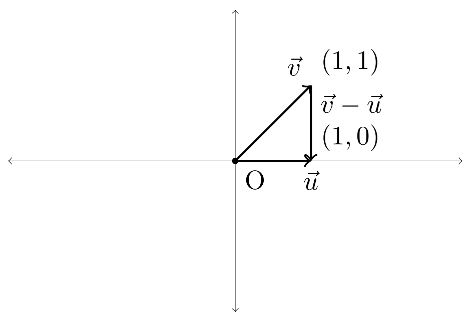
Calculus II

1 Vectors

$\mathbb{R}$ represents the set of real numbers.
$\mathbb{R}^2$ represents a 2 dimensional real plane.

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$



Normally elements of $\mathbb{R}$ are known as scalers.

· add (or subtract) two vectors
· if $c \in \mathbb{R}$ and $v \in \mathbb{R}^2$, $c\vec{v}$

1.1 Dot Product

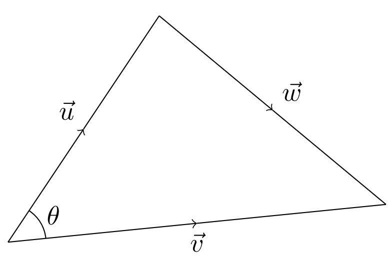
$u = (u_1, u_2)$ and $v = (v_1, v_2)$

$$u.v = u_1.v_1 + u_2.v_2$$

Theorem

$$u.v = |u||v|\cos(\theta)$$

where θ is the angle between $\vec{u}$ and $\vec{v}$ and $|u|$ is the length of vector $\vec{u}$



Proof:

$$
\begin{aligned}
w^2 &= u^2 + v^2 - 2|u||v|\cos(\theta) \dots (1) \\
\vec{w} &= \vec{v} - \vec{u} \\
w &= (v_1 - u_1, v_2 - u_2) \\
w^2 &= (v_1 - u_1)^2 + (v_2 - u_2)^2 \\
w^2 &= v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2 \\
w^2 &= v^2 + u^2 - 2v_1u_1 - 2v_2u_2 \dots (2)
\end{aligned}
$$

now, as (1) = (2)

$$
\begin{aligned}
u^2 + v^2 - 2|u||v|\cos(\theta) &= v^2 + u^2 - 2v_1u_1 - 2v_2u_2 \\
|u||v|\cos(\theta) &= v_1u_1 + v_2u_2 \\
|u||v|\cos(\theta) &= \vec{u}.\vec{v}
\end{aligned}
$$

Hence proved.

Extending the above theorem to $\mathbb{R}^n$: Consider $\vec{u} = (u_1, u_2, \dots, u_n), \vec{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}$

$$\vec{u}.\vec{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n$$

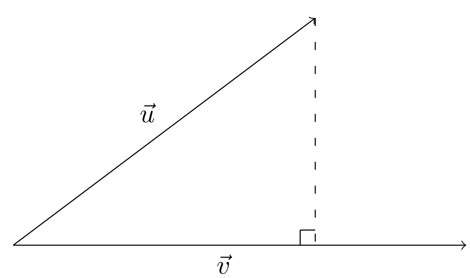
1.2 Unit Vectors

If $\vec{v} \in \mathbb{R}^2$ is a vector. then,

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

1.3 Projections

If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^2$ and θ is the angle between them:



let $\vec{w}$ be the projection of $\vec{u}$ on $\vec{v}$

$$\vec{w} = |u|\cos(\theta)\hat{v}$$
$$\vec{w} = \frac{|u|\cos(\theta)|v|\vec{v}}{|v|^2}$$
$$\vec{w} = \frac{\vec{u}.\vec{v}}{|\vec{v}|^2}\vec{v}$$

1.4 Cross Product

Consider the vectors, u, v. Then the cross-product of u and v is defined as:

$$u \times v = |u||v|\sin(\theta)\hat{n}$$

where $\hat{n}$ is the unit vector perpendicular to the plane containing u and v, and also $(\vec{u}, \vec{v}, \hat{n})$ form a right handed system.

Properties of Cross Product:

- $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- $(r\vec{u}) \times (s\vec{v}) = rs\vec{u} \times \vec{v}$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

$\vec{u}$ and $\vec{v}$ can also be represented as:

$$\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$$

$$\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\hat{i} + (u_3v_1 - u_1v_3)\hat{i} + (u_1v_2 - u_2v_1)\hat{k}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

the above formula is only symbolic and meant to represent a cross product.

2 Multi-Variable Calculus

let $r : \mathbb{R} \to \mathbb{R}^3$ be a function. (It could be to any $\mathbb{R}^n$)
for $t \in \mathbb{R}$, $r(t)$ is a vector in $\mathbb{R}^3$

$$r(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

2.1 Continuity

r is continuous at a if:

$$\lim_{t\to a} r(t) = r(a)$$

i.e. $\forall\epsilon > 0, \exists\delta > 0$ such that $|t - a| < \delta \implies |r(t) - r(a)| < \epsilon$

or, if f, g and h are continuous at a, r is continuous at a.

2.2 Differentiability

let $r : \mathbb{R} \to \mathbb{R}^3$ be a function. (We are taking $\mathbb{R}^3$ here, but it could be any $\mathbb{R}^n$) the derivative of r at a is defined as:

$$r'(a) = \lim_{t\to a} \frac{r(t) - r(a)}{t - a}$$

if

$$r(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$
$$r'(t) = f'(t)\hat{\imath} + g'(t)\hat{\jmath} + h'(t)\hat{k}$$

2.3 Integration

let $f : [a, b] \to \mathbb{R}$ be a function.

the integral of f from a to b is defined as:

$$\int_a^b f(x)dx = \text{Area under the curve } y = f(x) \text{ from } x = a \text{ to } x = b$$

Anti-derivative let $f : [a, b] \to \mathbb{R}$ be a function.

$$\int f(t)dt = F(t) + c$$

such that $F'(t) = f(t)$ and c is an arbitrary constant

Fundamental Theorem of Calculus let $f : [a, b] \to \mathbb{R}$ be a function and F be its anti-derivative.

$$\int_a^b f(t)dt = F(b) - F(a)$$

2.3.1 Extending to a Vector Valued Function

Consider a function $r : \mathbb{R} \to \mathbb{R}^3$ (we are taking $\mathbb{R}^3$ as an example, it could be any $\mathbb{R}^n$)

$$r(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$
$$\int_a^b r(t)dt = \int_a^b f(t)dt\hat{\imath} + \int_a^b g(t)dt\hat{\jmath} + \int_a^b h(t)dt\hat{k}$$

Anti-derivative

$$\int r(t)dt = \left(\int f(t)dt + c_1\right)\hat{i} + \left(\int g(t)dt + c_2\right)\hat{j} + \left(\int h(t)dt + c_3\right)\hat{k}$$

$$\int r(t)dt = R(t) + C$$

where $R'(t) = r(t)$ and C is an arbitrary constant vector.

$$R(t) = F(t)\hat{i} + G(t)\hat{j} + H(t)\hat{k}$$

where $F'(t) = f(t)$, $G'(t) = g(t)$ and $H'(t) = h(t)$

2.3.2 Length of a Curve

let $r : [a, b] \to \mathbb{R}^3$ be a function. (As usual, we are taking $\mathbb{R}^3$, but it can be any $\mathbb{R}^n$)
the length of the curve l from a to b is defined as:

$$l = \int_a^b |r'(t)|dt$$

3 Multi-Variable Functions

let $f : D \to \mathbb{R}$ be a function. Let $D \subseteq \mathbb{R}^n$. Then f is called a Multi Variable Function
Eg:

$$f(x, y) = \frac{1}{x^2 + y^2}$$

Domain of f is $\mathbb{R}^2 - \{(0, 0)\}$

3.1 Limits

let $f : D \to \mathbb{R}$ be a function. and $D \subseteq \mathbb{R}^2$. (we are taking $\mathbb{R}^2$ but it can be extended to any $\mathbb{R}^n$)

$$\lim_{(x,y)\to(x_1,y_1)} f(x, y) = L$$

if $\forall \epsilon > 0, \exists \delta > 0$ such that $|(x, y) - (x_1, y_1)| < \delta \implies |f(x, y) - L| < \epsilon$
Eg:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$$

$$\lim_{(x,y)\to(0,0)} \frac{x(x - y)}{\sqrt{x} - \sqrt{y}}$$

$$\lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y})$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = 0$$

Eg:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$$

this limit does not exist.

3.2 Derivatives

The usual definition of derivatives can't be extended to multi-variable functions.
Eg:

$$f(x,y) = x$$

$$f'(x) = \lim_{(x,y)\to(x_1,y_1)} \frac{f(x,y)-f(x_1,y_1)}{|(x,y)-(x_1,y_1)|}$$

if we keep y constant, Then the above limit becomes:

$$f'(x,y_0) = \lim_{(x,y_0)\to(x_1,y_0)} \frac{x-x_1}{|x-x_1|}$$

which does not exist.

Hence, we define partial derivatives.
let $f : D \to \mathbb{R}$ be a function. and $D \subseteq \mathbb{R}^2$. (we are taking $\mathbb{R}^2$ but it can be extended to any $\mathbb{R}^n$)
the partial derivative of f with respect to x at (x_0, y_0) is defined as:

$$\frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} = \frac{f(x,y)-f(x_0,y_0)}{x-x_0}$$

Example

$$f(x,y) = \begin{cases} 0, xy = 0 \\ 1, xy \neq 0 \end{cases}$$

The above function seems to be continuous if we approach $(0,0)$ along the x or y axes. But if we approach it along any other line passing through the origin, the limit doesn't exist. Therefore, $f(x,y)$ is not continuos at $(0,0)$.

But the partial derivatives wrt x and y still exist:

$$\frac{\partial f}{\partial x}\bigg|_(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = 0$$

$$\frac{\partial f}{\partial y}\bigg|_(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = 0$$

3.2.1 Chain Rule

If $w = f(x, y)$ is differentiable and $x = x(t)$ and $y = y(t)$ are differentiable functions of t, then the derivative of w with respect to t is given by:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Now, this can be extended to n variable function $w = \psi(a, b, c, \dots)$ by:

$$\frac{dw}{dt} = \frac{\partial \psi}{\partial a}\frac{da}{dt} + \frac{\partial \psi}{\partial b}\frac{db}{dt} + \frac{\partial \psi}{\partial c}\frac{dc}{dt} + \dots$$

3.3 Differentiability

Let $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be differentiable at (x_0, y_0) if Δz satisfies the following equation of the form:

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

in which both $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$

3.3.1 Directional Derivative

Let $f(x, y)$ be a function in 2 variables and $\vec{u} = u_1\hat{i} + u_2\hat{j}$ be a vector in $\mathbb{R}^2$.

The directional derivative of f along u at $P_0(x_0, y_0)$:

$$(D_u f)_{P_0} = \lim_{t \to 0} \frac{f\left(x_0 + tu_1, y_0 + tu_2\right) + f\left(x_0, y_0\right)}{t}$$

(The denominator should be $t|u|$ but we don't take it at such for some unknown reason)

Taking the same example as before: if $(u_1, u_2) \neq (0, 0)$

$$(D_u f)_{(0,0)} = \lim_{t \to 0} \frac{f\left(tu_1, tu_2\right) - f(0, 0)}{t}$$

$$= \lim_{t \to 0} = \frac{1}{t}$$

But this limit doesn't exist.

Another method to evaluate the directional derivative is by using gradients: let $u = u_1\hat{i} + u_2\hat{j}$

$$(D_u f)_{P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0} \frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \frac{dy}{ds}$$

now, $\frac{dx}{ds} = u_1$ and $\frac{dy}{ds} = u_2$

$$(D_u f)_{P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2$$

$$(D_u f)_{P_0} = (\nabla f)_{P_0} \cdot u$$

3.3.2 Gradients

The gradient vector of a function $f(x, y)$ at a point P_0 is given by:

$$\nabla_{P_0} f = \frac{\partial f}{\partial x}\bigg|_{P_0} \hat{i} + \frac{\partial f}{\partial y}\bigg|_{P_0} \hat{j}$$

Note: This formula can be extended to any $\mathbb{R}^n$. Though the geometric meaning may not remain the same.

Also, at every point P_0, the gradient vector is perpendicular to the tangent plane to that curve at that point P_0.

4 Tangent Planes and Normal Lines

The Tangent Plane at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 and perpendicular to $\nabla_{P_0} f$. The Normal Line of f at the point P_0 is the line through P_0 is the line parallel to $\nabla_{P_0} f$ and passing through P_0.

4.0.1 Estimation of change in a specific direction

let f be a function of 2 or more variables and u be a unit vector. Then the change in f in the direction of u is given by:

$$df = (\nabla_{P_0} f \cdot u)\, ds$$

where ds is the small change in the direction of u.

5 Extreme Values and Saddle Points

Local Minima: In 2D, $f(x_0, y_0)$ is a local minima if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in some open disk centered at (x_0, y_0)
Local Maxima: In 2D, $f(x_0, y_0)$ is a local maxima if $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in some open disk centered at (x_0, y_0)
Geometrically, local minima are valleys bottoms, and local maxima are peaks.

5.1 First Derivative Test

If $P_0(x_0, y_0)$ is a local extremum in the domain of f, and f is differentiable at P_0, and if the first partial derivatives of f exist at P_0, then:

$$\frac{\partial f}{\partial x} = 0 \qquad\qquad \frac{\partial f}{\partial y} = 0$$

A point where both the partial derivatives are zero is called a critical point.

5.2 Saddle Points

A differentiable function $f(x, y)$ has a saddle point at $P_0(x_0, y_0)$ if f has a critical point P_0 if in every open disk centered at P_0 there are domain points (x, y) such that $f(x, y) > f(x_0, y_0)$ and $f(x, y) < f(x_0, y_0)$

5.3 Second Derivative Test

Let $f(x, y)$ be a function of 2 variables and $P_0(x_0, y_0)$ be a critical point of f. If the second partial derivatives of f exist and are continuous in some open disk centered at P_0, then:

- f has a local minima at P_0, if $f_{xx}f_{yy} - {f_{xy}}^2 > 0$ and $f_{xx} > 0$ at point P_0
- f has a local maxima at P_0, if $f_{xx}f_{yy} - {f_{xy}}^2 > 0$ and $f_{xx} < 0$ at point P_0
- f has a saddle point at P_0, if $f_{xx}f_{yy} - {f_{xy}}^2 < 0$ at point P_0

6 Lagrange Multipliers

When we need to find the extremum points of a function whose domain is constrained to a particular subset of a plane

The method of Lagrange Multipliers is as follows:
The local extremum values of $f(x, y, z)$ whose variables are subject to a constraint $g(x, y, z) = 0$ are found to be on the surface $g(x, y, z) = 0$ and following the following differential equation:

$$\nabla f = \lambda \nabla g$$

7 Integration

We will consider $\mathbb{R}^2$ for now, but the same can be extended to $\mathbb{R}^n$
Consider the rectangle $R : [a, b] \times [c, d] = \{(x, y) \mid x \in [a, b], y \in [c, d]\}$ in $\mathbb{R}^2$.
Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a function.
The analogue of the Riemann Sum for a function of 2 variables is given by: Take a partition of the rectangle R into partitions:

$$P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

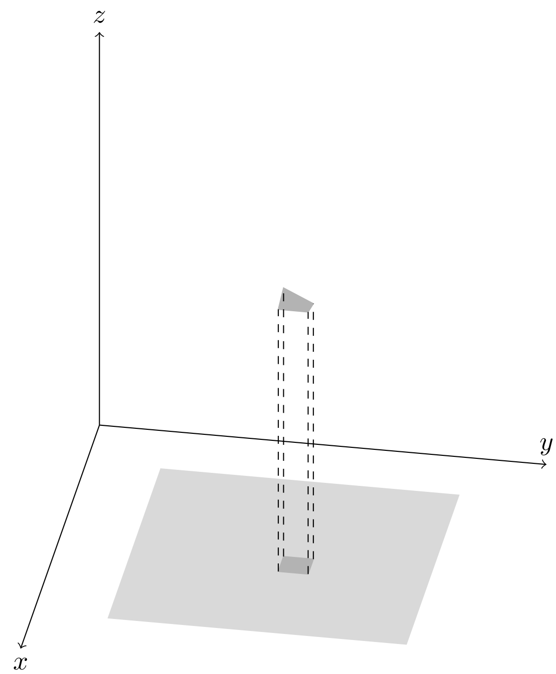
$$Q = \{c = y_0 < y_1 < \cdots < y_{m-1} < y_m = d\}$$

$$P \times Q = \{(x_i, y_j), i = \{1, 2, \ldots, n\}, j = \{1, 2, \ldots, m\}\}$$

The Riemann Sum is given by:

$$S = \sum_{\alpha=1}^{\eta} f(p_\alpha) \Delta A_\alpha$$

If the above summation converges to a Real Number as $n, m \to \infty$, regardless of the choice made in forming the Riemann sum, then the function f is Riemann Integrable over R.



Note: When the function f is continuous in R, then the Riemann sum will always converge.

Theorem If $f : [a, b] \times [c, d] \to \mathbb{R}$ is continuous, then:

$$\int_a^b \left(\int_c^d f(x, y)dy \right) dx = \iint_R f(x, y)dA = \int_c^d \left(\int_a^b f(x, y)dx \right) dy$$

7.1 Geometric Interpretation

for a function $f : [a, b] \times [c, d] \to \mathbb{R}$, the integral of f over the rectangle R is the volume of the solid bounded by the surface $z = f(x, y)$ and the rectangle R.

8 Integral over a General Bounded Region

The limits of the inner integral will now become a function of the outer integral's variable. This helps us achieve different shapes for our bounded region. This is known as Fubini's Theorem: Let $f(x, y)$ be continuous on a region R:

1. If R is defined by $a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 being continuos in $[a, b]$, then:

$$\iint_R f(x, y)dA = \int_a^b dx \int_{g_1(x)}^{g_2(x)} f(x, y)dy$$

2. If R is defined by $c \leq y \leq d, h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 being continuos in $[c, d]$, then:

$$\iint_R f(x,y)dA = \int_c^d dy \int_{h_1(x)}^{h_2(x)} f(x,y)dx$$

Area of Bouned Region When $f(x, y) = 1|\forall(x, y) \in R$, the integral will give us the area of the bounded plane region R.

9 Integrals in Polar Coordinates

When we divide the plane into n different smaller regions use the Riemann Summation to approximate the integral of a function $f(r, \theta)$ over a region R in polar coordinates:

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\Delta A_k$$

We can take dA such that the sides have either constant r or constant θ. Therefore, small area element ΔA is given by:

$$\begin{aligned}
\Delta A &= \text{Area of larger sector} - \text{area of smaller sector} \\
\Delta A &= \frac{\Delta\theta}{2}\left[\left(r + \frac{\Delta r}{2}\right) - \left(r - \frac{\Delta r}{2}\right)\right] \\
\Delta A &= \frac{\Delta\theta}{2}\left(2r\Delta r\right) \\
\Delta A &= r\Delta r\Delta\theta
\end{aligned}$$

Now, the Riemann Summation becomes:

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k) r_k \Delta r_k \Delta\theta_k$$

As $\Delta r, \Delta\theta \to 0$, $n \to \infty$, the Riemann Summation converges to the integral, assuming $f(r, \theta) is continuos in R$:

$$\lim_{n \to infty} S_n = \iint_R f(r, theta) r dr d\theta$$

Applying Fubini's theorem, we can write the integral as:

$$I = \int_{\theta_1}^{\theta_2} d\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta) r dr = \int_{r_1}^{r_2} r dr \int_{\theta_1(r)}^{\theta_2(r)} f(r,\theta) d\theta$$

Assuming, $r_1(\theta) \leq r \leq r_2(\theta)$ and $\theta_1(r) \leq \theta \leq \theta_2(r)$ and $r_1, r_2, \theta_1, \theta_2$ are continuous functions.