

SOLUTIONS TO PRINCIPLE OF MATHEMATICAL ANALYSIS  
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# 1 The Real and Complex Numbers

1. If  $r$  is rational ( $r \neq 0$ ) and  $x$  is irrational, prove that  $r + x$  and  $rx$  are irrational.

We will prove that  $r + x$  is irrational by *reductio ad impossibilem*, contradiction. That is,  $p \rightarrow q$  becomes  $p \wedge \neg q$ . Suppose  $r$  is rational ( $r \neq 0$ ) and  $x$  is irrational and  $r + x$  is rational. Since  $r$  is rational,  $-r$  is rational and exist by the field axioms of addition. The sum of two rational numbers is rational by the closure property of  $\mathbb{Q}$ . Then  $-r + (r + x) = (-r + r) + x = x$ . We have reached a contradiction since  $x$  is clearly irrational. Therefore,  $r + x$  is irrational.

For the second statement, we will again use the argument of *reductio ad impossibilem*. Since  $r \neq 0$  and rational,  $\frac{1}{r}$  is rational and exists by the field axioms of multiplication. The multiplication of two rational numbers is rational, again, by the closure property of  $\mathbb{Q}$ . Then  $\frac{1}{r}(rx) = (\frac{1}{r}r)(x) = x$ . We have reached a contradiction since  $x$  is irrational. That is,  $rx$  is irrational.

2. Prove that there is no rational number whose square is 12.

Suppose there is a rational number whose square is 12. Let  $\frac{a}{b}$  be this rational number. Then  $a^2 = 12b^2$ . By the Fundamental Theorem of Arithmetic, we can write  $a$ ,  $b$ , and 12 as a product of *unique* primes. Let  $p_i$  and  $q_i$  be prime numbers and  $\alpha_i, \beta_i \in \mathbb{Z}^{\geq 0}$  for  $i = 1, 2, \dots, n$ . Then  $a = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_n^{\alpha_n}$ ,  $b = q_1^{\beta_1} \cdot q_2^{\beta_2} \cdot \dots \cdot q_n^{\beta_n}$ , and  $12 = 2^2 \cdot 3$ . We now have

$$\begin{aligned} (p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_n^{\alpha_n})^2 &= 2^2 \cdot 3 (q_1^{\beta_1} \cdot q_2^{\beta_2} \cdot \dots \cdot q_n^{\beta_n})^2 \\ p_1^{2\alpha_1} \cdot p_2^{2\alpha_2} \cdot \dots \cdot p_n^{2\alpha_n} &= 2^2 \cdot 3 (q_1^{2\beta_1} \cdot q_2^{2\beta_2} \cdot \dots \cdot q_n^{2\beta_n}) \end{aligned} \quad (1)$$

Let  $p_k^{2\alpha_k}$  be  $3^{2\alpha_k}$  and  $q_m = 3^{2\beta_m}$ . Then by equation (1)

$$\begin{aligned} 3^{2\alpha_k} &= 3 \cdot 3^{2\beta_m} \\ &= 3^{2\beta_m+1} \end{aligned}$$

Therefore,  $2\alpha_k = 2\beta_m + 1$  which is a contradiction since an even number can never be an odd number. That is, there is no rational number whose square is 12.

3. Prove Proposition 1.15.

Proposition 1.15 states that the axioms for multiplication imply the following statements.

- (a) If  $x \neq 0$  and  $xy = xz$ , then  $y = z$ .

By the field axioms of multiplication, since  $x \neq 0$ ,

$$y = 1 \cdot y = \frac{1}{x}xy = \frac{1}{x}xz = \frac{1}{x}xz = z$$

as was needed to be shown.

- (b) If  $x \neq 0$  and  $xy = x$ , then  $y = 1$ .

Since  $x \neq 0$ , we have

$$y = 1 \cdot y = \frac{1}{x}xy = \frac{1}{x}x = 1$$

as was needed to be shown.

- (c) If  $x \neq 0$  and  $xy = 1$ , then  $y = 1/x$ .

Again, since we have that  $x \neq 0$ ,

$$y = 1 \cdot y = \frac{1}{x}xy = \frac{1}{x} \cdot 1 = \frac{1}{x}$$

as was needed to be shown.

- (d) If  $x \neq 0$ , then  $1/(1/x) = x$

Again, since we have that  $x \neq 0$ ,

$$\frac{1}{1/x} = 1 \cdot \frac{1}{1/x} = x \frac{1}{x} \frac{1}{1/x} = x \frac{1}{x} = x$$

as was needed to be shown.

4. Let  $E$  be a nonempty subset of an ordered set; suppose  $\alpha$  is a lower bound of  $E$  and  $\beta$  is an upper bound of  $E$ . Prove that  $\alpha \leq \beta$ .

Since  $E \neq \emptyset$ ,  $x \in E$ . Since  $\alpha$  is a lower bound,  $\alpha \leq x$ , and since  $\beta$  is an upper bound,  $\beta \geq x$ . By the transitivity property,  $\alpha \leq \beta$ .

5. Let  $A$  be a nonempty set of real numbers which is bounded below. Let  $-A$  be the set of all numbers  $-x$ , where  $x \in A$ . Prove that

$$\inf A = -\sup(-A).$$

Since  $A$  is nonempty and bounded below,  $A = \{x : x \in A\}$  and  $\inf(A) = \alpha$ . Now,  $-A = \{-x : x \in A\}$  is also nonempty. Since  $\alpha$  is the infimum of  $A$ ,  $\alpha \leq x$  for all  $x \in A$ . By multiplying by  $-1$ , we get the following inequality

$$\alpha \leq x \Rightarrow -\alpha \geq -x.$$

That is,  $-\alpha$  is an upper bound of  $-A$ . Suppose  $-\gamma = \sup(-A)$  and  $\varepsilon > 0$ . Then  $-\gamma + \varepsilon \notin -A$

$$-\alpha \geq -\gamma + \varepsilon \geq -\gamma \geq -x$$

Again, by multiplying by negative one, we have

$$\alpha \leq \gamma - \varepsilon \leq \gamma \leq x$$

but  $\gamma - \varepsilon \notin A$  so  $\gamma$  is a lower bound of  $A$  which would contradict the fact that  $\alpha$  is the greatest lower bound of  $A$ . In order for  $\gamma$  to be the lower bound,  $\gamma = \alpha$  since the infimum is unique. So  $-\alpha = \sup(-A)$ . Therefore,  $\alpha = \inf(A) = -\sup(-A) = -(-\alpha) = \alpha$ .

6. Fix  $b > 1$ .

- (a) If  $m, n, p, q$  are integers,  $n, q > 0$ , and  $r = m/n = p/q$ , prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define  $b^r = (b^m)^{1/n}$ .

Since  $n, q > 0$ ,  $nr = m = np/q$ .

$$(b^m)^{1/n} = (b^{np/q})^{1/n} = [(b^p)^{n/q}]^{1/n} = (b^p)^{1/q} = b^{p/q} = b^r$$

- (b) Prove that  $b^{r+s} = b^r b^s$  if  $r$  and  $s$  are rational.

Let  $r = \frac{a}{b}$  and  $s = \frac{c}{d}$ . Then

$$b^{r+s} = b^{(ad+bc)/(bd)} = (b^{ad+bc})^{1/(bd)} = (b^a)^{1/b} (b^c)^{1/d} = b^r b^s$$

- (c) If  $x$  is real, define  $B(x)$  to be the set of all numbers  $b^t$ , where  $t$  is rational and  $t \leq x$ . Prove that

$$b^r = \sup B(r)$$

when  $r$  is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real  $x$ .

From the statement  $b^r = \sup B(r)$ , we see that  $b^r \in B(r)$ . Let  $b^t \in B(r)$ . Then  $b^r = b^t b^{r-t}$ . Since  $b > 1$ ,  $b^t 1^{r-t} \leq b^t b^{r-t} = b^r$ ; therefore,  $b^t \leq b^r$  for all  $b^t \in B(r)$  so  $b^r = \sup B(r)$ .

- (d) Prove that  $b^{x+y} = b^x b^y$  for all real  $x$  and  $y$ .

7. Fix  $b > 1$ ,  $y > 0$ , and prove that there is a unique real  $x$  such that  $b^x = y$ , by completing the following outline. (This is called the logarithm of  $y$  to the base of  $b$ .)

- (a) For any positive integer  $n$ ,  $b^n - 1 \geq n(b - 1)$ .

From Theorem 1.21, we have that  $b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$ . Therefore, we now have

$$\begin{aligned} b^n - 1 &= (b - 1)(b^{n-1} + b^{n-2}1 + \cdots + ba^{n-2} + 1^{n-1}) \\ &\geq (b - 1)(1^{n-1} + 1^{n-2}1 + \cdots + (1)1^{n-2} + 1^{n-1}) \\ &= n(b - 1)1^{n-1} \\ &= n(b - 1) \end{aligned} \tag{2}$$

where equation (2) occurs from letting  $b = 1$ , and since  $b > 1$ , we get the less than or equal to inequality.

- (b) Hence  $b - 1 \geq n(b^{1/n} - 1)$ .

- (c) If  $t > 1$  and  $n > (b - 1)/(t - 1)$ , then  $b^{1/n} < t$ .

- (d) If  $w$  is such that  $b^w < y$ , then  $b^{w+1/n} < y$  for sufficiently large  $n$ ; to see this, apply part (c) with  $t = y \cdot b^{-w}$ .

- (e) If  $b^w > y$ , then  $b^{w-1/n} > y$  for sufficiently large  $n$ .

- (f) Let  $A$  be the set of all  $w$  such that  $b^w < y$ , and show that  $x = \sup(A)$  satisfies  $b^x = y$ .

- (g) Prove that  $x$  is unique.

8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:  $-1$  is a square*

Suppose that  $i > 0$ . Then  $i^2 = -1 \not> 0$ . Instead, let's suppose that  $i < 0$ . Then  $i^4 = 1 \not< 0$ . Therefore,  $\mathbb{C}$  is not ordered.

9. Suppose  $z = a + bi$ ,  $w = c + di$ . Define  $z < w$  if  $a < c$ , and also  $a = c$  but  $b < d$ . Prove that this turns the set of all complex numbers into an ordered set. (This type of relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least upper bound property?

The Law of Trichotomy states that a real number is either positive, negative, or zero. In other words, if  $x, y \in \mathbb{R}$ , then  $x < y$ ,  $x = y$ , or  $x > y$ . Let  $a, b, c, d \in \mathbb{R}$ . Then  $a < c$ ,  $a = c$ , or  $a > c$ . If  $a < c$ , then  $z < w$ . If  $a > c$ , then  $z > w$ . For  $a = c$ , we have either  $b < d$ ,  $b = d$ , or  $b > d$ . If  $b < d$ , then  $z < w$ . If  $b > d$ , then  $z > w$ . Finally, if  $b = d$ , then  $z = w$ . Let  $z, w, u \in \mathbb{C}$  and  $a, b, c, d, e, f \in \mathbb{R}$  such that  $z$  and  $w$  are defined as above and  $u = e + if$ . We need to show the transitive property. That is, if  $z < w$  and  $w < u$ , then  $z < u$ . Since  $z < w$  and  $w < u$ , we have that either  $a < c$  or  $a = c$  and  $b < d$  and  $c < e$  or  $c = e$  and  $d < f$ . If  $a < c$  and  $c < e$ , then  $a < e$  and  $z < u$ . If  $a < c$ ,  $c = e$ , and  $d < f$ , then  $z < u$  since  $b < d < f$ . If  $a = c$ ,  $b < d$ , and  $c < e$ , then  $a = c < e$  so  $z < u$ . If  $a = c$ ,  $b < d$ ,  $c = e$ , and  $d < f$ , then  $a = c = e$  and  $b < d < f$  so  $z < u$ . Thus,  $\mathbb{C}$  is an order set under the dictionary order. Since  $\mathbb{C}$  is an order set under the dictionary order, we have by the completeness axiom that  $\mathbb{C}$  with the dictionary order has the least upper bound property.

10. Suppose  $z = a + bi$ ,  $w = u + iv$ , and

$$a = \left( \frac{|w| + u}{2} \right)^{1/2}, \quad b = \left( \frac{|w| - u}{2} \right)^{1/2}.$$

Prove that  $z^2 = w$  if  $v \geq 0$  and that  $\bar{z}^2 = w$  if  $v \leq 0$ . Conclude that every complex number (with one exception!) has two complex square roots.

We have that  $z^2 = a^2 - b^2 + 2abi$  so  $a^2 - b^2 = u$ .

$$\begin{aligned} 2ab &= 2 \left( \frac{|w| + u}{2} \frac{|w| - u}{2} \right)^{1/2} \\ &= \pm \sqrt{|w|^2 - u^2} \\ &= \pm v \end{aligned}$$

For  $v \geq 0$ ,  $z^2 = u + iv = w$ . Now,  $\bar{z}^2 = a^2 - b^2 - 2abi$ , so again we have  $a^2 - b^2 = u$  and  $-2ab = \mp v$ . If  $v \leq 0$ , then  $\bar{z}^2 = u + iv = w$ . Therefore, all nonzero complex numbers have at least two complex square roots.

11. If  $z$  is a complex number, prove that there exists an  $r \geq 0$  and a complex number  $w$  with  $|w| = 1$  such that  $z = rw$ . Are  $w$  and  $r$  always uniquely determined by  $z$ ?

Since  $|w| = 1$ , we can write  $w$  as  $w = \frac{z}{|z|}$ . Then let  $r = |z|$  so  $z = rw$  where  $w$  and  $r$  are unique. If  $z = 0$ , then  $r = 0$  and  $w \in \mathbb{C}$  such that  $|w| = 1$ . Therefore,  $w$  is not unique.

12. If  $z_1, \dots, z_n$  are complex, prove that

$$|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n|.$$

First, we will show the triangle inequality is true for  $n = 2$  and use induction for  $n \geq 2$  and  $n \in \mathbb{Z}^+$ . For  $n = 2$ , we need to show  $|z_1 + z_2| \leq |z_1| + |z_2|$ .

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\ &= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 \\ &\leq |z_1|^2 + 2\operatorname{Re}(z_1\bar{z}_2) + |z_2|^2 \\ &= (|z_1| + |z_2|)^2 \end{aligned}$$

Taking square roots of the left and right sides, we have the desired results. Suppose this is true for  $k < n$ . Then

$$|z_1 + \dots + z_k| \leq |z_1| + \dots + |z_k|.$$

Now, we need to show it is true for  $k + 1$ .

$$\begin{aligned} |z_1 + \dots + z_{k+1}| &= |(z_1 + \dots + z_k) + z_{k+1}| \\ &\leq |z_1 + \dots + z_k| + |z_{k+1}| \\ &\leq |z_1| + \dots + |z_{k+1}| \end{aligned}$$

Therefore, by the principle of mathematical induction, the  $n$  dimensional triangle inequality is true.

13. If  $x, y$  are complex, prove that

$$||x| - |y|| \leq |x - y|.$$

Let  $x = x + y - y$ . Then by the triangle inequality, we have

$$\begin{aligned} |x + y - y| &\leq |x - y| + |y| \\ |x| &\leq |x - y| + |y| \\ |x| - |y| &\leq |x - y| \end{aligned}$$

Similarly, we could let  $y = y + x - x$  and conclude

$$|y| - |x| \leq |x - y|.$$

Thus,

$$||x| - |y|| \leq |x - y|.$$

14. If  $z$  is a complex number such that  $|z| = 1$ , that is, such that  $z\bar{z} = 1$ , compute

$$|1 + z|^2 + |1 - z|^2.$$

We have that  $|z|^2 = z\bar{z}$  so

$$\begin{aligned} |1 + z|^2 + |1 - z|^2 &= (1 + z)(1 + \bar{z}) + (1 - z)(1 - \bar{z}) \\ &= 2 + z + \bar{z} + 2 - z - \bar{z} \\ &= 4 \end{aligned}$$

15. Under what conditions does equality hold in the Schwarz inequality?

The Schwarz inequality (also known as the Cauchy-Schwarz inequality) is

$$\left| \sum_j^n a_j \bar{b}_j \right|^2 \leq \sum_j^n |a_j|^2 \sum_j^n |b_j|^2.$$

Let  $A = \sum |a_j|^2$ ,  $B = \sum |\bar{b}_j|^2$ , and  $C = \sum |a_j \bar{b}_j|^2$ . From the proof in the book, we have  $0 = B(AB - |C|^2)$ . Therefore, equality holds if  $B = 0$  or  $AB - |C|^2 = 0$ .

16. Suppose  $k \geq 3$ ,  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ ,  $|\mathbf{x} - \mathbf{y}| = d > 0$ , and  $r > 0$ . Prove:

(a) If  $2r > d$ , there are infinitely many  $\mathbf{z} \in \mathbb{R}^k$  such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

(b) If  $2r = d$ , there exactly one such  $\mathbf{z}$ .

(c) If  $2r < d$ , there is no such  $\mathbf{z}$

How must these statements be modified if  $k$  is 2 or 1?

17. Prove that

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

if  $\mathbf{x} \in \mathbb{R}^k$  and  $\mathbf{y} \in \mathbb{R}^k$ . Interpret this geometrically, as a statement about parallelograms.

We have that

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y})(\mathbf{x} + \mathbf{y}) \\ &= |\mathbf{x}|^2 + 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2 \end{aligned} \tag{3}$$

$$\begin{aligned} |\mathbf{x} - \mathbf{y}|^2 &= (\mathbf{x} - \mathbf{y})(\mathbf{x} - \mathbf{y}) \\ &= |\mathbf{x}|^2 - 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2 \end{aligned} \tag{4}$$

Then by adding equations (3) and (4), we have

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

Then  $\mathbf{x} + \mathbf{y}$  is the longer diagonal of the parallelogram and  $\mathbf{x} - \mathbf{y}$  is the shorter diagonal of the parallelogram see figure 1.

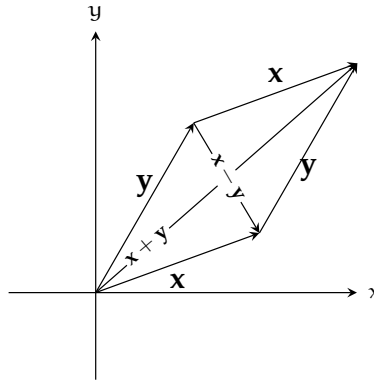


Figure 1: The parallelogram for vectors  $\mathbf{x}$  and  $\mathbf{y}$ .

Then the sum of squares of the diagonals of a parallelogram are equal to the sum of the squares of the sides of the parallelogram.

18. If  $k \geq 2$  and  $\mathbf{x} \in \mathbb{R}^k$ , prove that there exists  $\mathbf{y} \in \mathbb{R}^k$  such that  $\mathbf{y} \neq \mathbf{0}$  but  $\mathbf{x} \cdot \mathbf{y} = 0$ . Is this also true if  $k = 1$ ?

If  $\mathbf{x} = \mathbf{0}$ , then the components of  $\mathbf{y}$  can be any real numbers. If  $\mathbf{x} \neq \mathbf{0}$ , then  $\mathbf{y} = [-x_k \ -x_{k-1} \ \cdots \ -x_1]^T$ . For  $k = 1$ , this is not true since for the multiplication of any two nonzero real numbers is nonzero.

19. Suppose  $\mathbf{a} \in \mathbb{R}^k$ ,  $\mathbf{b} \in \mathbb{R}^k$ . Find  $\mathbf{c} \in \mathbb{R}^k$  and  $r > 0$  such that

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

if and only if  $|\mathbf{x} - \mathbf{c}| = r$ . (Solutions  $3\mathbf{c} = 4\mathbf{b} - \mathbf{a}$ ,  $3r = 2|\mathbf{b} - \mathbf{a}|$ .)

## 2 Basic Topology

1. Prove that the empty set is a subset of every set.

Let  $A$  be set. If  $x \notin A$ , then  $x \notin \emptyset$ . Since  $\emptyset$  is the empty set,  $x \notin \emptyset$  is a given. By contrapositive, if  $x \in \emptyset$ , then  $x \in A$ ; therefore,  $\emptyset \subset A$ .

2. A complex number  $z$  is said to be *algebraic* if there are integers  $a_0, \dots, a_n$ , not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. *Hint: For every positive integer  $N$  there are only finitely many equations with*

$$n + |a_0| + |a_1| + \dots + |a_n| = N.$$

Let  $N \in \mathbb{Z}^+$  and  $A_N$  be the set of algebraic equations for a given  $N$ . Since  $1 \leq n \leq N$ , each  $A_N$  is finite. The set of algebraic numbers is  $\bigcup_{N \in \mathbb{Z}^+} A_N$ . The union of countable sets is countable so the set of algebraic numbers is countable.

3. Prove that there exist real numbers which are not algebraic.

The set of algebraic numbers are countable. Therefore, the set of algebraic real numbers would also be countable. The real numbers are an uncountable set and the union of uncountable sets are not countable. We have reached a contradiction so there are real numbers which are not algebraic.

4. Is the set of all irrational real numbers countable?

No. Let  $\mathbb{I}$  be the set of irrational numbers and  $\mathbb{Q}$  be the set of rational numbers. Then  $\mathbb{R} = \mathbb{I} \cup \mathbb{Q}$ . The set of rational numbers is countable. If  $\mathbb{I}$  were countable, then  $\mathbb{R}$  would be countable as well.

5. Construct a bounded set of real numbers with exactly three limit points.

Let  $A_0 = \{1/n \mid n \in \mathbb{Z}^+\}$ ,  $A_1 = \{1 + 1/n \mid n \in \mathbb{Z}^+\}$ , and  $A_2 = \{2 + 1/n \mid n \in \mathbb{Z}^+\}$ . Then the limit point of  $A_1$  is 0, the limit point of  $A_2$  is 1, and the limit point of  $A_3$  is 2. Let  $S = A_1 \cup A_2 \cup A_3$ . Now  $S$  is bounded below by zero and above by three with limit points 0, 1, 2.

6. Let  $E'$  be the set of all limit points of a set  $E$ . Prove that  $E'$  is closed. Prove that  $E$  and  $\bar{E}$  have the same limit points. (Recall that  $\bar{E} = E \cup E'$ .) Do  $E$  and  $E'$  always have the same limit points?

Let  $x \in X \setminus E'$  and  $x' \in E'$ . Then  $x$  is not a limit point of  $E$ . Let  $\epsilon > 0$  be given and  $y \in E$ . Set  $\epsilon = \min\{|d(x, x') - d(x, y)|, |d(y, x') - d(x, y)|\}$ . Then  $N_\epsilon(x) = \{y \in E \mid d(x, y) < \epsilon\}$  is a neighborhood of  $x$  that contains  $y \neq x$  such that  $y \notin E'$ . Therefore,  $x$  is an interior point of  $X \setminus E'$  so  $X \setminus E'$  is open and its complement is closed, namely  $E'$ . Let  $x'$  be a limit point of  $E$ . Then  $x'$  is a limit point of  $\bar{E}$  since  $\bar{E} = E \cup E'$ . Now, let  $x'$  be a limit point of  $\bar{E}$ . Since  $x' \in \bar{E}$ , we can construct a neighborhood of  $x'$  such that for  $y \neq x'$   $y \in \bar{E}$ . If  $y \in E$ ,  $x'$  is a limit point of  $E$  and  $E'$ . Suppose  $y \in \bar{E} \setminus E$ .

7. Let  $A_1, A_2, \dots$  be subsets of a metric space.

- (a) If  $B_n = \bigcup_{i=1}^n A_i$ , prove that  $\bar{B}_n = \bigcup_{i=1}^n \bar{A}_i$  for  $n = 1, 2, \dots$
- (b) If  $B_n = \bigcup_{i=1}^n A_i$ , prove that  $\bar{B} \supset \bigcup_{i=1}^n \bar{A}_i$ .

8. Is every point of every open set  $E \subset \mathbb{R}^2$  a limit point of  $E$ ? Answer the same question for closed sets of  $\mathbb{R}^2$ .

9. Let  $E^\circ$  denote the set of all interior points of a set  $E$ .

- (a) Prove that  $E^\circ$  is always open.
- (b) Prove that  $E$  is open if and only if  $E^\circ = E$ .
- (c) If  $G \subset E$  and  $G$  is open, prove that  $G \subset E^\circ$ .
- (d) Prove that the complement of  $E^\circ$  is the closure of the complement of  $E$ .
- (e) Do  $E$  and  $\bar{E}$  always have the same interiors?
- (f) Do  $E$  and  $E^\circ$  always have the same closures?



10. Let  $X$  be an infinite set. For  $p \in X$  and  $q \in X$ , define

$$d(p, q) = \begin{cases} 1, & \text{if } p \neq q \\ 0, & \text{if } p = q \end{cases}$$

Prove that this is a metric space. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

11. For  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$ , define

$$\begin{aligned} d_1(x, y) &= (x - y)^2 \\ d_2(x, y) &= \sqrt{|x - y|} \\ d_3(x, y) &= |x^2 - y^2| \\ d_4(x, y) &= |x - 2y| \\ d_5(x, y) &= \frac{|x - y|}{1 + |x - y|} \end{aligned}$$

Determine for each of these, whether it is a metric or not.

12. Let  $K \subset \mathbb{R}$  consist of 0 and the numbers  $1/n$  for  $n = 1, 2, \dots$ . Prove that  $K$  is compact directly from the definition (without using the Heine-Borel theorem).

13. Construct a compact set of real numbers whose limit points form a countable set.

14. Give an example of an open cover of the segment  $(0, 1)$  which has no finite subcover.

15. Show that Theorem 2.36 and its Corollary become false (in  $\mathbb{R}$ , for example) if the word "compact" is replaced by "closed" or by "bounded".

16. Regard  $\mathbb{Q}$ , the set of all rational numbers, as a metric space, with  $d(p, q) = |p - q|$ . Let  $E$  be the set of all  $p \in \mathbb{Q}$  such that  $2 < p^2 < 3$ . Show that  $E$  is closed and bounded in  $\mathbb{Q}$ , but that  $E$  is not compact. Is  $E$  open in  $\mathbb{Q}$ ?

17. Let  $E$  be the set of all  $x \in [0, 1]$  whose decimal expansion contains only the digits 4 and 7. Is  $E$  countable? Is  $E$  dense in  $[0, 1]$ ? Is  $E$  compact? Is  $E$  perfect?

18. Is there a nonempty perfect set in  $\mathbb{R}$  which contains no rational number?

19. (a) If  $A$  and  $B$  are disjoint closed sets in some metric space  $X$ , prove that they are separated.

(b) Prove the same for disjoint open sets.

(c) Fix  $p \in X$ ,  $\delta > 0$ , define  $A$  to be the set of all  $q \in X$  for which  $d(p, q) < \delta$ , define  $B$  similarly, with  $>$  in place of  $<$ . Prove that  $A$  and  $B$  are separated.

(d) Prove that every connected metric space with at least two points is uncountable. *Hint: Use item 19 (c).*

20. Are closures and interiors of connected sets always connected? (Look at subsets of  $\mathbb{R}^2$ .)

21. Let  $A$  and  $B$  be separated subsets of some  $\mathbb{R}^k$ , suppose  $\mathbf{a} \in A$ ,  $\mathbf{b} \in B$ , and define

$$\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b}$$

for  $t \in \mathbb{R}$ . Put  $A_0 = \mathbf{p}^{-1}(A)$ ,  $B_0 = \mathbf{p}^{-1}(B)$ . (Thus  $t \in A_0$  if and only if  $\mathbf{p}(t) \in A$ .)

(a) Prove that  $A_0$  and  $B_0$  are separated subsets of  $\mathbb{R}$ .

(b) Prove that there exists  $t_0 \in (0, 1)$  such that  $\mathbf{p}(t_0) \notin A \cup B$ .

(c) Prove that every convex subset of  $\mathbb{R}^k$  is connected.

22. A metric space is *separable* if it contains a countable dense subset. Show that  $\mathbb{R}^k$  is separable. *Hint: Consider the set of points which have only rational coordinates*

23. A collection  $\{V_\alpha\}$  of open subsets of  $X$  is said to be a *base* for  $X$  if the following is true: For every  $x \in X$  and every open set  $G \subset X$  such that  $x \in G$ , we have  $x \in V_\alpha \subset G$  for some  $\alpha$ . In other words, every open set in  $X$  is the union of a subcollection of  $\{V_\alpha\}$ . Prove that every separable metric space has a *countable base*. *Hint: Take all neighborhoods with rational radius and center in some countable dense subset of  $X$ .*

24. Let  $X$  be a metric space in which every infinite subset has a limit point. Prove that  $X$  is separable. *Hint: Fix  $\delta > 0$ , and pick  $x_1 \in X$ . Having chosen  $x_1, \dots, x_j \in X$ , choose  $x_{j+1} \in X$ , if possible, so that  $d(x_i, x_{j+1}) \geq \delta$  for  $i = 1, \dots, j$ . Show that this process must stop after a finite number of steps, and that  $X$  can therefore be covered by finitely many neighborhoods of radius  $\delta$ . Take  $\delta = 1/n$  ( $n = 1, 2, \dots$ ), and consider the centers of the corresponding neighborhoods.*
25. Prove that every compact metric space  $K$  has a countable base, and that  $K$  is therefore separable. *Hint: For every positive integer  $n$ , there are finitely many neighborhoods of radius  $1/n$  whose union covers  $K$ .*
26. Let  $X$  be a metric space in which every infinite subset has a limit point. Prove that  $X$  is compact. *Hint: By exercise 23 and 24,  $X$  has a countable base. It follows that every open cover of  $X$  has a countable subcover  $\{G_n\}$ ,  $n = 1, 2, \dots$ . If no finite subcollection of  $\{G_n\}$  covers  $X$ , then the complement  $F_n$  of  $G_1 \cup \dots \cup G_n$  is nonempty for each  $n$ , but  $\bigcap F_n$  is empty. If  $E$  is a set which contains a point from each  $F_n$ , consider a limit point of  $E$ , and obtain a contradiction.*
27. Define a point  $p$  in a metric space  $X$  to be a *condensation point* of a set  $E \subset X$  if every neighborhood of  $p$  contains uncountably many points of  $E$ . Suppose  $E \subset \mathbb{R}^k$ ,  $E$  is uncountable, and let  $P$  be the set of all condensation points of  $E$ . Prove that  $P$  is perfect and that at most countably many points of  $E$  are not in  $P$ . In other words, show that  $P^c \cap E$  is at most countable. *Hint: Let  $\{V_n\}$  be a countable base of  $\mathbb{R}^k$ , let  $W$  be the union of those  $V_n$  for which  $E \cap V_n$  is at most countable, and show that  $P = W^c$ .*
28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (*Corollary: Every countable closed set in  $\mathbb{R}^k$  has isolated points.*) *Hint: Use exercise 27.*
29. Prove that every open set in  $\mathbb{R}$  is the union of an at most countable collection of disjoint segments. *Hint: Use exercise 22.*
30. Imitate the proof of Theorem 2.43 to obtain the following results:
- If  $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$ , where each  $F_n$  is a closed subset of  $\mathbb{R}^k$ , then at least one  $F_n$  has a nonempty interior.
- Equivalent statement:* If  $G_n$  is a dense open subset of  $\mathbb{R}^k$ , for  $n = 1, 2, \dots$  then  $\bigcap_{n=1}^{\infty} G_n$  is not empty (in fact, it is dense in  $\mathbb{R}^k$ ).
- (This is a special case of Baire's theorem; see exercise 22, chapter 3, for the general case.)