Tutorial 1

Question 1

We can split the planet into its mantle and core separately, calculate their moment of inertia and then add them together.
For the core:

$$I_c = \frac{2}{5} m_c r_c^2$$

where, I_c = Moment of Inertia of the core, and m_c = Mass of the core.

$$m_c = \frac{4}{3}\pi r_c^3 \rho_c$$

$$\therefore I_c = \frac{8}{15}\pi r_c^5 \rho_c$$

For the mantle:

$$I_m = \frac{2}{5} m_m \frac{R^5 - r_c^5}{R^3 - r_c^3}$$

where, I_c = Moment of Inertia of the mantle, and m_m = Mass of the mantle.

$$m_m = \frac{4}{3}\pi (R^3 - r_c^3)\rho_m$$

$$\therefore I_m = \frac{8}{15}\pi (R^5 - r_c^5)\rho_m$$

$$\therefore I = I_c + I_m = \frac{8}{15}\pi r_c^5 \rho_c + \frac{8}{15}\pi (R^5 - r_c^5)\rho_m$$

or

$$I = \frac{8\pi}{15}\left[\rho_m R^5 + (\rho_c - \rho_m) r_c^5\right]$$

(a)

We know that:

$$I = \frac{8\pi}{15}\left[\rho_m R^5 + (\rho_c - \rho_m) r_c^5\right]$$

$$I = \frac{8\pi}{15}\rho_m R^5 \left[1 + (\rho_c/\rho_m - 1)(r_c/R)^5\right]$$

also:

$$M = \rho_c \frac{4}{3}\pi r_c^3 + \rho_m \frac{4}{3}\pi R^3 - \rho_m \frac{4}{3}\pi r_c^3$$

$$M = \frac{4\pi}{3}\left(\rho_m R^3 + (\rho_c - \rho_m) r_c^3\right)$$

$$M = \frac{4\pi}{3}\rho_m R^3 \left(1 + (\rho_c/\rho_m - 1)(r_c/R)^3\right)$$

$$MR^2 = \frac{4\pi}{3}\rho_m R^5 \left(1 + \left(\rho_c/\rho_m - 1\right)\left(r_c/R\right)^3\right)$$

$$\therefore I/MR^2 = \frac{2}{5}\left[\frac{1 + \left(\rho_c/\rho_m - 1\right)\left(r_c/R\right)^5}{1 + \left(\rho_c/\rho_m - 1\right)\left(r_c/R\right)^3}\right]$$

(b)

For Earth:

$$\rho_c/\rho_m - 1 = 1.4$$
$$r_c/R = 0.547$$

$$I/MR^2 = \frac{2}{5}\left[\frac{1 + 1.4(0.547)^5}{1 + 1.4(0.547)^3}\right]$$
$$I/MR^2 \approx 3.48 \times 10^{-1}$$