Solutions to Principle of Mathematical Analysis

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Contents

1 The Real and Complex Numbers

1. If r is rational ($r \neq 0$) and x is irrational, prove that $r+x$ and rx are irrational.

   We will prove that $r+x$ is irrational by *reductio ad impossibilem*, contradiction. That is, $p \to q$ becomes $p \wedge \neg q$. Suppose r is rational ($r \neq 0$) and x is irrational and $r+x$ is rational. Since r is rational, $-r$ is rational and exist by the field axioms of addition. The sum of two rational numbers is rational by the closure property of $\mathbb{Q}$. Then $-r+(r+x) = (-r+r)+x = x$. We have reached a contradiction since x is clearly irrational. Therefore, $r+x$ is irrational.

   For the second statement, we will again use the argument of *reductio ad impossibilem*. Since $r \neq 0$ and rational, $\frac{1}{r}$ is rational and exists by the field axioms of multiplication. The multiplication of two rational numbers is rational, again, by the closure property of $\mathbb{Q}$. Then $\frac{1}{r}(rx) = \left(\frac{1}{r}r\right)(x) = x$. We have reached a contradiction since x is irrational. That is, rx is irrational.

2. Prove that there is no rational number whose square is 12.

   Suppose there is a rational number whose square is 12. Let $\frac{a}{b}$ be this rational number. Then $a^2 = 12b^2$. By the Fundamental Theorem of Arithmetic, we can write a, b, and 12 as a product of *unique* primes. Let p_i and q_i be prime numbers and $\alpha_i, \beta_i \in \mathbb{Z}^{\geqslant 0}$ for $i = 1, 2, \ldots, n$. Then $a = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, $b = q_1^{\beta_1} \cdot q_2^{\beta_2} \cdots q_n^{\beta_n}$, and $12 = 2^2 \cdot 3$. We now have

$$
\begin{aligned}
(p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_n^{\alpha_n})^2 &= 2^2 \cdot 3(q_1^{\beta_1} \cdot q_2^{\beta_2} \cdots q_n^{\beta_n})^2 \\
p_1^{2\alpha_1} \cdot p_2^{2\alpha_2} \cdots p_n^{2\alpha_n} &= 2^2 \cdot 3(q_1^{2\beta_1} \cdot q_2^{2\beta_2} \cdots q_n^{2\beta_n})
\end{aligned} \tag{1.1}
$$

   Let $p_k^{2\alpha_k}$ be $3^{2\alpha_k}$ and $q_m = 3^{2\beta_m}$. Then by equation (1.1)

$$
\begin{aligned}
3^{2\alpha_k} &= 3 \cdot 3^{2\beta_m} \\
&= 3^{2\beta_m+1}
\end{aligned}
$$

   Therefore, $2\alpha_k = 2\beta_m + 1$ which is a contradiction since an even number can never be an odd number. That is, there is no rational number whose square is 12.

3. Prove Proposition 1.15.

   Proposition 1.15 states that the axioms for multiplication imply the following statements.

   (a) If $x \neq 0$ and $xy = xz$, then $y = z$.

   By the field axioms of multiplication, since $x \neq 0$,

$$y = 1 \cdot y = \frac{1}{x}xy = \frac{1}{x}xz = \frac{1}{x}xz = z$$

   as was needed to be shown.

   (b) If $x \neq 0$ and $xy = x$, then $y = 1$.

   Since $x \neq 0$, we have

$$y = 1 \cdot y = \frac{1}{x}xy = \frac{1}{x}x = 1$$

   as was needed to shown.

   (c) If $x \neq 0$ and $xy = 1$, then $y = 1/x$.

   Again, since we have that $x \neq 0$,

$$y = 1 \cdot y = \frac{1}{x}xy = \frac{1}{x} \cdot 1 = \frac{1}{x}$$

   as was needed to be shown.

(d) If $x \neq 0$, then $1/(1/x) = x$

Again, since we have that $x \neq 0$,

$$\frac{1}{1/x} = 1 \cdot \frac{1}{1/x} = x\frac{1}{x}\frac{1}{1/x} = x\frac{1}{x}x = x$$

as was needed to be shown.

4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leqslant \beta$.

Since $E \neq \varnothing$, $x \in E$. Since α is a lower bound, $\alpha \leqslant x$, and since β is an upper bound, $\beta \geqslant x$. By the transitivity property, $\alpha \leqslant \beta$.

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Since A is nonempty and bounded below, $A = \{x : x \in A\}$ and $\inf(A) = \alpha$. Now, $-A = \{-x : x \in A\}$ is also nonempty. Since α is the infimum of A, $\alpha \leqslant x$ for all $x \in A$. By multiplying by -1, we get the following inequality

$$\alpha \leqslant x \Rightarrow -\alpha \geqslant -x.$$

That is, $-\alpha$ is an upper bound of $-A$. Suppose $-\gamma = \sup(-A)$ and $\varepsilon > 0$. Then $-\gamma + \varepsilon \notin -A$

$$-\alpha \geqslant -\gamma + \varepsilon \geqslant -\gamma \geqslant -x$$

Again, by multiplying by negative one, we have

$$\alpha \leqslant \gamma - \varepsilon \leqslant \gamma \leqslant x$$

but $\gamma - \varepsilon \notin A$ so γ is a lower bound of A which would contradict the fact that α is the greatest lower bound of A. In order for γ to be the lower bound, $\gamma = \alpha$ since the infimum is unique. So $-\alpha = \sup(-A)$. Therefore, $\alpha = \inf(A) = -\sup(-A) = -(-\alpha) = \alpha$.

6. Fix $b > 1$.

(a) If m, n, p, q are integers, $n, q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.

Since $n, q > 0$, $nr = m = np/q$.

$$(b^m)^{1/n} = (b^{np/q})^{1/n} = \left[(b^p)^{n/q}\right]^{1/n} = (b^p)^{1/q} = b^{p/q} = b^r$$

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

Let $r = \frac{a}{b}$ and $s = \frac{c}{d}$. Then

$$b^{r+s} = b^{(ad+bc)/(bd)} = (b^{ad+bc})^{1/(bd)} = (b^a)^{1/b}(b^c)^{1/d} = b^r b^s$$

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leqslant x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x.

From the statement $b^r = \sup B(r)$, we see that $b^r \in B(r)$. Let $b^t \in B(r)$. Then $b^r = b^t b^{r-t}$. Since $b > 1$, $b^t 1^{r-t} \leqslant b^t b^{r-t} = b^r$; therefore, $b^t \leqslant b^r$ for all $b^t \in B(r)$ so $b^r = \sup B(r)$.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

7. Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This is called the logarithm of y to the base of b.)

   (a) For any positive integer n, $b^n - 1 \geqslant n(b-1)$.

   From Theorem 1.21, we have that $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots a^{n-1})$. Therefore, we now have

$$
\begin{aligned}
b^n - 1 &= (b-1)(b^{n-1} + b^{n-2}1 + \cdots + ba^{n-2} + 1^{n-1}) \\
&\geqslant (b-1)(1^{n-1} + 1^{n-2}1 + \cdots + (1)1^{n-2} + 1^{n-1}) \\
&= n(b-1)1^{n-1} \\
&= n(b-1)
\end{aligned} \tag{1.2}
$$

   where equation (1.2) occurs from letting $b = 1$, and since $b > 1$, we get the less than or equal to inequality.

   (b) Hence $b - 1 \geqslant n(b^{1/n} - 1)$.

   (c) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

   (d) If w is such that $b^w < y$, then $b^{w+1/n} < y$ for sufficiently large n; to see this, apply part (c) with $t = y \cdot b^{-w}$.

   (e) If $b^w > y$, then $b^{w-1/n} > y$ for sufficiently large n.

   (f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup(A)$ satisfies $b^x = y$.

   (g) Prove that x is unique.

8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint: -1 is a square*

   Suppose that $i > 0$. Then $i^2 = -1 \not> 0$. Instead, let's suppose that $i < 0$. Then $i^4 = 1 \not< 0$. Therefore, $\mathbb{C}$ is not ordered.

9. Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least upper bound property?

   The Law of Trichotomy states that a real number is either positive, negative, or zero. In otherwords, if $x, y \in \mathbb{R}$, then $x < y$, $x = y$, or $x > y$. Let $a, b, c, d \in \mathbb{R}$. Then $a < c$, $a = c$, or $a > c$. If $a < c$, then $z < w$. If $a > c$, then $z > w$. For $a = c$, we have either $b < d$, $b = d$, or $b > d$. If $b < d$, then $z < w$. If $b > d$, them $z > w$. Finally, if $b = d$, then $z = w$. Let $z, w, u \in \mathbb{C}$ and $a, b, c, d, e, f \in \mathbb{R}$ such that z and w are defined as above and $u = e + if$. We need to show the tansitive property. That is, if $z < w$ and $w < u$, then $z < u$. Since $z < w$ and $w < u$, we have that either $a < c$ or $a = c$ and $b < d$ and $c < e$ or $c = e$ and $d < f$. If $a < c$ and $c < e$, then $a < e$ and $z < u$. If $a < c$, $c = e$, and $d < f$, then $z < u$ since $b < d < f$. If $a = c$, $b < d$, and $c < e$, then $a = c < e$ so $z < u$. If $a = c$, $b < d$, $c = e$, and $d < f$, then $a = c = e$ and $b < d < f$ so $z < u$. Thus, $\mathbb{C}$ is an order set under the dictionary order. Since $\mathbb{C}$ is an order set under the dictionary order, we have by the completeness axiom that $\mathbb{C}$ with the dictionary order has the least upper bound property.

10. Suppose $z = a + bi$, $w = u + iv$, and

$$
a = \Big(\frac{|w| + u}{2}\Big)^{1/2}, \qquad b = \Big(\frac{|w| - u}{2}\Big)^{1/2}.
$$

   Prove that $z^2 = w$ if $v \geqslant 0$ and that $\bar{z}^2 = w$ if $v \leqslant 0$. Conclude that every complex number (with one exception!) has two complex square roots.

   We have that $z^2 = a^2 - b^2 + 2abi$ so $a^2 - b^2 = u$.

$$
2ab = 2\Big(\frac{|w| + u}{2}\frac{|w| - u}{2}\Big)^{1/2}
$$

$$
\begin{aligned}
&= \pm\sqrt{|w|^2 - u^2} \\
&= \pm v
\end{aligned}
$$

For $v \geqslant 0$, $z^2 = u + iv = w$. Now, $\bar{z}^2 = a^2 - b^2 - 2abi$, so again we have $a^2 - b^2 = u$ and $-2ab = \mp v$. If $v \leqslant 0$, then $\bar{z}^2 = u + iv = w$. Therefore, all nonzero complex numbers have at least two complex square roots.

11. If z is a complex number, prove that there exists an $r \geqslant 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

    Since $|w| = 1$, we can write w as $w = \frac{z}{|z|}$. Then let $r = |z|$ so $z = rw$ where w and r are unique. If $z = 0$, then $r = 0$ and $w \in \mathbb{C}$ such that $|w| = 1$. Therefore, w is not unique.

12. If $z_1, \dots, z_n$ are complex, prove that

    $$|z_1 + z_2 + \cdots + z_n| \leqslant |z_1| + |z_2| + \cdots + |z_n|.$$

    First, we will show the triangle inequality is true for $n = 2$ and use induction for $n \geqslant 2$ and $n \in \mathbb{Z}^+$. For $n = 2$, we need to show $|z_1 + z_2| \leqslant |z_1| + |z_2|$.

    $$
    \begin{aligned}
    |z_1 + z_2|^2 &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\
    &= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 \\
    &\leqslant |z_1|^2 + 2\mathrm{Re}(z_1 z_2) + |z_2|^2 \\
    &= \big(|z_1| + |z_2|\big)^2
    \end{aligned}
    $$

    Taking square roots of the left and right sides, we have the desired results. Suppose this is true for $k < n$. Then

    $$|z_1 + \cdots + z_k| \leqslant |z_1| + \cdots + |z_k|.$$

    Now, we need to show it is true for $k + 1$.

    $$
    \begin{aligned}
    |z_1 + \cdots + z_{k+1}| &= |(z_1 + \cdots + z_k) + z_{k+1}| \\
    &\leqslant |z_1 + \cdots + z_k| + |z_{k+1}| \\
    &\leqslant |z_1| + \cdots + |z_{k+1}|
    \end{aligned}
    $$

    Therefore, by the principle of mathematical induction, the n dimensional triangle inequality is true.

13. If x, y are complex, prove that

    $$\big||x| - |y|\big| \leqslant |x - y|.$$

    Let $x = x + y - y$. Then by the triangle inequality, we have

    $$
    \begin{aligned}
    |x + y - y| &\leqslant |x - y| + |y| \\
    |x| &\leqslant |x - y| + |y| \\
    |x| - |y| &\leqslant |x - y|
    \end{aligned}
    $$

    Similarly, we could let $y = y + x - x$ and conclude

    $$|y| - |x| \leqslant |x - y|.$$

    Thus,

    $$\big||x| - |y|\big| \leqslant |x - y|.$$

14. If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute

    $$|1 + z|^2 + |1 - z|^2.$$

    We have that $|z|^2 = z\bar{z}$ so

    $$
    \begin{aligned}
    |1 + z|^2 + |1 - z|^2 &= (1 + z)(1 - \bar{z}) + (1 - z)(1 - \bar{z}) \\
    &= 2 + z + \bar{z} + 2 - z - \bar{z} \\
    &= 4
    \end{aligned}
    $$

15. Under what conditions does equality hold in the Schwarz inequality?

The Schwarz inequality (also known as the Cauchy-Schwarz inequality) is

$$\left|\sum_j^n a_j \bar{b}_j\right|^2 \leqslant \sum_j^n |a_j|^2 \sum_j^n |b_j|^2.$$

Let $A = \sum |a_j|^2$, $B = \sum |\bar{b}_j|^2$, and $C = \sum |a_j \bar{b}_j|^2$. From the proof in the book, we ahve $0 = B(AB - |C|^2)$. Therefore, equality holds if $B = 0$ or $AB - |C|^2 = 0$.

16. Suppose $k \geqslant 3$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:

   (a) If $2r > d$, there are infinitely many $\mathbf{z} \in \mathbb{R}^k$ such that

   $$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

   (b) If $2r = d$, there exactly one such $\mathbf{z}$.

   (c) If $2r < d$, there is no such $\mathbf{z}$

How must these statements be modified if k is 2 or 1?

17. Prove that

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

if $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{y} \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

We have that

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y})(\mathbf{x} + \mathbf{y}) \\ &= |\mathbf{x}|^2 + 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2 \qquad (1.3) \\ |\mathbf{x} - \mathbf{y}|^2 &= (\mathbf{x} - \mathbf{y})(\mathbf{x} - \mathbf{y}) \\ &= |\mathbf{x}|^2 - 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2 \qquad (1.4) \end{aligned}$$

Then by adding equations (1.3) and (1.4), we have

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

Then $\mathbf{x}+\mathbf{y}$ is the longer diagonal of the parallelogram and $\mathbf{x}-\mathbf{y}$ is the shorter diagonal of the parallelogram see figure 1.1.

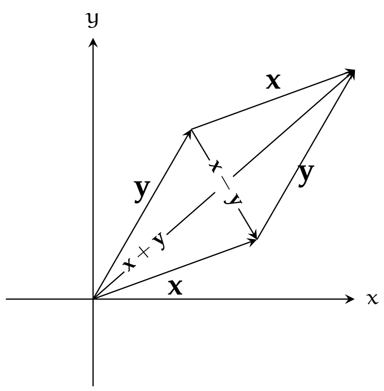

Figure 1.1: The parallelogram for vectors $\mathbf{x}$ and $\mathbf{y}$.

Then the sum of squares of the diagonals of a parallelogram are equal to the sum of the squares of the sides of the parallelogram.

18. If $k \geqslant 2$ and $\mathbf{x} \in \mathbb{R}^k$, prove that there exists $\mathbf{y} \in \mathbb{R}^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

If $\mathbf{x} = 0$, then the components of $\mathbf{y}$ can be any real numbers. If $\mathbf{x} \neq 0$, then $\mathbf{y} = \begin{bmatrix} -x_k & -x_{k-1} & \cdots & -x_1 \end{bmatrix}^\mathsf{T}$. For $k = 1$, this is not true since for the multiplication of any two nonzero real numbers is nonzero.

19. Suppose $\mathbf{a} \in \mathbb{R}^k$, $\mathbf{b} \in \mathbb{R}^k$. Find $\mathbf{c} \in \mathbb{R}^k$ and $r > 0$ such that

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

if and only if $|\mathbf{x} - \mathbf{c}| = r$. (Solutions $3\mathbf{c} = 4\mathbf{b} - \mathbf{a}$, $3r = 2|\mathbf{b} - \mathbf{a}|$.)

2 Basic Topology

1. Prove that the empty set is a subset of every set.

   Let A be set. If $x \notin A$, then $x \notin \varnothing$. Since $\varnothing$ is the empty set, $x \notin \varnothing$ is a given. By contrapositive, if $x \in \varnothing$, then $x \in A$; therefore, $\varnothing \subset A$.

2. A complex number z is said to be *algebraic* if there are integers $a_0, \ldots, a_n$, not all zero, such that

   $$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

   Prove that the set of all algebraic numbers is countable. *Hint: For every positive integer N there are only finitely many equations with*

   $$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

   Let $N \in \mathbb{Z}^+$ and A_N be the set of algebraic equations for a given N. Since $1 \leqslant n \leqslant N$, each A_N is finite. The set of algebraic numbers is $\bigcup_{N \in \mathbb{Z}^+} A_n$. The union of countable sets is countable so the set of algebraic numbers is countable.

3. Prove that there exist real numbers which are not algebraic.

   The set of algebraic numbers are countable. Therefore, the set of algebraic real numbers would also be countable. The real numbers are an uncountable set and the union of uncontable sets are not contable. We have reached a contradiction so there are real numbers which are not algebraic.

4. Is the set of all irrational real numbers countable?

   No. Let $\mathbb{I}$ be the set of irrational numbers and $\mathbb{Q}$ be the set of rational numbers. Then $\mathbb{R} = \mathbb{I} \cup \mathbb{Q}$. The set of rational numbers is countable. If $\mathbb{I}$ were countable, then $\mathbb{R}$ would be countable as well.

5. Construct a bounded set of real numbers with exactly three limit points.

   Let $A_0 = \{1/n \mid n \in \mathbb{Z}^+\}$, $A_1 = \{1 + 1/n \mid n \in \mathbb{Z}^+\}$, and $A_2 = \{2 + 1/n \mid n \in \mathbb{Z}^+\}$. Then the limit point of A_1 is 0, the limit point of A_2 is 1, and the limit point of A_2 is 2. Let $S = A_1 \cup A_2 \cup A_3$. Now S is bounded below by zero and above by three with limit points $0, 1, 2$.

6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do E and E' always have the same limit points?

   Let $x \notin E'$. Then x is not a limit point of E. Now x has a neighborhood which doesn't intersect with E' so the complement of E' is open; therefore, E' is closed. If x is a limit point of E then $x \in E'$ so x is a limit point of $\bar{E}$. Suppose x is a limit point of $\bar{E}$. Then $x \in \bar{E}$ since $\bar{E}$ is closed. Thus, $x \in E'$ or $x \in E$. If $x \in E'$, then x is a limit point of E so suppose x is in E. Then we have a neighborhood $N_r(x)$ for $r > 0$ such that $N \cap E = \{x\}$. Since E' is closed, x isn't a limit point of E'. Let $M_r(x)$ be a neighborhood of x such taht $M \cap E' = \varnothing$. Let $V = N \cap M$ so U is a neighborhood of x. Therefore, V is a neighborhood of x. Now $V \cap \bar{E} = (V \cap E) \cup (V \cap E') = \{x\} \cup \varnothing = \{x\}$ so x is not a limit point of $\bar{E}$ so $x \in E'$ and is a limit point of E. No. Consider $E = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$. Then $E' = \{0\}$ and the limit point of E' is $\varnothing$.

7. Let $A_1, A_2, \ldots$ be subsets of a metric space.

   (a) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\bar{B}_n = \bigcup_{i=1}^n \bar{A}_i$ for $n = 1, 2, \ldots$

   (b) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\bar{B} \supset \bigcup_{i=1}^n \bar{A}_i$.

8. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets of $\mathbb{R}^2$.

9. Let E° denote the set of all interior points of a set E.
   (a) Prove that E° is always open.
   (b) Prove that E is open if and only if $E^\circ = E$.
   (c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.
   (d) Prove that the complement of E° is the closure of the complement of E.
   (e) Do E and $\bar{E}$ always have the same interiors?
   (f) Do E and E° always have the same closures?

10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1, & \text{if } p \neq q \\ 0, & \text{if } p = q \end{cases}$$

    Prove that this a metric space. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

11. For $x \in \mathbb{R}$ and $y \in \mathbb{R}$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2 \\ d_2(x,y) &= \sqrt{|x-y|} \\ d_3(x,y) &= |x^2-y^2| \\ d_4(x,y) &= |x-2y| \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|} \end{aligned}$$

    Determine for each of these, whether it is a metric or not.

12. Let $K \subset \mathbb{R}$ consist of 0 and the numbers $1/n$ for $n = 1, 2, \ldots$ Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

13. Construct a compact set of real numbers whose limit points form a countable set.

14. Give an example of an open cover of the segment $(0,1)$ which has no finite subcover.

15. Show that Theorem 2.36 and its Corollary become false (in $\mathbb{R}$, for example) if the word "compact" is replaced by "closed" or by "bounded".

16. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p,q) = |p-q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

17. Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0,1]$? Is E compact? Is E perfect?

18. Is there a nonempty perfect set in $\mathbb{R}$ which contains no rational number?

19. (a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.
    (b) Prove the same for disjoint open sets.
    (c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p,q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.
    (d) Prove that every connected metric space with at least two points is uncountable. *Hint: Use item 19 (c).*

20. Are closures and interiors of connected sets always connected? (Look at subsets of $\mathbb{R}^2$.)

21. Let A and B be separated subsets of some $\mathbb{R}^k$, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for $t \in \mathbb{R}$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. (Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.)

 (a) Prove that A_0 and B_0 are separated subsets of $\mathbb{R}$.

 (b) Prove that there exists $t_0 \in (0,1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.

 (c) Prove that every convex subset of $\mathbf{R}^k$ is connected.

22. A metric space is *separable* if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable. *Hint: Consider the set of points which have only rational coordinates*

23. A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$. Prove that every separable metric space has a *countable* base. *Hint: Take all neighborhoods with rational radius and center in some countable dense subset of* X.

24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. *Hint: Fix* $\delta > 0$, *and pick* $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geqslant \delta$ for $i = 1, \ldots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = 1/n$ ($n = 1, 2, \ldots$), and consider the centers of the corresponding neighborhoods.

25. Prove that every compact metric space K has a countable base, and that K is therefore separable. *Hint: For every positive integer* n, *there are finitely many neighborhoods of radius* $1/n$ *whose union covers* K.

26. Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact. *Hint: By exercise* 23 *and* 24, X *has a countable base.* It follows that every open cover of X has a countable subcover $\{G_n\}$, $n = 1, 2, \ldots$ If no finite subcollection of $\{G_n\}$ covers X, then the complement F_n of $G_1 \cup \cdots \cup G_n$ is nonempty for each n, but $\bigcap F_n$ is empty. If E is a set which contains a point from each F_n, consider a limit point of E, and obtain a contradiction.

27. Define a point p in a metric space X to be *condensation point* of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E. Suppose $E \subset \mathbb{R}^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable. *Hint: Let* $\{V_n\}$ *be a countable base of* $\mathbb{R}^k$, *let* W *be the union of those* V_n *for which* $E \cap V_n$ *is at most countable, and show that* $P = W^c$.

28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. *(Corollary: Every countable close set in* $\mathbb{R}^k$ *has isolated points.) Hint: Use exercise* 27.

29. Prove that every open set in $\mathbb{R}$ is the union of an at most countable collection of disjoints segments. *Hint: Use exercise* 22.

30. Imitate the proof of Theorem 2.43 to obtain the following results:

 If $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$, where each F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n has a nonempty interior.

 Equivalent statement: If G_n is a dense open subset $\mathbb{R}^k$, for $n = 1, 2, \ldots$ then $\bigcap_{n=1}^{\infty} G_n$ is not empty (in fact, it is dense in $\mathbb{R}^k$).

 (This is a special case of Baire's theorem; see exercise 22, chapter 3, for the general case.)

3 Numerical Sequences and Series

1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

   Since $\{s_n\}$ converges, it is Cauchy. Let $\epsilon > 0$ be given. There exist $n, m > N$ such that $|s_n - s_m| < \epsilon$ since $\{s_n\}$ is Cauchy.

$$\begin{aligned} |s_n| &= |s_n - s_m + s_m| \\ &\leqslant |s_n - s_m| + |s_m| \\ |s_n| - |s_m| &\leqslant |s_n - s_m| \end{aligned}$$

   Similarly, we can show

$$\begin{aligned} |s_m| - |s_n| &\leqslant |s_m - s_n| \\ &= |s_n - s_m| \end{aligned}$$

   so

$$\big||s_n| - |s_m|\big| \leqslant |s_n - s_m| < \epsilon.$$

   No. Consider the sequence $\{s_n\} = (-1)^n$. Let $\epsilon = 1$. If $\{s_n\}$ converges, it will converge to ± 1. WLOG assume $s_n \to 1$. Let $n > N$ such that n is odd.

$$|(-1)^n - 1| = |-1 - 1| = 2 \not< \epsilon$$

   Therefore, the sequence $\{s_n\}$ doesn't converge. However, $\{|s_n|\}$ does converge to 1. Let $\epsilon > 0$ given. There exist an $n > N$ such that $\big||(-1)^n| - 1\big| < \epsilon$. For any $n > N$, $(-1)^n = \pm 1$ and $|\pm 1| = 1$.

$$\big||(-1)^n| - 1\big| = |1 - 1| = 0 < \epsilon$$

2. Caclulate $\lim_{n\to\infty} \sqrt{n^2 + n} - n$.

$$\begin{aligned} \lim_{n\to\infty} \sqrt{n^2 + n} - n &= \lim_{n\to\infty} \left(\sqrt{n^2 + n} - n\right) \frac{\sqrt{n^2 + n} + n}{\sqrt{n^2 + n} + n} \\ &= \lim_{n\to\infty} \frac{n}{\sqrt{n^2 + n} + n} \\ &= \lim_{n\to\infty} \frac{n}{\sqrt{n^2 + n} + n} \frac{1/n}{1/n} \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{1 + 1/n} + 1} \\ &= \frac{1}{2} \end{aligned}$$

3. If $s_1 = \sqrt{2}$ and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}}$$

   $n \in \mathbb{Z}^+$ prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n \in \mathbb{Z}^+$.

   Let $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$ be written as

$$s = \sqrt{2 + s} \Rightarrow s^2 - s - 2 = 0.$$

   Then $\sqrt{2 + \sqrt{s_n}} < \sqrt{2 + s}$. Since we are dealing with real numbers, we are only looking for positive s.

$$s^2 - s - 2 = (s - 2)(s + 1) = 0 \tag{3.1}$$

so $s = 2, -1$. Thus, $s_n < 2$ so $\{s_n\}$ is bounded above by two. Additionally, since $s_1 = \sqrt{2}$, we have that $\sqrt{2} < s_n < 2$. The parabola is concave up and symmetrical about $s = 1/2$. That is, s monotonically increases from $(1/2, \infty)$ so $\{s_n\}$ monotonoically increases on $\left[\sqrt{2}, 2\right)$. Therefore, $\{s_n\}$ converges and it converges to 2.

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0, \qquad s_{2m} = \frac{s_{2m-1}}{2}, \qquad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Let's determine a few of the terms. Then $s_{2m} = \{0, 1/4, 3/8, 7/16, \ldots\}$ and $s_{2m+1} = \{1/2, 3/4, 7/8, 15/16, \ldots\}$ or we can write them as

$$\begin{aligned} s_{2m} &= \frac{1}{2} - \frac{1}{2^m} \\ s_{2m+1} &= 1 - \frac{1}{2^m} \end{aligned}$$

The $\lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1$ and $\lim_{n\to\infty} \inf s_n = \lim_{n\to\infty} \frac{1}{2} - \frac{1}{2^n} = \frac{1}{2}$.

5. For any two real sequences $\{a_n\}, \{b_n\}$, prove that

$$\lim_{n\to\infty} \sup(a_n + b_n) \leqslant \lim_{n\to\infty} \sup a_n + \lim_{n\to\infty} \sup b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.