MT3244 - Calculus on Manifolds

Nachiketa Kulkarni

Contents

Chapter 1

Parameterized Curves and Surfaces (in $\mathbb{R}^n$)

1.1 Paramterized Curves

Definition A parameterized curve is a continiuous function

$$f : I \to \mathbb{R}^n$$

where I is an open set in $\mathbb{R}$.

1.1.1 Examples

1. Straight line: $f : \mathbb{R} \to \mathbb{R}^2$ defined as $f(\alpha) = (\alpha, m\alpha + c)$ where m and c are constants.
2. Parabola: $f : \mathbb{R} \to \mathbb{R}^2$ defined as $f(\alpha) = (\alpha, m\alpha^2)$ where m is a constant.
3. Circle: $f : \mathbb{R} \to \mathbb{R}^2$ defined as $f(\alpha) = (\cos\alpha, \sin\alpha)$.
    - Function is bounded.
    - There is no polynomial paramterization of this curve.
    - Rational Parameterizations exists:

    $$f(t) = \left(\frac{t^2 - 1}{t + 1}, \frac{2t}{t + 1} \right)$$

4. Not involving Modulus: $f : \mathbb{R} \to \mathbb{R}^2$ defined as $f(\alpha) = (\alpha^3, \alpha^2)$.

1.1.2 Differentiation Curves:

A function f which is differentiable for a $t \in I$ is called a Parameterized Differentiable Curves [for all possible paramterizations]. Let $f(t) = (f_1(t), f_2(t), \ldots, f_n(t))$, then its derivative is defined as: $f'(t) = (f_1'(t), f_2'(t), \ldots, f_n'(t))$. This is used to define the direction of the tangent at a point $f(t)$ as $f'(t)$.

Regular Curve: A differentiable curve such that the tangent vector is non-zero for all $t \in I$

1.2 Paramterized Surfaces

Definition A parameterized curve is a continuous function

$$f : I \to \mathbb{R}^n$$

where I is an open set in $\mathbb{R}^2$. In our case, $n = 3$.

1.2.1 Examples

1. Affine Linear Transformation: $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined as $f(u, v) = (p_1 + q_1 u + r_1 v, p_2 + q_2 u + r_2 v, p_3 + q_3 u + r_3 v)$. Let $q = (q_1, q_2, q_3)$ and $r = (r_1, r_2, r_3)$ are:
   - 0: The space is a fixed point.
   - Linearly Dependent: The space is a line.
   - Linearly Independent: The space is a plane.
2. Sphere: $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined as $f(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)$. There is no polynomial parameterization to a sphere.
3. Cone: $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined as $f(u, v) = (v \cos u, v \sin u, v)$. Polynomial Parameterization exists as follows:
$$(u, v) \to (u^2 - v^2, 2v, u^2 + v^2)$$

1.2.2 Differentiable Surfaces