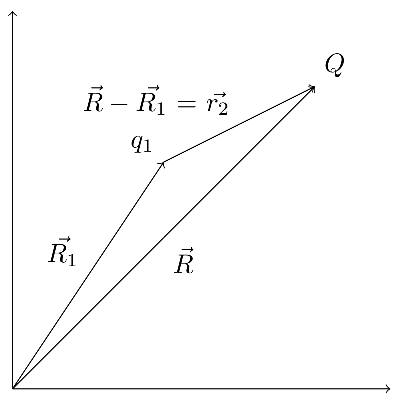
Introduction to Electromagnetism

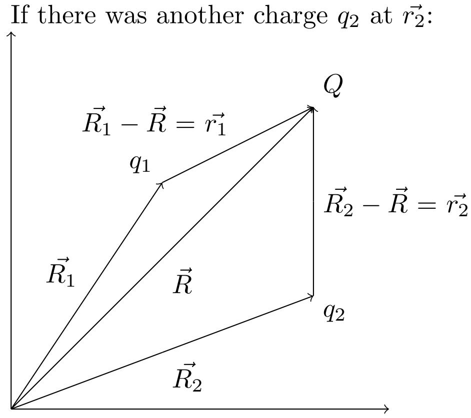
1 Coulomb's Law



the force on Q due to q_1 is given by:

$$\vec{F_1} = \frac{1}{4\pi\epsilon_0}\frac{q_1 Q}{r_1^2}\hat{r_1}$$

If there was another charge q_2 at $\vec{r_2}$:



The force on Q due to q_2 is given by:

$$\vec{F_2} = \frac{1}{4\pi\epsilon_0}\frac{q_2 Q}{r_2^2}\hat{r_2}$$

And the net charge on Q is given by:

$$\vec{F} = \vec{F_1} + \vec{F_2}$$

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 Q}{r_1^2}\hat{r_1} + \frac{1}{4\pi\epsilon_0}\frac{q_2 Q}{r_2^2}\hat{r_2}$$

If there were n charges, the net force on Q would be:

$$\vec{F} = \sum_{i=1}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_i Q}{r_i^2}\hat{r_i}$$

$$\vec{F} = Q\sum_{i=1}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_i}{r_i^2}\hat{r}_i$$

where $\sum_{i=1}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_i}{r_i^2}\hat{r}_i$ is the electric field $\vec{E}$ at $\vec{R}$ due to the n charges.

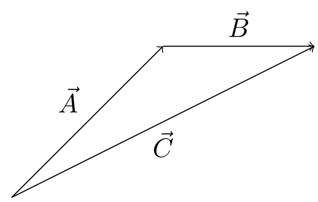
Field A value, vector or tensor, that is defined for every point in space and time.

2 Vector Calculus

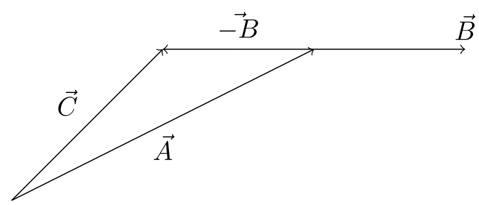
Vectors A vector is a quantity that is said to transform like a displacement.

2.1 Operations on Vectors

- **Addition**: $\vec{A} + \vec{B} = \vec{C}$



- **Subtraction:** $\vec{A} - \vec{B} = \vec{C}$



- **Multiplication by a scalar:** $c\vec{A} = \vec{C}$
- **Dot Product:** $\vec{A}.\vec{B} = AB\cos(\theta)$. Dot product is a scalar. It is commutative and distributive.
- **Cross Product:** $\vec{A} \times \vec{B} = AB\sin(\theta)\hat{n}$ where $\hat{n}$ is the unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.

Cross product is a vector. It is anti-commutative and distributive.

2.2 Component form of a Vector:

A vector $\vec{A}$ can be written as:

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the x, y, z axes respectively.

Given that $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$, we can perform the following operations:

- **Addition (and Subtraction):** $\vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$
- **Multiplication by a scalar:** $c\vec{A} = cA_x\hat{i} + cA_y\hat{j} + cA_z\hat{k}$, where c is a scalar.
- **Dot Product:** $\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$
- **Modulus:** $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$
- **Cross Product:** $\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$ This can also be written in a determinant form:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

2.3 Triple Product:

Scalar Triple Product: $\vec{A}.(\vec{B}\times\vec{C}) = \vec{B}.(\vec{C}\times\vec{A}) = \vec{C}.(\vec{A}\times\vec{B})$ Geometrically the scalar triple product is the volume of the parallelepiped formed by the three vectors.

Vector Triple Product: $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$

3 Differential Calculus

3.1 'Ordinary' Derivative:

$\frac{df}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ Geometrically, the derivative is the slope of the tangent to the curve at a point.

3.2 Gradient:

Consider a scalar, T, which exists at every point in space.

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

This can the written in the dot product form as:

$$dT = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z})$$

this is denoted as:

$$dT = \nabla T \cdot d\vec{r}$$

Here, we treat ∇ as an operator. It takes a scalar and returns a vector.

The ∇ operator:

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

3.3 Divergence:

Consider a vector, $\vec{A}$, which exists at every point in space. The divergence of $\vec{A}$ is defined as:

$$\nabla \cdot \vec{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})$$

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

It behaves like a dot product, and is a scalar.

3.4 Curl:

Consider a vector, $\vec{A}$, which exists at every point in space. The curl of $\vec{A}$ is defined as:

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

It behaves like a cross product, and is a vector.

3.5 Second Derivatives:

- **Divergence of a Gradient:** $\nabla \cdot (\nabla T)$

$$\nabla \cdot (\nabla T) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)$$

$$\nabla \cdot (\nabla T) = \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

It is denoted by $\nabla^2 T$ and is called the Laplacian of T.

Note: Laplacian of a vector is theoretically not defined. But when $\nabla^2 \vec{A}$ is written, it means:

$$\nabla^2 \vec{A} = \left(\nabla^2 A_x\right) \hat{x} + \left(\nabla^2 A_y\right) \hat{y} + \left(\nabla^2 A_z\right) \hat{z}$$

- **Curl of a Gradient:** $\nabla \times (\nabla T)$

$$\nabla \times (\nabla T) = 0$$

  Curl of a gradient is always zero.

- **Gradient of Divergence:** $\nabla \left(\nabla \cdot \vec{A}\right)$

- **Divergence of Curl:** $\nabla \cdot \left(\nabla \times \vec{A}\right)$

$$\nabla \cdot \left(\nabla \times \vec{A}\right) = 0$$

  Divergence of a Curl is always zero.

- **Curl of Curl:** $\nabla \times \left(\nabla \times \vec{A}\right)$

$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla \left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}$$

4 Integral Calculus

4.1 Line Integral:

Consider a vector field $\vec{V}$ and a curve C joining points a and b. The line integral of $\vec{V}$ along C is defined as:

$$\int_{aP}^{b} \vec{V} \cdot d\vec{l}$$

where $d\vec{l}$ is the differential displacement along the curve C.

when C is a closed figure, it is written as:

$$\oint \vec{V} \cdot d\vec{l}$$

4.2 Surface Integral:

Consider a vector field $\vec{V}$ and a surface S. The surface integral of $\vec{V}$ over S is defined as:

$$\int_S \vec{V} \cdot d\vec{a}$$

where $d\vec{a}$ is the differential area vector of the surface S.

When S is a closed surface:

$$\oint \vec{V} \cdot d\vec{a}$$

4.3 Volume Integral:

Consider a scalar field T and a volume V. The volume integral of T over V is defined as:

$$\int_V T dV$$

If we consider a vector field $\vec{A}$, then:

$$\int \vec{A} dV$$

$$\hat{x} \int A_x dV + \hat{y} \int A_y dV + \hat{z} \int A_z dV$$

In Cartesian coordinates:

$$dV = dxdydz$$

In Spherical Polar coordinates:

$$dV = r^2 \sin(\theta) dr d\theta d\phi$$

4.4 Fundamental Theorem of Calculus:

If $f(x)$ is a continuous function and $F(x)$ is the anti-derivative of $f(x)$, then:

$$\int_a^b f(x) = F(b) - F(a)$$

- **Fundamental Theorem for Gradients:**
  Consider a scalar field T and a curve C, from $\vec{a}$ to $\vec{b}$, then:
  $$\int_{\vec{a}}^{\vec{b}} \nabla T \cdot d\vec{l} = T(\vec{b}) - T(\vec{a})$$
  If the curve is a closed curve, then:
  $$\int_{\vec{a}}^{\vec{b}} \nabla T \cdot d\vec{l} = 0$$
- **Fundamental Theorem for Divergence:** a.k.a Gauss' Theorem, Green's Theorem or divergence theorem. Consider a vector field $\vec{A}$ and a volume V, then:
  $$\int_V \left(\vec{\nabla} \cdot \vec{v} \right) = \oint \vec{v} . d\vec{s}$$
  where ds is the differential area vector of the surface S.

- **Fundamental Theorem for Curl:** a.k.a Stokes' Theorem. Consider a vector field $\vec{A}$ and a surface S, then:

$$\int_S \left(\vec{\nabla} \times \vec{v}\right) \vec{da} = \oint \vec{v}.\vec{dl}$$

  where dl is the differential displacement vector along the boundary curve C of the surface S.

5 Dirac Delta Function:

5.1 One-Dimensional Dirac Delta Function:

It is a function that is zero everywhere except at $x = 0$ and has an integral of 1. It is defined as:

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

Mathematically, it is not a function, i.e., don't tell your maths profs that its a function.

5.2 Three-Dimensional Dirac Delta Function:

It is a function that is zero everywhere except at $\vec{r} = \vec{r_0}$ and has a volume integral of 1. It is defined as:

$$\delta^3(\vec{r}) = \begin{cases} \infty & (x, y, z) = (0, 0, 0) \\ 0 & \text{everywhere else} \end{cases}$$

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

and

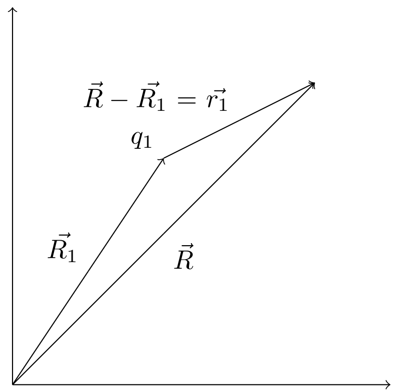
$$\int_{\text{all space}} \delta^3\left(\vec{r}\right) d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z) dx dy dz$$

Generalizing the function:

$$\int_{\text{all space}} f(r) \delta^3\left(r - a\right) d\tau = f(a)$$

6 Electrostatics

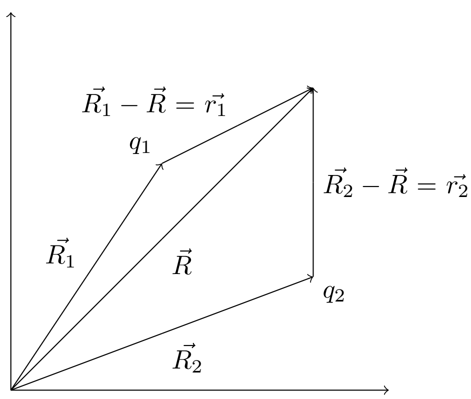
6.1 Electric Field



The Electric Field at $\vec{R}$ due to q_1 is given by:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1^2}\hat{r_1}$$

If there is another charge present:



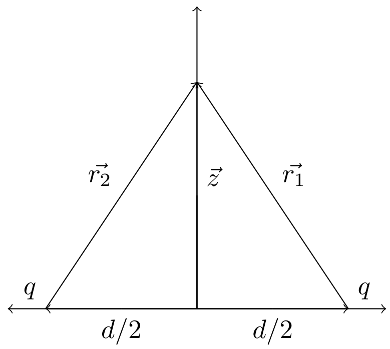
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_1^2}\hat{r_1} + \frac{q_2}{r_1^2}\hat{r_2}\right)$$

Principle of Superposition:

The net electric field at $\vec{R}$ due to n charges is given by:

$$E = \frac{1}{4\pi\epsilon_0}\left(\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r_i}\right)$$

Question:



Consider two charges, q, placed at $r = \pm\frac{d}{2}$ respectively. Find the electric field at z.

Answer

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\left(\frac{d}{2}\right)^2 + z^2}\hat{r_1} + \frac{q}{\left(\frac{d}{2}\right)^2 + +z^2}\hat{r_2}\right)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{z^2 + \frac{d^2}{4}}\right)\left(\frac{z}{\sqrt{z^2 + \frac{d^2}{4}}}\hat{z}\right)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{2qz}{\left(z^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}}\right)\hat{z}$$

Now, if z is very large (but not infinity):

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{z^2}\right)\left(1 + \frac{d^2}{4z^2}\right)^{\frac{-3}{2}}\hat{z}$$

this can be approximated to:

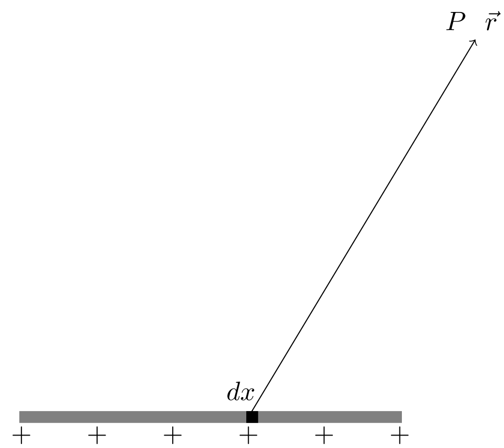
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{z^2}\right)\left(1 - \frac{3d^2}{8z^2}\right)\hat{z}$$

Similar methods can be used to find the electric field for any discrete charge distribution.

6.2 Continuous Charge Distribution

6.2.1 Linear Charge Distribution

Consider a linear Charge distribution, λ of length l, along the x axis. The electric field at point P is given by:



The Electric Field at point P due to dx is given by:

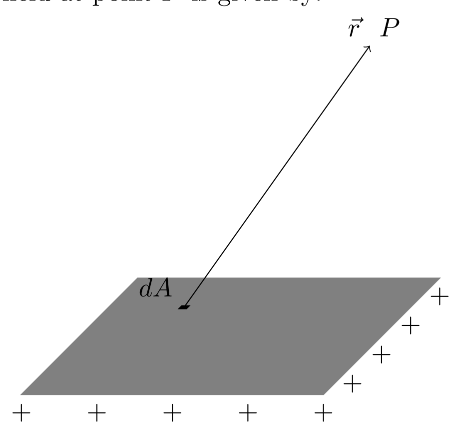
$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda dx}{r^2}\hat{r}$$

Hence, the electric field at point P due to the entire linear charge distribution is given by:

$$\vec{E} = \int_{\text{length } l} \frac{1}{4\pi\epsilon_0}\frac{\lambda dx}{r^2}\hat{r}$$

6.2.2 Surface Charge Distribution

Consider a surface charge distribution, σ of area A, on the xy plane. The electric field at point P is given by:



The Electric Field at point P due to dA is given by:

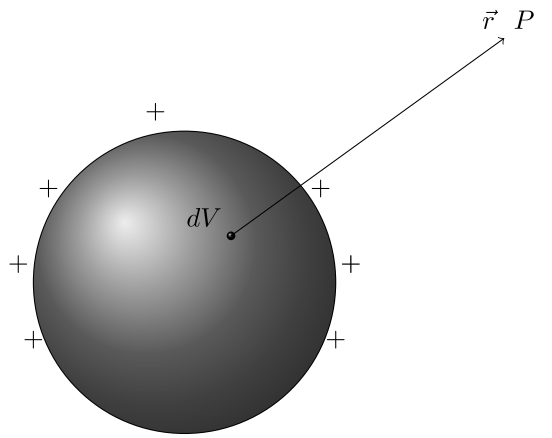
$$d\vec{E} = \frac{1}{4\pi\epsilon}\frac{\sigma dA}{r^2}\hat{r}$$

Hence, the electric field at point P due to the entire surface charge distribution is given by:

$$\vec{E} = \int_{\text{area } A} \frac{1}{4\pi\epsilon} \frac{\sigma dA}{r^2} \hat{r}$$

6.2.3 Volume Charge Distribution

Consider a volume charge distribution, ρ of volume V, in space. The electric field at point P is given by:



The electric field at point P due to dV is given by:

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\rho dV}{r^2} \hat{r}$$

Hence, the electric field at point P due to the entire volume charge distribution is given by:

$$\vec{E} = \int_{\text{volume } V} \frac{1}{4\pi\epsilon} \frac{\rho dV}{r^2} \hat{r}$$

7 Divergence and Curl of Electric Field

7.1 Flux and Gauss' Law

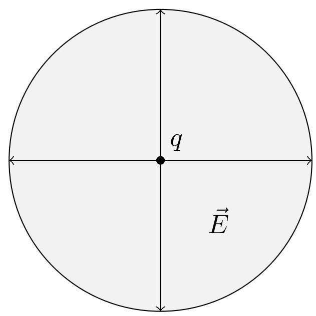
Flux The Electric Flux through a surface S is defined as:

$$\Phi_E = \int_S \vec{E} \cdot \vec{da}$$

Now, if the surface S is a closed surface, then:

$$\Phi_E = \oint \vec{E} \cdot \vec{da}$$

A charge in a sphere Consider a charge q in the centre of a spherical shell:



The flux through the spherical shell would be given by:

$$\Phi_E = \oint \vec{E} \cdot \vec{da}$$

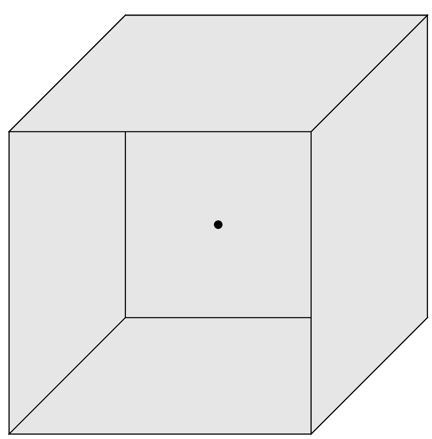
As $\vec{E}$ is constant and is always perpendicular to the surface:

$$\Phi_E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \int_{\text{shere}} dA$$

$$\Phi_E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} (4\pi r^2)$$

$$\Phi_E = \frac{q}{\epsilon_0}$$

A charge in a cube Consider a charge q in the centre of a cubical surface of side $2a$:



Now, the flux through each of the individual surfaces is equal by symmetry. The flux through the surface is:

$$d\Phi_E = \int_{surface} da \frac{1}{4\pi\epsilon_0} \frac{q}{a^2 \sec^2(\theta)} \cos(\theta)$$

$$\Phi_E = \frac{q}{6\epsilon_0}$$

Flux through the entire surface is:

$$\Phi_E = \frac{q}{\epsilon_0}$$

Gauss' Law As seen with the above two examples of the electric flux through a closed surface, the electric flux through a closed surface is proportional to the charge enclosed by the surface. This is known as Gauss' Law. Mathematically, it is written as:

$$\oint \vec{E} \cdot \vec{da} = \frac{Q_{enc}}{\epsilon_0}$$

where Q_{enc} is the charge enclosed by the surface. This is also know as the integral form of Gauss' Law or the integral form of Maxwell's first equation.

To write it in the form of a differential equation:

$$Q_{enc} = \int_V \rho d\tau$$

$$\int_V (\vec{\nabla} \cdot \vec{E}) dV = \oint \vec{E} \cdot \vec{da}$$

$$\int_V (\vec{\nabla} \cdot \vec{E}) dV = \int_V \rho d\tau$$

As this is true for all volumes, we can write:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

This is the differential form of Gauss' Law or Maxwell's first equation.

7.1.1 Divergence of Electric Field

The Electric Field at a point $\vec{r}$ due to a continuous charge distribution at $\vec{r'}$ is given by:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\hat{\gamma}}{\gamma^2} \rho(\vec{r'}) d\tau'$$

taking the divergence of the above function

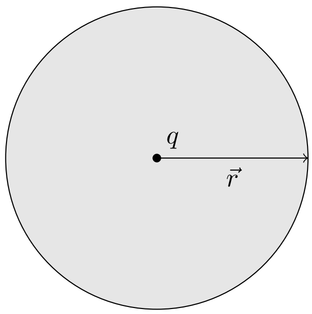
$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \vec{\nabla} \cdot \left(\frac{\hat{\gamma}}{\gamma^2} \right) \rho(\vec{r'}) d\tau'$$

$$\vec{\nabla} \cdot \frac{\hat{\gamma}}{\gamma^2} = 4\pi\delta^3(\gamma)$$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int 4\pi\delta^3(r - r')\rho(r')d\tau' = \frac{1}{\epsilon_0}\rho(r)$$

7.1.2 Application of Gauss' Law

- **Spherical Charge Distribution** Consider a spherical charge distribution of radius γ and net charge Q. To find the electric field at a point r :



We know that flux through a sphere of radius r is given by:

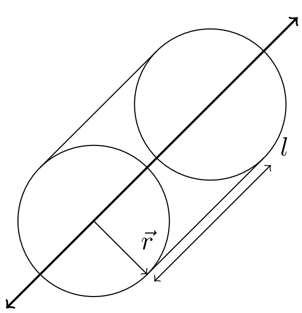
$$\Phi_E = \oint \vec{E} \cdot \vec{da} = \frac{q}{\epsilon_0}$$

$$E \int_{\text{surface}} da = \frac{q}{\epsilon_0}$$

$$E 4\pi r^2 = \frac{q}{\epsilon_0}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

- **Infinitely Long Linear Charge Distribution** Consider an infinitely long linear charge distribution with charge density λ. To find the electric field at a radial point r :



Now, the flux through the cylinder is given by:

$$\Phi_E = \oint \vec{E} \cdot \vec{da} = \frac{\lambda l}{\epsilon_0}$$

Also, the flux through the flat surface is zero as the field is perpendicular to the area at every point on the surface. Hence, the flux through the curved surface is equal to that of the cylinder

$$\Phi_E = \int_{\text{curved surface}} \vec{E} \cdot \vec{da}$$

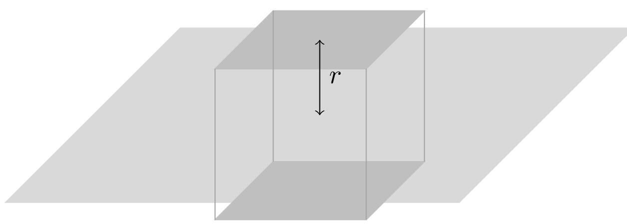
By symmetry, the electric field is constant and is always perpendicular to the surface.

$$\Phi_E = E \int_{\text{curved surface}} da = \frac{\lambda l}{\epsilon_0}$$

$$E 2\pi r l = \frac{\lambda l}{\epsilon_0}$$

$$\vec{E} = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{r} \hat{r}$$

- **Infinitely Large Surface Charge Distribution** Consider an infinitely large surface charge distribution with charge density σ. To find the electric field at a radial point r :



Now the flux through the cuboid is given by:

$$\Phi_E = \oint \vec{E} \cdot \vec{da} = \frac{\sigma A}{\epsilon_0}$$

As the electric field is always parallel to 4 of the surfaces, the flux through them is zero. As the electric field is constant and is always perpendicular to through the remaining surfaces:

$$\Phi_E = E \int_{\text{surface}} da = \frac{\sigma A}{\epsilon_0}$$

$$E 2A = \frac{\sigma A}{\epsilon_0}$$

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{r}$$