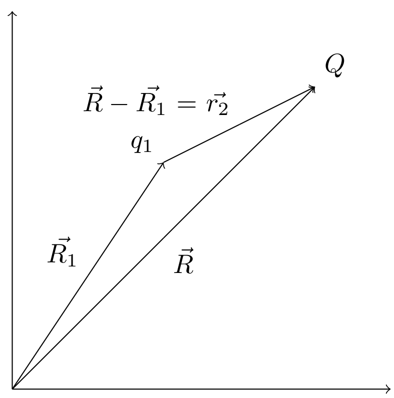
Introduction to Electromagnetism

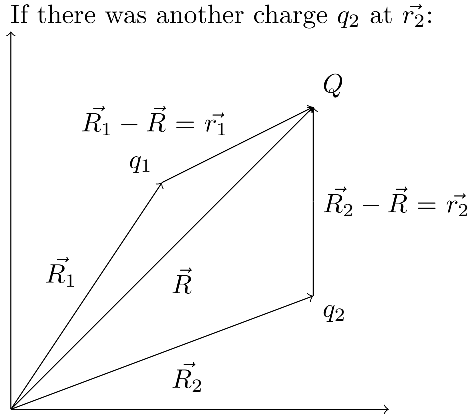
1 Coulomb's Law



the force on Q due to q_1 is given by:

$$\vec{F_1} = \frac{1}{4\pi\epsilon_0}\frac{q_1 Q}{r_1^2}\hat{r_1}$$

If there was another charge q_2 at $\vec{r_2}$:



The force on Q due to q_2 is given by:

$$\vec{F_2} = \frac{1}{4\pi\epsilon_0}\frac{q_2 Q}{r_2^2}\hat{r_2}$$

And the net charge on Q is given by:

$$\vec{F} = \vec{F_1} + \vec{F_2}$$

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 Q}{r_1^2}\hat{r_1} + \frac{1}{4\pi\epsilon_0}\frac{q_2 Q}{r_2^2}\hat{r_2}$$

If there were n charges, the net force on Q would be:

$$\vec{F} = \sum_{i=1}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_i Q}{r_i^2}\hat{r_i}$$

$$\vec{F} = Q \sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_i}{r_i^2} \hat{r}_i$$

where $\sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_i}{r_i^2} \hat{r}_i$ is the electric field $\vec{E}$ at $\vec{R}$ due to the n charges.

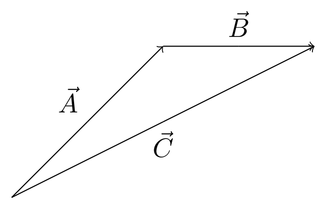
Field A value, vector or tensor, that is defined for every point in space and time.

2 Vector Calculus

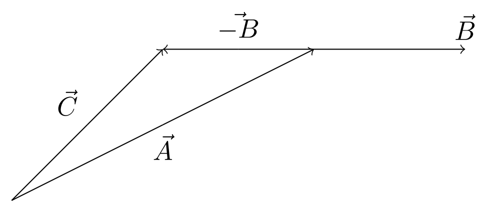
Vectors A vector is a quantity that is said to transform like a displacement.

2.1 Operations on Vectors

- **Addition**: $\vec{A} + \vec{B} = \vec{C}$



- **Subtraction:** $\vec{A} - \vec{B} = \vec{C}$



- **Multiplication by a scalar:** $c\vec{A} = \vec{C}$
- **Dot Product:** $\vec{A}.\vec{B} = AB\cos(\theta)$. Dot product is a scalar. It is commutative and distributive.
- **Cross Product:** $\vec{A} \times \vec{B} = AB\sin(\theta)\hat{n}$ where $\hat{n}$ is the unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.

Cross product is a vector. It is anti-commutative and distributive.

2.2 Component form of a Vector:

A vector $\vec{A}$ can be written as:

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

where $\hat{\imath}, \hat{\jmath}, \hat{k}$ are unit vectors along the x, y, z axes respectively.

Given that $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$, we can perform the following operations:

- **Addition (and Subtraction):** $\vec{A} + \vec{B} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + (A_z + B_z)\hat{k}$
- **Multiplication by a scalar:** $c\vec{A} = cA_x\hat{\imath} + cA_y\hat{\jmath} + cA_z\hat{k}$, where c is a scalar.
- **Dot Product:** $\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$
- **Modulus:** $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$
- **Cross Product:** $\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{\imath} + (A_zB_x - A_xB_z)\hat{\jmath} + (A_xB_y - A_yB_x)\hat{k}$ This can also be written in a determinant form:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

2.3 Triple Product:

Scalar Triple Product: $\vec{A}.(\vec{B}\times\vec{C}) = \vec{B}.(\vec{C}\times\vec{A}) = \vec{C}.(\vec{A}\times\vec{B})$ Geometrically the scalar triple product is the volume of the parallelopiped formed by the three vectors.

Vector Triple Product: $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$

3 Differential Calculus

3.1 "Ordinary" Derivative:

$\frac{df}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ Geometrically, the derivative is the slope of the tangent to the curve at a point.

3.2 Gradient:

Consider a scalar, T, which exists at every point in space.

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

This can the written in the dot product form as:

$$dT = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z})$$

this is denoted as:

$$dT = \nabla T \cdot d\vec{r}$$

Here, we treat ∇ as an operator. It takes a scalar and returns a vector.

The ∇ operator:

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

3.3 Divergence:

Consider a vector, $\vec{A}$, which exists at every point in space. The divergence of $\vec{A}$ is defined as:

$$\nabla \cdot \vec{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})$$

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

It behaves like a dot product, and is a scalar.

3.4 Curl:

Consider a vector, $\vec{A}$, which exists at every point in space. The curl of $\vec{A}$ is defined as:

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

It behaves like a cross product, and is a vector.

3.5 Second Derivatives:

- **Divergence of a Gradient:** $\nabla \cdot (\nabla T)$

$$\nabla \cdot (\nabla T) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)$$

$$\nabla \cdot (\nabla T) = \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

It is denoted by $\nabla^2 T$ and is called the Laplacian of T.

Note: Laplacian of a vector is theoretically not defined. But when $\nabla^2 \vec{A}$ is written, it means:

$$\nabla^2 \vec{A} = \left(\nabla^2 A_x\right) \hat{x} + \left(\nabla^2 A_y\right) \hat{y} + \left(\nabla^2 A_z\right) \hat{z}$$

- **Curl of a Gradient:** $\nabla \times (\nabla T)$

$$\nabla \times (\nabla T) = 0$$

  Curl of a gradient is always zero.

- **Gradient of Divergence:** $\nabla \left(\nabla \cdot \vec{A}\right)$

- **Divergence of Curl:** $\nabla \cdot \left(\nabla \times \vec{A}\right)$

$$\nabla \cdot \left(\nabla \times \vec{A}\right) = 0$$

  Divergence of a Curl is always zero.

- **Curl of Curl:** $\nabla \times \left(\nabla \times \vec{A}\right)$

$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla \left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}$$

4 Integral Calculus

4.1 Line Integral:

Consider a vector field $\vec{V}$ and a curve C joining points a and b. The line integral of $\vec{V}$ along C is defined as:

$$\int_{aP}^{b} \vec{V} \cdot d\vec{l}$$

where $d\vec{l}$ is the differential displacement along the curve C.

when C is a closed figure, it is written as:

$$\oint \vec{V} \cdot d\vec{l}$$

4.2 Surface Integral:

Consider a vector field $\vec{V}$ and a surface S. The surface integral of $\vec{V}$ over S is defined as:

$$\int_S \vec{V} \cdot d\vec{a}$$

where $d\vec{a}$ is the differential area vector of the surface S.

When S is a closed surface:

$$\oint \vec{V} \cdot d\vec{a}$$

4.3 Volume Integral:

Consider a scalar field T and a volume V. The volume integral of T over V is defined as:

$$\int_V T dV$$

If we consider a vector field $\vec{A}$, then:

$$\int \vec{A} dV$$

$$\hat{x} \int A_x dV + \hat{y} \int A_y dV + \hat{z} \int A_z dV$$

In Cartesian coordinates:

$$dV = dxdydz$$

In Spherical Polar coordinates:

$$dV = r^2 \sin(\theta) dr d\theta d\phi$$

4.4 Fundamental Theorem of Calculus:

If $f(x)$ is a continuous function and $F(x)$ is the anti-derivative of $f(x)$, then:

$$\int_a^b f(x) = F(b) - F(a)$$

- **Fundamental Theorem for Gradients:**
  Consider a scalar field T and a curve C, from $\vec{a}$ to $\vec{b}$, then:

  $$\int_{\vec{a}}^{\vec{b}} \nabla T \cdot d\vec{l} = T(\vec{b}) - T(\vec{a})$$

  If the curve is a closed curve, then:

  $$\int_{\vec{a}}^{\vec{b}} \nabla T \cdot d\vec{l} = 0$$

- **Fundamental Theorem for Divergence:** a.k.a Gauss' Theorem, Green's Theorem or divergence theorem. Consider a vector field $\vec{A}$ and a volume V, then:

  $$\int_V \left(\vec{\nabla} \cdot \vec{v}\right) = \oint \vec{v} . \vec{ds}$$

  where ds is the differential area vector of the surface S.

- **Fundamental Theorem for Curl:** a.k.a Stokes' Theorem. Consider a vector field $\vec{A}$ and a surface S, then:

$$\int_S \left(\vec{\nabla} \times \vec{v}\right) \vec{da} = \oint \vec{v}.\vec{dl}$$

where dl is the differential displacement vector along the boundary curve C of the surface S.

5 Dirac Delta Function:

5.1 One-Dimensional Dirac Delta Function:

It is a function that is zero everywhere except at $x = 0$ and has an integral of 1. It is defined as:

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

Mathematically, it is not a function. i.e. don't tell your maths profs that its a function.

5.2 Three-Dimensional Dirac Delta Function:

It is a function that is zero everywhere except at $\vec{r} = \vec{r_0}$ and has a volume integral of 1. It is defined as:

$$\delta^3(\vec{r}) = \begin{cases} \infty & (x, y, z) = (0, 0, 0) \\ 0 & \text{everywhere else} \end{cases}$$

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

and

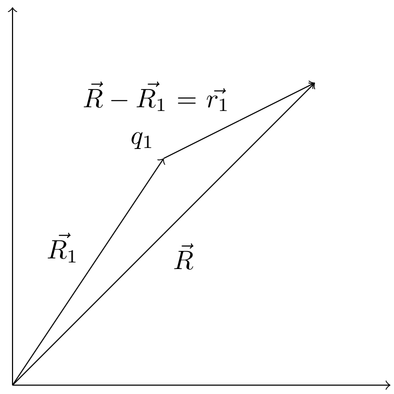
$$\int_{\text{all space}} \delta^3 \left(\vec{r}\right) d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z) dxdydz$$

Generalizing the function:

$$\int_{\text{all space}} f(r)\delta^3 \left(r - a\right) d\tau = f(a)$$

6 Electrostatics

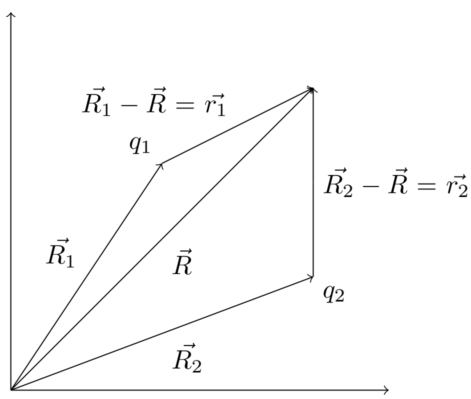
6.1 Electric Field



The Electric Field at $\vec{R}$ due to q_1 is given by:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1^2}\hat{r_1}$$

If there is another charge present:



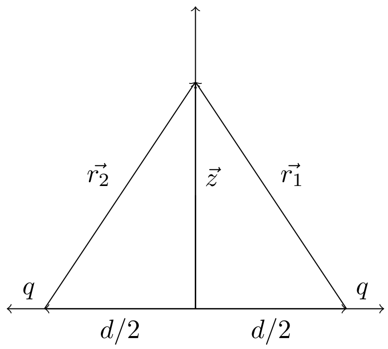
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_1^2}\hat{r_1} + \frac{q_2}{r_1^2}\hat{r_2}\right)$$

Principle of Superposition:

The net electric field at $\vec{R}$ due to n charges is given by:

$$E = \frac{1}{4\pi\epsilon_0}\left(\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r_i}\right)$$

Question:



Consider two charges, q, placed at $r = \pm\frac{d}{2}$ respectively. Find the electric field at z.

Answer

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\left(\frac{d}{2}\right)^2 + z^2}\hat{r_1} + \frac{q}{\left(\frac{d}{2}\right)^2 + +z^2}\hat{r_2}\right)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{z^2 + \frac{d^2}{4}}\right)\left(\frac{z}{\sqrt{z^2 + \frac{d^2}{4}}}\hat{z}\right)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{2qz}{\left(z^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}}\right)\hat{z}$$

Now, if z is very large (but not infinity):

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{z^2}\right)\left(1 + \frac{d^2}{4z^2}\right)^{\frac{-3}{2}}\hat{z}$$

this can be approximated to:

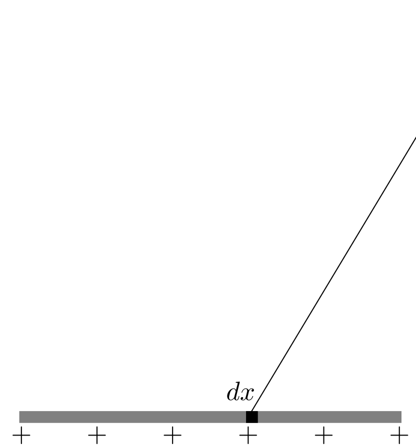
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{2q}{z^2}\right)\left(1 - \frac{3d^2}{8z^2}\right)\hat{z}$$

Similar methods can be used to find the electric field for any discrete charge distribution.

6.2 Continuous Charge Distribution

6.2.1 Linear Charge Distribution

Consider a linear Charge distribution, λ of length l, along the x axis. The electric field at point P is given by:



The Electric Field at point P due to dx is given by:

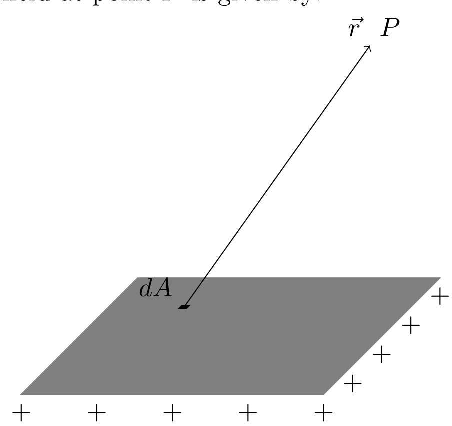
$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda dx}{r^2}\hat{r}$$

Hence, the electric field at point P due to the entire linear charge distribution is given by:

$$\vec{E} = \int_{\text{length } l} \frac{1}{4\pi\epsilon_0}\frac{\lambda dx}{r^2}\hat{r}$$

6.2.2 Surface Charge Distribution

Consider a surface charge distribution, σ of area A, on the xy plane. The electric field at point P is given by:



The Electric Field at point P due to dA is given by:

$$d\vec{E} = \frac{1}{4\pi\epsilon}\frac{\sigma dA}{r^2}\hat{r}$$

Hence, the electric field at point P due to the entire surface charge distribution is given by:

$$\vec{E} = \int_{\text{area } A} \frac{1}{4\pi\epsilon} \frac{\sigma dA}{r^2} \hat{r}$$

6.2.3 Volume Charge Distribution