

Introduction to Electromagnetism

1 Coulomb's Law



$$\vec{F}_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1 Q}{r_1^2} \hat{r}_1$$

If there was another charge q_2 at \vec{r}_2 :



The force on Q due to q_2 is given by:

$$\vec{F}_2 = \frac{1}{4\pi\epsilon_0} \frac{q_2 Q}{r_2^2} \hat{r}_2$$

And the net force on Q is given by:

$$\vec{F} = \vec{F}_1 + \vec{F}_2$$

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 Q}{r_1^2} \hat{r}_1 + \frac{1}{4\pi\epsilon_0} \frac{q_2 Q}{r_2^2} \hat{r}_2$$

If there were n charges, the net force on Q would be:

$$\vec{F} = \sum_{i=1}^n \frac{1}{4\pi\epsilon_0} \frac{q_i Q}{r_i^2} \hat{r}_i$$

$$\vec{F} = Q \sum_{i=1}^n \frac{1}{4\pi\epsilon_0} \frac{q_i}{r_i^2} \hat{r}_i$$

where $\sum_{i=1}^n \frac{1}{4\pi\epsilon_0} \frac{q_i}{r_i^2} \hat{r}_i$ is the electric field \vec{E} at \vec{R} due to the n charges.

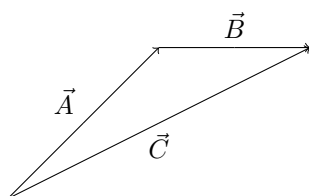
Field: A value, vector or tensor, that is defined for every point in space and time.

2 Vector Calculus

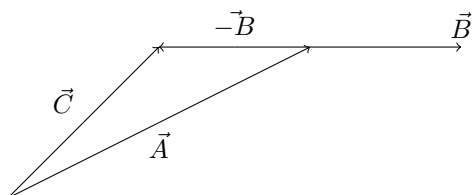
Vectors A vector is a quantity that is said to transform like a displacement.

2.1 Operations on Vectors

- **Addition:** $\vec{A} + \vec{B} = \vec{C}$



- **Subtraction:** $\vec{A} - \vec{B} = \vec{C}$



- **Multiplication by a scalar:** $c\vec{A} = \vec{C}$
- **Dot Product:** $\vec{A} \cdot \vec{B} = AB \cos(\theta)$. Dot product is a scalar. It is commutative and distributive.
- **Cross Product:** $\vec{A} \times \vec{B} = AB \sin(\theta) \hat{n}$ where \hat{n} is the unit vector perpendicular to the plane containing \vec{A} and \vec{B} .

Cross product is a vector. It is anti-commutative and distributive.

2.2 Component form of a Vector:

A vector \vec{A} can be written as:

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the x, y, z axes respectively.

Given that $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ and $\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$, we can perform the following operations:

- **Addition (and Subtraction):** $\vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$
- **Multiplication by a scalar:** $c\vec{A} = cA_x\hat{i} + cA_y\hat{j} + cA_z\hat{k}$, where c is a scalar.
- **Dot Product:** $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$
- **Modulus:** $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$
- **Cross Product:** $\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$ This can also be written in a determinant form:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

2.3 Triple Product:

Scalar Triple Product: $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$ Geometrically the scalar triple product is the volume of the parallelepiped formed by the three vectors.

Vector Triple Product: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

3 Differential Calculus

3.1 "Ordinary" Derivative:

$\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ Geometrically, the derivative is the slope of the tangent to the curve at a point.

3.2 Gradient:

Consider a scalar, T , which exists at every point in space.

$$dT = \left(\frac{\partial T}{\partial x} \right) dx + \left(\frac{\partial T}{\partial y} \right) dy + \left(\frac{\partial T}{\partial z} \right) dz$$

This can be written in the dot product form as:

$$dT = \left(\frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} \right) \cdot (dx \hat{x} + dy \hat{y} + dz \hat{z})$$

this is denoted as:

$$dT = \nabla T \cdot d\vec{r}$$

Here, we treat ∇ as an operator. It takes a scalar and returns a vector.

The ∇ operator:

$$\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$$

3.3 Divergence:

Consider a vector, \vec{A} , which exists at every point in space. The divergence of \vec{A} is defined as:

$$\begin{aligned} \nabla \cdot \vec{A} &= \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \\ \nabla \cdot \vec{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \end{aligned}$$

It behaves like a dot product, and is a scalar.

3.4 Curl:

Consider a vector, \vec{A} , which exists at every point in space. The curl of \vec{A} is defined as:

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

It behaves like a cross product, and is a vector.

3.5 Second Derivatives:

- **Divergence of a Gradient:** $\nabla \cdot (\nabla T)$

$$\begin{aligned} \nabla \cdot (\nabla T) &= \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} \right) \\ \nabla \cdot (\nabla T) &= \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \end{aligned}$$

It is denoted by $\nabla^2 T$ and is called the Laplacian of T .

Note: Laplacian of a vector is theoretically not defined. But when $\nabla^2 \vec{A}$ is written, it means:

$$\nabla^2 \vec{A} = (\nabla^2 A_x) \hat{x} + (\nabla^2 A_y) \hat{y} + (\nabla^2 A_z) \hat{z}$$

- **Curl of a Gradient:** $\nabla \times (\nabla T)$

$$\nabla \times (\nabla T) = 0$$

Curl of a gradient is always zero.

- **Gradient of Divergence:** $\nabla (\nabla \cdot \vec{A})$

- **Divergence of Curl:** $\nabla \cdot (\nabla \times \vec{A})$

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

Divergence of a Curl is always zero.

- **Curl of Curl:** $\nabla \times (\nabla \times \vec{A})$

$$\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

4 Integral Calculus