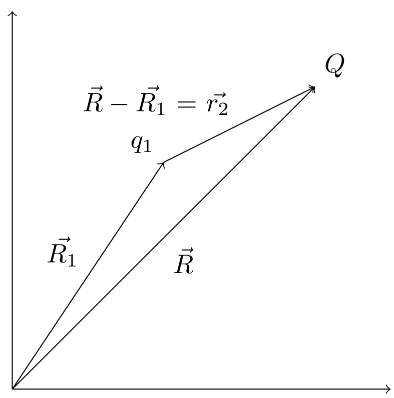
Introduction to Electromagnetism

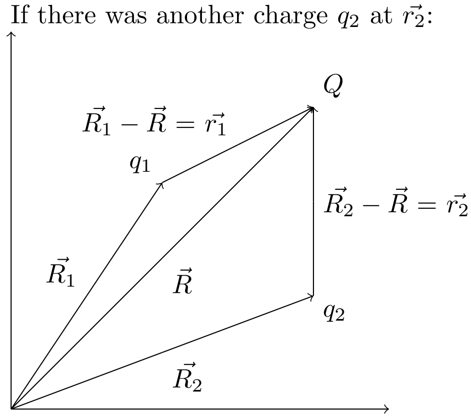
1 Coulomb's Law



the force on Q due to q_1 is given by:

$$\vec{F_1} = \frac{1}{4\pi\epsilon_0}\frac{q_1 Q}{r_1^2}\hat{r_1}$$

If there was another charge q_2 at $\vec{r_2}$:



The force on Q due to q_2 is given by:

$$\vec{F_2} = \frac{1}{4\pi\epsilon_0}\frac{q_2 Q}{r_2^2}\hat{r_2}$$

And the net charge on Q is given by:

$$\vec{F} = \vec{F_1} + \vec{F_2}$$

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 Q}{r_1^2}\hat{r_1} + \frac{1}{4\pi\epsilon_0}\frac{q_2 Q}{r_2^2}\hat{r_2}$$

If there were n charges, the net force on Q would be:

$$\vec{F} = \sum_{i=1}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_i Q}{r_i^2}\hat{r_i}$$

$$\vec{F} = Q \sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_i}{r_i^2} \hat{r}_i$$

where $\sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_i}{r_i^2} \hat{r}_i$ is the electric field $\vec{E}$ at $\vec{R}$ due to the n charges.

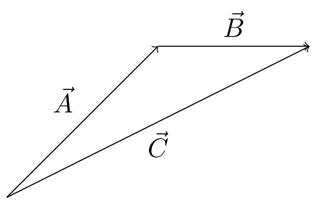
Field A value, vector or tensor, that is defined for every point in space and time.

2 Vector Calculus

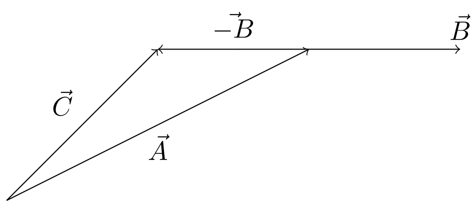
Vectors A vector is a quantity that is said to transform like a displacement.

2.1 Operations on Vectors

- **Addition**: $\vec{A} + \vec{B} = \vec{C}$



- **Subtraction:** $\vec{A} - \vec{B} = \vec{C}$



- **Multiplication by a scalar:** $c\vec{A} = \vec{C}$
- **Dot Product:** $\vec{A}.\vec{B} = AB\cos(\theta)$. Dot product is a scalar. It is commutative and distributive.
- **Cross Product:** $\vec{A} \times \vec{B} = AB\sin(\theta)\hat{n}$ where $\hat{n}$ is the unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.

Cross product is a vector. It is anti-commutative and distributive.

2.2 Component form of a Vector:

A vector $\vec{A}$ can be written as:

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the x, y, z axes respectively.

Given that $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$, we can perform the following operations:

- **Addition (and Subtraction):** $\vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$
- **Multiplication by a scalar:** $c\vec{A} = cA_x\hat{i} + cA_y\hat{j} + cA_z\hat{k}$, where c is a scalar.
- **Dot Product:** $\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$
- **Modulus:** $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$
- **Cross Product:** $\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$ This can also be written in a determinant form:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

2.3 Triple Product:

Scalar Triple Product: $\vec{A}.(\vec{B}\times\vec{C}) = \vec{B}.(\vec{C}\times\vec{A}) = \vec{C}.(\vec{A}\times\vec{B})$ Geometrically the scalar triple product is the volume of the parallelepiped formed by the three vectors.

Vector Triple Product: $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$

3 Differential Calculus

3.1 'Ordinary' Derivative:

$\frac{df}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ Geometrically, the derivative is the slope of the tangent to the curve at a point.

3.2 Gradient:

Consider a scalar, T, which exists at every point in space.

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

This can the written in the dot product form as:

$$dT = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z})$$

this is denoted as:

$$dT = \nabla T \cdot d\vec{r}$$

Here, we treat ∇ as an operator. It takes a scalar and returns a vector.

The ∇ operator:

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

3.3 Divergence:

Consider a vector, $\vec{A}$, which exists at every point in space. The divergence of $\vec{A}$ is defined as:

$$\nabla \cdot \vec{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})$$

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

It behaves like a dot product, and is a scalar.

3.4 Curl:

Consider a vector, $\vec{A}$, which exists at every point in space. The curl of $\vec{A}$ is defined as:

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

It behaves like a cross product, and is a vector.

3.5 Second Derivatives:

- **Divergence of a Gradient:** $\nabla \cdot (\nabla T)$

$$\nabla \cdot (\nabla T) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)$$

$$\nabla \cdot (\nabla T) = \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

It is denoted by $\nabla^2 T$ and is called the Laplacian of T.

Note: Laplacian of a vector is theoretically not defined. But when $\nabla^2 \vec{A}$ is written, it means:

$$\nabla^2 \vec{A} = \left(\nabla^2 A_x\right) \hat{x} + \left(\nabla^2 A_y\right) \hat{y} + \left(\nabla^2 A_z\right) \hat{z}$$

- **Curl of a Gradient:** $\nabla \times (\nabla T)$

$$\nabla \times (\nabla T) = 0$$

  Curl of a gradient is always zero.

- **Gradient of Divergence:** $\nabla \left(\nabla \cdot \vec{A}\right)$

- **Divergence of Curl:** $\nabla \cdot \left(\nabla \times \vec{A}\right)$

$$\nabla \cdot \left(\nabla \times \vec{A}\right) = 0$$

  Divergence of a Curl is always zero.

- **Curl of Curl:** $\nabla \times \left(\nabla \times \vec{A}\right)$

$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla \left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}$$

4 Integral Calculus

4.1 Line Integral:

Consider a vector field $\vec{V}$ and a curve C joining points a and b. The line integral of $\vec{V}$ along C is defined as:

$$\int_{aP}^{b} \vec{V} \cdot d\vec{l}$$

where $d\vec{l}$ is the differential displacement along the curve C.

when C is a closed figure, it is written as:

$$\oint \vec{V} \cdot d\vec{l}$$

4.2 Surface Integral:

Consider a vector field $\vec{V}$ and a surface S. The surface integral of $\vec{V}$ over S is defined as:

$$\int_S \vec{V} \cdot d\vec{a}$$

where $d\vec{a}$ is the differential area vector of the surface S.

When S is a closed surface:

$$\oint \vec{V} \cdot d\vec{a}$$

4.3 Volume Integral:

Consider a scalar field T and a volume V. The volume integral of T over V is defined as:

$$\int_V TdV$$

If we consider a vector field $\vec{A}$, then:

$$\int \vec{A}dV$$

$$\hat{x}\int A_x dV + \hat{y}\int A_y dV + \hat{z}\int A_z dV$$

In Cartesian coordinates:

$$dV = dxdydz$$

In Spherical Polar coordinates:

$$dV = r^2 \sin(\theta) dr d\theta d\phi$$

4.4 Fundamental Theorem of Calculus:

If $f(x)$ is a continuous function and $F(x)$ is the anti-derivative of $f(x)$, then:

$$\int_a^b f(x) = F(b) - F(a)$$

- **Fundamental Theorem for Gradients:**
  Consider a scalar field T and a curve C, from $\vec{a}$ to $\vec{b}$, then:

  $$\int_{\vec{a}}^{\vec{b}} \nabla T \cdot d\vec{l} = T(\vec{b}) - T(\vec{a})$$

  If the curve is a closed curve, then:

  $$\int_{\vec{a}}^{\vec{b}} \nabla T \cdot d\vec{l} = 0$$

- **Fundamental Theorem for Divergence:** a.k.a Gauss' Theorem, Green's Theorem or divergence theorem. Consider a vector field $\vec{A}$ and a volume V, then:

  $$\int_V \left(\vec{\nabla} \cdot \vec{v}\right) = \oint \vec{v}.\vec{ds}$$

  where ds is the differential area vector of the surface S.

- **Fundamental Theorem for Curl:** a.k.a Stokes' Theorem. Consider a vector field $\vec{A}$ and a surface S, then:

$$\int_S \left(\vec{\nabla} \times \vec{v}\right) \vec{da} = \oint \vec{v}.\vec{dl}$$

where dl is the differential displacement vector along the boundary curve C of the surface S.

5 Dirac Delta Function:

5.1 One-Dimensional Dirac Delta Function:

It is a function that is zero everywhere except at $x = 0$ and has an integral of 1. It is defined as:

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

Mathematically, it is not a function, i.e., don't tell your maths profs that its a function.

5.2 Three-Dimensional Dirac Delta Function:

It is a function that is zero everywhere except at $\vec{r} = \vec{r_0}$ and has a volume integral of 1. It is defined as:

$$\delta^3(\vec{r}) = \begin{cases} \infty & (x, y, z) = (0, 0, 0) \\ 0 & \text{everywhere else} \end{cases}$$

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

and

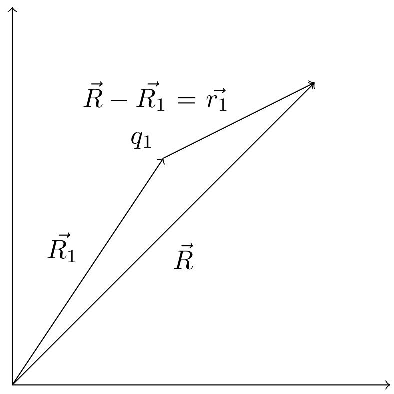
$$\int_{\text{all space}} \delta^3\left(\vec{r}\right) d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z) dxdydz$$

Generalizing the function:

$$\int_{\text{all space}} f(r)\delta^3\left(r - a\right) d\tau = f(a)$$

6 Electrostatics

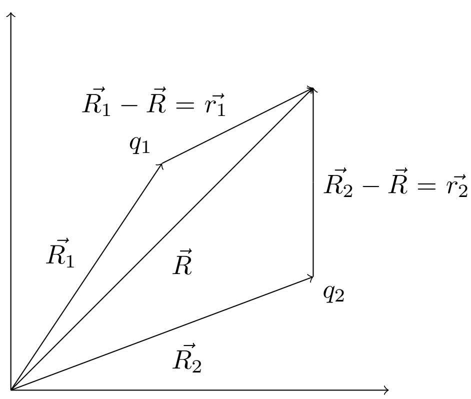
6.1 Electric Field



The Electric Field at $\vec{R}$ due to q_1 is given by:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1^2}\hat{r_1}$$

If there is another charge present:



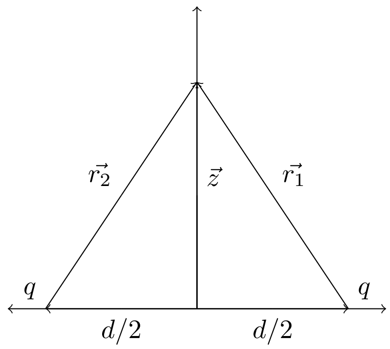
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_1^2}\hat{r_1} + \frac{q_2}{r_1^2}\hat{r_2}\right)$$

Principle of Superposition:

The net electric field at $\vec{R}$ due to n charges is given by:

$$E = \frac{1}{4\pi\epsilon_0}\left(\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r_i}\right)$$

Question:



Consider two charges, q, placed at $r = \pm\frac{d}{2}$ respectively. Find the electric field at z.

Answer

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\left(\frac{d}{2}\right)^2 + z^2}\hat{r_1} + \frac{q}{\left(\frac{d}{2}\right)^2 + +z^2}\hat{r_2}\right)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{z^2 + \frac{d^2}{4}}\right)\left(\frac{z}{\sqrt{z^2 + \frac{d^2}{4}}}\hat{z}\right)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{2qz}{\left(z^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}}\right)\hat{z}$$

Now, if z is very large (but not infinity):

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{z^2}\right)\left(1 + \frac{d^2}{4z^2}\right)^{\frac{-3}{2}}\hat{z}$$

this can be approximated to:

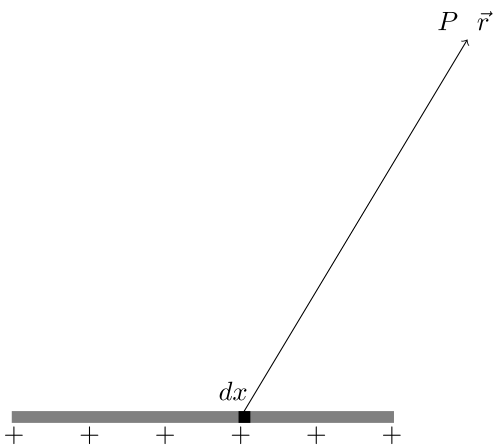
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{z^2}\right)\left(1 - \frac{3d^2}{8z^2}\right)\hat{z}$$

Similar methods can be used to find the electric field for any discrete charge distribution.

6.2 Continuous Charge Distribution

6.2.1 Linear Charge Distribution

Consider a linear Charge distribution, λ of length l, along the x axis. The electric field at point P is given by:



The Electric Field at point P due to dx is given by:

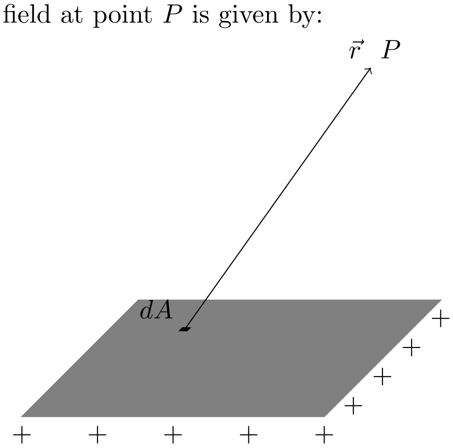
$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda dx}{r^2}\hat{r}$$

Hence, the electric field at point P due to the entire linear charge distribution is given by:

$$\vec{E} = \int_{\text{length } l} \frac{1}{4\pi\epsilon_0}\frac{\lambda dx}{r^2}\hat{r}$$

6.2.2 Surface Charge Distribution

Consider a surface charge distribution, σ of area A, on the xy plane. The electric field at point P is given by:



The Electric Field at point P due to dA is given by:

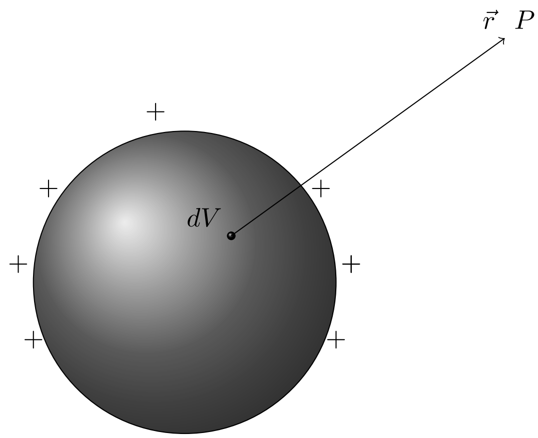
$$d\vec{E} = \frac{1}{4\pi\epsilon}\frac{\sigma dA}{r^2}\hat{r}$$

Hence, the electric field at point P due to the entire surface charge distribution is given by:

$$\vec{E} = \int_{\text{area } A} \frac{1}{4\pi\epsilon} \frac{\sigma dA}{r^2} \hat{r}$$

6.2.3 Volume Charge Distribution

Consider a volume charge distribution, ρ of volume V, in space. The electric field at point P is given by:



The electric field at point P due to dV is given by:

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\rho dV}{r^2} \hat{r}$$

Hence, the electric field at point P due to the entire volume charge distribution is given by:

$$\vec{E} = \int_{\text{volume } V} \frac{1}{4\pi\epsilon} \frac{\rho dV}{r^2} \hat{r}$$

7 Divergence and Curl of Electric Field

7.1 Flux and Gauss' Law

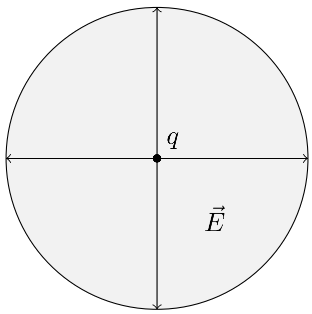
Flux The Electric Flux through a surface S is defined as:

$$\Phi_E = \int_S \vec{E} \cdot \vec{da}$$

Now, if the surface S is a closed surface, then:

$$\Phi_E = \oint \vec{E} \cdot \vec{da}$$

A charge in a sphere Consider a charge q in the centre of a spherical shell:



The flux through the spherical shell would be given by:

$$\Phi_E = \oint \vec{E} \cdot \vec{da}$$

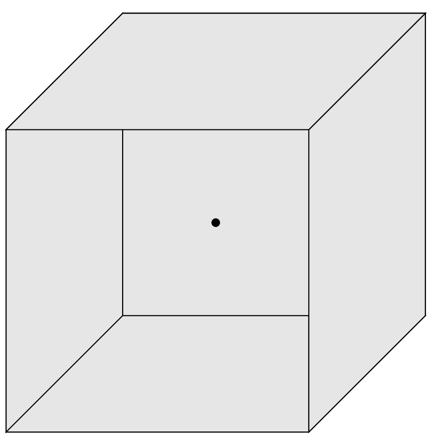
As $\vec{E}$ is constant and is always perpendicular to the surface:

$$\Phi_E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \int_{\text{shere}} dA$$

$$\Phi_E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} (4\pi r^2)$$

$$\Phi_E = \frac{q}{\epsilon_0}$$

A charge in a cube Consider a charge q in the centre of a cubical surface of side $2a$:



Now, the flux through each of the individual surfaces is equal by symmetry. The flux through the surface is:

$$d\Phi_E = \int_{surface} da \frac{1}{4\pi\epsilon_0} \frac{q}{a^2 \sec^2(\theta)} \cos(\theta)$$

$$\Phi_E = \frac{q}{6\epsilon_0}$$

Flux through the entire surface is:

$$\Phi_E = \frac{q}{\epsilon_0}$$

Gauss' Law As seen with the above two examples of the electric flux through a closed surface, the electric flux through a closed surface is proportional to the charge enclosed by the surface. This is known as Gauss' Law. Mathematically, it is written as:

$$\oint \vec{E} \cdot \vec{da} = \frac{Q_{enc}}{\epsilon_0}$$

where Q_{enc} is the charge enclosed by the surface. This is also know as the integral form of Gauss' Law or the integral form of Maxwell's first equation.

To write it in the form of a differential equation:

$$Q_{enc} = \int_V \rho d\tau$$

$$\int_V (\vec{\nabla} \cdot \vec{E}) dV = \oint \vec{E} \cdot \vec{da}$$

$$\int_V (\vec{\nabla} \cdot \vec{E}) dV = \int_V \rho d\tau$$

As this is true for all volumes, we can write:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

This is the differential form of Gauss' Law or Maxwell's first equation.

7.2 Divergence of Electric Field

The Electric Field at a point $\vec{r}$ due to a continuous charge distribution at $\vec{r'}$ is given by:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\hat{\gamma}}{\gamma^2} \rho(\vec{r'}) d\tau'$$

taking the divergence of the above function

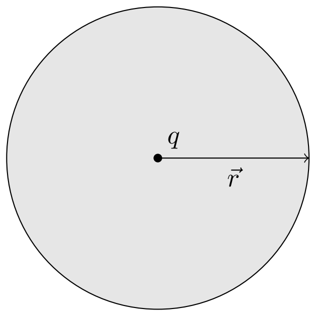
$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \vec{\nabla} \cdot \left(\frac{\hat{\gamma}}{\gamma^2} \right) \rho(\vec{r'}) d\tau'$$

$$\vec{\nabla} \cdot \frac{\hat{\gamma}}{\gamma^2} = 4\pi\delta^3(\gamma)$$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{4\pi\epsilon_0} \int 4\pi\delta^3(r - r')\rho(r') d\tau' = \frac{1}{\epsilon_0}\rho(r)$$

7.2.1 Application of Gauss' Law

- **Spherical Charge Distribution** Consider a spherical charge distribution of radius γ and net charge Q. To find the electric field at a point r :



We know that flux through a sphere of radius r is given by:

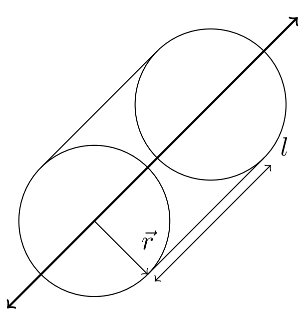
$$\Phi_E = \oint \vec{E} \cdot \vec{da} = \frac{q}{\epsilon_0}$$

$$E \int_{\text{surface}} da = \frac{q}{\epsilon_0}$$

$$E 4\pi r^2 = \frac{q}{\epsilon_0}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

- **Infinitely Long Linear Charge Distribution** Consider an infinitely long linear charge distribution with charge density λ. To find the electric field at a radial point r :



Now, the flux through the cylinder is given by:

$$\Phi_E = \oint \vec{E} \cdot \vec{da} = \frac{\lambda l}{\epsilon_0}$$

Also, the flux through the flat surface is zero as the field is perpendicular to the area at every point on the surface. Hence, the flux through the curved surface is equal to that of the cylinder

$$\Phi_E = \int_{\text{curved surface}} \vec{E} \cdot \vec{da}$$

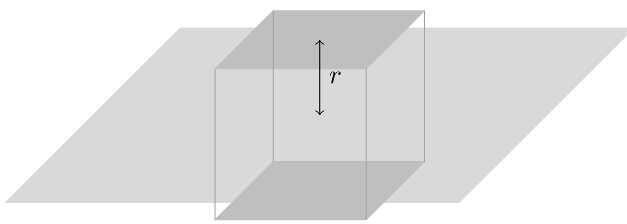
By symmetry, the electric field is constant and is always perpendicular to the surface.

$$\Phi_E = E \int_{\text{curved surface}} da = \frac{\lambda l}{\epsilon_0}$$

$$E 2\pi r l = \frac{\lambda l}{\epsilon_0}$$

$$\vec{E} = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{r} \hat{r}$$

- **Infinitely Large Surface Charge Distribution** Consider an infinitely large surface charge distribution with charge density σ. To find the electric field at a radial point r :



Now the flux through the cuboid is given by:

$$\Phi_E = \oint \vec{E} \cdot \vec{da} = \frac{\sigma A}{\epsilon_0}$$

As the electric field is always parallel to 4 of the surfaces, the flux through them is zero. As the electric field is constant and is always perpendicular to through the remaining surfaces:

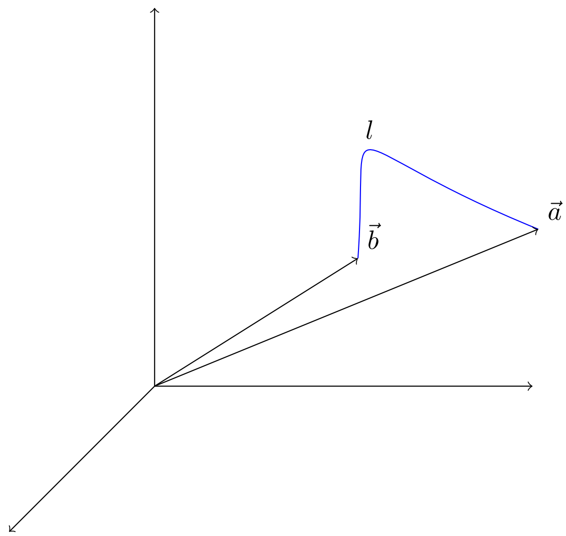
$$\Phi_E = E \int_{\text{surface}} da = \frac{\sigma A}{\epsilon_0}$$

$$E 2A = \frac{\sigma A}{\epsilon_0}$$

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{r}$$

7.3 Curl of Electric Field

Let us consider a path l and an electric field $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$ which exists in 3d space.



taking the line integral of $\vec{E}$ from $\vec{a}$ to $\vec{b}$:

$$\int_a^b \vec{E} \cdot \vec{dl}$$

using spherical coordinates:

$$\vec{E} \cdot \vec{dl} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr$$

$$\int_a^b \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \cdot \vec{dl} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{b} - \frac{q}{a}\right)$$

Now, if we consider a closed path, i.e., $a = b$:

$$\oint \vec{E} \cdot \vec{dl} = 0$$

Hence, by stokes theorem:

$$\vec{\nabla} \times \vec{E} 0$$

8 Electric Potential

As the line integral is independent of the path taken, we can define a function $V(\vec{r})$ such that:

$$V(\vec{r}) = -\int_O^{\vec{r}} \vec{E} \cdot \vec{dl}$$

Here, O is a reference point.

Potential difference between two points $\vec{a}$ and $\vec{b}$ is given by:

$$V(\vec{b}) - V(\vec{a}) = -\int_O^b \vec{E} \cdot \vec{dl} + \int_O^a \vec{E} \cdot \vec{dl} = -\int_a^b \vec{E} \cdot \vec{dl}$$

Now, from Fundamental Theorem of Gradients:

$$V(\vec{b}) - V(\vec{a}) = -\int_a^b \vec{E} \cdot \vec{dl} = \int_a^b \vec{\nabla} V \cdot \vec{dl}$$

$$-\vec{\nabla} V = \vec{E}$$

Combining it with the differential form of Gauss' Law:

$$\vec{\nabla} \cdot \left(\vec{\nabla} V\right) = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

8.1 Properties of Electric Potential

- Potential and Potential Energy are completely different.
- Working with a scalar (as in only one component) is simpler than working with three different components as in the case of Electric Field.
- The reference point O is arbitrary and changing it will only add a constant k to the potential. While Subtracting, the will get cancelled out.
- Potential also obeys the Principle of Superposition
- Unit: Joule per Coulomb a.k.a. Volt

8.2 Electric Potential due to Continuous Charge distribution

8.2.1 Line Charge

Consider a line charge distribution l with uniform charge distribution λ

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_l \frac{\lambda}{r} dl$$

8.2.2 Surface Charge

Consider a surface charge distribution S with uniform charge distribution σ

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma}{r}$$

8.2.3 Volume Charge

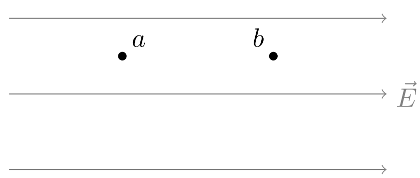
Consider a volume charge distribution V with uniform charge distribution ρ

$$V(r) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho}{r}$$

9 Work and Energy in Electrostatics

9.1 Work done by Electric Field

Consider a charge q slowly move from point a to point b in an electric field $\vec{E}$.



The work done to move from a to b is given by:

$$W = \int_a^b \vec{F}\cdot\vec{dl}$$

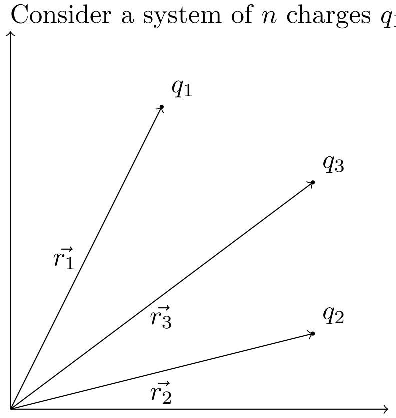
$$W = -q\int_a^b \vec{E}\cdot\vec{dl}$$

$$W = -q\left[V(a) - V(b)\right]$$

As shown above, the work done is independent of the path taken. Therefore, Electrostatic force is a conservative force.

9.2 Energy of a System of Charges

Consider a system of n charges q_1, q_2, q_3 at points $\vec{r_1}, \vec{r_2}, \vec{r_3}$ respectively.



The work done to bring all the charges from ∞ to their respective positions is given by:

$$W = \sum_{i=1}^{n} \sum_{j-1, j>i}^{n} \frac{q_i q_j}{r_{ij}}$$

9.2.1 Energy of a Continuous charge distribution

Consider a continuous charge distribution ρ in space.
The work done to bring all the charges from ∞ to their respective positions is given by:

$$W = \frac{1}{2} \int \rho V d\tau$$

This can also be written as:

$$W = \frac{\epsilon_0}{2} \left(\int_V E^2 d\tau + \oint_S V E \cdot da \right)$$

If we apply this to all space, the surface integral vanishes and the Volume integral remains finite:

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 d\tau$$

Example To find the self energy of a sphere with radius R and uniform q

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 d\tau$$

$$W = \frac{\epsilon_0}{2} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} E^2 r^2 \sin(\theta) dr d\theta d\phi$$

$$W = \frac{4\pi\epsilon_0}{2} \int_0^{\infty} E^2 r^2 dr$$

$$W = \frac{4\pi\epsilon_0}{2} \left[\int_R^{\infty} \left(\frac{\rho 4\pi R^3}{3 \cdot 4\pi\epsilon_0 r^2} \right)^2 r^2 dr + \int_0^R \left(\frac{\rho r}{3\epsilon_0} \right)^2 r^2 dr \right]$$

$$W = \frac{4\pi\epsilon_0}{2} \left[\frac{\rho^2 R^6}{3^2 \epsilon_0^2} \int_R^{\infty} \frac{1}{r^4} r^2 dr + \frac{\rho^2}{3^2 \epsilon_0^2} \int_0^R r^2 \cdot r^2 dr \right]$$

$$W = \frac{4\pi\epsilon_0}{2} \left[\frac{\rho^2 R^6}{3^2 \epsilon_0^2} \int_R^{\infty} \frac{1}{r^2} dr + \frac{\rho^2}{3^2 \epsilon_0^2} \int_0^R r^4 dr \right]$$

$$W = \frac{4\pi\epsilon_0}{2} \left[\frac{\rho^2 R^6}{3^2 \epsilon_0^2} \left. \frac{-1}{r} \right|_R^{\infty} + \frac{\rho^2}{3^2 \epsilon_0^2} \left. \frac{r^5}{5} \right|_0^R \right]$$

$$W = \frac{4\pi\epsilon_0}{2} \left[\frac{-\rho^2 R^6}{3^2 \epsilon_0^2} (0 - \frac{1}{R}) + \frac{\rho^2}{3^2 \epsilon_0^2} \left(\frac{R^5}{5} - 0 \right) \right]$$

$$W = \frac{4\pi\epsilon_0}{2}\left[\frac{\rho^2 R^5}{3^2\epsilon_0^2} + \frac{\rho^2}{3^2\epsilon_0^2}\frac{R^5}{5}\right]$$

$$W = \frac{4\pi\epsilon_0}{2}\frac{\rho^2 R^5}{3^2\epsilon_0^2}\left(1 + \frac{1}{5}\right)$$

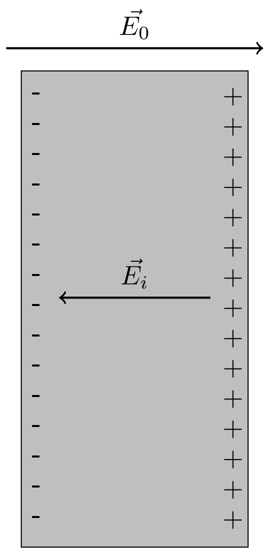
$$W = \frac{4\pi\epsilon_0}{2}\frac{q^2}{\frac{4^2}{3^2}\pi^2 R^6}\frac{R^5}{3^2\epsilon_0^2}\frac{6}{5}$$

$$W = \frac{3}{5}\frac{1}{4\pi\epsilon_0}\frac{q^2}{R}$$

10 Conductors

Properties of an ideal conductor:

- **E=0 inside the conductor** If there was an electric field, the charges (electrons) inside the conductor would move such that there is a positive charge on one part of the conductor and negative charge on the other side of the conductor. This would create an electric field opposite to that of the external field and we would get no net field in the bulk of the conductor



  Here, $\vec{E}_0$ is the external field and $\vec{E}_i$ is the induced field.

- $\rho = 0$ **inside the conductor** From Gauss' Law, if there is no field inside the conductor, then there is no charge inside the conductor.

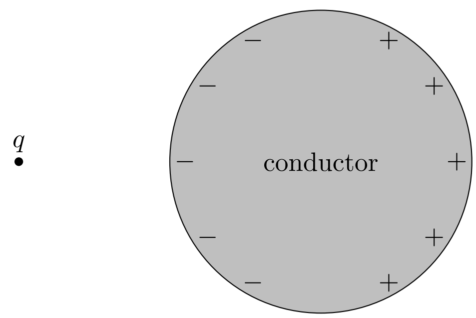
- **Any net charge resides on the surface** If there was a net charge inside the conductor, there would be an electric field inside the conductor. This would cause the charges to move until there is no net field inside the conductor.

- **A conductor is equipotential** Consider two points in or on the surface of the conductor. $V(b) - V(a) = \int_a^b -E \cdot dl = 0$. $\therefore V(a) = V(b)$.

- $\vec{E}$ **is perpendicular to the surface just outside the conductor**. If otherwise, charge will flow along the surface of the conductor and will nullify any tangential component of the electric field.

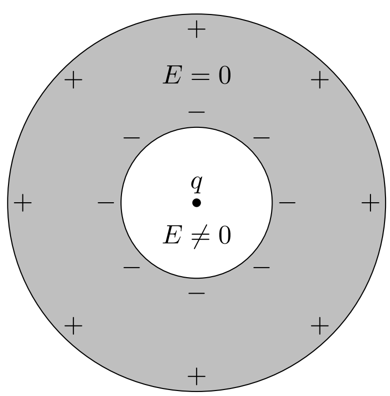
10.1 Induced charges on a conductor

If we hold a point charge q near an uncharged conductor:



Since the negative charges are closer to the charge q, the conductor will get attracted to it.

If there is a cavity inside the conductor, then it will be electrically isolated from the outside of the conductor. Now, if we have a conductor with a cavity, and place a charge q inside the cavity:



11 Capacitors

When we have two conductors, and give a charge $+q$ to one and $-q$ to the other:

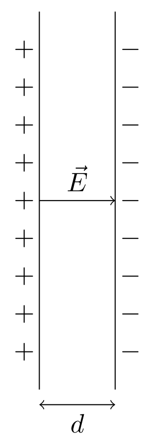
$$V = V_+ - V_- = \int_{(-)}^{(+)} -E \cdot dl$$

now, E is proportional to q. Hence, V is proportional to q.
This proportionality constant is called capacitance C.

$$C = \frac{q}{V}$$

C depends completely on the geometry of the conductors and the medium between them. It is independent of the charge on the conductors. The SI unit of capacitance is Farad (F).

Parallel Plate Capacitor: Consider two conducting parallel parallel plates of area A and separation d. If we place a charge $+q$ on one plate and $-q$ on the other, the electric field between the plates is given by:



The field between the plates is uniform and is given by:

$$E = \frac{\sigma}{\epsilon_0}$$

Therefore, the potential difference between the plates is given by:

$$V = \int_{(-)}^{(+)} -E \cdot dl = -E \cdot d = -\frac{\sigma d}{\epsilon_0}$$

The capacitance C is given by:

$$C = \frac{q}{V} = \frac{\sigma A}{\frac{\sigma d}{\epsilon_0}} = \frac{\epsilon_0 A}{d}$$

Spacial Techniques

12 Laplace's Equation

Many times, its a pain in the ass to find the electric field, potential, or charge density using the normal equations. Therefore we use the differential form of the problem to get an analytical view of the problem:

$$\nabla^2 V = -\frac{1}{\epsilon}\rho$$

This is Poisson's Equation. If we take a region in space such that $\rho = 0$ (obviously not 0 everywhere), we get Laplace's Equation:

$$\nabla^2 V = 0$$

12.1 Laplace's Equation in One-dimension

$$\frac{d^2V}{dx^2} = 0$$
$$V(x) = mx + b$$

This is an equation of a straight line in One-dimension.

- It means that $V(x)$ is the average of $V(x-a)$ and $V(x+a)$.
- V has no local minima or maxima

12.2 Laplace's Equation in Two-dimension

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = 0$$

The properties are similar to those in One-dimension. We define average here over a circle of radius R:

$$V(x,y) = \frac{1}{2\pi R}\oint_{circle} V dl$$

12.3 Laplace's Equation in Three-dimension

The properties are the same as 1D Average is defined as:

$$V(x,y,z) = \frac{1}{4\pi R^2}\oint_{sphere} V da$$

13 Boundary Conditions and Uniqueness Theorem

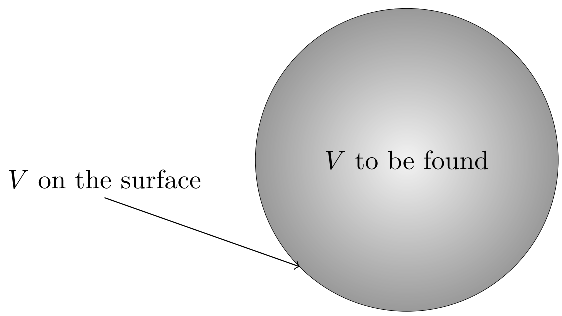
With a few boundary conditions with the Laplace's Equation, we can solve for V. THis leads to the Uniqueness Theorems.

13.1 First Uniqueness Theorem

The solution to the Laplace's Equation in some volume v is uniquely dereminable if the value of V is specified on the boundary surface S.

Proof

Consider a closed figure and the potentials on its boundary is known:



If there were 2 different solutions:

$$\nabla^2 V_1 = 0$$
$$\nabla^2 V_2 = 0$$

taking their difference:

$$V_1 - V_2 \equiv V_3$$
$$\nabla^2 V_3 = 0$$

at the boundary of the figure: now, as by Laplace Equation, we can't have local extremum, the only solution is $V_3 = 0$. Hence, $V_1 = V_2$

13.2 Second Uniqueness Theorem

In a volume v surrounded by conductor containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given.
Proof is similar to the first uniqueness theorem.

14 Polarization