MT3434 - Topics in Number Theory

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Contents

Chapter 1

Basics

1.1 Arithmetic Functions

An Arithmetical function is any function defined as $f : \mathbb{N} \to \mathbb{C}$.

Additive Arithmetic Funtion: An Arithmetic function is Additive if for all relatively primes $m, n \in \mathbb{N}$:

$$f(m \cdot n) = f(m) + f(n)$$

If the above function holds for all $m, n \in \mathbb{N}$ then f is completely additive.

Multiplicative Arithmetic Funtion: An Arithmetic function is Multiplicative if for all relatively primes $m, n \in \mathbb{N}$:

$$f(m \cdot n) = f(m) \cdot f(n)$$

If the above function holds for all $m, n \in \mathbb{N}$ then f is completely multiplicative.

1.1.1 Examples:

1. $\omega(x) =$ No. of distinct Prime divisors of x.
   $\Rightarrow$ Additive, but not completely.

2. $\Omega(x) =$ No. of Prime divisors of x, counted with multiplicity.
   $\Rightarrow$ Completely Additive.

3. $\mu(x) = \begin{cases} (-1)^k & \text{, if } x = p_1 \cdot p_2 \cdots p_n \text{ are distinct primes} \\ 0 & \text{, otherwise, i.e., x is not a sqaure-free} \end{cases}$
   $Rightarrow$ Multiplicative, but not completely.

4. $I : \mathbb{N} \to \mathbb{C}$ such that: $I(N) = \lfloor \frac{1}{N} \rfloor$

   Theorem: If $n \geq 1$:

   $$\sum_{d|n} \mu(d) = I(n)$$

Proof: Case 1: $n = 1$: $\mu(1) = 1 = I(n)$, trivially.
Case 2: $n > 1$. Let $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_m^{\alpha_m} \cdot$. Observe that $\mu(d)$ is zero for all non-sqaure-free divisors. Therefore:

$$\begin{aligned}
\sum_{d|n} \mu(d) &= \sum_{d|n \text{ \& } d \text{ is square-free}} \mu(d) \\
&= \sum_{d|N} \mu(d) \qquad [N = p_1 \cdot p_2 \cdots p_m] \\
&= 1 + \sum
\end{aligned}$$