Rings and Modules

Nachiketa Kulkarni

Contents

Chapter 1

Introduction to Rings

1.1 Definition of a Ring

A ring R is a set with two binary operations, $+$ and $\times$, satisfying the following conditions:

- $(R, +)$ is an abelian group.
- $\times$ is associative.
- $\times$ distributes over $+$.

A Ring is said to be commutative if $a \times b = b \times a$ for all $a, b \in R$.
A Ring is said to have a multiplicative identity if there exists an element $1 \in R$ such that $1 \times a = a \times 1 = a$ for all $a \in R$.

Subrings: A subset S of a ring R is called a subring if:

- S is closed under addition and multiplication.
- S contains the additive identity 0 of R.
- For every $a \in S$, $-a \in S$.

1.1.1 Examples

- **Trivial Ring:** Take any abelian group $(G, +)$ and define multiplication as $a \times b = 0$ for all $a, b \in G$, where 0 is the identity of the group.
- **Integers:** The set of integers $\mathbb{Z}$ with usual addition and multiplication forms a ring. Also, the quotient group $\mathbb{Z}/n\mathbb{Z}$ is a ring for any integer n.
- **Hamiltonian Quaternions:** The set of quaternions $\mathbb{H} = 1, i, j, k$, where $i^2 = j^2 = k^2 = -1$.
- **Polynomial Rings:** Fix a commutative ring R. The set of polynomials with coefficients in R, denoted $R[x]$, forms a ring with addition and multiplication defined as usual.

1.2 Properties of Rings

Proposition: If R is a ring, then the following hold:

1. $0a = a0 = 0$ for all $a \in R$.
2. $(-a)b = a(-b) = -(ab)$ for all $a, b \in R$.
3. If the ring has a multiplicative identity 1, then it is unique.
4. $(-1)a = -a$ for all $a \in R$.

More Definitions: Consider a ring R:

- A non-zero element $a \in R$ is called a **zero divisor** if there exists a non-zero $b \in R$ such that either $ab = 0$ or $ba = 0$.
- Assume R has a multiplicative identity 1. An element $a \in R$ is called a **unit** if there exists an element $b \in R$ such that $ab = ba = 1$. The set of all units in R is denoted by $R^{\times}$.
- A Ring R with identity is called an **integral domain** if it has no zero divisors and $1 \neq 0$.

Proposition: If R is an integral domain, then the following hold:

1. $R^{\times}$ is a group under multiplication.
2. R is a field if multiplication is commutative and every non-zero element is a unit, i.e., $R^{\times} = R - \{0\}$.
3. A zero divisor cannot be a unit and vice versa.

   Proof: If a is a zero divisor, then there exists a non-zero b such that $ab = 0$. Now, assume a is a unit, then there exists c such that $ac = 1$. But:

   $$b = (ca)b = c(ab) = c0 = 0$$

1.3 Homomorphisms and Isomorphisms

Let R and S be rings. A **ring homomorphism** is a function $\phi : R \to S$ such that:

1. The map ϕ preserves addition: $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in R$.
2. The map ϕ preserves multiplication: $\phi(ab) = \phi(a) + \phi(b)$ for all $a, b \in R$.

The kernel of a ring homomorphism ϕ, $\ker \phi$, is the set of elements in R that map to 0 in S. A bijective ring homomorphism is called a **ring isomorphism**, denoted by $R \cong S$.