MT4214 - Algebraic Geometry

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Contents

Chapter 1

Motivation - Cayley–Hamilton theorem

Statement: Every Square Matrix over a commutative ring satisfies its own Charectoristic Polynomial.

Proof: Step 1: Let A be a diagonal matrix with $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ as the diagonal elements. Trivially, we can show that the Charecteristic Polynomial will be evaluated as follows:

$$\begin{aligned}\chi_A(x) &= \text{det}(A - xI_n)\\ &= (\lambda_1 - x)(\lambda_2 - x)\ldots(\lambda_n - x)\\ &= 0\end{aligned}$$

Step 2: A is diagonalizable. Then there exists matrices B, D such that $A = BDB^{-1}$. A property that will be used is as follows: $\chi_A(x) = \chi_D(x)$. Now, if we calculate the Charecteristic Polynomial for A:

$$\begin{aligned}\chi_A(A) &= \text{det}(A - xI_n)\\ &= \chi_D(A)\\ &= B\chi_D(D)B^{-1}\\ &= 0\end{aligned}$$

Step 3: General A. We know that diagonalizable matrices are dense in $M_{n\times n}(\mathbb{C})$.
Consider the following function:

$$\phi : M_{n\times n}(\mathbb{C}) \to M_{n\times n}(\mathbb{C})$$

such that $\phi(A) = \chi_A(A) = 0\ \forall A \in$ Diagonal Matrices. The above function is a continuous function [Trust me bro]. Now, $\{0\}$ is a closed set. Therefore the pullback of a closed set will have to be a closed set as well. But diagonal elements are dense in $M_{n\times n}(\mathbb{C})$. Therefore we use this to extend this to the entire topological space, $M_{n\times n}(\mathbb{C})$.

$$\phi(A) = 0\ \forall\ A$$

But this above arguement only for fields which are Cauchy Complete. What about the charecteristic p fields. There is no obvious topology, and hence no dense set.

1.1 Zariski Topology on K^n

Let K be an algebraicaly closed field. We want to define a topology on K^n.

Define a ring $A = K[X_1, X_2, \dots, X_n]$ is the ring of polynomial in n variables. Now, choose an element $f \in A$.

$$f : K^n \to K \text{ where } (a_1, a_2, \dots, a_n) \to f(a_1, a_2, \dots, a_n)$$

Now, we define a set function as follows:

$$Z(f) = \{(a_1, a_2, \dots, a_n) \,|\, f(a_1, a_2, \dots, a_n) = 0\}$$

Here, $Z(f)$ can be empty. Extending this to multiple functions:

$$Z(f_1, f_2, \dots, f_m) = \bigcap_{1 \leq i \leq m} Z(f_i)$$

Let $I \subseteq A$ be an ideal.

$$f(I) = \bigcap_{p \in I} Z(p)$$

Noetherian Ring: Let R be a commutative ring with unity. R is Noetherian if every ideal of R is finitely generated.

OR

R is Noetherian if every increasing sequence $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ of ideals has a largest element.