Linear Algebra

1 Introduction to Linear Systems

They are system of equations that have variables that are linear. Example:

$$x + y = 2$$

and

$$2x - y = 1$$

Normally the coefficients are real numbers, but they can be complex numbers as well.
System of linear equations of m equations and n variables:

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m$$

What a linear equation in n variables represents is a given space in $\mathbb{R}^n$.

Question: What it the line passing through $(1, 1)$ and $(-1, -3)$?

Answer: We will use first principles to find the equation of the line.
Let the equation of the line be $y = mx + c$.
We know that $(1, 1)$ and $(-1, -3)$ lie on the line.

$$1 = m * 1 + c \ldots (1)$$

$$-3 = m * (-1) + c \ldots (2)$$

Subtracting (2) from (1), we get:

$$4 = 2m$$

$$m = 2$$

Substituting $m = 2$ in (1), we get:

$$1 = 2 + c$$

$$c = -1$$

2 Matrices and Vectors

Matrix: Group of numbers(or equations, expressions, etc.) in rows and columns.

Example:

$$A = \begin{pmatrix} 1 & 0 & -1 & 4 \\ 2 & 9 & 3 & 5 \\ 5 & 2 & 10 & 6 \end{pmatrix}$$

The above matrix has 3 rows and 4 columns.
Entry in 2nd row, 3rd column = 3

It can also be represented as

$$A = (a_{ij})$$

where a_{ij} refers to the element in A at the ith row and jth column

Special Matrices: Some special matrices:

- **Zero Matrix:**

$$A = O_{m\times n} = \begin{bmatrix} 0 & . & . & . & 0 \\ . & & & & . \\ . & & & & . \\ 0 & . & . & . & 0 \end{bmatrix}$$

- **Square Matrix:** here $m = n$

$$A_3 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

- **Identity Matrix:** A square matrix with all diagonal element equal to 1 and non-diagonal elements equal to 0.

$$I_n = \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & . & & & \\ & & & . & & \\ & & & & . & \\ 0 & & & & & 1 \end{bmatrix}$$

- **Diagonal Matrix:** A square matrix with all non-diagonal elements equal to 0.

$$D_n = \begin{bmatrix} d_1 & & & & & 0 \\ & d_2 & & & & \\ & & . & & & \\ & & & . & & \\ & & & & . & \\ 0 & & & & & d_n \end{bmatrix}$$

where d_i are generally non-zero, but not necessarily.

- **Upper Triangular Matrix:** A square matrix with all non-diagonal elements below the diagonal equal to 0

$$B = \begin{bmatrix} b_1 & & & & \star \\ & b_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & d_n \end{bmatrix}$$

where $\star$ can be any number.

- **Lower Triangular Matrix:** A square matrix with all non-diagonal elements above the diagonal equal to 0

$$C = \begin{bmatrix} c_1 & & & & 0 \\ & c_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ \star & & & & c_n \end{bmatrix}$$

where $\star$ can be any number.

Augmented Matrix

Given a system of linear equations:

$$3x_1 - 2x_2 + 4x_3 = 0$$

$$2x_1 + x_2 + 3x_3 = 1$$

$$5x_1 + x_2 - 2x_3 = -1$$

The augmented matrix is:

$$\left[\begin{array}{ccc|c} 3 & -2 & 4 & 0 \\ 2 & 1 & 3 & 1 \\ 5 & 1 & -2 & -1 \end{array}\right]$$

for a system of m equations and n variables:

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m$$

The augmented matrix would be:

$$\left[\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m \end{array}\right]$$

Elementary Row Operations

- Interchange two rows: if R_i and R_j are two rows of a matrix A, then $R_i \leftrightarrow R_j$
- Multiply a row by a constant: if R_i is a row of a matrix A and c is a constant, $cR_i \to R_i$
- Multiply a row by a constant and add it to another row: if R_i and R_j are two rows of a matrix A and c is a constant, $R_i + cR_j \to R_i$

Reduced Row - Echelon Form

A matrix is said to be in reduced row-echelon form if:

- All rows consisting entirely of zeros are at the bottom of the matrix
- The first non-zero entry in each row is a 1 (called a leading 1)
- Each leading 1 is the only non-zero entry in its column
- Each leading 1 is to the right of the leading 1 in the row above it

Example:

- $$\left[\begin{array}{ccccc|c} \mathbf{1} & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & \mathbf{1} & 0 & -1 & 4 \\ 0 & 0 & 0 & \mathbf{1} & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- If the augmented matrix is:

  $$\left[\begin{array}{cccc|c} 1 & -3 & 0 & -5 & -7 \\ 3 & -12 & -2 & -27 & -33 \\ -2 & 10 & 2 & 24 & 29 \\ -1 & 6 & 1 & 14 & 17 \end{array}\right]$$

  Then its reduced row-echelon form is:

  $$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

3 Solutions of Linear Systems

3.1 Number of Solutions

Note: all the following matrices are in reduced row-echelon form.

- **No Solution:** If the system has no solution, then the system is said to be *inconsistent*.
  Eg:
  $$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
  This has no solution as the last row is $0 = 4$ which is not possible.

- **Infinitely Many Solutions:** If the system has infinitely many solutions, then the system is said to be *consistent with infinitely many solutions*.
  Eg:
  $$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
  This has a free variable x_2 and hence has infinitely many solutions.

- **Unique Solution:** If the system has a unique solution, then the system is said to be **consistent with unique solution**.
  Eg:
  $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$
  This has no free variables and hence has a unique solution.

3.2 Rank of A Matrix

The rank of a matrix is the number of leading 1's in the reduced row-echelon form of the matrix.
Eg:

$$\text{rank} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 2$$

$$\text{Reduced Row Echelon Form of } \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & 0 & -1 \\ 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Consider a system of m linear equations in n variables, and the coefficient matrix is A.

- $\text{rank}(A) \leq m$ and $\text{rank}(A) \leq n$

- If the system is inconsistent, then $\text{rank}(A) < m$
- If the system is consistent with unique solution, then $\text{rank}(A) = n$
- If the system is consistent with infinitely many solutions, then $\text{rank}(A) < n$

The contrapositive of the above statements are also true.

4 Vector space

A vector space over $mathbbR$ is a set of Vectors along with rules for addition and scalar multiplication

- u+v = v+u
- u+(v+w) = (u+v)+w
- there exists a zero vector 0 such that u+0 = u
- $\forall u \in V$, there exists a vector -u such that u+(-u) = 0
- c(dv) = (cd)v
- 1v = v
- c(u+v) = cu + cv

5 Linear Transformation

A function $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation if there exists a $n \times m$ matrix A such that $T(\vec{x}) = A\vec{x}$

This shows that T is a function from a vector space to another vector space.

$$T(\vec{x}) = \vec{y}$$
$$\vec{y} = A\vec{x}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_1 1 & a_1 2 & \cdots & a_1 m \\ a_2 1 & a_2 2 & \cdots & a_2 m \\ \vdots & \vdots & \ddots & \vdots \\ a_n 1 & a_n 2 & \cdots & a_n m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

Each $\mathbb{R}^m$ has some special vectors: $e_i \forall i = 1, 2, 3, .., m$.

$$e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

where 1 is in i^{th} position.

Theorem Any linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ of the form T_A where
$A = \begin{bmatrix} | & | & \cdots & | \\ T(e_1) & T(e_2) & \cdots & T(e_m) \\ | & | & \cdots & | \end{bmatrix}$