MT4214 - Algebraic Geometry

Nachiketa Kulkarni

Contents

Chapter 1

Motivation - Cayley–Hamilton theorem

Statement: Every Square Matrix over a commutative ring satisfies its own Charectoristic Polynomial.

Proof: Step 1: Let A be a diagonal matrix with $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ as the diagonal elements. Trivially, we can show that the Charecteristic Polynomial will be evaluated as follows:

$$\begin{aligned}\chi_A(x) &= \det(A - xI_n) \\ &= (\lambda_1 - x)(\lambda_2 - x) \dots (\lambda_n - x) \\ &= 0\end{aligned}$$

Step 2: A is diagonalizable. Then there exists matrices B, D such that $A = BDB^{-1}$. A property that will be used is as follows: $\chi_A(x) = \chi_D(x)$. Now, if we calculate the Charecteristic Polynomial for A:

$$\begin{aligned}\chi_A(A) &= \det(A - xI_n) \\ &= \chi_D(A) \\ &= B\chi_D(D)B^{-1} \\ &= 0\end{aligned}$$

Step 3: General A. We know that diagonalizable matrices are dense in $M_{n\times n}(\mathbb{C})$.

Consider the following function:

$$\phi : M_{n\times n}(\mathbb{C}) \to M_{n\times n}(\mathbb{C})$$

such that $\phi(A) = \chi_A(A) = 0 \; \forall A \in$ Diagonal Matrices. The above function is a continuous function [Trust me bro]. Now, $\{0\}$ is a closed set. Therefore the pullback of a closed set will have to be a closed set as well. But diagonal elements are dense in $M_{n\times n}(\mathbb{C})$. Therefore we use this to extend this to the entire topological space, $M_{n\times n}(\mathbb{C})$.

$$\phi(A) = 0 \; \forall \; A$$

But this above arguement only for fields which are Cauchy Complete. What about the charecteristic p fields. There is no obvious topology, and hence no dense set.

1.1 Zariski Topology on K^n

Let K be an algebraicaly closed field. We want to define a topology on K^n.

Define a ring $A = K[X_1, X_2, \ldots, X_n]$ is the ring of polynomial in n variables. Now, choose an element $f \in A$.

$$f : K^n \to K \text{ where } (a_1, a_2, \ldots, a_n) \to f(a_1, a_2, \ldots, a_n)$$

Now, we define a set function as follows:

$$Z(f) = \{(a_1, a_2, \ldots, a_n) \,|\, f(a_1, a_2, \ldots, a_n) = 0\}$$

Here, $Z(f)$ can be empty. Extending this to multiple functions:

$$Z(f_1, f_2, \ldots, f_m) = \bigcap_{1 \le i \le m} Z(f_i)$$

Let $I \subseteq A$ be an ideal.

$$f(I) = \bigcap_{p \in I} Z(p)$$

Lemma: Let $I \subseteq A$ be an ideal of A generated as follows: $I = \langle f_1, f_2, \ldots, f_n \rangle$. Then:

$$\begin{aligned} Z(I) &= Z(\langle f_1, f_2, \ldots, f_n \rangle) \\ &= Z(\{f_1, f_2, \ldots, f_n\}) \end{aligned}$$

Proof: We will prove this by showing that one set contains the other and vise=versa.

Part 1: $Z(\{f_1, f_2, \ldots, f_n\}) \subseteq Z(I)$ as each of the f_i is always contained in the I.

Part 2: Consider an element $f \in I$. As the ideal I is generated by f_i, f can be written as a linear combination of f_is, with coefficients in A:

$$f = \sum_{i=1}^{n} c_i f_i$$

where, all $c_i \in A$. Now, if $\bar{a} \in Z(\{f_1, f_2, \ldots, f_n\})$, $f_i(\bar{a}) = 0$ for all i. Therefore:

$$\begin{aligned} f(\bar{a}) &= \sum_{i=1}^{n} c_i f_i(\bar{a}) \\ &= 0 \end{aligned}$$

Hence, $Z(\{f_1, f_2, \ldots, f_n\}) \subseteq Z(I)$. Therefore, $Z(\{f_1, f_2, \ldots, f_n\}) = Z(I)$.

1.1.1 $Z(I)$ form a Topology on K^n

We define a topology on K^n by claiming that the closed sets in K^n are defined by the sets $Z(I)$. It satisifies the axioms of Topology as follows:

1. $\emptyset$ is in the topology: A is an ideal of itself. $Z(A) = \emptyset$.

2. K^n is in the topology: 0 polynomial also forms an ideal of A. $Z(0) = K^n$

3. Finite Union of $Z(I)$ belong to the topology: Let I, J, IJ be Ideals of A.

   Proposition: $Z(I) \cup Z(J) = Z(IJ)$.

   Proof: Consider an $a \in Z(I)$. This implies that $f(a) = 0\ \forall\ f \in I$. Using that we can say that a is a solution for any polynomial of the form $f \cdot g$ where, $f \in I$ and $g \in A$. Now, similarly, consider a b in $Z(J)$. It would be a solution for any polynomial of the form $f \cdot g$ where, $f \in J$ and $g \in A$. Hence, $a \in Z(I) \cup Z(J) \Rightarrow a \in Z(IJ)$

   Consider an $a \in Z(IJ)$. This implies that for all $f \in I$ and $g \in J$, the product $f(a)g(a) = 0$. As A is an integral domain, one of the factors must be 0. Let $a \notin I$. Then, there exists an $f \in I$ such that $f(a) \neq 0$. But, as $a \in Z(IJ)$, for every $g \in J$, $f(a)g(a) = 0$. This implies $g(a) = 0$ for every $g \in J$. Hence, $a \in Z(IJ) \Rightarrow Z(I) \cup Z(J)$

4. Arbitrary Intersection of $Z(I)$ belong to the topology: let $I, J, I + J$ be Ideals of A.

   Proposition: $Z(I) \cap Z(J) = Z(I + J)$.

   Proof: Consider an $a \in Z(I) \cap Z(J)$. This implies that for any two polynomials $f \in I$ and $g \in J$, the sum $(f + g)(a) = 0$. Therefore, $a \in Z(I + J)$.

   Now, let $a \in Z(I + J)$. Trivially, $a \in Z(I)$ and $a \in Z(J)$.

Therefore, $Z(I)$ forms a topology on K^n. $\mathbb{A}_k^n$ is defined as the vector space K^n with the Zariski Topology.

Exercise: Let X be a Compact Hausdorff Topological Space.

$$C(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous.}\}$$

Show that $C(X)$ is an $\mathbb{R}$-Algebra on X and that it forms a Commutative Ring.

1.1.2 Category Theory

Definition: A Category C is a collection of objects $\mathrm{ob}(C)$ and the following conditions:

1. For any two elements $x_1, y_1 \in \mathrm{ob}(C)$, there is a set of morphisms $\mathrm{Hom}_C(x_1, y_1)$. (We can think of $\mathrm{ob}(C)$ as dots and $\mathrm{Hom}_C(x_1, y_1)$ an arrow (could be bidirectional) from x_1 to y_1).

2. Composition of Morphisms is defined as follows: For any $x, y, z \in \mathrm{ob}(C)$

$$\begin{aligned} \mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) &\to \mathrm{Hom}(x, z) \\ f \times g &\mapsto g \circ f \end{aligned}$$

Which follow these axioms:

1. If $x, x', y, y' \in \mathrm{ob}(C)$, and $x \neq x'$, $y \neq y'$, then $f \in \mathrm{Hom}(x, y)$ does not belong to $\mathrm{Hom}(x', y')$.
2. For any $x, y, z \in ob(C)$, $x \circ (y \circ z) = (x \circ y) \circ z$.
3. For all $x \in \mathrm{ob}(C)$, there exists identity map $I_x \in \mathrm{Hom}(x, x)$ such that:

with the following properties:

1. If $x \neq x'$ or $y \neq y'$, then:

$$\mathrm{Hom}_C(x, y) \cap \mathrm{Hom}_C(x', y') = \emptyset$$

2. Composition is associative:

$$(f \circ (g \circ h)) = ((f \circ g) \circ h) = (f \circ g \circ h)$$

3. Identity maps: for every $x, y \in \mathrm{ob}(C)$, there exist $I_x(x, x) \in \mathrm{Hom}_C(x, x)$ and $I_y(y, y) \in \mathrm{Hom}_C(y, y)$:

$$f \circ I_x = f \text{ and } I_y \circ f = f$$

   And $I_x \circ f$ doesn't make any sense as the set maps don't align.

Example: Groups, Rings, k-Algebra, C_{opp}.
C_{opp} is constructed as:

- Consider a Category C.
- C_{opp} contains the same points a C.
- $\mathrm{Hom}_{C_{\mathrm{opp}}(y,x)} = \mathrm{Hom}_C(x, y)$

1.1.3 Commutative Algebra

Noetherian Ring: Let R be a commutative ring with unity. R is Noetherian if every ideal of R is finitely generated.

OR

R is Noetherian if every increasing sequence $I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$ of ideals has a largest element.

Theroem: Show that the two definitions of a Noetherian Ring are equivalent. I.e., If R has Ascending Chain Condition $\Leftrightarrow$ All ideals of R are finitely generated.

Part 1: Consider a Commutative Ring R with a 1 and the Ascending Chain Condition. Let I be an Ideal which is not finitely generated. We can form the following chain:

$$
\begin{aligned}
&I_0 = 0 \\
&I_1 = \langle f_1 \rangle && f_1 \in I \setminus I_0 \\
&I_2 = \langle f_1, f_2 \rangle && f_2 \in I \setminus I_1 \\
&\;\vdots \\
&I_n = \langle f_1, f_2, \ldots, f_n \rangle && f_n \in I \setminus I_n
\end{aligned}
$$

$$I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_n \subsetneq \ldots$$

Now, as R has Ascending Chain Condition, I_n stabilizes, i.e., there will be no f to pick from $I \setminus I_m$ for some finite m. Therefore, I is finitely generated.

Part 2: Consider a Commutative Ring R with a 1 and every ideal in R is finitely generated. Let there exist an Infinite Ascending Chain of Ideals, which does not stabilize:

$$I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_n \subsetneq \ldots$$

As every ideal is finitely generated, consider the ideal:

$$I = \bigcup_{n=0}^{\infty} I_n$$

Let I be generated by $\{f_1, f_2, \ldots, f_m\}$. As each $f_i \in I$, there exists a N_i such that $f_i \in I_{N_i}$. Let $N = \max(N_1, N_2, \ldots, N_m)$. Then, for all i, $f_i \in I_N$. Therefore, $I \subseteq I_N$. But, by construction, $I_N \subseteq I$. Hence, $I = I_N$. But, as the chain is strictly increasing, there exists a I_{N+1} such that $I_N \subsetneq I_{N+1} \subseteq I$. This is a contradiction. Therefore, the chain stabilizes.

Hilbert Basis Theorem: Let R be a Noetherian Ring. Then, the Polynomial Ring $R[X]$ is also Noetherian.