

SOLUTIONS TO PRINCIPLES OF MATHEMATICAL ANALYSIS
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1 The Real and Complex Numbers

1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

We will prove that $r + x$ is irrational by *reductio ad impossibilem*, contradiction. That is, $p \rightarrow q$ becomes $p \wedge \neg q$. Suppose r is rational ($r \neq 0$) and x is irrational and $r + x$ is rational. Since r is rational, $-r$ is rational and exist by the field axioms of addition. The sum of two rational numbers is rational by the closure property of \mathbb{Q} . Then $-r + (r + x) = (-r + r) + x = x$. We have reached a contradiction since x is clearly irrational. Therefore, $r + x$ is irrational.

For the second statement, we will again use the argument of *reductio ad impossibilem*. Since $r \neq 0$ and rational, $\frac{1}{r}$ is rational and exists by the field axioms of multiplication. The multiplication of two rational numbers is rational, again, by the closure property of \mathbb{Q} . Then $\frac{1}{r}(rx) = (\frac{1}{r}r)(x) = x$. We have reached a contradiction since x is irrational. That is, rx is irrational.

2. Prove that there is no rational number whose square is 12.

Suppose there is a rational number whose square is 12. Let $\frac{a}{b}$ be this rational number. Then $a^2 = 12b^2$. By the Fundamental Theorem of Arithmetic, we can write a , b , and 12 as a product of *unique* primes. Let p_i and q_i be prime numbers and $\alpha_i, \beta_i \in \mathbb{Z}^{\geq 0}$ for $i = 1, 2, \dots, n$. Then $a = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, $b = q_1^{\beta_1} \cdot q_2^{\beta_2} \cdots q_n^{\beta_n}$, and $12 = 2^2 \cdot 3$. We now have

$$\begin{aligned} (p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_n^{\alpha_n})^2 &= 2^2 \cdot 3 (q_1^{\beta_1} \cdot q_2^{\beta_2} \cdots q_n^{\beta_n})^2 \\ p_1^{2\alpha_1} \cdot p_2^{2\alpha_2} \cdots p_n^{2\alpha_n} &= 2^2 \cdot 3 (q_1^{2\beta_1} \cdot q_2^{2\beta_2} \cdots q_n^{2\beta_n}) \end{aligned} \quad (1.1)$$

Let $p_k^{2\alpha_k}$ be $3^{2\alpha_k}$ and $q_m = 3^{2\beta_m}$. Then by equation (1.1)

$$\begin{aligned} 3^{2\alpha_k} &= 3 \cdot 3^{2\beta_m} \\ &= 3^{2\beta_m+1} \end{aligned}$$

Therefore, $2\alpha_k = 2\beta_m + 1$ which is a contradiction since an even number can never be an odd number. That is, there is no rational number whose square is 12.

3. Prove Proposition 1.15.

Proposition 1.15 states that the axioms for multiplication imply the following statements.

- (a) If $x \neq 0$ and $xy = xz$, then $y = z$.

By the field axioms of multiplication, since $x \neq 0$,

$$y = 1 \cdot y = \frac{1}{x}xy = \frac{1}{x}xz = \frac{1}{x}xz = z$$

as was needed to be shown.

- (b) If $x \neq 0$ and $xy = x$, then $y = 1$.

Since $x \neq 0$, we have

$$y = 1 \cdot y = \frac{1}{x}xy = \frac{1}{x}x = 1$$

as was needed to shown.

- (c) If $x \neq 0$ and $xy = 1$, then $y = 1/x$.

Again, since we have that $x \neq 0$,

$$y = 1 \cdot y = \frac{1}{x}xy = \frac{1}{x} \cdot 1 = \frac{1}{x}$$

as was needed to be shown.

(d) If $x \neq 0$, then $1/(1/x) = x$

Again, since we have that $x \neq 0$,

$$\frac{1}{1/x} = 1 \cdot \frac{1}{1/x} = x \frac{1}{x} \frac{1}{1/x} = x \frac{1}{x} x = x$$

as was needed to be shown.

4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E . Prove that $\alpha \leq \beta$.

Since $E \neq \emptyset$, $x \in E$. Since α is a lower bound, $\alpha \leq x$, and since β is an upper bound, $\beta \geq x$. By the transitivity property, $\alpha \leq \beta$.

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Since A is nonempty and bounded below, $A = \{x : x \in A\}$ and $\inf(A) = \alpha$. Now, $-A = \{-x : x \in A\}$ is also nonempty. Since α is the infimum of A , $\alpha \leq x$ for all $x \in A$. By multiplying by -1 , we get the following inequality

$$\alpha \leq x \Rightarrow -\alpha \geq -x.$$

That is, $-\alpha$ is an upper bound of $-A$. Suppose $-\gamma = \sup(-A)$ and $\varepsilon > 0$. Then $-\gamma + \varepsilon \notin -A$

$$-\alpha \geq -\gamma + \varepsilon \geq -\gamma \geq -x$$

Again, by multiplying by negative one, we have

$$\alpha \leq \gamma - \varepsilon \leq \gamma \leq x$$

but $\gamma - \varepsilon \notin A$ so γ is a lower bound of A which would contradict the fact that α is the greatest lower bound of A . In order for γ to be the lower bound, $\gamma = \alpha$ since the infimum is unique. So $-\alpha = \sup(-A)$. Therefore, $\alpha = \inf(A) = -\sup(-A) = -(-\alpha) = \alpha$.

6. Fix $b > 1$.

(a) If m, n, p, q are integers, $n, q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.

Since $n, q > 0$, $nr = m = np/q$.

$$(b^m)^{1/n} = (b^{np/q})^{1/n} = [(b^p)^{n/q}]^{1/n} = (b^p)^{1/q} = b^{p/q} = b^r$$

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

Let $r = \frac{a}{b}$ and $s = \frac{c}{d}$. Then

$$b^{r+s} = b^{(ad+bc)/(bd)} = (b^{ad+bc})^{1/(bd)} = (b^a)^{1/b} (b^c)^{1/d} = b^r b^s$$

(c) If x is real, define $B(x)$ to be the set of all numbers b^t , where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x .

From the statement $b^r = \sup B(r)$, we see that $b^r \in B(r)$. Let $b^t \in B(r)$. Then $b^r = b^t b^{r-t}$. Since $b > 1$, $b^t 1^{r-t} \leq b^t b^{r-t} = b^r$; therefore, $b^t \leq b^r$ for all $b^t \in B(r)$ so $b^r = \sup B(r)$.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y .

7. Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This is called the logarithm of y to the base of b .)

(a) For any positive integer n , $b^n - 1 \geq n(b - 1)$.

From Theorem 1.21, we have that $b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$. Therefore, we now have

$$\begin{aligned} b^n - 1 &= (b - 1)(b^{n-1} + b^{n-2}1 + \cdots + b1^{n-2} + 1^{n-1}) \\ &\geq (b - 1)(1^{n-1} + 1^{n-2}1 + \cdots + (1)1^{n-2} + 1^{n-1}) \\ &= n(b - 1)1^{n-1} \\ &= n(b - 1) \end{aligned} \tag{1.2}$$

where equation (1.2) occurs from letting $b = 1$, and since $b > 1$, we get the less than or equal to inequality.

(b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

(c) If $t > 1$ and $n > (b - 1)/(t - 1)$, then $b^{1/n} < t$.

(d) If w is such that $b^w < y$, then $b^{w+1/n} < y$ for sufficiently large n ; to see this, apply part (c) with $t = y \cdot b^{-w}$.

(e) If $b^w > y$, then $b^{w-1/n} > y$ for sufficiently large n .

(f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup(A)$ satisfies $b^x = y$.

(g) Prove that x is unique.

8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint: -1 is a square*

Suppose that $i > 0$. Then $i^2 = -1 \not> 0$. Instead, let's suppose that $i < 0$. Then $i^4 = 1 \not< 0$. Therefore, \mathbb{C} is not ordered.

9. Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least upper bound property?

The Law of Trichotomy states that a real number is either positive, negative, or zero. In other words, if $x, y \in \mathbb{R}$, then $x < y$, $x = y$, or $x > y$. Let $a, b, c, d \in \mathbb{R}$. Then $a < c$, $a = c$, or $a > c$. If $a < c$, then $z < w$. If $a > c$, then $z > w$. For $a = c$, we have either $b < d$, $b = d$, or $b > d$. If $b < d$, then $z < w$. If $b > d$, then $z > w$. Finally, if $b = d$, then $z = w$. Let $z, w, u \in \mathbb{C}$ and $a, b, c, d, e, f \in \mathbb{R}$ such that z and w are defined as above and $u = e + if$. We need to show the transitive property. That is, if $z < w$ and $w < u$, then $z < u$. Since $z < w$ and $w < u$, we have that either $a < c$ or $a = c$ and $b < d$ and $c < e$ or $c = e$ and $d < f$. If $a < c$ and $c < e$, then $a < e$ and $z < u$. If $a < c$, $c = e$, and $d < f$, then $z < u$ since $b < d < f$. If $a = c$, $b < d$, and $c < e$, then $a = c < e$ so $z < u$. If $a = c$, $b < d$, $c = e$, and $d < f$, then $a = c = e$ and $b < d < f$ so $z < u$. Thus, \mathbb{C} is an order set under the dictionary order. Since \mathbb{C} is an order set under the dictionary order, we have by the completeness axiom that \mathbb{C} with the dictionary order has the least upper bound property.

10. Suppose $z = a + bi$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2} \right)^{1/2}, \quad b = \left(\frac{|w| - u}{2} \right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $\bar{z}^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

We have that $z^2 = a^2 - b^2 + 2abi$ so $a^2 - b^2 = u$.

$$2ab = 2 \left(\frac{|w| + u}{2} \frac{|w| - u}{2} \right)^{1/2}$$

$$= \pm \sqrt{|w|^2 - u^2}$$

$$= \pm v$$

For $v \geq 0$, $z^2 = u + iv = w$. Now, $\bar{z}^2 = a^2 - b^2 - 2abi$, so again we have $a^2 - b^2 = u$ and $-2ab = \mp v$. If $v \leq 0$, then $\bar{z}^2 = u + iv = w$. Therefore, all nonzero complex numbers have at least two complex square roots.

11. If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z ?

Since $|w| = 1$, we can write w as $w = \frac{z}{|z|}$. Then let $r = |z|$ so $z = rw$ where w and r are unique. If $z = 0$, then $r = 0$ and $w \in \mathbb{C}$ such that $|w| = 1$. Therefore, w is not unique.

12. If z_1, \dots, z_n are complex, prove that

$$|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n|.$$

First, we will show the triangle inequality is true for $n = 2$ and use induction for $n \geq 2$ and $n \in \mathbb{Z}^+$. For $n = 2$, we need to show $|z_1 + z_2| \leq |z_1| + |z_2|$.

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\ &= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 \\ &\leq |z_1|^2 + 2\operatorname{Re}(z_1\bar{z}_2) + |z_2|^2 \\ &= (|z_1| + |z_2|)^2 \end{aligned}$$

Taking square roots of the left and right sides, we have the desired results. Suppose this is true for $k < n$. Then

$$|z_1 + \dots + z_k| \leq |z_1| + \dots + |z_k|.$$

Now, we need to show it is true for $k + 1$.

$$\begin{aligned} |z_1 + \dots + z_{k+1}| &= |(z_1 + \dots + z_k) + z_{k+1}| \\ &\leq |z_1 + \dots + z_k| + |z_{k+1}| \\ &\leq |z_1| + \dots + |z_{k+1}| \end{aligned}$$

Therefore, by the principle of mathematical induction, the n dimensional triangle inequality is true.

13. If x, y are complex, prove that

$$||x| - |y|| \leq |x - y|.$$

Let $x = x + y - y$. Then by the triangle inequality, we have

$$\begin{aligned} |x + y - y| &\leq |x - y| + |y| \\ |x| &\leq |x - y| + |y| \\ |x| - |y| &\leq |x - y| \end{aligned}$$

Similarly, we could let $y = y + x - x$ and conclude

$$|y| - |x| \leq |x - y|.$$

Thus,

$$||x| - |y|| \leq |x - y|.$$

14. If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2.$$

We have that $|z|^2 = z\bar{z}$ so

$$\begin{aligned} |1 + z|^2 + |1 - z|^2 &= (1 + z)(1 - \bar{z}) + (1 - z)(1 - \bar{z}) \\ &= 2 + z + \bar{z} + 2 - z - \bar{z} \\ &= 4 \end{aligned}$$

15. Under what conditions does equality hold in the Schwarz inequality?

The Schwarz inequality (also known as the Cauchy-Schwarz inequality) is

$$\left| \sum_{j=1}^n a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2.$$

Let $A = \sum |a_j|^2$, $B = \sum |\bar{b}_j|^2$, and $C = \sum |a_j \bar{b}_j|^2$. From the proof in the book, we have $0 = B(AB - |C|^2)$. Therefore, equality holds if $B = 0$ or $AB - |C|^2 = 0$.

16. Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:

- (a) If $2r > d$, there are infinitely many $\mathbf{z} \in \mathbb{R}^k$ such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

- (b) If $2r = d$, there is exactly one such \mathbf{z} .

- (c) If $2r < d$, there is no such \mathbf{z}

How must these statements be modified if k is 2 or 1?

17. Prove that

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

if $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{y} \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

We have that

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) \\ &= |\mathbf{x}|^2 + 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2 \end{aligned} \tag{1.3}$$

$$\begin{aligned} |\mathbf{x} - \mathbf{y}|^2 &= (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \\ &= |\mathbf{x}|^2 - 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2 \end{aligned} \tag{1.4}$$

Then by adding equations (1.3) and (1.4), we have

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

Then $\mathbf{x} + \mathbf{y}$ is the longer diagonal of the parallelogram and $\mathbf{x} - \mathbf{y}$ is the shorter diagonal of the parallelogram see figure 1.1.

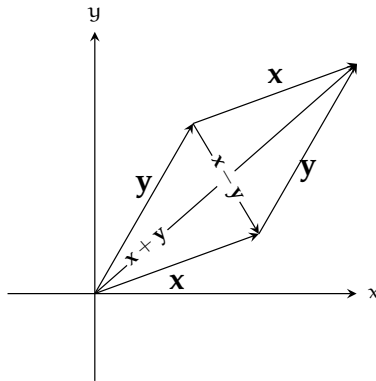


Figure 1.1: The parallelogram for vectors \mathbf{x} and \mathbf{y} .

Then the sum of squares of the diagonals of a parallelogram are equal to the sum of the squares of the sides of the parallelogram.

18. If $k \geq 2$ and $\mathbf{x} \in \mathbb{R}^k$, prove that there exists $\mathbf{y} \in \mathbb{R}^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

If $\mathbf{x} = \mathbf{0}$, then the components of \mathbf{y} can be any real numbers. If $\mathbf{x} \neq \mathbf{0}$, then

$$\mathbf{y} = \begin{bmatrix} -x_k & -x_{k-1} & \cdots & -x_1 \end{bmatrix}^T.$$

For $k = 1$, this is not true since for the multiplication of any two nonzero real numbers is nonzero.

19. Suppose $\mathbf{a} \in \mathbb{R}^k$, $\mathbf{b} \in \mathbb{R}^k$. Find $\mathbf{c} \in \mathbb{R}^k$ and $r > 0$ such that

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

if and only if $|\mathbf{x} - \mathbf{c}| = r$. (Solutions $3\mathbf{c} = 4\mathbf{b} - \mathbf{a}$, $3r = 2|\mathbf{b} - \mathbf{a}|$.)

2 Basic Topology

1. Prove that the empty set is a subset of every set.

Let A be set. If $x \notin A$, then $x \notin \emptyset$. Since \emptyset is the empty set, $x \notin \emptyset$ is a given. By contrapositive, if $x \in \emptyset$, then $x \in A$; therefore, $\emptyset \subset A$.

2. A complex number z is said to be *algebraic* if there are integers a_0, \dots, a_n , not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. *Hint: For every positive integer N there are only finitely many equations with*

$$n + |a_0| + |a_1| + \dots + |a_n| = N.$$

Let $N \in \mathbb{Z}^+$ and A_N be the set of algebraic equations for a given N . Since $1 \leq n \leq N$, each A_N is finite. The set of algebraic numbers is $\bigcup_{N \in \mathbb{Z}^+} A_N$. The union of countable sets is countable so the set of algebraic numbers is countable.

3. Prove that there exist real numbers which are not algebraic.

The set of algebraic numbers are countable. Therefore, the set of algebraic real numbers would also be countable. The real numbers are an uncountable set and the union of uncountable sets are not countable. We have reached a contradiction so there are real numbers which are not algebraic.

4. Is the set of all irrational real numbers countable?

No. Let \mathbb{I} be the set of irrational numbers and \mathbb{Q} be the set of rational numbers. Then $\mathbb{R} = \mathbb{I} \cup \mathbb{Q}$. The set of rational numbers is countable. If \mathbb{I} were countable, then \mathbb{R} would be countable as well.

5. Construct a bounded set of real numbers with exactly three limit points.

Let $A_0 = \{1/n \mid n \in \mathbb{Z}^+\}$, $A_1 = \{1 + 1/n \mid n \in \mathbb{Z}^+\}$, and $A_2 = \{2 + 1/n \mid n \in \mathbb{Z}^+\}$. Then the limit point of A_1 is 0, the limit point of A_2 is 1, and the limit point of A_2 is 2. Let $S = A_1 \cup A_2 \cup A_3$. Now S is bounded below by zero and above by three with limit points 0, 1, 2.

6. Let E' be the set of all limit points of a set E . Prove that E' is closed. Prove that E and \bar{E} have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do E and E' always have the same limit points?

Let $x \notin E'$. Then x is not a limit point of E . Now x has a neighborhood which doesn't intersect with E' so the complement of E' is open; therefore, E' is closed. If x is a limit point of E then $x \in E'$ so x is a limit point of \bar{E} . Suppose x is a limit point of \bar{E} . Then $x \in \bar{E}$ since \bar{E} is closed. Thus, $x \in E'$ or $x \in E$. If $x \in E'$, then x is a limit point of E so suppose x is in E . Then we have a neighborhood $N_r(x)$ for $r > 0$ such that $N \cap E = \{x\}$. Since E' is closed, x isn't a limit point of E' . Let $M_r(x)$ be a neighborhood of x such that $M \cap E' = \emptyset$. Let $V = N \cap M$ so V is a neighborhood of x . Therefore, V is a neighborhood of x . Now $V \cap \bar{E} = (V \cap E) \cup (V \cap E') = \{x\} \cup \emptyset = \{x\}$ so x is not a limit point of \bar{E} so $x \in E'$ and is a limit point of E . No. Consider $E = \{1/n \mid n \in \mathbb{Z}^+\}$. Then $E' = \{0\}$ and the limit point of E' is \emptyset .

7. Let A_1, A_2, \dots be subsets of a metric space.

(a) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\bar{B}_n = \bigcup_{i=1}^n \bar{A}_i$ for $n = 1, 2, \dots$

(b) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\bar{B} \supset \bigcup_{i=1}^n \bar{A}_i$.

8. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E ? Answer the same question for closed sets of \mathbb{R}^2 .

9. Let E° denote the set of all interior points of a set E .

- (a) Prove that E° is always open.
- (b) Prove that E is open if and only if $E^\circ = E$.
- (c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.
- (d) Prove that the complement of E° is the closure of the complement of E .
- (e) Do E and \bar{E} always have the same interiors?
- (f) Do E and E° always have the same closures?

10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p, q) = \begin{cases} 1, & \text{if } p \neq q \\ 0, & \text{if } p = q \end{cases}$$

Prove that this is a metric space. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

11. For $x \in \mathbb{R}$ and $y \in \mathbb{R}$, define

$$\begin{aligned} d_1(x, y) &= (x - y)^2 \\ d_2(x, y) &= \sqrt{|x - y|} \\ d_3(x, y) &= |x^2 - y^2| \\ d_4(x, y) &= |x - 2y| \\ d_5(x, y) &= \frac{|x - y|}{1 + |x - y|} \end{aligned}$$

Determine for each of these, whether it is a metric or not.

- 12. Let $K \subset \mathbb{R}$ consist of 0 and the numbers $1/n$ for $n = 1, 2, \dots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).
- 13. Construct a compact set of real numbers whose limit points form a countable set.
- 14. Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.
- 15. Show that Theorem 2.36 and its Corollary become false (in \mathbb{R} , for example) if the word "compact" is replaced by "closed" or by "bounded".
- 16. Regard \mathbb{Q} , the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in \mathbb{Q} , but that E is not compact. Is E open in \mathbb{Q} ?
- 17. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?
- 18. Is there a nonempty perfect set in \mathbb{R} which contains no rational number?
- 19. (a) If A and B are disjoint closed sets in some metric space X , prove that they are separated.
(b) Prove the same for disjoint open sets.
(c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.
(d) Prove that every connected metric space with at least two points is uncountable. *Hint: Use item 19 (c).*
- 20. Are closures and interiors of connected sets always connected? (Look at subsets of \mathbb{R}^2 .)
- 21. Let A and B be separated subsets of some \mathbb{R}^k , suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b}$$

for $t \in \mathbb{R}$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. (Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.)

- (a) Prove that A_0 and B_0 are separated subsets of \mathbb{R} .
- (b) Prove that there exists $t_0 \in (0, 1)$ such that $p(t_0) \notin A \cup B$.
- (c) Prove that every convex subset of \mathbb{R}^k is connected.
22. A metric space is *separable* if it contains a countable dense subset. Show that \mathbb{R}^k is separable. *Hint: Consider the set of points which have only rational coordinates*
23. A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α . In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$. Prove that every separable metric space has a *countable* base. *Hint: Take all neighborhoods with rational radius and center in some countable dense subset of X .*
24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. *Hint: Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \dots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \dots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ . Take $\delta = 1/n$ ($n = 1, 2, \dots$), and consider the centers of the corresponding neighborhoods.*
25. Prove that every compact metric space K has a countable base, and that K is therefore separable. *Hint: For every positive integer n , there are finitely many neighborhoods of radius $1/n$ whose union covers K .*
26. Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact. *Hint: By exercise 23 and 24, X has a countable base. It follows that every open cover of X has a countable subcover $\{G_n\}$, $n = 1, 2, \dots$. If no finite subcollection of $\{G_n\}$ covers X , then the complement F_n of $G_1 \cup \dots \cup G_n$ is nonempty for each n , but $\bigcap F_n$ is empty. If E is a set which contains a point from each F_n , consider a limit point of E , and obtain a contradiction.*
27. Define a point p in a metric space X to be a *condensation point* of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E . Suppose $E \subset \mathbb{R}^k$, E is uncountable, and let P be the set of all condensation points of E . Prove that P is perfect and that at most countably many points of E are not in P . In other words, show that $P^c \cap E$ is at most countable. *Hint: Let $\{V_n\}$ be a countable base of \mathbb{R}^k , let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.*
28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (*Corollary: Every countable closed set in \mathbb{R}^k has isolated points.*) *Hint: Use exercise 27.*
29. Prove that every open set in \mathbb{R} is the union of an at most countable collection of disjoint segments. *Hint: Use exercise 22.*
30. Imitate the proof of Theorem 2.43 to obtain the following results:
- If $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$, where each F_n is a closed subset of \mathbb{R}^k , then at least one F_n has a nonempty interior.
- Equivalent statement:* If G_n is a dense open subset of \mathbb{R}^k , for $n = 1, 2, \dots$ then $\bigcap_{n=1}^{\infty} G_n$ is not empty (in fact, it is dense in \mathbb{R}^k).
- (This is a special case of Baire's theorem; see exercise 22, chapter 3, for the general case.)

3 Numerical Sequences and Series

1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

Since $\{s_n\}$ converges, it is Cauchy. Let $\epsilon > 0$ be given. There exist $n, m > N$ such that $|s_n - s_m| < \epsilon$ since $\{s_n\}$ is Cauchy.

$$\begin{aligned} |s_n| &= |s_n - s_m + s_m| \\ &\leq |s_n - s_m| + |s_m| \\ |s_n| - |s_m| &\leq |s_n - s_m| \end{aligned}$$

Similarly, we can show

$$\begin{aligned} |s_m| - |s_n| &\leq |s_m - s_n| \\ &= |s_n - s_m| \end{aligned}$$

so

$$||s_n| - |s_m|| \leq |s_n - s_m| < \epsilon.$$

No. Consider the sequence $\{s_n\} = (-1)^n$. Let $\epsilon = 1$. If $\{s_n\}$ converges, it will converge to ± 1 . WLOG assume $s_n \rightarrow 1$. Let $n > N$ such that n is odd.

$$|(-1)^n - 1| = |-1 - 1| = 2 \not< \epsilon$$

Therefore, the sequence $\{s_n\}$ doesn't converge. However, $\{|s_n|\}$ does converge to 1. Let $\epsilon > 0$ given. There exist an $n > N$ such that $|(-1)^n - 1| < \epsilon$. For any $n > N$, $(-1)^n = \pm 1$ and $|\pm 1| = 1$.

$$|(-1)^n - 1| = |1 - 1| = 0 < \epsilon$$

2. Calculate $\lim_{n \rightarrow \infty} \sqrt{n^2 + n} - n$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt{n^2 + n} - n &= \lim_{n \rightarrow \infty} \left(\sqrt{n^2 + n} - n \right) \frac{\sqrt{n^2 + n} + n}{\sqrt{n^2 + n} + n} \\ &= \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + n} + n} \\ &= \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + n} + n} \cdot \frac{1/n}{1/n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + 1/n} + 1} \\ &= \frac{1}{2} \end{aligned}$$

3. If $s_1 = \sqrt{2}$ and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}}$$

$n \in \mathbb{Z}^+$ prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n \in \mathbb{Z}^+$.

Let $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$ be written as

$$s = \sqrt{2 + s} \Rightarrow s^2 - s - 2 = 0.$$

Then $\sqrt{2 + \sqrt{s_n}} < \sqrt{2 + s}$. Since we are dealing with real numbers, we are only looking for positive s .

$$s^2 - s - 2 = (s - 2)(s + 1) = 0 \tag{3.1}$$

so $s = 2, -1$. Thus, $s_n < 2$ so $\{s_n\}$ is bounded above by two. Additionally, since $s_1 = \sqrt{2}$, we have that $\sqrt{2} \leq s_n < 2$. The parabola is concave up and symmetrical about $s = 1/2$. That is, s monotonically increases from $(1/2, \infty)$ so $\{s_n\}$ monotonically increases on $[\sqrt{2}, 2)$. Theorem 3.14 states that monotonic sequences converge if and only if it is bounded. Therefore, since $\{s_n\}$ is bounded and monotonic, $\{s_n\}$ converges and it converges to 2.

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0, \quad s_{2m} = \frac{s_{2m-1}}{2}, \quad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Let's determine a few of the terms. Then $s_{2m} = \{0, 1/4, 3/8, 7/16, \dots\}$ and $s_{2m+1} = \{1/2, 3/4, 7/8, 15/16, \dots\}$ or we can write them as

$$s_{2m} = \frac{1}{2} - \frac{1}{2^m}$$

$$s_{2m+1} = 1 - \frac{1}{2^m}$$

The $\lim_{n \rightarrow \infty} \sup s_n = \lim_{n \rightarrow \infty} 1 - \frac{1}{2^n} = 1$ and $\lim_{n \rightarrow \infty} \inf s_n = \lim_{n \rightarrow \infty} \frac{1}{2} - \frac{1}{2^n} = \frac{1}{2}$.

5. For any two real sequences $\{a_n\}, \{b_n\}$, prove that

$$\lim_{n \rightarrow \infty} \sup(a_n + b_n) \leq \lim_{n \rightarrow \infty} \sup a_n + \lim_{n \rightarrow \infty} \sup b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

Let $\lim_{n \rightarrow \infty} \sup a_n = a$, $\lim_{n \rightarrow \infty} \sup b_n = b$, and $\lim_{n \rightarrow \infty} \sup(a_n + b_n) = c$ where $a_n + b_n = c_n$. Let $\epsilon > 0$ be given. Then there exist $N_1, N_2 \in \mathbb{Z}^+$ such that for $n \geq N_1$ and $n \geq N_2$

$$|a_n - a| < \frac{\epsilon}{2}$$

$$|b_n - b| < \frac{\epsilon}{2}$$

Let $N = \max\{N_1, N_2\}$. Then when $n \geq N$,

$$|c_n - c| = |a_n + b_n - (a + b)| \tag{3.2}$$

$$\leq |a_n - a| + |b_n - b| \tag{3.3}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \epsilon$$

From equations (3.2) and (3.3), we have that

$$\limsup c_n \leq \limsup a_n + \limsup b_n,$$

and since $c_n = a_n + b_n$, the identity follows.

6. Investigate the behavior (convergence or divergence) of $\sum a_n$ if

(a) $a_n = \sqrt{n+1} - \sqrt{n}$

Let S_N be the N th partial sum. Then

$$S_N = \sum_{n=0}^N (\sqrt{n+1} - \sqrt{n}) = 1 + \sqrt{2} - 1 + \sqrt{3} - \sqrt{2} + \dots + \sqrt{N} - \sqrt{N-1} + \sqrt{N+1} - \sqrt{N}$$

Therefore, the $S_N = \sqrt{N+1}$.

$$\lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \sqrt{N+1} = \infty$$

so the series doesn't converge.

(b) $a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$

Let's re-write the series by multiplying by the conjugate.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n(\sqrt{n+1} + \sqrt{n})}$$

Now

$$\sum_{n=1}^{\infty} \frac{1}{n(\sqrt{n+1} + \sqrt{n})} \leq \sum_{n=1}^{\infty} \frac{1}{2n\sqrt{n}}$$

By theorem 3.28, $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{2n\sqrt{n}} < \infty$$

since $p = 3/2$ so

$$\sum_{n=1}^{\infty} \frac{1}{n(\sqrt{n+1} + \sqrt{n})} < \infty.$$

(c) $a_n = (\sqrt[n]{n} - 1)^n$

By the root test,

$$\lim_{n \rightarrow \infty} \sqrt[n]{|\sqrt[n]{n} - 1|^n} = \lim_{n \rightarrow \infty} |\sqrt[n]{n} - 1|.$$

Let $x_n = \sqrt[n]{n} - 1$. Then

$$n = (x_n + 1)^n = \sum_{k=0}^n \binom{n}{k} x_n^k = 1 + nx_n + \frac{n(n-1)}{2} x_n^2 + \dots$$

$$\text{so } \frac{n(n-1)}{2} x_n^2 < n \Rightarrow x_n^2 < \frac{2}{n-1} \Rightarrow x_n < \sqrt{\frac{2}{n-1}}.$$

$$\begin{aligned} \lim_{n \rightarrow \infty} |\sqrt[n]{n} - 1| &= \lim_{n \rightarrow \infty} |x_n| \\ &< \lim_{n \rightarrow \infty} \sqrt{\frac{2}{n-1}} \\ &= 0 \end{aligned}$$

Since $\sqrt{\frac{2}{n-1}}$ converges and $x_n < \sqrt{\frac{2}{n-1}}$, x_n converges. By the root and comparison test,

$$\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$$

converges.

(d) $a_n = \frac{1}{1+z^n}$ for complex values of z .

$$\sum_{n=0}^{\infty} \frac{1}{1+z^n} \leq \sum_{n=0}^{\infty} \frac{1}{z^n}$$

and $\sum_{n=0}^{\infty} \frac{1}{z^n}$ converges for $|\frac{1}{z}| < 1$.

7. Prove that the convergence of $\sum a_n$ implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n \geq 0$.

For $a_n \geq 1$, $\sqrt{a_n} \leq a_n$. By the comparison test,

$$\sum \frac{\sqrt{a_n}}{n} < \sum a_n < \infty$$

so for $a_n \geq 1$, the series converges. For $0 \leq a_n < 1$, $a_n \leq \sqrt{a_n}$. We can write all rationals and irrational numbers in $[0, 1)$ as b/n for $b \in \mathbb{R}^{\geq 0}$. Then

$$\sum \frac{\sqrt{a_n}}{n} = \frac{\sqrt{b}}{n\sqrt{n}} < \infty$$

since $p = 3/2 > 1$ so the series converges.

8. If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

By theorem 3.14, we know that monotonic bounded sequences converge. Let $\{b_n\} \rightarrow M$ for some number $M < \infty$ or $|b_n| \leq M$. Since $\sum a_n$ converges, for a given $\epsilon > 0$ and $k \geq N$, $m \geq k \geq N$ implies that

$$\left| \sum_{n=k}^m a_n \right| \leq \sum_{n=k}^m |a_n| \leq \epsilon.$$

Take $\epsilon = \frac{\epsilon}{M}$. Then

$$\left| \sum a_n b_n \right| \leq \sum |a_n| |b_n| \leq \sum |a_n| M.$$

Since $\sum |a_n| \leq \epsilon/M$, the result follows; that is,

$$\sum |a_n| M \leq \epsilon$$

so $\sum a_n b_n$ converges.

9. Find the radius of convergence of each of the following power series:

(a) $\sum n^3 z^n$

Here will use the ratio test.

$$\limsup_{n \rightarrow \infty} \left| \frac{(n+1)^3 z^{n+1}}{n^3 z^n} \right| = |z| \limsup_{n \rightarrow \infty} \left| \frac{n+1}{n} \right|^3 = |z| \limsup_{n \rightarrow \infty} \left| \frac{n}{n} \right|^3 = |z|$$

Then $\limsup = \alpha$ and $R = \frac{1}{\alpha}$ so $R = 1$ and $|z| < 1$ for convergence.

(b) $\sum \frac{2^n}{n!} z^n$

Again, we use the ratio test.

$$\limsup_{n \rightarrow \infty} \left| \frac{2^{n+1} z^{n+1} n!}{2^n z^n (n+1)!} \right| = 2|z| \limsup_{n \rightarrow \infty} \left| \frac{1}{n+1} \right| = 0$$

Thus, $R = \infty$.

(c) $\sum \frac{2^n}{n^2} z^n$

Following the same test, we have

$$\limsup_{n \rightarrow \infty} \left| \frac{2^{n+1} z^{n+1} n^2}{2^n z^n (n+1)^2} \right| = 2|z| \limsup_{n \rightarrow \infty} \left| \frac{n}{n+1} \right|^2 = 2|z|$$

so $R = 1/2$ and $|z| < 1/2$ for convergence.

(d) $\sum \frac{n^3}{3^n} z^n$

$$\limsup_{n \rightarrow \infty} \left| \frac{(n+1)^3 z^{n+1} 3^n}{3^{n+1} z^n n^3} \right| = \frac{|z|}{3} \limsup_{n \rightarrow \infty} \left| \frac{n}{n} \right|^3 = \frac{|z|}{3}$$

so $R = 3$ and $|z| < 3$ for convergence.

10. Suppose the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

11. Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

- (a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

Theorem 3.23 states that if $\sum a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$. Since $\sum a_n$ doesn't converge, there exist no M such that $a_n \leq M$ for $M \in \mathbb{R}$. Then

$$\lim_{n \rightarrow \infty} \frac{a_n}{1+a_n} \geq \lim_{n \rightarrow \infty} \frac{M}{1+M} = 1$$

for some $M \gg 10^8$. Therefore, the limit is greater than or equal to one so $\sum \frac{a_n}{1+a_n}$ doesn't converge.

(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \dots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

Each partial sum s_N increase so

$$\frac{a_{N+1}}{s_{N+1}} + \dots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{1}{s_{N+k}}(a_{N+1} + \dots + a_{N+k}) = 1 - \frac{s_N}{s_{N+k}}$$

since $a_{N+1} + \dots + a_{N+k} = s_{N+k} - s_N$. Now,

$$\sum \frac{a_n}{s_n} = \frac{a_1}{a_1} + \frac{a_2}{a_1 + a_2} + \dots + \frac{a_n}{\sum a_n} + \dots$$

Let $\epsilon > 0$ be given. Then

$$\left| 1 - \frac{s_N}{s_{N+k}} \right| > \epsilon$$

since for k sufficiently large, $s_{N+k} \rightarrow \infty$. That is, $|1 - s_N/s_{N+k}|$ can be made larger than $1/2$. Take $\epsilon = 0.1$ and the series falls to converge.

(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

We can write $\frac{1}{s_{n-1}} - \frac{1}{s_n}$ as

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_n s_{n-1}}$$

where $s_n = a_n + \sum_{i=1}^{n-1} a_i$ and $s_{n-1} = \sum_{i=1}^{n-1} a_i$ so $s_n - s_{n-1} = a_n$. Now $s_n^2 \geq s_n s_{n-1}$ so $\frac{1}{s_n^2} \leq \frac{1}{s_n s_{n-1}}$.

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{a_n}{s_n s_{n-1}} \geq \frac{a_n}{s_n^2}$$

The telescoping series

$$\sum_{n=2}^{\infty} \frac{1}{s_{n-1}} - \frac{1}{s_n} \geq \sum_{n=1}^{\infty} \frac{a_n}{s_n^2}.$$

Since

$$\sum_{n=2}^N \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{1}{s_1} - \frac{1}{s_2} + \frac{1}{s_2} - \frac{1}{s_3} + \dots + \frac{1}{s_{N-1}} - \frac{1}{s_N} + \frac{1}{s_N} - \frac{1}{s_{N+1}} = \frac{1}{s_1} - \frac{1}{s_{N+1}}$$

and $\sum a_n \rightarrow \infty$, $\lim_{N \rightarrow \infty} \frac{-1}{s_{N+1}} = \lim_{N \rightarrow \infty} \frac{-1}{a_{N+1}} = \frac{-1}{\infty} = 0$ so

$$\sum_{n=2}^{\infty} \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{1}{a_1}$$

and $\sum \frac{a_n}{s_n^2} < \infty$.

(d) What can be said about

$$\sum \frac{a_n}{1 + na_n} \quad \text{and} \quad \sum \frac{a_n}{1 + n^2 a_n}?$$

For the second series, we have

$$\sum \frac{a_n}{1 + n^2 a_n} \leq \sum \frac{1}{n^2} < \infty.$$

For the first series, suppose $a_n \in \mathbb{R}$, then

$$\sum \frac{a_n}{1 + na_n} \leq \sum \frac{1}{n} \rightarrow \infty$$

Suppose $a_n = 1/n^{1+p}$. Then

$$\sum \frac{a_n}{1 + na_n} = \sum \frac{1/n^{1+p}}{1 + n(1/n^{1+p})} = \sum \frac{1}{n^{1+p} + n} \leq \sum \frac{1}{n^{1+p}} < \infty$$

for $p > 0$. Otherwise, the series diverges to infinity.

12. Suppose $a_n > 0$ and $\sum a_n$ converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) Prove that

$$\frac{a_m}{r_m} + \dots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

$$\frac{a_m}{r_m} + \dots + \frac{a_n}{r_n} > \frac{a_m + \dots + a_n}{r_m} = \frac{r_m - r_n}{r_m} = 1 - \frac{r_n}{r_m}$$

By the same reasoning as item 11 (b), the series doesn't converge.

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Consider $\sqrt{r_n} - \sqrt{r_{n+1}}$.

$$\begin{aligned} \sqrt{r_n} - \sqrt{r_{n+1}} &= \sqrt{r_n} - \sqrt{r_{n+1}} \frac{\sqrt{r_n} + \sqrt{r_{n+1}}}{\sqrt{r_n} + \sqrt{r_{n+1}}} \\ &= \frac{r_n - r_{n+1}}{\sqrt{r_n} + \sqrt{r_{n+1}}} \\ &= \frac{a_n}{\sqrt{r_n} + \sqrt{r_{n+1}}} \\ &> \frac{a_n}{2\sqrt{r_n}} \\ 2(\sqrt{r_n} - \sqrt{r_{n+1}}) &> \frac{a_n}{\sqrt{r_n}} \end{aligned}$$

Let's consider the series of

$$\begin{aligned} 2 \sum_{n=1}^{\infty} (\sqrt{r_n} - \sqrt{r_{n+1}}) &= 2 \lim_{N \rightarrow \infty} \sum_{n=1}^N (\sqrt{r_n} - \sqrt{r_{n+1}}) \\ &= 2 \lim_{N \rightarrow \infty} [\sqrt{r_1} - \sqrt{r_2} + \sqrt{r_2} - \sqrt{r_3} + \dots + \sqrt{r_N} - \sqrt{r_{N+1}}] \\ &= 2 \lim_{N \rightarrow \infty} (\sqrt{r_1} - \sqrt{r_{N+1}}) \\ &= 2\sqrt{r_1} - \lim_{N \rightarrow \infty} \left(\sum_{m=N+1}^{\infty} a_m \right)^{1/2} \end{aligned}$$

Since $\sum a_n$ converges, $\sum_{m=N+1}^{\infty} a_m$ can be made less than $\epsilon > 0$.

$$= 2\sqrt{r_1}$$

By the comparison test, $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

13. Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series. Then $\sum |a_n| < M$ and $\sum |b_n| < n$. Let $c_n = \sum_k^m a_k b_{n-k}$.

$$\begin{aligned} \sum_{n=0}^m |c_n| &= \sum_{n=0}^m \left| \sum_{k=0}^n a_k b_{n-k} \right| \\ &\leq \sum_{n=0}^m \sum_{k=0}^n |a_k b_{n-k}| \\ &= |a_0 b_0| + |a_0 b_1| + |a_1 b_0| + \dots + |a_0 b_m| + |a_1 b_{m-1}| + \dots + |a_{m-1} b_1| + |a_m b_0| \end{aligned}$$

$$\begin{aligned}
&= |a_0||b_0| + |a_0||b_1| + |a_1||b_0| + \cdots + |a_0||b_m| + |a_1||b_{m-1}| + \cdots + |a_{m-1}||b_1| + |a_m||b_0| \\
&= \sum_{n=0}^m |a_n| \sum_{k=0}^{m-n} |b_k| \\
&< M \sum_{k=0}^{m-n} |b_k| \\
&< MN
\end{aligned}$$

Therefore, the Cauchy product of two absolutely convergent series converge.

14. If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + \cdots + s_n}{n+1}$$

for $n = 0, 1, \dots$

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.

Since $\{s_n\} \rightarrow s$, for $\epsilon > 0$, there exists $n > N$ such that $|s_n - s| < \epsilon/2$. Let $N_0 = \max\left\{N, \frac{4(N+1)|s|}{\epsilon}\right\}$. For $n > N_0$,

$$\begin{aligned}
|\sigma_n - s| &= \left| \frac{s_0 + \cdots + s_n}{n+1} - s \right| \\
&= \left| \frac{s_0 - s + \cdots + s_n - s}{n+1} \right| \\
&\leq \left| \frac{s_0 - s + \cdots + s_N - s}{n+1} \right| + \left| \frac{s_{N+1} - s + \cdots + s_n - s}{n+1} \right| \\
&< \frac{m_1}{n+1} |s_N - s| + \frac{m_2}{n+1} |s_n - s|
\end{aligned}$$

where $m_1, m_2 < n+1$

$$\begin{aligned}
&< |s_N - s| + |s_n - s| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Thus, $\lim \sigma_n = s$.

(b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.

Let $\{s_n\} = (-1)^n$. Then

$$\sigma_n = \frac{1 - 1 + 1 - \cdots + 1}{n+1} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{1}{n+1}, & \text{if } n \text{ is even} \end{cases}$$

Now taking the limit of σ_n , we have that $\lim \sigma_n = 0$.

(c) Can it happen that $s_n > 0$ for all n and that $\limsup = \infty$, although $\lim \sigma_n = 0$.

Yes. Let $\{s_n\} = \log(\log(n+1))$ for $n \geq 2$. Then

$$\limsup_{n \rightarrow \infty} s_n = \infty.$$

We can write σ_n as

$$\sigma_n = \frac{\log(\log(3)) + \log(\log(4)) + \cdots + \log(\log(n+1))}{n+1} \leq \frac{\log(n \log(n))}{n+1}$$

Now taking the limit of σ_n , we have

$$\lim_{n \rightarrow \infty} \sigma_n = \lim_{n \rightarrow \infty} \frac{s_n}{n+1}$$

$$\begin{aligned}
&\leq \lim_{n \rightarrow \infty} \frac{\log(n \log(n))}{n+1} \\
&= \lim_{n \rightarrow \infty} \frac{\log(n)}{n+1} + \lim_{n \rightarrow \infty} \frac{\log(\log(n))}{n+1} \\
&= \lim_{n \rightarrow \infty} \frac{\log(\log(n))}{n+1} \\
&= 0
\end{aligned}$$

(d) Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^n k a_k.$$

Assume that $\lim n a_n = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives converse of item 14 (a), but under the additional assumption that $n a_n \rightarrow 0$.]

Recall that $s_n = \sum_{k=0}^n a_k$. Then the left hand side can be written as

$$\begin{aligned}
s_n - \sigma_n &= a_0 + \cdots + a_n - \frac{s_0 + \cdots + s_n}{n+1} \\
&= a_0 + \cdots + a_n - \frac{(n+1)a_0 + n a_1 + \cdots + a_n}{n+1} \\
&= \frac{a_1 + 2a_2 + \cdots + n a_n}{n+1} \\
&= \frac{1}{n+1} \sum_{k=1}^n k a_k
\end{aligned}$$

as was needed to be shown.

(e) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|n a_n| \leq M$ for all n , and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$, by completing the following outline:

If $m < n$, then

$$s_n - \sigma_n = \frac{m+1}{n-m} (\sigma_n - \sigma_m) + \sum_{i=m+1}^n (s_n - s_i).$$

For these i ,

$$|s_n - s_i| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}.$$

Fix $\epsilon > 0$ and associate with each n the integer m that satisfies

$$m \leq \frac{n-\epsilon}{1+\epsilon} < m+1$$

Then $(m+1)/(n-m) \leq 1/\epsilon$ and $|s_n - s_i| < M\epsilon$. Hence

$$\limsup_{n \rightarrow \infty} |s_n - \sigma| \leq M\epsilon.$$

Since ϵ was arbitrary, $\lim s_n = \sigma$.

15. Definition 3.21 can be extended to the case in which the a_n lie in some fixed \mathbb{R}^k . Absolute convergence is defined as convergence of $\sum |a_n|$. Show that Theorems 3.22, 3.23, 3.25(a), 3.33, 3.34, 3.42, 3.45, 3.47, and 3.55 are true in this more general setting. (Only slight modifications are required in any of the proofs.)

16. Fix a positive number α . Choose $x_1 > \sqrt{\alpha}$, and define x_1, x_2, \dots , by the recursive formula

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{\alpha}{x_n} \right).$$

(a) Prove that $\{x_n\}$ decrease monotonically and the $\lim x_n = \sqrt{\alpha}$.

(b) Put $\epsilon = x_n - \sqrt{\alpha}$, and show that

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\epsilon_{n+1} < \beta \left(\frac{\epsilon_1}{\beta} \right)^{2^n}$$

for $n = 1, 2, \dots$

(c) That is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\epsilon_1/\beta < \frac{1}{10}$ and that therefore

$$\epsilon_5 < 4 \cdot 10^{-16}, \quad \epsilon_6 < 4 \cdot 10^{-23}.$$

17. Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$, and define

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}.$$

(a) Prove that $x_1 > x_3 > x_5 > \dots$.

(b) Prove that $x_2 < x_4 < x_6 < \dots$.

(c) Prove that $\lim x_n = \sqrt{\alpha}$.

(d) Compare the rapidity of convergence of this process with the one described in item 16.