Solutions to Principles of Mathematical Analysis

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Contents

1 The Real and Complex Numbers

1. If r is rational ($r \neq 0$) and x is irrational, prove that $r+x$ and rx are irrational.

   We will prove that $r+x$ is irrational by *reductio ad impossibilem*, contradiction. That is, $p \to q$ becomes $p \wedge \neg q$. Suppose r is rational ($r \neq 0$) and x is irrational and $r+x$ is rational. Since r is rational, $-r$ is rational and exist by the field axioms of addition. The sum of two rational numbers is rational by the closure property of $\mathbb{Q}$. Then $-r+(r+x) = (-r+r)+x = x$. We have reached a contradiction since x is clearly irrational. Therefore, $r+x$ is irrational.

   For the second statement, we will again use the argument of *reductio ad impossibilem*. Since $r \neq 0$ and rational, $\frac{1}{r}$ is rational and exists by the field axioms of multiplication. The multiplication of two rational numbers is rational, again, by the closure property of $\mathbb{Q}$. Then $\frac{1}{r}(rx) = \left(\frac{1}{r}r\right)(x) = x$. We have reached a contradiction since x is irrational. That is, rx is irrational.

2. Prove that there is no rational number whose square is 12.

   Suppose there is a rational number whose square is 12. Let $\frac{a}{b}$ be this rational number. Then $a^2 = 12b^2$. By the Fundamental Theorem of Arithmetic, we can write a, b, and 12 as a product of *unique* primes. Let p_i and q_i be prime numbers and $\alpha_i, \beta_i \in \mathbb{Z}^{\geqslant 0}$ for $i = 1, 2, \ldots, n$. Then $a = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, $b = q_1^{\beta_1} \cdot q_2^{\beta_2} \cdots q_n^{\beta_n}$, and $12 = 2^2 \cdot 3$. We now have

$$\begin{aligned}(p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_n^{\alpha_n})^2 &= 2^2 \cdot 3(q_1^{\beta_1} \cdot q_2^{\beta_2} \cdots q_n^{\beta_n})^2 \\ p_1^{2\alpha_1} \cdot p_2^{2\alpha_2} \cdots p_n^{2\alpha_n} &= 2^2 \cdot 3(q_1^{2\beta_1} \cdot q_2^{2\beta_2} \cdots q_n^{2\beta_n}) \qquad (1.1)\end{aligned}$$

   Let $p_k^{2\alpha_k}$ be $3^{2\alpha_k}$ and $q_m = 3^{2\beta_m}$. Then by equation (1.1)

$$\begin{aligned}3^{2\alpha_k} &= 3 \cdot 3^{2\beta_m} \\ &= 3^{2\beta_m+1}\end{aligned}$$

   Therefore, $2\alpha_k = 2\beta_m + 1$ which is a contradiction since an even number can never be an odd number. That is, there is no rational number whose square is 12.

3. Prove Proposition 1.15.

   Proposition 1.15 states that the axioms for multiplication imply the following statements.

   (a) If $x \neq 0$ and $xy = xz$, then $y = z$.

   By the field axioms of multiplication, since $x \neq 0$,

$$y = 1 \cdot y = \frac{1}{x}xy = \frac{1}{x}xz = \frac{1}{x}xz = z$$

   as was needed to be shown.

   (b) If $x \neq 0$ and $xy = x$, then $y = 1$.

   Since $x \neq 0$, we have

$$y = 1 \cdot y = \frac{1}{x}xy = \frac{1}{x}x = 1$$

   as was needed to shown.

   (c) If $x \neq 0$ and $xy = 1$, then $y = 1/x$.

   Again, since we have that $x \neq 0$,

$$y = 1 \cdot y = \frac{1}{x}xy = \frac{1}{x} \cdot 1 = \frac{1}{x}$$

   as was needed to be shown.

(d) If $x \neq 0$, then $1/(1/x) = x$

Again, since we have that $x \neq 0$,

$$\frac{1}{1/x} = 1 \cdot \frac{1}{1/x} = x\frac{1}{x}\frac{1}{1/x} = x\frac{1}{x}x = x$$

as was needed to be shown.

4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leqslant \beta$.

Since $E \neq \varnothing$, $x \in E$. Since α is a lower bound, $\alpha \leqslant x$, and since β is an upper bound, $\beta \geqslant x$. By the transitivity property, $\alpha \leqslant \beta$.

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Since A is nonempty and bounded below, $A = \{x : x \in A\}$ and $\inf(A) = \alpha$. Now, $-A = \{-x : x \in A\}$ is also nonempty. Since α is the infimum of A, $\alpha \leqslant x$ for all $x \in A$. By multiplying by -1, we get the following inequality

$$\alpha \leqslant x \Rightarrow -\alpha \geqslant -x.$$

That is, $-\alpha$ is an upper bound of $-A$. Suppose $-\gamma = \sup(-A)$ and $\varepsilon > 0$. Then $-\gamma + \varepsilon \notin -A$

$$-\alpha \geqslant -\gamma + \varepsilon \geqslant -\gamma \geqslant -x$$

Again, by multiplying by negative one, we have

$$\alpha \leqslant \gamma - \varepsilon \leqslant \gamma \leqslant x$$

but $\gamma - \varepsilon \notin A$ so γ is a lower bound of A which would contradict the fact that α is the greatest lower bound of A. In order for γ to be the lower bound, $\gamma = \alpha$ since the infimum is unique. So $-\alpha = \sup(-A)$. Therefore, $\alpha = \inf(A) = -\sup(-A) = -(-\alpha) = \alpha$.

6. Fix $b > 1$.

(a) If m, n, p, q are integers, $n, q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.

Since $n, q > 0$, $nr = m = np/q$.

$$(b^m)^{1/n} = (b^{np/q})^{1/n} = \left[(b^p)^{n/q}\right]^{1/n} = (b^p)^{1/q} = b^{p/q} = b^r$$

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

Let $r = \frac{a}{b}$ and $s = \frac{c}{d}$. Then

$$b^{r+s} = b^{(ad+bc)/(bd)} = (b^{ad+bc})^{1/(bd)} = (b^a)^{1/b}(b^c)^{1/d} = b^r b^s$$

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leqslant x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x.

From the statement $b^r = \sup B(r)$, we see that $b^r \in B(r)$. Let $b^t \in B(r)$. Then $b^r = b^t b^{r-t}$. Since $b > 1$, $b^t 1^{r-t} \leqslant b^t b^{r-t} = b^r$; therefore, $b^t \leqslant b^r$ for all $b^t \in B(r)$ so $b^r = \sup B(r)$.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

7. Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This is called the logarithm of y to the base of b.)

(a) For any positive integer n, $b^n - 1 \geqslant n(b-1)$.

From Theorem 1.21, we have that $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots a^{n-1})$. Therefore, we now have

$$\begin{aligned} b^n - 1 &= (b-1)(b^{n-1} + b^{n-2}1 + \cdots + ba^{n-2} + 1^{n-1}) \\ &\geqslant (b-1)(1^{n-1} + 1^{n-2}1 + \cdots + (1)1^{n-2} + 1^{n-1}) \qquad (1.2) \\ &= n(b-1)1^{n-1} \\ &= n(b-1) \end{aligned}$$

where equation (1.2) occurs from letting $b = 1$, and since $b > 1$, we get the less than or equal to inequality.

(b) Hence $b - 1 \geqslant n(b^{1/n} - 1)$.

(c) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

(d) If w is such that $b^w < y$, then $b^{w+1/n} < y$ for sufficiently large n; to see this, apply part (c) with $t = y \cdot b^{-w}$.

(e) If $b^w > y$, then $b^{w-1/n} > y$ for sufficiently large n.

(f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup(A)$ satisfies $b^x = y$.

(g) Prove that x is unique.

8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint: -1 is a square*

Suppose that $i > 0$. Then $i^2 = -1 \not> 0$. Instead, let's suppose that $i < 0$. Then $i^4 = 1 \not< 0$. Therefore, $\mathbb{C}$ is not ordered.

9. Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least upper bound property?

The Law of Trichotomy states that a real number is either positive, negative, or zero. In otherwords, if $x, y \in \mathbb{R}$, then $x < y$, $x = y$, or $x > y$. Let $a, b, c, d \in \mathbb{R}$. Then $a < c$, $a = c$, or $a > c$. If $a < c$, then $z < w$. If $a > c$, then $z > w$. For $a = c$, we have either $b < d$, $b = d$, or $b > d$. If $b < d$, then $z < w$. If $b > d$, them $z > w$. Finally, if $b = d$, then $z = w$. Let $z, w, u \in \mathbb{C}$ and $a, b, c, d, e, f \in \mathbb{R}$ such that z and w are defined as above and $u = e + if$. We need to show the tansitive property. That is, if $z < w$ and $w < u$, then $z < u$. Since $z < w$ and $w < u$, we have that either $a < c$ or $a = c$ and $b < d$ and $c < e$ or $c = e$ and $d < f$. If $a < c$ and $c < e$, then $a < e$ and $z < u$. If $a < c$, $c = e$, and $d < f$, then $z < u$ since $b < d < f$. If $a = c$, $b < d$, and $c < e$, then $a = c < e$ so $z < u$. If $a = c$, $b < d$, $c = e$, and $d < f$, then $a = c = e$ and $b < d < f$ so $z < u$. Thus, $\mathbb{C}$ is an order set under the dictionary order. Since $\mathbb{C}$ is an order set under the dictionary order, we have by the completeness axiom that $\mathbb{C}$ with the dictionary order has the least upper bound property.

10. Suppose $z = a + bi$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \qquad b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geqslant 0$ and that $\bar{z}^2 = w$ if $v \leqslant 0$. Conclude that every complex number (with one exception!) has two complex square roots.

We have that $z^2 = a^2 - b^2 + 2abi$ so $a^2 - b^2 = u$.

$$2ab = 2\left(\frac{|w| + u}{2}\frac{|w| - u}{2}\right)^{1/2}$$

$$= \pm\sqrt{|w|^2 - u^2}$$
$$= \pm v$$

For $v \geqslant 0$, $z^2 = u + iv = w$. Now, $\bar{z}^2 = a^2 - b^2 - 2abi$, so again we have $a^2 - b^2 = u$ and $-2ab = \mp v$. If $v \leqslant 0$, then $\bar{z}^2 = u + iv = w$. Therefore, all nonzero complex numbers have at least two complex square roots.

11. If z is a complex number, prove that there exists an $r \geqslant 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

    Since $|w| = 1$, we can write w as $w = \frac{z}{|z|}$. Then let $r = |z|$ so $z = rw$ where w and r are unique. If $z = 0$, then $r = 0$ and $w \in \mathbb{C}$ such that $|w| = 1$. Therefore, w is not unique.

12. If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leqslant |z_1| + |z_2| + \cdots + |z_n|.$$

First, we will show the triangle inequality is true for $n = 2$ and use induction for $n \geqslant 2$ and $n \in \mathbb{Z}^+$. For $n = 2$, we need to show $|z_1 + z_2| \leqslant |z_1| + |z_2|$.

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\
&= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 \\
&\leqslant |z_1|^2 + 2\mathrm{Re}(z_1 z_2) + |z_2|^2 \\
&= \big(|z_1| + |z_2|\big)^2
\end{aligned}$$

Taking square roots of the left and right sides, we have the desired results. Suppose this is true for $k < n$. Then

$$|z_1 + \cdots + z_k| \leqslant |z_1| + \cdots + |z_k|.$$

Now, we need to show it is true for $k + 1$.

$$\begin{aligned}
|z_1 + \cdots + z_{k+1}| &= |(z_1 + \cdots + z_k) + z_{k+1}| \\
&\leqslant |z_1 + \cdots + z_k| + |z_{k+1}| \\
&\leqslant |z_1| + \cdots + |z_{k+1}|
\end{aligned}$$

Therefore, by the principle of mathematical induction, the n dimensional triangle inequality is true.

13. If x, y are complex, prove that

$$\big||x| - |y|\big| \leqslant |x - y|.$$

Let $x = x + y - y$. Then by the triangle inequality, we have

$$\begin{aligned}
|x + y - y| &\leqslant |x - y| + |y| \\
|x| &\leqslant |x - y| + |y| \\
|x| - |y| &\leqslant |x - y|
\end{aligned}$$

Similarly, we could let $y = y + x - x$ and conclude

$$|y| - |x| \leqslant |x - y|.$$

Thus,

$$\big||x| - |y|\big| \leqslant |x - y|.$$

14. If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2.$$

We have that $|z|^2 = z\bar{z}$ so

$$\begin{aligned}
|1 + z|^2 + |1 - z|^2 &= (1 + z)(1 - \bar{z}) + (1 - z)(1 - \bar{z}) \\
&= 2 + z + \bar{z} + 2 - z - \bar{z} \\
&= 4
\end{aligned}$$

15. Under what conditions does equality hold in the Schwarz inequality?

The Schwarz inequality (also known as the Cauchy-Schwarz inequality) is

$$\left|\sum_{j}^{n} a_j \bar{b}_j\right|^2 \leqslant \sum_{j}^{n} |a_j|^2 \sum_{j}^{n} |b_j|^2.$$

Let $A = \sum |a_j|^2$, $B = \sum |\bar{b}_j|^2$, and $C = \sum |a_j \bar{b}_j|^2$. From the proof in the book, we have $0 = B(AB - |C|^2)$. Therefore, equality holds if $B = 0$ or $AB - |C|^2 = 0$.

16. Suppose $k \geqslant 3$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:

   (a) If $2r > d$, there are infinitely many $\mathbf{z} \in \mathbb{R}^k$ such that

   $$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

   (b) If $2r = d$, there exactly one such $\mathbf{z}$.

   (c) If $2r < d$, there is no such $\mathbf{z}$

How must these statements be modified if k is 2 or 1?

17. Prove that

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

if $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{y} \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.

We have that

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y})(\mathbf{x} + \mathbf{y}) \\ &= |\mathbf{x}|^2 + 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2 \qquad (1.3) \\ |\mathbf{x} - \mathbf{y}|^2 &= (\mathbf{x} - \mathbf{y})(\mathbf{x} - \mathbf{y}) \\ &= |\mathbf{x}|^2 - 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2 \qquad (1.4) \end{aligned}$$

Then by adding equations (1.3) and (1.4), we have

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

Then $\mathbf{x} + \mathbf{y}$ is the longer diagonal of the parallelogram and $\mathbf{x} - \mathbf{y}$ is the shorter diagonal of the parallelogram see figure 1.1.

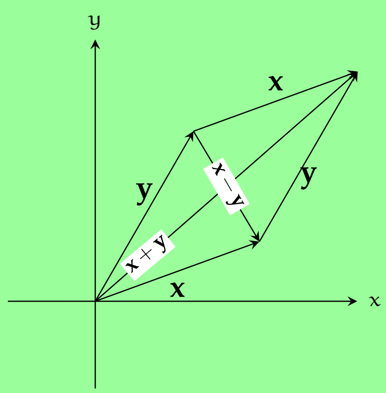

Figure 1.1: The parallelogram for vectors $\mathbf{x}$ and $\mathbf{y}$.

Then the sum of squares of the diagonals of a parallelogram are equal to the sum of the squares of the sides of the parallelogram.

18. If $k \geqslant 2$ and $\mathbf{x} \in \mathbb{R}^k$, prove that there exists $\mathbf{y} \in \mathbb{R}^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

If $\mathbf{x} = 0$, then the components of $\mathbf{y}$ can be any real numbers. If $\mathbf{x} \neq 0$, then

$$\mathbf{y} = \begin{bmatrix} -x_k & -x_{k-1} & \cdots & -x_1 \end{bmatrix}^T.$$

For $k = 1$, this is not true since for the multiplication of any two nonzero real numbers is nonzero.

19. Suppose $\mathbf{a} \in \mathbb{R}^k$, $\mathbf{b} \in \mathbb{R}^k$. Find $\mathbf{c} \in \mathbb{R}^k$ and $r > 0$ such that

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

if and only if $|\mathbf{x} - \mathbf{c}| = r$. (Solutions $3\mathbf{c} = 4\mathbf{b} - \mathbf{a}$, $3r = 2|\mathbf{b} - \mathbf{a}|$.)

2 Basic Topology

1. Prove that the empty set is a subset of every set.

   Let A be set. If $x \notin A$, then $x \notin \varnothing$. Since $\varnothing$ is the empty set, $x \notin \varnothing$ is a given. By contrapositive, if $x \in \varnothing$, then $x \in A$; therefore, $\varnothing \subset A$.

2. A complex number z is said to be *algebraic* if there are integers $a_0, \dots, a_n$, not all zero, such that

   $$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

   Prove that the set of all algebraic numbers is countable. *Hint: For every positive integer N there are only finitely many equations with*

   $$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

   Let $N \in \mathbb{Z}^+$ and A_N be the set of algebraic equations for a given N. Since $1 \leqslant n \leqslant N$, each A_N is finite. The set of algebraic numbers is $\bigcup_{N \in \mathbb{Z}^+} A_n$. The union of countable sets is countable so the set of algebraic numbers is countable.

3. Prove that there exist real numbers which are not algebraic.

   The set of algebraic numbers are countable. Therefore, the set of algebraic real numbers would also be countable. The real numbers are an uncountable set and the union of uncountable sets are not countable. We have reached a contradiction so there are real numbers which are not algebraic.

4. Is the set of all irrational real numbers countable?

   No. Let $\mathbb{I}$ be the set of irrational numbers and $\mathbb{Q}$ be the set of rational numbers. Then $\mathbb{R} = \mathbb{I} \cup \mathbb{Q}$. The set of rational numbers is countable. If $\mathbb{I}$ were countable, then $\mathbb{R}$ would be countable as well.

5. Construct a bounded set of real numbers with exactly three limit points.

   Let $A_0 = \{1/n \mid n \in \mathbb{Z}^+\}$, $A_1 = \{1 + 1/n \mid n \in \mathbb{Z}^+\}$, and $A_2 = \{2 + 1/n \mid n \in \mathbb{Z}^+\}$. Then the limit point of A_1 is 0, the limit point of A_2 is 1, and the limit point of A_2 is 2. Let $S = A_1 \cup A_2 \cup A_3$. Now S is bounded below by zero and above by three with limit points $0, 1, 2$.

6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do E and E' always have the same limit points?

   Let $x \notin E'$. Then x is not a limit point of E. Now x has a neighborhood which doesn't intersect with E' so the complement of E' is open; therefore, E' is closed. If x is a limit point of E then $x \in E'$ so x is a limit point of $\bar{E}$. Suppose x is a limit point of $\bar{E}$. Then $x \in \bar{E}$ since $\bar{E}$ is closed. Thus, $x \in E'$ or $x \in E$. If $x \in E'$, then x is a limit point of E so suppose x is in E. Then we have a neighborhood $N_r(x)$ for $r > 0$ such that $N \cap E = \{x\}$. Since E' is closed, x isn't a limit point of E'. Let $M_r(x)$ be a neighborhood of x such that $M \cap E' = \varnothing$. Let $V = N \cap M$ so U is a neighborhood of x. Therefore, V is a neighborhood of x. Now $V \cap \bar{E} = (V \cap E) \cup (V \cap E') = \{x\} \cup \varnothing = \{x\}$ so x is not a limit point of $\bar{E}$ so $x \in E'$ and is a limit point of E. No. Consider $E = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$. Then $E' = \{0\}$ and the limit point of E' is $\varnothing$.

7. Let $A_1, A_2, \dots$ be subsets of a metric space.

   (a) If $B_n = \bigcup_{i=1}^{n} A_i$, prove that $\bar{B}_n = \bigcup_{i=1}^{n} \bar{A}_i$ for $n = 1, 2, \dots$

   For $n = 2$, $\bar{B} = \overline{A_1 \cup A_2} = \bar{A}_1 \cup \bar{A}_2$. Suppose $x \in \overline{A_1 \cup A_2}$. Then $x \in A_1 \cup A_1' \cup A_2 \cup A_2'$ since $\bar{E} = E \cup E'$. Therefore, $x \in \bar{A}_1 \cup \bar{A}_2$ so $\overline{A_1 \cup A_2} \subseteq \bar{A}_1 \cup \bar{A}_2$. Suppose $x \in \bar{A}_1 \cup \bar{A}_2$. Then $x \in A_1 \cup A_2 \cup (A_1 \cup A_2)' = \overline{A_1 \cup A_2}$. Thus, we have that $\bar{A}_1 \cup \bar{A}_2 \subseteq \overline{A_1 \cup A_2}$ and that $\bar{A}_1 \cup \bar{A}_2 = \overline{A_1 \cup A_2}$. Now we can show the closure of the union of n subsets is the union of closure of the subsets.

   $$\bar{B}_n = \overline{\bigcup_{i=1}^{n} A_i}$$

$$
\begin{aligned}
&= \overline{A_1 \cup \bigcup_{i=2}^{n} A_i} \\
&= \bar{A}_1 \cup \bigcup_{i=2}^{n} \bar{A}_i \\
&= \bigcup_{i=1}^{n} \bar{A}_i
\end{aligned}
$$

(b) If $B_n = \bigcup_{i=1}^{\infty} A_i$, prove that $\bar{B} \supset \bigcup_{i=1}^{\infty} \bar{A}_i$.

From the premise, we have that $B_n \subseteq \bigcup_{i=1}^{\infty} A_i$ and $B_n \supseteq \bigcup_{i=1}^{\infty} A_i$.

8. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets of $\mathbb{R}^2$.

Let $x \in E$. Let $\epsilon > 0$ be given. Let $N_\epsilon(x)$ be a neighborhood about x of radius ϵ. Now $N \cap E \subset \mathbb{R}^2$ and the intersection of a finite number of open sets is open. Therefore, $N \cap E$ open neighborhood about x. Thus, x is a limit point of E.

Let the closed set E consist of only the point $p = (0, 0)$. Every open neighborhood of p contains no points of E except p. Thus, p is not a limit point of E.

9. Let E° denote the set of all interior points of a set E.

(a) Prove that E° is always open.

Let $x \in E^\circ$ and $\epsilon > 0$. There exists $y \in E$ such that $d(x, y) < \epsilon$. Let $r = \epsilon - d(x, y) > 0$. If $d(z, y) < r = \epsilon - d(x, y)$, then $d(z, y) + d(x, y) < \epsilon$. By the triangle inequality, $d(x, z) \leqslant d(z, y) + d(x, y) < \epsilon$. Therefore, $z \in E$. Now, $y \in E^\circ$ if there is a neighborhood of y such that $N_\delta(y) \subset E$. Let $\delta < \epsilon/2$. Then $N_\delta(y) = d(x, y) \subset E$. Thus, y is interior point and E° is always open.

(b) Prove that E is open if and only if $E^\circ = E$.

Suppose E is open. By definition, E is open if every point of E is an interior point of E or $E = E^\circ$. Suppose $E = E^\circ$. Then every point of E is an interior point of E so it follows that E is open by definition.

(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

Since G is open, $G = G^\circ$. Therefore, $G = G^\circ \subseteq E^\circ \subset E$.

(d) Prove that the complement of E° is the closure of the complement of E.

(e) Do E and $\bar{E}$ always have the same interiors?

(f) Do E and E° always have the same closures?

10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$
d(p, q) = \begin{cases} 1, & \text{if } p \neq q \\ 0, & \text{if } p = q \end{cases}
$$

Prove that this a metric space. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

By definition, the separation and coincidence axioms are satisfied. That is,

$$
d(p, q) \geqslant 0
$$

for $p \neq q$ and zero when $p = q$. For $p \neq q$, $d(p, q) = 1 = d(q, p)$, and when $p = q$, $d(p, q) = d(p, p) = 0$. Thus, symmetry is satisfied $d(p, q) = d(q, p)$. For the triangle inequality, if $p = q = r$, then we have $d(p, q) \leqslant d(p, r) + d(q, r) \Rightarrow 0 \leqslant 0$. If $p \neq q \neq r$, then $1 \leqslant 2$, and if $p = q$, then $0 \leqslant 2$.

11. For $x \in \mathbb{R}$ and $y \in \mathbb{R}$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2 \\ d_2(x,y) &= \sqrt{|x-y|} \\ d_3(x,y) &= |x^2-y^2| \\ d_4(x,y) &= |x-2y| \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|} \end{aligned}$$

Determine for each of these, whether it is a metric or not.

For d_1, note that squaring is $\geqslant 0$ for all $x, y \in \mathbb{R}$ and only when $x = y \Rightarrow (x-x)^2 = 0$. For symmetry, it is easy to show that $(x-y)^2 = [(-1)(y-x)]^2 = (y-x)^2$. For the triangle inequality, assume that $x \neq y \neq z$ because if they are we have $0 \leqslant 0$ and the identity holds.

$$\begin{aligned} d_1(x,z) &\leqslant d_1(x,y) + d_1(z,y) \\ x^2 - 2xz + z^2 &\leqslant x^2 - 2xy + 2y^2 - 2yz + z^2 \\ y(x+z) &\leqslant y^2 + xz \end{aligned}$$

Take $y = 0$. Then $0 \leqslant xz$. As long as either x or y are different signs $\pm$, the inequality doesn't hold. For instance, let $y = 0$, $x = -1$, and $z = 2$.

$$9 \nleqslant 1 + 4 = 5$$

Therefore, d_1 is not a metric. For d_2, $|x-y| \geqslant 0$ and $|x-y| = 0$ iff $x = y$. Therefore, $d_2(x,y) = \sqrt{|x-y|} \geqslant 0$ and zero iff $x = y$.

$$d_2(x,y) = \sqrt{|x-y|} = \sqrt{|(-1)(y-x)|} = \sqrt{|y-x|} = d_2(y,x)$$

For the triangle inequality, it is vacuously true when $x = y = z$.

$$\begin{aligned} d_2(x,z) &\leqslant d_2(x,y) + d_2(y,z) \\ |x-z| &\leqslant |x-y| + |y-z| + 2\sqrt{|x-y|}\sqrt{|y-z|} \end{aligned}$$

By the triangle inequality, we can write $|x-z|$ as

$$|x-z| = |x-y+y-z| \leqslant |x-y| + |y-z|$$

Since $2\sqrt{|x-y|}\sqrt{|y-z|} > 0$ for $x \neq y \neq z$, $d_2(x,z) \leqslant d_2(x,y) + d_2(y,z)$ and d_2 is a metric. d_3 is not a metric since $|x^2-y^2| = 0$ if $x = -y$ or $y = -x$. For example, let $x = 1$ and $y = -1$. Then $|1-(-1)^2| = 0$. d_4 is not metric since $|x-2x| = |-x| = |x|$ which is only zero when $x = 0$. Therefore, for all $x, y \in \mathbb{R}$, $x = y$ doesn't yield zero. For d_5, we have already established that $|x-y| \geqslant 0$ and zero iff $x = y$. Since the numerator is $|x-y|$,

$$\frac{|x-y|}{1+|x-y|} \geqslant 0$$

and zero iff $x = y$.

$$\begin{aligned} d_5(x,y) &= \frac{|x-y|}{1+|x-y|} \\ &= \frac{|(-1)(y-x)|}{1+|(-1)(y-x)|} \\ &= \frac{|y-x|}{1+|y-x|} \\ &= d_5(y,x) \end{aligned}$$

For the triangle inequality, we will multiple through by $(1+|x-z|)(1+|x-y|)(1+|y-z|)$. After simplifying, we will be left with

$$|x-z| \leqslant |x-y| + |y-z| + 2|x-y||y-z| + |x-y||y-z||x-z|$$

By the triangle inequality, we have that $|x-z| \leqslant |x-y| + |y-z|$. Since the other terms are strictly greater than or equal to zero with equality only when $x = y = z$, we can see that d_5 is a metric.

12. Let $K \subset \mathbb{R}$ consist of 0 and the numbers $1/n$ for $n = 1, 2, \ldots$ Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

13. Construct a compact set of real numbers whose limit points form a countable set.

14. Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

15. Show that Theorem 2.36 and its Corollary become false (in $\mathbb{R}$, for example) if the word "compact" is replaced by "closed" or by "bounded".

16. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

17. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?

18. Is there a nonempty perfect set in $\mathbb{R}$ which contains no rational number?

19. (a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.
    (b) Prove the same for disjoint open sets.
    (c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.
    (d) Prove that every connected metric space with at least two points is uncountable. *Hint: Use item 19 (c).*

20. Are closures and interiors of connected sets always connected? (Look at subsets of $\mathbb{R}^2$.)

21. Let A and B be separated subsets of some $\mathbb{R}^k$, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b}$$

    for $t \in \mathbb{R}$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. (Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.)
    (a) Prove that A_0 and B_0 are separated subsets of $\mathbb{R}$.
    (b) Prove that there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.
    (c) Prove that every convex subset of $\mathbf{R}^k$ is connected.

22. A metric space is *separable* if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable. *Hint: Consider the set of points which have only rational coordinates*

23. A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$. Prove that every separable metric space has a *countable* base. *Hint: Take all neighborhoods with rational radius and center in some countable dense subset of* X.

24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. *Hint: Fix* $\delta > 0$, *and pick* $x_1 \in X$. Having chosen $x_1, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geqslant \delta$ for $i = 1, \ldots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = 1/n$ $(n = 1, 2, \ldots)$, and consider the centers of the corresponding neighborhoods.

25. Prove that every compact metric space K has a countable base, and that K is therefore separable. *Hint: For every positive integer* n, *there are finitely many neighborhoods of radius* $1/n$ *whose union covers* K.

26. Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact. *Hint: By exercise 23 and 24,* X *has a countable base.* It follows that every open cover of X has a countable subcover $\{G_n\}$, $n = 1, 2, \ldots$ If no finite subcollection of $\{G_n\}$ covers X, then the complement F_n of $G_1 \cup \cdots \cup G_n$ is nonempty for each n, but $\bigcap F_n$ is empty. If E is a set which contains a point from each F_n, consider a limit point of E, and obtain a contradiction.

27. Define a point p in a metric space X to be *condensation point* of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E. Suppose $E \subset \mathbb{R}^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable. *Hint: Let* $\{V_n\}$ *be a countable base of* $\mathbb{R}^k$, *let* W *be the union of those* V_n *for which* $E \cap V_n$ *is at most countable, and show that* $P = W^c$.

28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. *(Corollary: Every countable close set in* $\mathbb{R}^k$ *has isolated points.) Hint: Use exercise 27.*

29. Prove that every open set in $\mathbb{R}$ is the union of an at most countable collection of disjoints segments. *Hint: Use exercise 22.*

30. Imitate the proof of Theorem 2.43 to obtain the following results:

    If $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$, where each F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n has a nonempty interior.

    Equivalent statement: If G_n is a dense open subset $\mathbb{R}^k$, for $n = 1, 2, \ldots$ then $\bigcap_{n=1}^{\infty} G_n$ is not empty (in fact, it is dense in $\mathbb{R}^k$).

    (This is a special case of Baire's theorem; see exercise 22, chapter 3, for the general case.)

3 Numerical Sequences and Series

1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

   Since $\{s_n\}$ converges, it is Cauchy. Let $\epsilon > 0$ be given. There exist $n, m > N$ such that $|s_n - s_m| < \epsilon$ since $\{s_n\}$ is Cauchy.

$$\begin{aligned} |s_n| &= |s_n - s_m + s_m| \\ &\leqslant |s_n - s_m| + |s_m| \\ |s_n| - |s_m| &\leqslant |s_n - s_m| \end{aligned}$$

   Similarly, we can show

$$\begin{aligned} |s_m| - |s_n| &\leqslant |s_m - s_n| \\ &= |s_n - s_m| \end{aligned}$$

   so

$$\big||s_n| - |s_m|\big| \leqslant |s_n - s_m| < \epsilon.$$

   No. Consider the sequence $\{s_n\} = (-1)^n$. Let $\epsilon = 1$. If $\{s_n\}$ converges, it will converge to ± 1. WLOG assume $s_n \to 1$. Let $n > N$ such that n is odd.

$$|(-1)^n - 1| = |-1 - 1| = 2 \not< \epsilon$$

   Therefore, the sequence $\{s_n\}$ doesn't converge. However, $\{|s_n|\}$ does converge to 1. Let $\epsilon > 0$ given. There exist an $n > N$ such that $\big||(-1)^n| - 1\big| < \epsilon$. For any $n > N$, $(-1)^n = \pm 1$ and $|\pm 1| = 1$.

$$\big||(-1)^n| - 1\big| = |1 - 1| = 0 < \epsilon$$

2. Caclulate $\lim_{n\to\infty} \sqrt{n^2+n} - n$.

$$\begin{aligned} \lim_{n\to\infty} \sqrt{n^2+n} - n &= \lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right) \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} \\ &= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} \\ &= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} \frac{1/n}{1/n} \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}+1} \\ &= \frac{1}{2} \end{aligned}$$

3. If $s_1 = \sqrt{2}$ and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}}$$

   $n \in \mathbb{Z}^+$ prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n \in \mathbb{Z}^+$.

   Let $s_{n+1} = \sqrt{2+\sqrt{s_n}}$ be written as

$$s = \sqrt{2+s} \Rightarrow s^2 - s - 2 = 0.$$

   Then $\sqrt{2+\sqrt{s_n}} < \sqrt{2+s}$. Since we are dealing with real numbers, we are only looking for positive s.

$$s^2 - s - 2 = (s-2)(s+1) = 0 \tag{3.1}$$

   so $s = 2, -1$. Thus, $s_n < 2$ so $\{s_n\}$ is bounded above by two. Additionally, since $s_1 = \sqrt{2}$, we have that $\sqrt{2} \leqslant s_n < 2$. The parabola is concave up and symmetrical about $s = 1/2$. That is, s monotonically increases from $(1/2, \infty)$ so $\{s_n\}$ monotonoically increases on $\left[\sqrt{2}, 2\right)$. Theorem 3.14 states that monotonic sequences converge if and only if it is bounded. Therefore, since $\{s_n\}$ is bounded and monotonic, $\{s_n\}$ converges and it converges to 2.

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0, \qquad s_{2m} = \frac{s_{2m-1}}{2}, \qquad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Let's determine a few of the terms. Then $s_{2m} = \{0, 1/4, 3/8, 7/16, \ldots\}$ and $s_{2m+1} = \{1/2, 3/4, 7/8, 15/16, \ldots\}$ or we can write them as

$$\begin{aligned} s_{2m} &= \frac{1}{2} - \frac{1}{2^m} \\ s_{2m+1} &= 1 - \frac{1}{2^m} \end{aligned}$$

The $\lim_{n\to\infty} \sup s_n = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1$ and $\lim_{n\to\infty} \inf s_n = \lim_{n\to\infty} \frac{1}{2} - \frac{1}{2^n} = \frac{1}{2}$.

5. For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that

$$\lim_{n\to\infty} \sup(a_n + b_n) \leqslant \lim_{n\to\infty} \sup a_n + \lim_{n\to\infty} \sup b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

Let $\lim_{n\to\infty} \sup a_n = a$, $\lim_{n\to\infty} \sup b_n = b$, and $\lim_{n\to\infty} \sup(a_n + b_n) = c$ where $a_n + b_n = c_n$. Let $\epsilon > 0$ be given. Then there exist $N_1, N_2 \in \mathbb{Z}^+$ such that for $n \geqslant N_1$ and $n \geqslant N_2$

$$\begin{aligned} |a_n - a| &< \frac{\epsilon}{2} \\ |b_n - b| &< \frac{\epsilon}{2} \end{aligned}$$

Let $N = \max\{N_1, N_2\}$. Then when $n \geqslant N$,

$$\begin{aligned} |c_n - c| &= |a_n + b_n - (a + b)| && (3.2) \\ &\leqslant |a_n - a| + |b_n - b| && (3.3) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

From equations (3.2) and (3.3), we have that

$$\lim \sup c_n \leqslant \lim \sup a_n + \lim \sup b_n,$$

and since $c_n = a_n + b_n$, the identity follows.

6. Investigate the behavior (convergence or divergence) of $\sum a_n$ if

(a) $a_n = \sqrt{n+1} - \sqrt{n}$

Let S_N be the Nth partial sum. Then

$$S_N = \sum_{n=0}^{N} (\sqrt{n+1} - \sqrt{n}) = 1 + \sqrt{2} - 1 + \sqrt{3} - \sqrt{2} + \cdots + \sqrt{N} - \sqrt{N-1} + \sqrt{N+1} - \sqrt{N}$$

Therefore, the $S_N = \sqrt{N+1}$.

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sqrt{N+1} = \infty$$

so the series doesn't converge.

(b) $a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$

Let's re-write the series by multiplying by the conjugate.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n} \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n(\sqrt{n+1} + \sqrt{n})}$$

Now

$$\sum_{n=1}^{\infty} \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \leqslant \sum_{n=1}^{\infty} \frac{1}{2n\sqrt{n}}$$

By theorem 3.28, $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leqslant 1$. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{2n\sqrt{n}} < \infty$$

since $p = 3/2$ so

$$\sum_{n=1}^{\infty} \frac{1}{n(\sqrt{n+1}+\sqrt{n})} < \infty.$$

(c) $a_n = \left(\sqrt[n]{n}-1\right)^n$

By the root test,

$$\lim_{n\to\infty} \sqrt[n]{|\sqrt[n]{n}-1|^n} = \lim_{n\to\infty} |\sqrt[n]{n}-1|.$$

Let $x_n = \sqrt[n]{n}-1$. Then

$$n = (x_n-1)^n = \sum_{k=0}^{n} \binom{n}{k} x_n^k = 1 + nx_n + \frac{n(n-1)}{2}x_n^2 + \cdots$$

so $\frac{n(n-1)}{2}x_n^2 < n \Rightarrow x_n^2 < \frac{2}{n-1} \Rightarrow x_n < \sqrt{\frac{2}{n-1}}$.

$$\begin{aligned}\lim_{n\to\infty} |\sqrt[n]{n}-1| &= \lim_{n\to\infty} |x_n| \\ &< \lim_{n\to\infty} \sqrt{\frac{2}{n-1}} \\ &= 0\end{aligned}$$

Since $\sqrt{\frac{2}{n-1}}$ converges and $x_n < \sqrt{\frac{2}{n-1}}$, x_n converges. By the root and comparison test,

$$\sum_{n=1}^{\infty} \left(\sqrt[n]{n}-1\right)^n$$

converges.

(d) $a_n = \frac{1}{1+z^n}$ for complex values of z.

$$\sum_{n=0}^{\infty} \frac{1}{1+z^n} \leqslant \sum_{n=0}^{\infty} \frac{1}{z^n}$$

and $\sum_{n=0}^{\infty} \frac{1}{z^n}$ converges for $\left|\frac{1}{z}\right| < 1$.

7. Prove that the convergence of $\sum a_n$ implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n \geqslant 0$.

For $a_n \geqslant 1$, $\sqrt{a_n} \leqslant a_n$. By the comparison test,

$$\sum \frac{\sqrt{a_n}}{n} < \sum a_n < \infty$$

so for $a_n \geqslant 1$, the series converges. For $0 \leqslant a_n < 1$, $a_n \leqslant \sqrt{a_n}$. We can write all rationals and irrational numbers in $[0, 1)$ as b/n for $b \in \mathbb{R}^{\geqslant 0}$. Then

$$\sum \frac{\sqrt{a_n}}{n} = \frac{\sqrt{b}}{n\sqrt{n}} < \infty$$

since $p = 3/2 > 1$ so the series converges.

8. If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

   By theorem 3.14, we know that monotonic bounded sequences converge. Let $\{b_n\} \to M$ for some number $M < \infty$ or $|b_n| \leqslant M$. Since $\sum a_n$ converges, for a given $\epsilon > 0$ and $k \geqslant N$, $m \geqslant k \geqslant N$ implies that
$$\left|\sum_{n=k}^{m} a_n\right| \leqslant \sum_{n=k}^{m} |a_n| \leqslant \epsilon.$$
   Take $\epsilon = \frac{\epsilon}{M}$. Then
$$\left|\sum a_n b_n\right| \leqslant \sum |a_n||b_n| \leqslant \sum |a_n| M.$$
   Since $\sum |a_n| \leqslant \epsilon/M$, the result follows; that is,
$$\sum |a_n| M \leqslant \epsilon$$
   so $\sum a_n b_n$ converges.

9. Find the radius of convergence of each of the following power series:

   (a) $\sum n^3 z^n$

   Here will use the ratio test.
$$\limsup_{n\to\infty}\left|\frac{(n+1)^3 z^{n+1}}{n^3 z^n}\right| = |z| \limsup_{n\to\infty}\left|\frac{n+1}{n}\right|^3 = |z| \limsup_{n\to\infty}\left|\frac{n}{n}\right|^3 = |z|$$
   Then $\limsup = \alpha$ and $R = \frac{1}{\alpha}$ so $R = 1$ and $|z| < 1$ for convergence.

   (b) $\sum \frac{2^n}{n!} z^n$

   Again, we use the ratio test.
$$\limsup_{n\to\infty}\left|\frac{2^{n+1} z^{n+1} n!}{2^n z^n (n+1)!}\right| = 2|z| \limsup_{n\to\infty}\left|\frac{1}{n+1}\right| = 0$$
   Thus, $R = \infty$.

   (c) $\sum \frac{2^n}{n^2} z^n$

   Following the same test, we have
$$\limsup_{n\to\infty}\left|\frac{2^{n+1} z^{n+1} n^2}{2^n z^n (n+1)^2}\right| = 2|z| \limsup_{n\to\infty}\left|\frac{n}{n}\right|^2 = 2|z|$$
   so $R = 1/2$ and $|z| < 1/2$ for convergence.

   (d) $\sum \frac{n^3}{3^n} z^n$
$$\limsup_{n\to\infty}\left|\frac{(n+1)^3 z^{n+1} 3^n}{3^{n+1} z^n n^3}\right| = \frac{|z|}{3} \limsup_{n\to\infty}\left|\frac{n}{n}\right|^3 = \frac{|z|}{3}$$
   so $R = 3$ and $|z| < 3$ for convergence.

10. Suppose the the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

11. Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

    (a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

    Theorem 3.23 states that if $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Since $\sum a_n$ doesn't converge, there exist no M such that $a_n \leqslant M$ for $M \in \mathbb{R}$. Then
$$\lim_{n\to\infty} \frac{a_n}{1+a_n} \geqslant \lim_{n\to\infty} \frac{M}{1+M} = 1$$
    for some $M \gg 10^8$. Therefore, the limit is greater than or equal to one so $\sum \frac{a_n}{1+a_n}$ doesn't converge.

(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geqslant 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

Each partial sum s_N increase so

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geqslant \frac{1}{s_{N+k}}(a_{N+1} + \cdots + a_{N+k}) = 1 - \frac{s_N}{s_{N+k}}$$

since $a_{N+1} + \cdots + a_{N+k} = s_{N+k} - s_N$. Now,

$$\sum \frac{a_n}{s_n} = \frac{a_1}{a_1} + \frac{a_2}{a_1 + a_2} + \cdots + \frac{a_n}{\sum a_n} + \cdots$$

Let $\epsilon > 0$ be given. Then

$$\left|1 - \frac{s_N}{s_{N+k}}\right| > \epsilon$$

since for k sufficiently large, $s_{N+k} \to \infty$. That is, $|1 - s_N/s_{N+k}|$ can be made larger than $1/2$. Take $\epsilon = 0.1$ and the series falls to converge.

(c) Prove that

$$\frac{a_n}{s_n^2} \leqslant \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

We can write $\frac{1}{s_{n-1}} - \frac{1}{s_n}$ as

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_n s_{n-1}}$$

where $s_n = a_n + \sum_{i=1}^{n-1} a_i$ and $s_{n-1} = \sum_{i=1}^{n-1} a_i$ so $s_n - s_{n-1} = a_n$. Now $s_n^2 \geqslant s_n s_{n-1}$ so $\frac{1}{s_n^2} \leqslant \frac{1}{s_n s_{n-1}}$.

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{a_n}{s_n s_{n-1}} \geqslant \frac{a_n}{s_n^2}$$

The telescoping series

$$\sum_{n=2}^{\infty} \frac{1}{s_{n-1}} - \frac{1}{s_n} \geqslant \sum_{n=1}^{\infty} \frac{a_n}{s_n^2}.$$

Since

$$\sum_{n=2}^{N} \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{1}{s_1} - \frac{1}{s_2} + \frac{1}{s_2} - \frac{1}{s_3} + \cdots + \frac{1}{s_{N-1}} - \frac{1}{s_N} + \frac{1}{s_N} - \frac{1}{s_{N+1}} = \frac{1}{s_1} - \frac{1}{s_{N+1}}$$

and $\sum a_n \to \infty$, $\lim_{N\to\infty} \frac{-1}{s_{N+1}} = \lim_{N\to\infty} \frac{-1}{a_{N+1}} = \frac{-1}{\infty} = 0$ so

$$\sum_{n=2}^{\infty} \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{1}{a_1}$$

and $\sum \frac{a_n}{s_n^2} < \infty$.

(d) What can be said about

$$\sum \frac{a_n}{1 + na_n} \quad \text{and} \quad \sum \frac{a_n}{1 + n^2 a_n}?$$

For the second series, we have

$$\sum \frac{a_n}{1 + n^2 a_n} \leqslant \sum \frac{1}{n^2} < \infty.$$

For the first series, suppose $a_n \in \mathbb{R}$, then

$$\sum \frac{a_n}{1 + na_n} \leqslant \sum \frac{1}{n} \to \infty$$

Suppose $a_n = 1/n^{1+p}$. Then

$$\sum \frac{a_n}{1 + na_n} = \sum \frac{1/n^{1+p}}{1 + n(1/n^{1+p})} = \sum \frac{1}{n^{1+p} + n} \leqslant \sum \frac{1}{n^{1+p}} < \infty$$

for $p > 0$. Otherwise, the series diverges to infinity.

12. Suppose $a_n > 0$ and $\sum a_n$ converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) Prove that

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{a_m + \cdots + a_n}{r_m} = \frac{r_m - r_n}{r_m} = 1 - \frac{r_n}{r_m}$$

By the same reasoning as item 11 (b), the series doesn't converge.

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Consider $\sqrt{r_n} - \sqrt{r_{n+1}}$.

$$\begin{aligned}
\sqrt{r_n} - \sqrt{r_{n+1}} &= \sqrt{r_n} - \sqrt{r_{n+1}} \frac{\sqrt{r_n} + \sqrt{r_{n+1}}}{\sqrt{r_n} + \sqrt{r_{n+1}}} \\
&= \frac{r_n - r_{n+1}}{\sqrt{r_n} + \sqrt{r_{n+1}}} \\
&= \frac{a_n}{\sqrt{r_n} + \sqrt{r_{n+1}}} \\
&> \frac{a_n}{2\sqrt{r_n}} \\
2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right) &> \frac{a_n}{\sqrt{r_n}}
\end{aligned}$$

Let's consider the series of

$$\begin{aligned}
2\sum_{n=1}^{\infty}\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right) &= 2\lim_{N\to\infty}\sum_{n=1}^{N}\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right) \\
&= 2\lim_{N\to\infty}\left[\sqrt{r_1} - \sqrt{r_2} + \sqrt{r_2} - \sqrt{r_3} + \cdots + \sqrt{r_N} - \sqrt{r_{N+1}}\right] \\
&= 2\lim_{N\to\infty}\left(\sqrt{r_1} - \sqrt{r_{N+1}}\right) \\
&= 2\sqrt{r_1} - \lim_{N\to\infty}\left(\sum_{m=N+1}^{\infty} a_m\right)^{1/2}
\end{aligned}$$

Since $\sum a_n$ converges, $\sum_{m=N+1}^{\infty} a_n$ can be made less than $\epsilon > 0$.

$$= 2\sqrt{r_1}$$

By the comparison test, $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

13. Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series. Then $\sum |a_n| < M$ and $\sum |b_n| < n$. Let $c_n = \sum_k^m a_k b_{n-k}$.

$$\begin{aligned}
\sum_{n=0}^{m} |c_n| &= \sum_{n=0}^{m}\left|\sum_{k=0}^{n} a_k b_{n-k}\right| \\
&\leqslant \sum_{n=0}^{m}\sum_{k=0}^{n} |a_k b_{n-k}| \\
&= |a_0 b_0| + |a_0 b_1| + |a_1 b_0| + \cdots + |a_0 b_m| + |a_1 b_{m-1}| + \cdots + |a_{m-1} b_1 + a_m b_0|
\end{aligned}$$

$$
\begin{aligned}
&= |a_0||b_0| + |a_0||b_1| + |a_1||b_0| + \cdots + |a_0||b_m| + |a_1||b_{m-1}| + \cdots + |a_{m-1}||b_1| + |a_m||b_0| \\
&= \sum_{n=0}^{m} |a_n| \sum_{k=0}^{m-n} |b_k| \\
&< M \sum_{k=0}^{m-n} |b_k| \\
&< MN
\end{aligned}
$$

Therefore, the Cauchy product of two absolutely convergent series converge.

14. If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + \cdots + s_n}{n+1}$$

for $n = 0, 1, \ldots$.

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.

Since $\{s_n\} \to s$, for $\epsilon > 0$, there exists $n > N$ such that $|s_n - s| < \epsilon/2$. Let $N_0 = \max\left\{N, \frac{4(N+1)|s|}{\epsilon}\right\}$. For $n > N_0$,

$$
\begin{aligned}
|\sigma_n - s| &= \left| \frac{s_0 + \cdots + s_n}{n+1} - s \right| \\
&= \left| \frac{s_0 - s + \cdots + s_n - s}{n+1} \right| \\
&\leqslant \left| \frac{s_0 - s + \cdots + s_N - s}{n+1} \right| + \left| \frac{s_{N+1} - s + \cdots + s_n - s}{n+1} \right| \\
&< \frac{m_1}{n+1}|s_N - s| + \frac{m_2}{n+1}|s_n - s|
\end{aligned}
$$

where $m_1, m_s < n+1$

$$
\begin{aligned}
&< |s_N - s| + |s_n - s| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
$$

Thus, $\lim \sigma_n = s$.

(b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.

Let $\{s_n\} = (-1)^n$. Then

$$\sigma_n = \frac{1 - 1 + 1 - \cdots + 1}{n+1} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{1}{n+1}, & \text{if } n \text{ is even} \end{cases}$$

Now taking the limit of σ_n, we have that $\lim \sigma_n = 0$.

(c) Can it happen that $s_n > 0$ for all n and that $\lim\sup = \infty$, although $\lim \sigma_n = 0$.

Yes. Let $\{s_n\} = \log(\log(n+1))$ for $n \geqslant 2$. Then

$$\limsup_{n\to\infty} s_n = \infty.$$

We can write σ_n as

$$\sigma_n = \frac{\log(\log(3)) + \log(\log(4)) + \cdots + \log(\log(n+1))}{n+1} \leqslant \frac{\log(n\log(n))}{n+1}$$

Now taking the limit of σ_n, we have

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \frac{s_n}{n+1}$$

$$
\begin{aligned}
&\leqslant \lim_{n\to\infty} \frac{\log(n\log(n))}{n+1} \\
&= \lim_{n\to\infty} \frac{\log(n)}{n+1} + \lim_{n\to\infty} \frac{\log(\log(n))}{n+1} \\
&= \lim_{n\to\infty} \frac{\log(\log(n))}{n+1} \\
&= 0
\end{aligned}
$$

(d) Put $a_n = s_n - s_{n-1}$, for $n \geqslant 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1}\sum_{k=1}^{n} k a_k.$$

Assume that $\lim n a_n = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives converse of item 14 (a), but under the additional assumption that $n a_n \to 0$.]

Recall that $s_n = \sum_{k=0}^{n} a_k$. Then the left hand side can be written as

$$
\begin{aligned}
s_n - \sigma_n &= a_0 + \cdots + a_n - \frac{s_0 + \cdots + s_n}{n+1} \\
&= a_0 + \cdots + a_n - \frac{(n+1)a_0 + n a_1 + \cdots + a_n}{n+1} \\
&= \frac{a_1 + 2a_2 + \cdots + n a_n}{n+1} \\
&= \frac{1}{n+1}\sum_{k=0}^{n} k a_k
\end{aligned}
$$

as was needed to be shown.

(e) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|n a_n| \leqslant M$ for all n, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$, by completing the following outline:

If $m < n$, then

we have that

$$
\begin{aligned}
\sigma_n - \sigma_m &= \frac{s_0 + \cdots + s_n}{n+1} - \frac{s_0 + \cdots + s_m}{m+1} \\
&= (s_0 + \cdots + s_n)\frac{m-n}{(n+1)(m+1)} + \frac{1}{m+1}\sum_{i=m+1}^{n} s_i \\
&= \frac{m-n}{m+1}\sigma_n + \frac{1}{m+1}\sum_{i=m+1}^{n} s_i
\end{aligned}
$$

Let's multiple through by $\frac{m+1}{m-n}$.

$$
\begin{aligned}
(\sigma_n - \sigma_m)\frac{m+1}{m-n} &= \sigma_n - \frac{1}{n-m}\sum_{i=m+1}^{n} s_i \\
-\sigma_n &= (\sigma_n - \sigma_m)\frac{m+1}{n-m} - \frac{1}{n-m}\sum_{i=m+1}^{n} s_i
\end{aligned}
$$

Finally, we just need to add s_n to both sides. Note that $\sum_{i=m+1}^{n} 1 = 1$ so take $s_n = \sum_{i=m+1}^{n} s_n$. Then we obtain the desired result.

$$s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{i=m+1}^{n}(s_n - s_i).$$

For these i,

we have

$$|s_n - s_i| = |a_n + a_{n-1} + \cdots + a_{i+1}|$$

$$\leqslant |a_n| + \cdots + |a_{i+1}|$$

By the hypothesis, $|na_n| \leqslant M$ so $|a_n| \leqslant M/n$.

$$\begin{aligned} &\leqslant \frac{M}{n} + \cdots + \frac{M}{i+1} \\ &= M\left(\frac{1}{n} + \cdots + \frac{1}{i+1}\right) \end{aligned}$$

Now, $i+1$ is the smallest indices so $\frac{1}{i+1}$ is the largest fraction and we have $n-i$ fractions.

$$\leqslant \frac{M(n-i)}{i+1}$$

Plugging in $i = m+1$, we achieve the desired results.

$$|s_n - s_i| \leqslant \frac{(n-i)M}{i+1} \leqslant \frac{(n-m-1)M}{m+2}.$$

Fix $\epsilon > 0$ and associate with each n the integer m that satifies

$$m \leqslant \frac{n-\epsilon}{1+\epsilon} < m+1$$

Then $(m+1)/(n-m) \leqslant 1/\epsilon$ and $|s_n - s_i| < M\epsilon$. Hence

$$\limsup_{n\to\infty} |s_n - \sigma| \leqslant M\epsilon.$$

Since ϵ was arbitrary, $\lim s_n = \sigma$.

15. Definition 3.21 can be extended to the case in which the a_n lie in some fixed $\mathbb{R}^k$. Absolute convergence is defined as convergence of $\sum |a_n|$. Show that Theorems $3.22, 3.23, 3.25(a), 3.33, 3.34, 3.42, 3.45, 3.47$, and 3.55 are true in this more general setting. (Only slight modifications are required in any of the proofs.)

Theorem 3.22: $\sum \mathbf{a}_n$ converges if and only if for every $\epsilon > 0$ there is an integer N such that

$$\left|\sum_{k=n}^{m} \mathbf{a}_k\right| \leqslant \epsilon$$

if $m \geqslant n \geqslant N$.

For $|a_i - b_i| \leqslant |\mathbf{a} - \mathbf{b}| \leqslant \sum_{i=1}^{k} |a_i - b_i|$, the sequence $\{\mathbf{a}_n\}$ converges if and only if each subsequence $\{a_{n_j}\}$ converges for $j = 1, \ldots, k$. That is, the sequences converge if they are Cauchy; therefore, the vector sequence is Cauchy.

Theorem 3.23: If $\sum \mathbf{a}_n$ converges, then $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{0}$.

From theorem 3.22, we have that $\sum \mathbf{a}_n$ converges if each $\{a_{n_j}\}$ converges for $j = 1, \ldots, k$. Thus, $a_{n_j} \to 0$ for each j so $\mathbf{a}_n \to 0$ or $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{0}$.

Theorem 3.25(a): If $|\mathbf{a}_n| \leqslant c_n$ for $n \geqslant N_0$, where N_0 is some fixed integer, and if $\sum c_n$ converges, then $\sum \mathbf{a}_n$ converges.

By the hypothesis, a_{n_j} converges for each j, and since each subsequences converges, $\sum \mathbf{a}_n$ converges.

Theorem 3.33: Given $\sum \mathbf{a}_n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|\mathbf{a}_n|}$. Then

(a) if $\alpha < 1$, $\sum \mathbf{a}_n$ converges;

From the previous theorems, we have that $\sqrt[n]{|a_{n_j}|} \leqslant \sqrt[n]{|\mathbf{a}_n|}$. Now, if $\alpha < 1$, then each subsequence converges; therefore, $\sum \mathbf{a}_n$ converges.

(b) if $\alpha > 1$, $\sum \mathbf{a}_n$ diverges; and

When $\alpha > 1$, $|\mathbf{a}_n| > 1$ for infinitely many n. Therefore, the series diverges.

(c) if $\alpha = 1$, the test gives no information.

Theorem 3.34: The series $\sum a_n$

(a) converges if $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$,

The limes superior inequality means that for some $\epsilon > 0$ and constant M, $|a_n| M\epsilon^n$. Thus, $\sum a_n$ converges absolutely so series converges by theorem 3.25.

(b) diverges if $\frac{|a_{n+1}|}{|a_n|} \geqslant 1$ for $n \leqslant n_0$, whenever n_0 is some fixed integer.

From the inequality, we get that a_n doesn't go to zero. Therefore, the series doesn't converge.

Theorem 3.42: Suppose

(a) the partial sum A_n of $\sum a_n$ form a bounded sequence;

(b) $b_0 \geqslant b_1 \geqslant b_2 \geqslant \cdots$;

(c) $\lim_{n\to\infty} b_n = 0$

Then $\sum b_n a_n$ converges.

Choose M such that $|A_n| \leqslant M$ for all n. Given $\epsilon > 0$, there is an integer N such that $b_N \leqslant \frac{\epsilon}{2M}$. For $N \leqslant p \leqslant q$, we have

$$\begin{aligned}\Big|\sum_{n=p}^{q} a_n b_n\Big| &= \Big|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\Big| \\ &\leqslant M\Big(\sum_{n=p}^{q-1} |b_n - b_{n+1}| + b_q + b_p\Big) \\ &\leqslant 2Mb_p \\ &\leqslant \epsilon\end{aligned}$$

The partial sums form a Cauchy sequence. Thus, $\sum b_n a_n$ converges.

Theorem 3.45: If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Let $c_n = |a_n|$. Then by theorem 3.25, $\sum a_n$ converges.

Theorem 3.47: If $\sum a_n = A$ and $\sum b_n = B$, then $\sum a_n + b_n = A + B$ and $\sum ca_n = cA$ for any fixed c.

By the previous theorems, we know that if for each component, the theorem holds, then the theorem holds for the vector itself.

Theorem 3.55: If $\sum a_n$ is a series of vectors which converges absolutely, then every rearrangement of $\sum a_n$ converges, and they all converge to the same sum.

16. Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_1, x_2, \ldots,$ by the recursive formula

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{\alpha}{x_n}\Big).$$

(a) Prove that $\{x_n\}$ decrease monotonically and the $\lim x_n = \sqrt{\alpha}$.

If $\{x_n\}$ decrease monotnically, then $x_n - x_{n+1} > 0$ for all n.

$$\begin{aligned}x_n - x_{n+1} &= x_n - \frac{1}{2}\Big(x_n + \frac{\alpha}{x_n}\Big) \\ &= \frac{x_n^2 - \alpha}{2x_n} \qquad (3.4)\end{aligned}$$

Suppose, on the contrary, that equation (3.4) is less than zero. Then $x_n < \sqrt{\alpha}$ which contradicts the fact that $x_1 > \sqrt{\alpha}$. Thus, equation (3.4) is less then zero and $\{x_n\}$ decreases monotonically. Now, $\{x_n\}$ is bounded above by $\sqrt{\alpha}$ and below by zero so $\{x_n\}$ converges. Suppose the limit is x. Then

$$\lim_{n\to\infty} x_n = \frac{1}{2}\Big(x_n + \frac{\alpha}{x_n}\Big)$$

$$x = \frac{1}{2}\left(x + \frac{\alpha}{x}\right)$$
$$x^2 = \alpha$$
$$x = \sqrt{\alpha}$$

Thus, $\{x_n\} \to \sqrt{\alpha}$.

(b) Put $\epsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\epsilon_{n+1} < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n}$$

for $n = 1, 2, \ldots$.

Let $\epsilon_{n+1} = x_{n+1} - \sqrt{\alpha}$. Then

$$\begin{aligned} x_{n+1} - \sqrt{\alpha} &= \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} \\ &= \frac{x_n^2 - 2x_n\sqrt{\alpha} + \alpha}{2x_n} \\ &= \frac{(x_n - \sqrt{\alpha})^2}{2x_n} \\ &= \frac{\epsilon_n^2}{2x_n} \end{aligned}$$

Since $x_n > \sqrt{\alpha}$, $1/x_n < 1/\sqrt{\alpha}$. Therefore,

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}} \tag{3.5}$$

From equation (3.5), we have that $\epsilon_{n+1} < \epsilon_n^2/\beta$. For $n = 1$, we obtain

$$\epsilon_2 < \frac{\epsilon_1^2}{\beta},$$

and when $n = 2$, we have

$$\epsilon_3 < \frac{\epsilon_2^2}{\beta} < \frac{\epsilon_1^4}{\beta^2\beta} = \beta\left(\frac{\epsilon_1}{\beta}\right)^4 = \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^{3-1}}.$$

Assume this is true for $k < n$. Then $\epsilon_k < \beta(\epsilon_1/\beta)^{2^{k-1}}$.

$$\epsilon_{k+1} < \frac{\epsilon_k^2}{\beta} < \frac{\beta^2}{\beta}\left(\frac{\epsilon_1^{2^{k-1}}}{\beta^{2^{k-1}}}\right)^2 = \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^k}$$

By the principle of mathematical induction, $\epsilon_{n+1} = \beta(\epsilon_1/\beta)^{2^n}$ for $n \in \mathbb{Z}^+$.

(c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\epsilon_1/\beta < \frac{1}{10}$ and that therefore

$$\epsilon_5 < 4 \cdot 10^{-16}, \qquad \epsilon_6 < 4 \cdot 10^{-23}.$$

$\epsilon_1 = x_1 - \sqrt{\alpha}$ and $\beta = 2\sqrt{\alpha}$ so

$$\frac{\epsilon_1}{\beta} = \frac{2 - \sqrt{3}}{2\sqrt{3}} \approx 0.077 < \frac{1}{10}$$

For ϵ_5 and ϵ_6, we have

$$\begin{aligned} \epsilon_5 &< \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^k} \\ &\approx 5.69 \times 10^{-18} \end{aligned}$$

$$
\begin{aligned}
&< 4 \cdot 10^{-16} \\
\epsilon_6 &< \beta \left(\frac{\epsilon_1}{\beta} \right)^{2^k} \\
&\approx 9.34 \times 10^{-36} \\
&< 4 \cdot 10^{-23}
\end{aligned}
$$

17. Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$, and define

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}.$$

   (a) Prove that $x_1 > x_3 > x_5 > \cdots$.

   (b) Prove that $x_2 < x_4 < x_6 < \cdots$.

   (c) Prove that $\lim x_n = \sqrt{\alpha}$.

   (d) Compare the rapidity of convergence of this process with the one described in item 16.

18. Replace the recursion formula of item 16 by

$$x_{n+1} = \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1}$$

where p is a fixed positive integer, and describe the behavior of the resulting sequences $\{x_n\}$.

19. Associate to each sequence $a = \{\alpha_n\}$, in which α_n is 0 or 2, the real number

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Prove that the set of all $x(a)$ is precisely the Cantor set described in section 2.44.

20. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

21. Prove the following analogue of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a *complete* metric space X, if $E_n \supset E_{n+1}$, and if

$$\lim_{n \to \infty} \operatorname{diam} E_n = 0,$$

then $\bigcap_1^\infty E_n$ consists of exactly one point.

22. Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap_1^\infty G_n$ is not empty. (In fact, it is dense in X.) *Hint: Find a shrinking sequence of neighborhoods E_n such that $\bar{E}_n \subset G_n$, and apply item 21.*

23. Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. *Hint: For any m, n*

$$d(p_n, q_n) \leqslant d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if m and n are large.

24. Let X be a metric space.

   (a) Call two Cauchy sequences $\{p_n\}, \{q_n\}$ in X *equivalent* if

$$\lim_{n \to \infty} d(p_n, q_n) = 0.$$

   Prove that this is an equivalence relation.

(b) Let X^* be the set of all equivalence classes so obtained . If $P, Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q_n$, define

$$\Delta(P, Q) = \lim_{n \to \infty} d(p_n, q_n);$$

by item 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence Δ is a distance function in X^*.

(c) Prove that the resulting metric space X^* is complete.

(d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, the mapping φ defined by $\varphi(p) = P_p$ is an isometry (that is, a distance-preserving mapping) of X into X^*.

(e) Prove that $\varphi(X)$ is dence in X^*, and that $\varphi(X) = X^*$ if X is complete. By item 24 (d), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the *completion* of X.

25. Let X be the metric space whose points are the rational numbers, with the metric $d(x, y) = |x - y|$. What is the completion of this space?