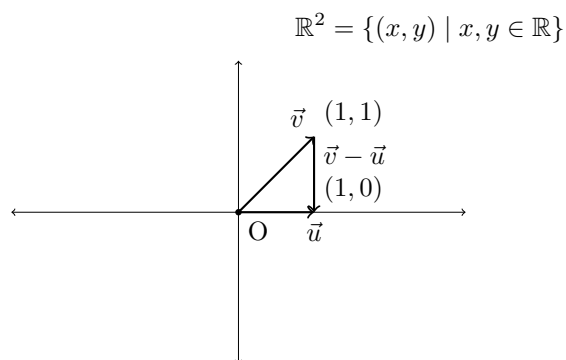


# Calculus II

## 1 Vectors

$\mathbb{R}$  represents the set of real numbers.

$\mathbb{R}^2$  represents a 2 dimensional real plane.



Normally elements of  $\mathbb{R}$  are known as scalars.

- add (or subtract) two vectors
- if  $c \in \mathbb{R}$  and  $v \in \mathbb{R}^2$ ,  $c\vec{v}$

### 1.1 Dot Product

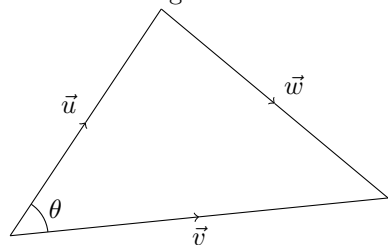
$u = (u_1, u_2)$  and  $v = (v_1, v_2)$

$$u.v = u_1.v_1 + u_2.v_2$$

#### Theorem

$$u.v = |u||v| \cos(\theta)$$

where  $\theta$  is the angle between  $\vec{u}$  and  $\vec{v}$  and  $|u|$  is the length of vector  $\vec{u}$



Proof:

$$w^2 = u^2 + v^2 - 2|u||v| \cos(\theta) \dots (1)$$

$$\vec{w} = \vec{v} - \vec{u}$$

$$w = (v_1 - u_1, v_2 - u_2)$$

$$w^2 = (v_1 - u_1)^2 + (v_2 - u_2)^2$$

$$w^2 = v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2$$

$$w^2 = v^2 + u^2 - 2v_1u_1 - 2v_2u_2 \dots (2)$$

now, as (1) = (2)

$$u^2 + v^2 - 2|u||v|\cos(\theta) = v^2 + u^2 - 2v_1u_1 - 2v_2u_2$$

$$|u||v|\cos(\theta) = v_1u_1 + v_2u_2$$

$$|u||v|\cos(\theta) = \vec{u} \cdot \vec{v}$$

Hence proved.

Extending the above theorem to  $\mathbb{R}^n$ : Consider  $\vec{u} = (u_1, u_2, \dots, u_n), \vec{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n$$

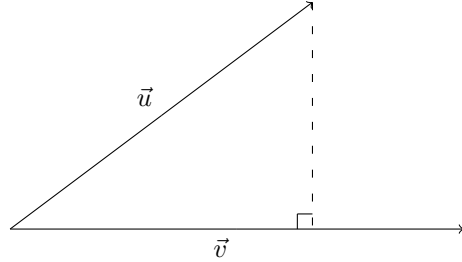
## 1.2 Unit Vectors

If  $\vec{v} \in \mathbb{R}^2$  is a vector. then,

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

## 1.3 Projections

If  $\vec{u}$  and  $\vec{v}$  are vectors in  $\mathbb{R}^2$  and  $\theta$  is the angle between them:



let  $\vec{w}$  be the projection of  $\vec{u}$  on  $\vec{v}$

$$\vec{w} = |u|\cos(\theta)\hat{v}$$

$$\vec{w} = \frac{|u|\cos(\theta)|v|\vec{v}}{|v|^2}$$

$$\vec{w} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$$

## 1.4 Cross Product

Consider the vectors,  $u, v$ . Then the cross-product of  $u$  and  $v$  is defined as:

$$u \times v = |u||v| \sin(\theta) \hat{n}$$

where  $\hat{n}$  is the unit vector perpendicular to the plane containing  $u$  and  $v$ , and also  $(\vec{u}, \vec{v}, \hat{n})$  form a right handed system.

### Properties of Cross Product:

- $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- $(r\vec{u}) \times (s\vec{v}) = rs\vec{u} \times \vec{v}$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

$\vec{u}$  and  $\vec{v}$  can also be represented as:

$$\vec{u} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k}$$

$$\vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$$

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2) \hat{i} + (u_3 v_1 - u_1 v_3) \hat{j} + (u_1 v_2 - u_2 v_1) \hat{k}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

the above formula is only symbolic and meant to represent a cross product.

## 2 Multi-Variable Calculus

let  $r : \mathbb{R} \rightarrow \mathbb{R}^3$  be a function. (It could be to any  $\mathbb{R}^n$ )  
for  $t \in \mathbb{R}$ ,  $r(t)$  is a vector in  $\mathbb{R}^3$

$$r(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$

### 2.1 Continuity

$r$  is continuous at  $a$  if:

$$\lim_{t \rightarrow a} r(t) = r(a)$$

i.e.  $\forall \epsilon > 0, \exists \delta > 0$  such that  $|t - a| < \delta \implies |r(t) - r(a)| < \epsilon$   
or, if  $f, g$  and  $h$  are continuous at  $a$ ,  $r$  is continuous at  $a$ .

## 2.2 Differentiability

let  $r : \mathbb{R} \rightarrow \mathbb{R}^3$  be a function. (We are taking  $\mathbb{R}^3$  here, but it could be any  $\mathbb{R}^n$ )  
the derivative of  $r$  at  $a$  is defined as:

$$r'(a) = \lim_{t \rightarrow a} \frac{r(t) - r(a)}{t - a}$$

if

$$\begin{aligned} r(t) &= f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k} \\ r'(t) &= f'(t)\hat{i} + g'(t)\hat{j} + h'(t)\hat{k} \end{aligned}$$

## 2.3 Integration

let  $f : [a, b] \rightarrow \mathbb{R}$  be a function.

the integral of  $f$  from  $a$  to  $b$  is defined as:

$$\int_a^b f(x)dx = \text{Area under the curve } y = f(x) \text{ from } x = a \text{ to } x = b$$

**Anti-derivative** let  $f : [a, b] \rightarrow \mathbb{R}$  be a function.

$$\int f(t)dt = F(t) + c$$

such that  $F'(t) = f(t)$  and  $c$  is an arbitrary constant

**Fundamental Theorem of Calculus** let  $f : [a, b] \rightarrow \mathbb{R}$  be a function and  $F$  be its anti-derivative.

$$\int_a^b f(t)dt = F(b) - F(a)$$

### 2.3.1 Extending to a Vector Valued Function

Consider a function  $r : \mathbb{R} \rightarrow \mathbb{R}^3$  (we are taking  $\mathbb{R}^3$  as an example, it could be any  $\mathbb{R}^n$ )

$$\begin{aligned} r(t) &= f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k} \\ \int_a^b r(t)dt &= \int_a^b f(t)dt\hat{i} + \int_a^b g(t)dt\hat{j} + \int_a^b h(t)dt\hat{k} \end{aligned}$$

### Anti-derivative

$$\int r(t)dt = \left( \int f(t)dt + c_1 \right) \hat{i} + \left( \int g(t)dt + c_2 \right) \hat{j} + \left( \int h(t)dt + c_3 \right) \hat{k}$$

$$\int r(t)dt = R(t) + C$$

where  $R'(t) = r(t)$  and  $C$  is an arbitrary constant vector.

$$R(t) = F(t)\hat{i} + G(t)\hat{j} + H(t)\hat{k}$$

where  $F'(t) = f(t)$ ,  $G'(t) = g(t)$  and  $H'(t) = h(t)$

### 2.3.2 Length of a Curve

let  $r : [a, b] \rightarrow \mathbb{R}^3$  be a function. (As usual, we are taking  $\mathbb{R}^3$ , but it can be any  $\mathbb{R}^n$ )

the length of the curve  $l$  from  $a$  to  $b$  is defined as:

$$l = \int_a^b |r'(t)|dt$$

## 3 Multi-Variable Functions

let  $f : D \rightarrow \mathbb{R}$  be a function. Let  $D \subseteq \mathbb{R}^n$ . Then  $f$  is called a Multi Variable Function

Eg:

$$f(x, y) = \frac{1}{x^2 + y^2}$$

Domain of  $f$  is  $\mathbb{R}^2 - \{(0, 0)\}$

### 3.1 Limits

let  $f : D \rightarrow \mathbb{R}$  be a function. and  $D \subseteq \mathbb{R}^2$ . (we are taking  $\mathbb{R}^2$  but it can be extended to any  $\mathbb{R}^n$ )

$$\lim_{(x,y) \rightarrow (x_1, y_1)} f(x, y) = L$$

if  $\forall \epsilon > 0, \exists \delta > 0$  such that  $|(x, y) - (x_1, y_1)| < \delta \implies |f(x, y) - L| < \epsilon$

Eg:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$$
$$\lim_{(x,y) \rightarrow (0,0)} \frac{x(x-y)}{\sqrt{x} - \sqrt{y}}$$

$$\lim_{(x,y) \rightarrow (0,0)} x(\sqrt{x} + \sqrt{y})$$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = 0$$

Eg:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^4 + y^2}$$

this limit does not exist.

### 3.2 Derivatives

The usual definition of derivatives can't be extended to multi-variable functions.

Eg:

$$f(x, y) = x$$

$$f'(x) = \lim_{(x,y) \rightarrow (x_1, y_1)} \frac{f(x, y) - f(x_1, y_1)}{|(x, y) - (x_1, y_1)|}$$

if we keep y constant, Then the above limit becomes:

$$f'(x, y_0) = \lim_{(x, y_0) \rightarrow (x_1, y_0)} \frac{x - x_1}{|x - x_1|}$$

which does not exist.

Hence, we define partial derivatives.

let  $f : D \rightarrow \mathbb{R}$  be a function. and  $D \subseteq \mathbb{R}^2$ . (we are taking  $\mathbb{R}^2$  but it can be extended to any  $\mathbb{R}^n$ )

the partial derivative of  $f$  with respect to  $x$  at  $(x_0, y_0)$  is defined as:

$$\left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} = \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

#### Example

$$f(x, y) = \begin{cases} 0, & xy = 0 \\ 1, & xy \neq 0 \end{cases}$$

The above function seems to be continuous if we approach  $(0, 0)$  along the x or y axes. But if we approach it along any other line passing through the origin, the limit doesn't exist. Therefore,  $f(x, y)$  is not continuous at  $(0, 0)$ .

But the partial derivatives wrt  $x$  and  $y$  still exist:

$$\left. \frac{\partial f}{\partial x} \right|_{(0,0)} = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x} = 0$$

$$\left. \frac{\partial f}{\partial y} \right|_{(0,0)} = \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y} = 0$$

### 3.2.1 Chain Rule

If  $w = f(x, y)$  is differentiable and  $x = x(t)$  and  $y = y(t)$  are differentiable functions of  $t$ , then the derivative of  $w$  with respect to  $t$  is given by:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Now, this can be extended to  $n$  variable function  $w = \psi(a, b, c, \dots)$  by:

$$\frac{dw}{dt} = \frac{\partial \psi}{\partial a} \frac{da}{dt} + \frac{\partial \psi}{\partial b} \frac{db}{dt} + \frac{\partial \psi}{\partial c} \frac{dc}{dt} + \dots$$

### 3.3 Differentiability

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is said to be differentiable at  $(x_0, y_0)$  if  $\Delta z$  satisfies the following equation of the form:

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

in which both  $\epsilon_1, \epsilon_2 \rightarrow 0$  as both  $\Delta x, \Delta y \rightarrow 0$

#### 3.3.1 Directional Derivative

Let  $f(x, y)$  be a function in 2 variables and  $\vec{u} = u_1\hat{i} + u_2\hat{j}$  be a vector in  $\mathbb{R}^2$ .

The directional derivative of  $f$  along  $u$  at  $P_0(x_0, y_0)$ :

$$(D_u f)_{P_0} = \lim_{t \rightarrow 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t}$$

(The denominator should be  $t|u|$  but we don't take it at such for some unknown reason)

Taking the same example as before: if  $(u_1, u_2) \neq (0, 0)$

$$\begin{aligned}(D_u f)_{(0,0)} &= \lim_{t \rightarrow 0} \frac{f(tu_1, tu_2) - f(0, 0)}{t} \\ &= \lim_{t \rightarrow 0} \frac{1}{t}\end{aligned}$$

But this limit doesn't exist.

Another method to evaluate the directional derivative is by using gradients:  
let  $u = u_1\hat{i} + u_2\hat{j}$

$$(D_u f)_{P_0} = \left( \frac{\partial f}{\partial x} \right)_{P_0} \frac{dx}{ds} + \left( \frac{\partial f}{\partial y} \right)_{P_0} \frac{dy}{ds}$$

now,  $\frac{dx}{ds} = u_1$  and  $\frac{dy}{ds} = u_2$

$$\begin{aligned}(D_u f)_{P_0} &= \left( \frac{\partial f}{\partial x} \right)_{P_0} u_1 + \left( \frac{\partial f}{\partial y} \right)_{P_0} u_2 \\ (D_u f)_{P_0} &= (\nabla f)_{P_0} \cdot u\end{aligned}$$

### 3.3.2 Gradients

The gradient vector of a function  $f(x, y)$  at a point  $P_0$  is given by:

$$\nabla_{P_0} f = \left. \frac{\partial f}{\partial x} \right|_{P_0} \hat{i} + \left. \frac{\partial f}{\partial y} \right|_{P_0} \hat{j}$$

Note: This formula can be extended to any  $\mathbb{R}^n$ . Though the geometric meaning may not remain the same.

Also, at every point  $P_0$ , the gradient vector is perpendicular to the tangent plane to that curve at that point  $P_0$ .

## 4 Tangent Planes and Normal Lines

The Tangent Plane at the point  $P_0(x_0, y_0, z_0)$  on the level surface  $f(x, y, z) = c$  of a differentiable function  $f$  is the plane through  $P_0$  and perpendicular to  $\nabla_{P_0} f$ . The Normal Line of  $f$  at the point  $P_0$  is the line through  $P_0$  is the line parallel to  $\nabla_{P_0} f$  and passing through  $P_0$ .

### 4.0.1 Estimation of change in a specific direction

let  $f$  be a function of 2 or more variables and  $u$  be a unit vector. Then the change in  $f$  in the direction of  $u$  is given by:

$$df = (\nabla_{P_0} f \cdot u) ds$$

where  $ds$  is the small change in the direction of  $u$ .

## 5 Extreme Values and Saddle Points

**Local Minima:** In 2D,  $f(x_0, y_0)$  is a local minima if  $f(x_0, y_0) \leq f(x, y)$  for all  $(x, y)$  in some open disk centered at  $(x_0, y_0)$

**Local Maxima:** In 2D,  $f(x_0, y_0)$  is a local maxima if  $f(x_0, y_0) \geq f(x, y)$  for all  $(x, y)$  in some open disk centered at  $(x_0, y_0)$

Geometrically, local minima are valleys bottoms, and local maxima are peaks.

### 5.1 First Derivative Test

If  $P_0(x_0, y_0)$  is a local extremum in the domain of  $f$ , and  $f$  is differentiable at  $P_0$ , and if the first partial derivatives of  $f$  exist at  $P_0$ , then:

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

A point where both the partial derivatives are zero is called a critical point.



## 5.2 Saddle Points

A differentiable function  $f(x, y)$  has a saddle point at  $P_0(x_0, y_0)$  if  $f$  has a critical point  $P_0$  if in every open disk centered at  $P_0$  there are domain points  $(x, y)$  such that  $f(x, y) > f(x_0, y_0)$  and  $f(x, y) < f(x_0, y_0)$

## 5.3 Second Derivative Test

Let  $f(x, y)$  be a function of 2 variables and  $P_0(x_0, y_0)$  be a critical point of  $f$ . If the second partial derivatives of  $f$  exist and are continuous in some open disk centered at  $P_0$ , then:

- $f$  has a local minima at  $P_0$ , if  $f_{xx}f_{yy} - f_{xy}^2 > 0$  and  $f_{xx} > 0$  at point  $P_0$
- $f$  has a local maxima at  $P_0$ , if  $f_{xx}f_{yy} - f_{xy}^2 > 0$  and  $f_{xx} < 0$  at point  $P_0$
- $f$  has a saddle point at  $P_0$ , if  $f_{xx}f_{yy} - f_{xy}^2 < 0$  at point  $P_0$

## 6 Lagrange Multipliers

When we need to find the extremum points of a function whose domain is constrained to a particular subset of a plane

The method of Lagrange Multipliers is as follows:

The local extremum values of  $f(x, y, z)$  whose variables are subject to a constraint  $g(x, y, z) = 0$  are found to be on the surface  $g(x, y, z) = 0$  and following the following differential equation:

$$\nabla f = \lambda \nabla g$$

## 7 Integration

We will consider  $\mathbb{R}^2$  for now, but the same can be extended to  $\mathbb{R}^n$

Consider the rectangle  $R : [a, b] \times [c, d] = \{(x, y) \mid x \in [a, b], y \in [c, d]\}$  in  $\mathbb{R}^2$ .

Let  $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$  be a function.

The analogue of the Riemann Sum for a function of 2 variables is given by: Take a partition of the rectangle  $R$  into partitions:

$$P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

$$Q = \{c = y_0 < y_1 < \cdots < y_{m-1} < y_m = d\}$$

$$P \times Q = \{(x_i, y_j), i = \{1, 2, \dots, n\}, j = \{1, 2, \dots, m\}\}$$

The Riemann Sum is given by:

$$S = \sum_{\alpha=1}^{\eta} f(p_{\alpha}) \Delta A_{\alpha}$$

If the above summation converges to a Real Number as  $n, m \rightarrow \infty$ , regardless of the choice made in forming the Riemann sum, then the function  $f$  is Riemann Integrable over  $R$ .

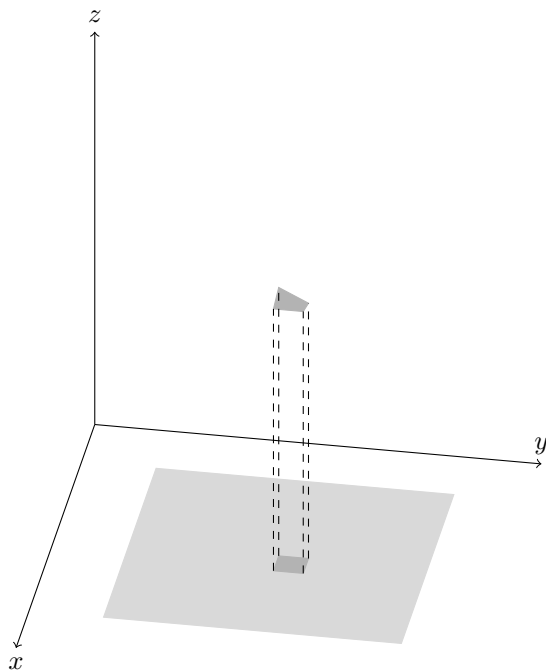
**Note:** When the function  $f$  is continuous in  $R$ , then the Riemann sum will always converge.

**Theorem** If  $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$  is continuous, then:

$$\int_a^b \left( \int_c^d f(x, y) dy \right) dx = \iint_R f(x, y) dA = \int_c^d \left( \int_a^b f(x, y) dx \right) dy$$

## 7.1 Geometric Interpretation

for a function  $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ , the integral of  $f$  over the rectangle  $R$  is the volume of the solid bounded by the surface  $z = f(x, y)$  and the rectangle  $R$ .



## 8 Integral over a General Bounded Region