MT4214 - Algebraic Geometry

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Contents

Chapter 1

Motivation - Cayley–Hamilton theorem

Statement: Every Square Matrix over a commutative ring satisfies its own Charectoristic Polynomial.

Proof: Step 1: Let A be a diagonal matrix with $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ as the diagonal elements. Trivially, we can show that the Charecteristic Polynomial will be evaluated as follows:

$$\begin{aligned}\chi_A(x) &= \text{det}(A - xI_n)\\ &= (\lambda_1 - x)(\lambda_2 - x)\ldots(\lambda_n - x)\\ &= 0\end{aligned}$$

Step 2: A is diagonalizable. Then there exists matrices B, D such that $A = BDB^{-1}$. A property that will be used is as follows: $\chi_A(x) = \chi_D(x)$. Now, if we calculate the Charecteristic Polynomial for A:

$$\begin{aligned}\chi_A(A) &= \text{det}(A - xI_n)\\ &= \chi_D(A)\\ &= B\chi_D(D)B^{-1}\\ &= 0\end{aligned}$$

Step 3: General A. We know that diagonalizable matrices are dense in $M_{n\times n}(\mathbb{C})$.

Consider the following function:

$$\phi : M_{n\times n}(\mathbb{C}) \to M_{n\times n}(\mathbb{C})$$

such that $\phi(A) = \chi_A(A) = 0\ \forall A \in$ Diagonal Matrices. The above function is a continuous function [Trust me bro]. Now, $\{0\}$ is a closed set. Therefore the pullback of a closed set will have to be a closed set as well. But diagonal elements are dense in $M_{n\times n}(\mathbb{C})$. Therefore we use this to extend this to the entire topological space, $M_{n\times n}(\mathbb{C})$.

$$\phi(A) = 0\ \forall\ A$$

But this above arguement only for fields which are Cauchy Complete. What about the charecteristic p fields. There is no obvious topology, and hence no dense set.

1.1 Zariski Topology on K^n

Let K be an algebraicaly closed field. We want to define a topology on K^n.

Define a ring $A = K[X_1, X_2, \dots, X_n]$ is the ring of polynomial in n variables. Now, choose an element $f \in A$.

$$f : K^n \to K \text{ where } (a_1, a_2, \dots, a_n) \to f(a_1, a_2, \dots, a_n)$$

Now, we define a set function as follows:

$$Z(f) = \{(a_1, a_2, \dots, a_n) \,|\, f(a_1, a_2, \dots, a_n) = 0\}$$

Here, $Z(f)$ can be empty. Extending this to multiple functions:

$$Z(f_1, f_2, \dots, f_m) = \bigcap_{1 \le i \le m} Z(f_i)$$

Let $I \subseteq A$ be an ideal.

$$f(I) = \bigcap_{p \in I} Z(p)$$

Noetherian Ring: Let R be a commutative ring with unity. R is Noetherian if every ideal of R is finitely generated.

OR

R is Noetherian if every increasing sequence $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ of ideals has a largest element.

Lemma: Let $I \subseteq A$ be an ideal of A generated as follows: $I = \langle f_1, f_2, \dots, f_n \rangle$. Then:

$$\begin{aligned} Z(I) &= Z(\langle f_1, f_2, \dots, f_n \rangle) \\ &= Z(\{f_1, f_2, \dots, f_n\}) \end{aligned}$$

Proof: We will prove this by showing that one set contains the other and vise=versa.

Part 1: $Z(\{f_1, f_2, \dots, f_n\}) \subseteq Z(I)$ as each of the f_i is always contained in the I.

Part 2: Consider an element $f \in I$. As the ideal I is generated by f_i, f can be written as a linear combination of f_is, with coefficients in A:

$$f = \sum_{i=1}^{n} c_i f_i$$

where, all $c_i \in A$. Now, if $\bar{a} \in Z(\{f_1, f_2, \dots, f_n\})$, $f_i(\bar{a}) = 0$ for all i. Therefore:

$$\begin{aligned} f(\bar{a}) &= \sum_{i=1}^{n} c_i f_i(\bar{a}) \\ &= 0 \end{aligned}$$

Hence, $Z(\{f_1, f_2, \dots, f_n\}) \subseteq Z(I)$. Therefore, $Z(\{f_1, f_2, \dots, f_n\}) = Z(I)$.

1.1.1 $Z(I)$ form a Topology on K^n

We define a topology on K^n by claiming that the closed sets in K^n are defined by the sets $Z(I)$. It satisifies the axioms of Topology as follows:

1. $\varnothing$ is in the topology: A is an ideal of itself. $Z(A) = \varnothing$.

2. K^n is in the topology: 0 polynomial also forms an ideal of A. $Z(0) = K^n$

3. Finite Union of $Z(I)$ belong to the topology: Let I, J, IJ be Ideals of A.

   Proposition: $Z(I) \cup Z(J) = Z(IJ)$.

   Proof: Consider an $a \in Z(I)$. This implies that $f(a) = 0 \; \forall \; f \in I$. Using that we can say that a is a solution for any polynomial of the form $f \cdot g$ where, $f \in I$ and $g \in A$. Now, similarly, consider a b in $Z(J)$. It would be a solution for any polynomial of the form $f \cdot g$ where, $f \in J$ and $g \in A$. Hence, $a \in Z(I) \cup Z(J) \Rightarrow a \in Z(IJ)$

   Consider an $a \in Z(IJ)$. This implies that for all $f \in I$ and $g \in J$, the product $f(a)g(a) = 0$. As A is an integral domain, one of the factors must be 0. Let $a \notin I$. Then, there exists an $f \in I$ such that $f(a) \neq 0$. But, as $a \in Z(IJ)$, for every $g \in J$, $f(a)g(a) = 0$. This implies $g(a) = 0$ for every $g \in J$. Hence, $a \in Z(IJ) \Rightarrow Z(I) \cup Z(J)$

4. Arbitrary Intersection of $Z(I)$ belong to the topology: let $I, J, I + J$ be Ideals of A.

   Proposition: $Z(I) \cap Z(J) = Z(I + J)$.

   Proof: Consider an $a \in Z(I) \cap Z(J)$. This implies that for any two polynomials $f \in I$ and $g \in J$, the sum $(f + g)(a) = 0$. Therefore, $a \in Z(I + J)$.

   Now, let $a \in Z(I + J)$. Trivially, $a \in Z(I)$ and $a \in Z(J)$.

Therefore, $Z(I)$ forms a topology on K^n. $\mathbb{A}^n_k$ is defined as the vector space K^n with the Zariski Topology.

Exercise: Let X be a Compact Hausdorff Topological Space.

$$C(X) = \{f : X \to \mathbb{R} \,|\, f \text{ is continuous.}\}$$

Show that $C(X)$ is an $\mathbb{R}$-Algebra on X and that it forms a Commutative Ring.