Rings and Modules

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Chapter 1

Introduction to Rings

1.1 Definition of a Ring

A ring R is a set with two binary operations, $+$ and $\times$, satisfying the following conditions:

- $(R, +)$ is an abelian group.
- $\times$ is associative.
- $\times$ distributes over $+$.

A Ring is said to be commutative if $a \times b = b \times a$ for all $a, b \in R$.
A Ring is said to have a multiplicative identity if there exists an element $1 \in R$ such that $1 \times a = a \times 1 = a$ for all $a \in R$.

1.1.1 Examples

- **Trivial Ring:** Take any abelian group $(G, +)$ and define multiplication as $a \times b = 0$ for all $a, b \in G$, where 0 is the identity of the group.
- **Integers:** The set of integers $\mathbb{Z}$ with usual addition and multiplication forms a ring. Also, the quotient group $\mathbb{Z}/n\mathbb{Z}$ is a ring for any integer n.
- **Hamiltonian Quaternions:** The set of quaternions $\mathbb{H} = 1, i, j, k$, where $i^2 = j^2 = k^2 = -1$.
- **Polynomial Rings:**