Ejercicio: $X_1, \ldots, X_n$ m.a. con distribución $\mathcal{N}(\mu, \sigma^2)$ donde el parámetro $\underline{\theta} = (\mu, \sigma^2)$ es desconocido. Encontrar el estimador de máxima verosimilitud para $\underline{\theta}$.

Resolución: Como $X_i \sim \mathcal{N}(\mu, \sigma^2)$ $\forall i = 1, \ldots, n$ $\Rightarrow$ $f_{X_i}(x_i; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $\forall x \in \mathbb{R}$ $\forall i = 1, \ldots, n$

- Sea $\underline{x} \in \mathbb{R}^n$. Calculemos la función de densidad conjunta de $\underline{X} = (X_1, \ldots, X_n)$:

$$f_{\underline{X}}(\underline{x}; \mu, \sigma^2) = \prod_{i=1}^{n} f_{X_i}(x_i; \mu, \sigma^2) = \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \right) = \frac{1}{(\sqrt{2\pi\sigma^2})^n} e^{-\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2\sigma^2}} = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2\sigma^2}}$$

X_i's independ.

- Para encontrar los estimadores M.V. de (μ, σ^2) debo maximizar $f_{\underline{X}}(\underline{x}; \mu, \sigma^2)$.

Como la función ln es continua y creciente, con el fin de simplificar las cuentas, vamos a maximizar $\ln(f_{\underline{X}}(\underline{x}; \mu, \sigma^2))$ en lugar de $f_{\underline{X}}(\underline{x}; \mu, \sigma^2)$.

$$\ln(f_{\underline{X}}(\underline{x}; \mu, \sigma^2)) = \ln\left(\frac{e^{-\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2\sigma^2}}}{(2\pi\sigma^2)^{n/2}} \right) = -\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2\sigma^2} - \frac{n}{2}\ln(2\pi\sigma^2)$$

$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

y $\ln(a^n) = n\ln(a)$

- Para maximizar $\ln(f_{\underline{X}}(\underline{x}; \mu, \sigma^2))$ vamos a calcular las derivadas de $\ln(f_{\underline{X}}(\underline{x}; \mu, \sigma^2))$ respecto de μ y σ^2, igualaremos a cero para encontrar los puntos críticos (posibles máximos, mínimos o puntos de inflexión) y finalmente, a través de la derivada segunda encontraremos μ y σ^2 que hacen máxima la función $\ln(f_{\underline{X}}(\underline{x}; \mu, \sigma^2))$.

* $$\frac{\partial}{\partial\mu}\left[\ln(f_{\underline{X}}(\underline{x}; \mu, \sigma^2))\right] = -\sum_{i=1}^{n} \frac{\cancel{2}(x_i-\mu)\cdot(-1)}{\cancel{2}\sigma^2} = \sum_{i=1}^{n}(x_i-\mu) = \sum_{i=1}^{n}(x_i) - n\mu = 0$$

$$\Leftrightarrow \sum_{i=1}^{n} x_i = n\mu$$

$$\Leftrightarrow \mu = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$$

$\Rightarrow$ $\hat{\mu} = \bar{x}$ es el candidato a ser el E.M.V. para μ.

* $\frac{\partial}{\partial\sigma^2}\left[\ln\left(f_{\underline{X}}(\underline{x};\mu,\sigma^2)\right)\right] = -\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2}\cdot\left(-\frac{1}{(\sigma^2)^2}\right) - \frac{n}{2}\,\frac{1}{(\cancel{2\pi}\sigma^2)}\cdot\cancel{2\pi} = \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{n}{2\sigma^2} =$

$$= \frac{1}{2\sigma^2}\left(\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 - n\right) = 0 \iff \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 - n = 0$$

$$\iff \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$$

Luego, $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\mu})^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 = S_n^2$

$\hat{\mu} = \bar{x}$

$\therefore\ \hat{\sigma}^2 = S_n^2$ es candidato a maximizar $\ln\left(f_{\underline{X}}(\underline{x};\mu,\sigma^2)\right)$.

Veamos que $(\hat{\mu},\hat{\sigma}^2) = (\bar{X}, S_n^2)$ es un punto de máximo. Para ello estudiaremos la matriz Hessiana y verificaremos que:

1) $\left.\frac{\partial^2}{(\partial\mu)^2}\ln\left(f_{\underline{X}}(\underline{x};\mu,\sigma^2)\right)\right|_{(\mu,\sigma^2)=(\bar{x},S_n^2)} < 0$ ó $\left.\frac{\partial^2}{(\partial\sigma^2)^2}\ln\left(f_{\underline{X}}(\underline{x};\mu,\sigma^2)\right)\right|_{(\mu,\sigma^2)=(\bar{x};S_n^2)} < 0$

2) El determinante de la matriz Hessiana en $(\mu,\sigma^2)=(\bar{x},S_n^2)$ tiene que ser positivo.

Verifiquemos:

1) $\frac{\partial^2}{(\partial\mu)^2}\ln\left(f_{\underline{X}}(\underline{x};\mu,\sigma^2)\right) = \frac{\partial}{\partial\mu}\left(\frac{1}{\sigma^2}\left(\sum_{i=1}^{n}x_i - n\mu\right)\right) = -\frac{n}{\sigma^2}$

$\Rightarrow \left.\frac{\partial^2}{(\partial\mu)^2}\ln\left(f_{\underline{X}}(\underline{x};\mu,\sigma^2)\right)\right|_{(\mu,\sigma^2)=(\bar{x},S_n^2)} = -\frac{n}{S_n^2} < 0$ ✓

2) $$\text{Hessiana} = \begin{bmatrix} \left.\frac{\partial^2}{(\partial\mu)^2}\ln\left(f_{\underline{X}}(\underline{x};\mu,\sigma^2)\right)\right|_{(\mu,\sigma^2)=(\bar{x},S_n^2)} & \left.\frac{\partial^2}{\partial\mu\,\partial\sigma^2}\ln\left(f_{\underline{X}}(\underline{x};\mu,\sigma^2)\right)\right|_{(\mu,\sigma^2)=(\bar{x},S_n^2)} \\ \left.\frac{\partial^2}{\partial\sigma^2\,\partial\mu}\ln\left(f_{\underline{X}}(\underline{x};\mu,\sigma^2)\right)\right|_{(\mu,\sigma^2)=(\bar{x},S_n^2)} & \left.\frac{\partial^2}{(\partial\sigma^2)^2}\ln\left(f_{\underline{X}}(\underline{x};\mu,\sigma^2)\right)\right|_{(\mu,\sigma^2)=(\bar{x},S_n^2)} \end{bmatrix}$$

$$\rightarrow \frac{\partial^2}{(\partial\sigma^2)^2} \ln\left(f_{\underline{X}}(\underline{x},\mu,\sigma^2)\right)\Big|_{(\mu,\sigma^2)=(\bar{x},S_n^2)} = \frac{\partial}{\partial\sigma^2}\left(\frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{n}{2\sigma^2}\right)\Big|_{(\mu,\sigma^2)=(\bar{x},S_n^2)} =$$

$$= \frac{1}{\cancel{2}}\sum_{i=1}^{n}(x_i-\mu)^2\cdot\left(-\frac{\cancel{2}}{(\sigma^2)^3}\right) - \frac{n}{2}\left(-\frac{1}{(\sigma^2)^2}\right)\Big|_{(\mu,\sigma^2)=(\bar{x},S_n^2)} =$$

$$= -\frac{1}{(\sigma^2)^3}\sum_{i=1}^{n}(x_i-\mu)^2 + \frac{n}{2(\sigma^2)^2}\Big|_{(\mu,\sigma^2)=(\bar{x},S_n^2)} = -\frac{1}{(S_n^2)^3}\sum_{i=1}^{n}(x_i-\bar{x})^2 + \frac{n}{2(S_n^2)^2} =$$

$$= -\frac{1}{(S_n^2)^3}\, n\cdot\underbrace{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}_{S_n^2} + \frac{n}{2(S_n^2)^2} = -\frac{n\,\cancel{S_n^2}}{(S_n^2)^{\cancel{3}}} + \frac{n}{2(S_n^2)^2} = -\frac{n}{(S_n^2)^2} + \frac{n}{2(S_n^2)^2} = -\frac{n}{2(S_n^2)^2}$$

$$\rightarrow \frac{\partial^2}{\partial\sigma^2\partial\mu} \ln\left(f_{\underline{X}}(\underline{x},\mu,\sigma^2)\right)\Big|_{(\mu,\sigma^2)=(\bar{x},S_n^2)} = \frac{\partial}{\partial\sigma^2}\left(\frac{1}{\sigma^2}\left(\sum_{i=1}^{n}x_i - n\mu\right)\right)\Big|_{(\mu,\sigma^2)=(\bar{x},S_n^2)} =$$

$$= \left(\sum_{i=1}^{n}x_i - n\mu\right)\left(-\frac{1}{(\sigma^2)^2}\right)\Big|_{(\mu,\sigma^2)=(\bar{x},S_n^2)} = -\frac{1}{(S_n^2)^2}\underbrace{\left(\sum_{i=1}^{n}x_i - n\bar{x}\right)}_{=0} = 0$$

$$= \frac{\partial^2}{\partial\mu\partial\sigma^2} \ln\left(f_{\underline{X}}(\underline{x},\mu,\sigma^2)\right)\Big|_{(\mu,\sigma^2)=(\bar{x},S_n^2)}$$

$$\therefore \det(\text{Hessiana}) = \det\begin{pmatrix} -\frac{n}{S_n^2} & 0 \\ 0 & -\frac{n}{2(S_n^2)^2}\end{pmatrix} = \left(-\frac{n}{S_n^2}\right)\cdot\left(-\frac{n}{2(S_n^2)^2}\right) = \frac{n^2}{2(S_n^2)^3} > 0 \quad \checkmark$$

$\therefore$ $\hat{\underline{\theta}} = (\hat{\mu},\hat{\sigma}^2) = (\bar{X}, S_n^2)$ es el estimador de máxima verosimilitud para $\theta = (\mu,\sigma^2)$.