MT2213 - Group Theory

Nachiketa Kulkarni

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Chapter 1

Definitions

1.1 Groups

A non-empty set G is a group, is considered to be a group with an operation $\star$ if to every pair $(x, y) \in G \times G$ and element $x \star y \in G$ is assigned, satisfying the following axioms:

1. **Associativity:** $\forall\ x, y, z \in G$, $x \star (y \star z) = (x \star y) \star z = x \star y \star z$
2. **Existence of Identity:** There exists an element $e \in G$ such that $e \star g = g \star e = g$
3. **Existence of Inverse:** For every element $x \in G$ there exists an element $x^{-1} \in G$ such that $x \star x^{-1} = e = x^{-1} \star x$, where $e \in G$ is the identity element of the group.

It is represented as $(G, \star)$. Some properties of groups:

1. **Uniqueness of Identity:** The identity element of a group is unique. Consider $e_1, e_2 \in G, e_1 \neq e_2$ and both are identity elements. Let $x \in G$, then $e_1 \star x = e_2 \star x = x$. This also implies that $e_1 = e_2$, hence the identity element is unique.
2. **Uniqueness of Inverse:** The inverse of an element in a group is unique. Consider $x \in G$, and $y_1, y_2 \in G$ are inverses of x. Then, $x \star y_1 = e = y_1 \star x$ and $x \star y_2 = e = y_2 \star x$. Now, $y_1 = y_1 \star e = y_1 \star (x \star y_2) = (y_1 \star x) \star y_2 = e \star y_2 = y_2$. Hence, the inverse of an element is unique.

1.1.1 Examples:

1. $(\mathbb{Z}, +)$ is a group:
   (a) Associativity: Addition is associative.
   (b) Identity: 0 is the identity. Let $x \in Z$. Now $0 + x = x + 0 = x$. Hence, it is an identity.
   (c) Inverse: Let $x \in \mathbb{Z}$. Now, $x + (-x) = (-x) + x = 0$, where 0 is the additive identity.
2. $(\mathbb{Q}^+, \times)$ is a group:
   (a) Associativity: Multiplication is associative.
   (b) Identity: 1 is the identity: Let $x \in \mathbb{Q}^+$. Now, $1 \times x = x \times 1 = x$. Hence, it is an identity.
   (c) Inverse: Let $x \in \mathbb{Q}^+$, Now, $x \times \frac{1}{x} = \frac{1}{x} \times x = 1$, where 1 is the multiplicative identity.

3. $(GL(n, \mathbb{R}), \times)$ is a group, where $\times$ is matrix multiplication (or combination of linear transformations):
   (a) Associativity: Matrix multiplication is associative.
   (b) Identity: I_n is the identity matrix.
   (c) Inverse: Let $A \in GL(n, \mathbb{R})$, then $A \times A^{-1} = A^{-1} \times A = I_n$.

Check if:

1. $(\mathbb{R}, \times)$ is a group or not.
   $0 \in \mathbb{R}$, 0 does not have an inverse. Hence, it is not a group.
2. $(\mathbb{C}, \times)$ is a group or not.
   $0 \in \mathbb{C}$, 0 does not have an inverse. Hence, it is not a group.
3. $(\mathbb{R}/\{0\}, \times)$ is a group or not.
   Yes its a group:
   (a) Associativity: Multiplication is associative.
   (b) Identity: 1 is an identity: Let $x \in \mathbb{R}/\{0\}$. Now, $1 \times x = x \times 1 = x$. Hence, it is an identity.
   (c) Inverse: Let $x \in \mathbb{R}/\{0\}$, Now, $x \times \frac{1}{x} = \frac{1}{x} \times x = 1$, where 1 is the multiplicative inverse.
4. $(\mathbb{C}/\{0\}, \times)$ is a group or not.
   Yes it is a group:
   (a) Associativity: Multiplication is associative.
   (b) Identity: 1 is an identity: Let $x \in \mathbb{C}/\{0\}$. Now, $1 \times x = x \times 1 = x$. Hence, it is an identity.
   (c) Inverse: Let $x \in \mathbb{C}/\{0\}$, Now, $x \times \frac{1}{x} = \frac{1}{x} \times x = 1$, where 1 is the multiplicative inverse.

1.1.2 Sub-groups

Consider a group $(G, \star)$. A non-empty subset H is a subgroup of G if H is a group with the same operation $\star$ as G.

A few properties of subgroups:

1. **Identity:** The identity element of G is also the identity element of H.
2. **Inverse:** If $x \in H$, then $x^{-1} \in H$.

1.1.3 Abelian Groups

A group $(G, \star)$ is said to be abelian if the operation $\star$ is commutative, i.e., $x \star y = y \star x$, $\forall x, y \in G$.