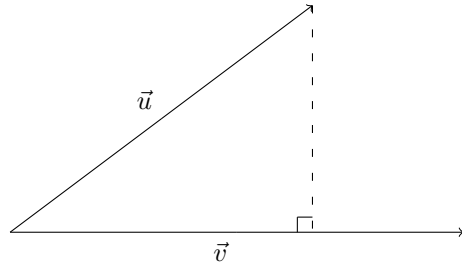
Unit Vectors

If $\vec{v} \in \mathbb{R}^2$ is a vector. then,

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

Projections

If \vec{u} and \vec{v} are vectors in \mathbb{R}^2 and θ is the angle between them:



let \vec{w} be the projection of \vec{u} on \vec{v}

$$\begin{aligned}
\vec{w} &= |u|\cos(\theta)\hat{v} \\
\vec{w} &= \frac{|u|\cos(\theta)|v|\vec{v}}{|v|^2} \\
\vec{w} &= \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}
\end{aligned}$$

Cross Product

Consider the vectors, u, v . Then the cross-product of u and v is defined as:

$$u \times v = |u||v| \sin(\theta) \hat{n}$$

where \hat{n} is the unit vector perpendicular to the plane containing u and v , and also $(\vec{u}, \vec{v}, \hat{n})$ form a right handed system.

Properties of Cross Product:

- $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- $(r\vec{u}) \times (s\vec{v}) = rs\vec{u} \times \vec{v}$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

\vec{u} and \vec{v} can also be represented as:

$$\vec{u} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k}$$

$$\vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$$

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2) \hat{i} + (u_3 v_1 - u_1 v_3) \hat{j} + (u_1 v_2 - u_2 v_1) \hat{k}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

the above formula is only symbolic and meant to represent a cross product.