MT2223 - Real Analysis I

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Chapter 1

Real Number System

1.1 Natural Numbers

$\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of all natural numbers. They can also be referred to as Positive Integers. The set of Natural Numbers is important because it is the smallest set which is closed under addition and multiplication.

1.1.1 Peano Axioms

1. The number $1 \in \mathbb{N}$
2. For every Natural Number n, there exists another Natural Number m which is known as the successor of n.
3. 1 is not the successor for any number in the set of Natural Number set.
4. If $m, n \in \mathbb{N}$ have the same successor, then $m = n$.
5. If $A \subset \mathbb{N}$ such that $1 \in A$ and $n \in A \implies n + 1 \in A$, then $A = \mathbb{N}$.

1.2 Integers

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is the set of all integers. This set is important because it is the smallest set which is closed under subtraction.

1.3 Rational Numbers

The set of Rational Numbers is defined as:

$$\mathbb{Q} = \left\{ \frac{m}{n} \middle| m, n \in \mathbb{Z}, n \neq 0 \right\}$$

This set is important because it is the smallest set which is closed under division. testChange

1.4 Real Numbers

We can see that the set of Rational Numbers is incomplete. There is no Rational Number whose square is 2.

Proof: Consider a rational number whose square is 2.

$$\left(\frac{p}{q}\right)^2 = 2$$

Here, p and q are both integers and co-prime.

$$\frac{p^2}{q^2} = 2$$
$$p^2 = 2q^2$$

Now, as p^2 is even, p must be even. Let $p = 2k$. then, $p^2 = 4k^2$:

$$4k^2 = 2q^2$$
$$2k^2 = q^2$$

Now, as q^2 is even, q must be even.

But this contradicts the fact that p and q are co-prime. Hence, by proof of contradiction, there is no Rational Number whose square is 2.

Question: $A = \{p \in \mathbb{Q} \mid p^2 < 2\}$. Can we find a number $q \in A$ such that $p \leq q, \forall p \in A$?

Answer: Claim: given $p \in A$, we can find $q \in A$ such that $p < q$. Consider the following number:

$$q = p + \frac{2 - p^2}{2 + p}$$
$$q = \frac{p\,(2 + p) + 2 - p^2}{p + 2}$$
$$q = \frac{2 + 2p}{2 + p}$$
$$q^2 - 2 = \frac{2\,(p^2 - 2)}{(p + 2)^2}$$

Looking at the first equation $q > p$. If $p^2 < 2$, then the final equation shows that $q^2 < 2$.

1.5 Functions

A function from a set A to a set B is a subset of $A \times B$ such that for every element in A, there exists a unique element in B. The set $A \times B$ is known as the Cartesian Product of A and B or Relation from A to B. A function from A to B is denoted as $f : A \to B$.

1.6 Ordered Sets

Let S be a ordered set. An order on S is a relation, denoted by $<$, such that:

1. For every $x, y \in S$, exactly one of the following is true:
    - $x < y$
    - $x = y$
    - $y > x$
2. $\forall\, x, y, z \in S$ if $x < y$ and $y < z$, then $x < z$.

1.6.1 Upper and Lower Bounds

Let S be an ordered set and $E \subset S$.

1. E is said to be bounded above if there exists $a \in S$, such that for every $x \in E$, $x \leq a$. Here, a is called an Upper Bound for E.
2. E is said to be bounded below if there exists $b \in S$, such that for every $x \in E$, $x \geq b$. Here, b is called a Lower Bound for E.

Supremum and Infimum

Supremum: Assume E to be bounded above. Suppose there exists a number $\alpha \in S$ such that:

- α is an Upper Bound for E.
- If $\gamma < \alpha$, then γ is not an Upper Bound for E.

Then α is known as the least Upper Bound or Supremum of E.

Infimum: Assume E to be bounded below. Suppose there exists a number $\beta \in S$ such that:

- β is a Lower Bound for E.
- If $\gamma > \beta$, then γ is not a Lower Bound for E.

Then β is known as the greatest Lower Bound or Infimum of E.

1.6.2 Least Upper Bound Property

An ordered set S is said to have the Least Upper Bound Property if $E \subset S$ is non-empty and bounded above, then $\sup E$ exists in S.

Theorem

Let S be an ordered set with the Least Upper Bound Property. Let $B \subset S$ be non-empty and bounded below. Let L be the set of all Lower Bounds of B. Then, $\alpha = \sup L$ exists in S, and $\alpha = \inf B$.

Proof: As B is bounded below, L is non-empty. The set L is bounded above by any element of B.

Let $\alpha \in S$ & $\alpha = \sup L$. Now, if $\gamma < \alpha$, then γ is not an Upper Bound for L, and hence $\gamma \notin B$. It is also true that $\alpha \leq x$ for every $x \in B$. Hence, $\alpha \in L$.

If $\alpha < \beta$ then $\beta \notin L$, since α is an upper bound of L. We have shown that $\alpha \in$ but $\beta \notin L$ if $\alpha < \beta$. Therefore, α is a lower bound for B and β is not. This implies that $\alpha = \inf B$.

1.7 Fields

A field is a set F with two operations, addition and multiplication, which satisfy the following field axioms:

1. Axioms for Addition:
   (a) Closure: if $a, b \in F$, then $a + b \in F$.
   (b) Commutativity: $a + b = b + a$, $\forall\, a, b \in F$
   (c) Associativity: $a + (b + c) = (a + b) + c$, $\forall\, a, b, c \in F$
   (d) Additive Identity: There exists an element $0 \in F$ such that $a + 0 = a = 0 + a$, $\forall\, a \in F$
   (e) Additive Inverse: For every $a \in F$, there exists a unique $b \in F$ such that $a + b = 0 = b + a$.
2. Axioms for Multiplication:
   (a) Closure: if $a, b \in F$, then $a \cdot b \in F$.
   (b) Commutativity: $a \cdot b = b \cdot a$, $\forall a, b \in F$
   (c) Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, $\forall\, a, b, c \in F$
   (d) Multiplicative Identity: There exists an element $1 \in F$ such that $a \cdot 1 = a = 1 \cdot a$, $\forall\, a \in F$
   (e) Multiplicative Inverse: For every $a \in F$, there exists a unique $b \in F$ such that $a \cdot b = 1 = b \cdot a$.
3. Distributive Law: $a \cdot (b + c) = a \cdot b + a \cdot c$, $\forall\, a, b, c \in F$

1.7.1 Ordered Fields

An ordered field is a field F, which is also an ordered set with a relation $<$ such that:

1. if $x, y, z \in F$ and $x < y$, then $x + z < y + z$.
2. if $x, y \in F$ and $x > 0$ and $y > 0$, then $x \cdot y > 0$.

Properties of Ordered Fields

1. If $x > 0$, then $-x < 0$.
2. If $x > 0$ and $y < z$, then $x \cdot y < x \cdot z$.
3. If $x \neq 0$, then $x^2 > 0$. This implies that $1 > 0$.
4. If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.

1.7.2 The Real Field

There exists an ordered field $\mathbb{R}$ which has the Least Upper Bound Property, and contains $\mathbb{Q}$ as a subfield.

Archimedean Property

If $x, y \in \mathbb{R}$ and $x > 0$, then there exists a positive integer n such that $nx > y$.

Proof: Let A be the set of all mx, where m is a positive integer. If the Archimedean Property is false, then y is an upper bound for A. But then A has a least upper bound in $\mathbb{R}$, say b. Since x is positive, $b-x < b$, and $b-x$ is not an upper bound for A. Hence, there exists an element m_0x such that $b - x < m_0x$. This implies that $b < (m_0 + 1)x$, which contradicts the fact that b is the least upper bound for A. Hence, by proof by contradiction, the Archimedean Property is true.

Density of $\mathbb{Q}$ in $\mathbb{R}$

If $x, y \in \mathbb{R}$ and $x < y$, then there exists a rational number p such that $x < p < y$.

Proof: Since $x < y$,

$$(y - x > 0$$

. By the Archimedean Property, there exists a positive integer n such that $n(y - x) > 1$. Again, by Archimedean Property, there exists positive integers m_1 and m_2 such that $m_1 > nx$ and $m_2 > -nx$. Then:

$$-m_2 < nx < m_1$$

Hence, there is an integer m (with $-m_2 < m < m_1$ such that:

$$m - 1 \leq nx < m$$

Using this:

$$nx < m \leq 1 + nx < ny$$

Since n is positive:

$$x < \frac{m}{n} < y$$

Hence, $\frac{m}{n}$ is a rational number between x and y.

1.7.3 The Complex Field

A Complex Number is defined as an ordered pair of Real Numbers, (a, b). If we consider two complex numbers, $x = (a, b)$ and $y = (c, d)$, then:

1. $x = y$, iff $a = c$ and $b = d$.
2. $x + y = (a + c, b + d)$
3. $x \cdot y = (ac - bd, ad + bc)$

By the above definitions, the set of Complex Numbers is a field, denoted by $\mathbb{C}$, with $(0, 0)$ as additive identity and $(1, 0)$ as multiplicative identity.

Schwarz Inequality

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are complex numbers, then:

$$\left| \sum_{j=1}^{n} a_j \bar{b_j} \right| \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

Proof: Consider $A = \sum |a_j|^2$, $B = \sum |b_j|^2$ and $\sum a_j \bar{b_j}$. Now, if either $A = 0$ or $B = 0$, then all of $a_j = 0$ or $b_j = 0$ respectively. Therefore, assume that $A, B > 0$.

$$\begin{aligned} \sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j) \left(B\bar{a_j} - \bar{C}\bar{b_j} \right) \\ &= B^2 \sum |a_j| - B\bar{C} \sum a_j \bar{b_j} - BC \sum \bar{a_j} b_j + |C|^2 \sum |b_j| \\ &= B^2 A - B |C|^2 && = B \left(AB - |C|^2 \right) \end{aligned}$$

As every term in the summation is non-negative:

$$B \left(AB - |C|^2 \right) \geq 0$$

Finally, as $B > 0$:

$$AB \geq |C|^2$$

1.8 Construction of $\mathbb{R}$ from $\mathbb{Q}$

Theorem: There exists an ordered field $\mathbb{R}$ which has the least upper bound property, and contains $\mathbb{Q}$ as a subfield.

Proof: There will be multiple sections of the proof.

1.8.1 Dedikind Cuts

We define Cuts to be as follows: A cut is a set $\alpha \subset \mathbb{Q}$ with the following properties:

1. α is not empty, and α is not equal to $\mathbb{Q}$.
2. If $x \in \alpha$, $y \in \mathbb{Q}$, and $y < x$ then $y \in \alpha$.
3. If $x \in \alpha$, then there exists $y \in \alpha$ such that $y > x$.