Linear Algebra

1 Introduction to Linear Systems

They are system of equations that have variables that are linear. Example:

$$x + y = 2$$

and

$$2x - y = 1$$

Normally the coefficients are real numbers, but they can be complex numbers as well.
System of linear equations of m equations and n variables:

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m$$

What a linear equation in n variables represents is a given space in $\mathbb{R}^n$.

Question: What it the line passing through $(1, 1)$ and $(-1, -3)$?

Answer: We will use first principles to find the equation of the line.
Let the equation of the line be $y = mx + c$.
We know that $(1, 1)$ and $(-1, -3)$ lie on the line.

$$1 = m * 1 + c \ldots (1)$$

$$-3 = m * (-1) + c \ldots (2)$$

Subtracting (2) from (1), we get:

$$4 = 2m$$

$$m = 2$$

Substituting $m = 2$ in (1), we get:

$$1 = 2 + c$$

$$c = -1$$

Matrix

Group of numbers(or equations, expressions, etc.) in rows and columns.

Example:

$$A = \begin{pmatrix} 1 & 0 & -1 & 4 \\ 2 & 9 & 3 & 5 \\ 5 & 2 & 10 & 6 \end{pmatrix}$$

The above matrix has 3 rows and 4 columns.
Entry in 2nd row, 3rd column = 3

It can also be represented as

$$A = (a_{ij})$$

where a_{ij} refers to the element in A at the ith row and jth column

Special Matrices: Some special matrices:

· **Zero Matrix:**

$$A = O_{m\times n} = \begin{bmatrix} 0 & . & . & . & 0 \\ . & & & & . \\ . & & & & . \\ 0 & . & . & . & 0 \end{bmatrix}$$

· **Square Matrix:** here $m = n$

$$A_3 = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

· **Identity Matrix:** A square matrix with all diagonal element equal to 1 and non-diagonal elements equal to 0.

$$I_n = \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & . & & & \\ & & & . & & \\ & & & & . & \\ 0 & & & & & 1 \end{bmatrix}$$

· **Diagonal Matrix:** A square matrix with all non-diagonal elements equal to 0.

$$D_n = \begin{bmatrix} d_1 & & & & & 0 \\ & d_2 & & & & \\ & & . & & & \\ & & & . & & \\ & & & & . & \\ 0 & & & & & d_n \end{bmatrix}$$

where d_i are generally non-zero, but not necessarily.

· **Upper Triangular Matrix:** A square matrix with all non-diagonal elements below the diagonal equal to 0

$$B = \begin{bmatrix} b_1 & & & & & \star \\ & b_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & d_n \end{bmatrix}$$

where $\star$ can be any number.

· **Lower Triangular Matrix:** A square matrix with all non-diagonal elements above the diagonal equal to 0

$$C = \begin{bmatrix} c_1 & & & & & 0 \\ & c_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ \star & & & & & c_n \end{bmatrix}$$

where $\star$ can be any number.

Augmented Matrix

Given a system of linear equations:

$$3x_1 - 2x_2 + 4x_3 = 0$$
$$2x_1 + x_2 + 3x_3 = 1$$
$$5x_1 + x_2 - 2x_3 = -1$$

The augmented matrix is:

$$\begin{bmatrix} 3 & -2 & 4 & : & 0 \\ 2 & 1 & 3 & : & 1 \\ 5 & 1 & -2 & : & -1 \end{bmatrix}$$

for a system of m equations and n variables:

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m$$

The augmented matrix would be:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & : & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & : & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & : & b_m \end{bmatrix}$$

Elementary Row Operations

- Interchange two rows: if R_i and R_j are two rows of a matrix A, then $R_i \leftrightarrow R_j$
- Multiply a row by a constant: if R_i is a row of a matrix A and c is a constant, $cR_i \to R_i$
- Multiply a row by a constant and add it to another row: if R_i and R_j are two rows of a matrix A and c is a constant, $R_i + cR_j \to R_i$

change with new rules