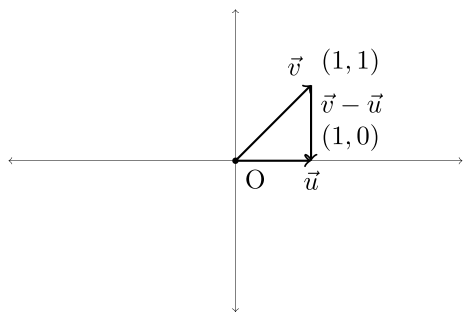
Calculus II

1 Vectors

$\mathbb{R}$ represents the set of real numbers.
$\mathbb{R}^2$ represents a 2 dimensional real plane.

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$



Normally elements of $\mathbb{R}$ are known as scalers.

· add (or subtract) two vectors
· if $c \in \mathbb{R}$ and $v \in \mathbb{R}^2$, $c\vec{v}$

1.1 Dot Product

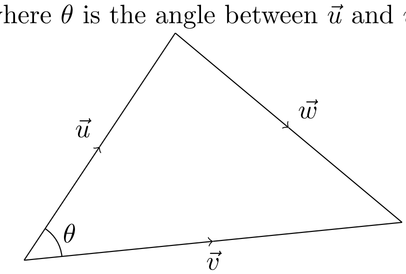
$u = (u_1, u_2)$ and $v = (v_1, v_2)$

$$u.v = u_1.v_1 + u_2.v_2$$

Theorem

$$u.v = |u||v|\cos(\theta)$$

where θ is the angle between $\vec{u}$ and $\vec{v}$ and $|u|$ is the length of vector $\vec{u}$



Proof:

$$w^2 = u^2 + v^2 - 2|u||v|\cos(\theta) \ldots (1)$$

$$\vec{w} = \vec{v} - \vec{u}$$

$$w = (v_1 - u_1, v_2 - u_2)$$

$$w^2 = (v_1 - u_1)^2 + (v_2 - u_2)^2$$

$$w^2 = v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2$$

$$w^2 = v^2 + u^2 - 2v_1u_1 - 2v_2u_2 \ldots (2)$$

now, as (1) = (2)

$$u^2 + v^2 - 2|u||v|\cos(\theta) = v^2 + u^2 - 2v_1u_1 - 2v_2u_2$$

$$|u||v|\cos(\theta) = v_1u_1 + v_2u_2$$

$$|u||v|\cos(\theta) = \vec{u}.\vec{v}$$

Hence proved.

Extending the above theorem to $\mathbb{R}^n$: Consider $\vec{u} = (u_1, u_2, \ldots, u_n), \vec{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}$

$$\vec{u}.\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

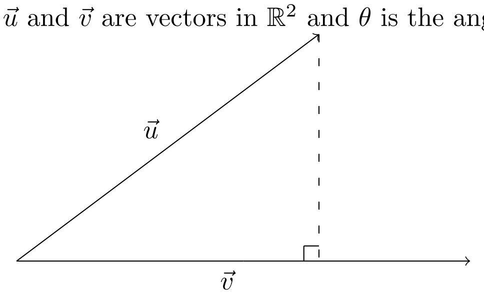
1.2 Unit Vectors

If $\vec{v} \in \mathbb{R}^2$ is a vector. then,

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

1.3 Projections

If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^2$ and θ is the angle between them:



let $\vec{w}$ be the projection of $\vec{u}$ on $\vec{v}$

$$\vec{w} = |u|\cos(\theta)\hat{v}$$

$$\vec{w} = \frac{|u|\cos(\theta)|v|\vec{v}}{|v|^2}$$

$$\vec{w} = \frac{\vec{u}.\vec{v}}{|\vec{v}|^2}\vec{v}$$

1.4 Cross Product

Consider the vectors, u, v. Then the cross-product of u and v is defined as:

$$u \times v = |u||v|\sin(\theta)\hat{n}$$

where $\hat{n}$ is the unit vector perpendicular to the plane containing u and v, and also $(\vec{u}, \vec{v}, \hat{n})$ form a right handed system.

Properties of Cross Product:

- $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- $(r\vec{u}) \times (s\vec{v}) = rs\vec{u} \times \vec{v}$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

$\vec{u}$ and $\vec{v}$ can also be represented as:

$$\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$$

$$\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\hat{i} + (u_3v_1 - u_1v_3)\hat{i} + (u_1v_2 - u_2v_1)\hat{k}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

the above formula is only symbolic and meant to represent a cross product.

2 Multi-Variable Calculus

let $r : \mathbb{R} \to \mathbb{R}^3$ be a function. (It could be to any $\mathbb{R}^n$)
for $t \in \mathbb{R}$, $r(t)$ is a vector in $\mathbb{R}^3$

$$r(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

2.1 Continuity

r is continuous at a if:

$$\lim_{t \to a} r(t) = r(a)$$

i.e. $\forall\epsilon > 0, \exists\delta > 0$ such that $|t - a| < \delta \implies |r(t) - r(a)| < \epsilon$
or, if f, g and h are continuous at a, r is continuous at a.

2.2 Differentiability

let $r : \mathbb{R} \to \mathbb{R}^3$ be a function. (We are taking $mathbbR^3$ here, but it could be any $\mathbb{R}^n$)
the derivative of r at a is defined as:

$$r'(a) = \lim_{t \to a} \frac{r(t) - r(a)}{t - a}$$

if

$$r(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$
$$r'(t) = f'(t)\hat{i} + g'(t)\hat{j} + h'(t)\hat{k}$$

2.3 Integration

let $f : [a, b] \to \mathbb{R}$ be a function.

the integral of f from a to b is defined as:

$$\int_a^b f(x)dx = \text{Area under the curve } y = f(x) \text{ from } x = a \text{ to } x = b$$

Anti-derivative let $f : [a, b] \to \mathbb{R}$ be a function.

$$\int f(t)dt = F(t) + c$$

such that $F'(t) = f(t)$ and c is an arbitrary constant

Fundamental Theorem of Calculus let $f : [a, b] \to \mathbb{R}$ be a function and F be its anti-derivative.

$$\int_a^b f(t)dt = F(b) - F(a)$$

2.3.1 Extending to a Vector Valued Function

Consider a function $r : \mathbb{R} \to \mathbb{R}^3$ (we are taking $\mathbb{R}^3$ as an example, it could be any $\mathbb{R}^n$)

$$r(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$
$$\int_a^b r(t)dt = \int_a^b f(t)dt\hat{i} + \int_a^b g(t)dt\hat{j} + \int_a^b h(t)dt\hat{k}$$

Anti-derivative

$$\int r(t)dt = \left(\int f(t)dt + c_1\right)\hat{i} + \left(\int g(t)dt + c_2\right)\hat{j} + \left(\int h(t)dt + c_3\right)\hat{k}$$

$$\int r(t)dt = R(t) + C$$

where $R'(t) = r(t)$ and C is an arbitrary constant vector.

$$R(t) = F(t)\hat{i} + G(t)\hat{j} + H(t)\hat{k}$$

where $F'(t) = f(t)$, $G'(t) = g(t)$ and $H'(t) = h(t)$

2.3.2 Length of a Curve

let $r : [a,b] \to \mathbb{R}^3$ be a function. (As usual, we are taking $\mathbb{R}^3$, but it can be any $\mathbb{R}^n$)
the length of the curve l from a to b is defined as:

$$l = \int_a^b |r'(t)|dt$$

3 Multi-Variable Functions

let $f : D \to \mathbb{R}$ be a function. Let $D \subseteq \mathbb{R}^n$. Then f is called a Multi Variable Function
Eg:

$$f(x,y) = \frac{1}{x^2 + y^2}$$

Domain of f is $\mathbb{R}^2 - \{(0,0)\}$

3.1 Limits

let $f : D \to \mathbb{R}$ be a function. and $D \subseteq \mathbb{R}^2$. (we are taking $\mathbb{R}^2$ but it can be extended to any $\mathbb{R}^n$)

$$\lim_{(x,y)\to(x_1,y_1)} f(x,y) = L$$

if $\forall \epsilon > 0, \exists \delta > 0$ such that $|(x,y) - (x_1,y_1)| < \delta \implies |f(x,y) - L| < \epsilon$
Eg:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$$

$$\lim_{(x,y)\to(0,0)} \frac{x(x-y)}{\sqrt{x} - \sqrt{y}}$$

$$\lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y})$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = 0$$

Eg:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$$

this limit does not exist.

3.2 Derivatives

The usual definition of derivatives can't be extended to multi-variable functions.
Eg:

$$f(x,y) = x$$

$$f'(x) = \lim_{(x,y)\to(x_1,y_1)} \frac{f(x,y)-f(x_1,y_1)}{|(x,y)-(x_1,y_1)|}$$

if we keep y constant, Then the above limit becomes:

$$f'(x,y_0) = \lim_{(x,y_0)\to(x_1,y_0)} \frac{x-x_1}{|x-x_1|}$$

which does not exist.

Hence, we define partial derivatives.
let $f : D \to \mathbb{R}$ be a function. and $D \subseteq \mathbb{R}^2$. (we are taking $\mathbb{R}^2$ but it can be extended to any $\mathbb{R}^n$)
the partial derivative of f with respect to x at (x_0, y_0) is defined as:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \frac{f(x,y)-f(x_0,y_0)}{x-x_0}$$

Example

$$f(x,y) = \begin{cases} 0, xy = 0 \\ 1, xy \neq 0 \end{cases}$$

The above function seems to be continuous if we approach $(0,0)$ along the x or y axes. But if we approach it along any other line passing through the origin, the limit doesn't exist. Therefore, $f(x,y)$ is not continuos at $(0,0)$.

But the partial derivatives wrt x and y still exist:

$$\left.\frac{\partial f}{\partial x}\right|_{(}0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(}0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = 0$$

3.2.1 Chain Rule

If $w = f(x, y)$ is differentiable and $x = x(t)$ and $y = y(t)$ are differentiable functions of t, then the derivative of w with respect to t is given by:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Now, this can be extended to n variable function $w = \psi(a, b, c, \dots)$ by:

$$\frac{dw}{dt} = \frac{\partial \psi}{\partial a}\frac{da}{dt} + \frac{\partial \psi}{\partial b}\frac{db}{dt} + \frac{\partial \psi}{\partial c}\frac{dc}{dt} + \dots$$

3.3 Differentiability

Let $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be differentiable at (x_0, y_0) if Δz satisfies the following equation of the form:

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

in which both $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$

3.3.1 Directional Derivative

Let $f(x, y)$ be a function in 2 variables and $\vec{u} = u_1\hat{i} + u_2\hat{j}$ be a vector in $\mathbb{R}^2$.

The directional derivative of f along u at $P_0(x_0, y_0)$:

$$(D_u f)_{P_0} = \lim_{t\to 0} \frac{f\left(x_0 + tu_1, y_0 + tu_2\right) + f\left(x_0, y_0\right)}{t}$$

(The denominator should be $t|u|$ but we don't take it at such for some unknown reason)

Taking the same example as before: if $(u_1, u_2) \neq (0, 0)$

$$(D_u f)_{(0,0)} = \lim_{t\to 0} \frac{f\left(tu_1, tu_2\right) - f(0,0)}{t}$$

$$= \lim_{t\to 0} = \frac{1}{t}$$

But this limit doesn't exist.

Another method to evaluate the directional derivative is by using gradients: let $u = u_1\hat{i} + u_2\hat{j}$

$$(D_u f)_{P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0}\frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\frac{dy}{ds}$$

now, $\frac{dx}{ds} = u_1$ and $\frac{dy}{ds} = u_2$

$$(D_u f)_{P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2$$

$$(D_u f)_{P_0} = (\nabla f)_{P_0} \cdot u$$