Probabilidad y Estadística

Problema 2

a) $X_1, \dots, X_n$ m.a , $X_i \sim Be(p) = B(1,p)$ $\forall i=1,\dots,n$

$$\Rightarrow \sum_{i=1}^{n} X_i \sim B(n,p)$$

Sol.

$$X_i = \begin{cases} 1 & \text{con prob. } p \\ 0 & \text{con prob } 1-p \end{cases} \quad , \; Im(X_i) = \{0,1\} \quad \forall i=1,\dots,n$$

$$P_{X_i}(x) = \begin{cases} p^x (1-p)^{1-x} & \text{si } x \in \{0,1\} \\ 0 & \text{c.c.} \end{cases} \qquad \forall i=1,\dots,n$$

$$W = \sum_{i=1}^{n} X_i = h(X_1,\dots,X_n) \sim \text{¿?} \qquad W: \Omega \to \mathbb{R} \qquad P_W(x) = P(W=x)$$

Calculemos la f.p.m.c de $(X_1,\dots,X_n)$, $P_{(X_1,\dots,X_n)}: \mathbb{R}^n \to [0,1]$

Sea $\underline{x} = (x_1,\dots,x_n) \in \mathbb{R}^n$,

- Si $\exists\, i=1,\dots,n$ tal que $x_i \notin \{0,1\} = Im(X_i) \Rightarrow P_{\underline{X}}(\underline{x}) = P_{X_1,\dots,X_n}(x_1,\dots,x_n) =$

$$= P\big((X_1=x_1)\cap(X_2=x_2)\cap\dots\cap(X_n=x_n)\big) = 0$$

def. de f.p.m.c.

- si $x_i \in \{0,1\}$ $\forall i=1,\dots,n$

$$P_{\underline{X}}(\underline{x}) = \prod_{i=1}^{n} P_{X_i}(x_i) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i}$$

X_i's son indep.

$$\Rightarrow \therefore \quad p_{\underline{X}}(\underline{x}) = \begin{cases} p^{\sum_{i=1}^{n} x_i} (1-p)^{n-\sum x_i} & \text{si } x_i \in \{0,1\} \ \forall i=1,\dots,n \\ 0 & \text{c.c} \end{cases}$$

$W \sim$ ¿? con $W = \sum_{i=1}^{n} X_i$

$I_m(W) = \{0, 1, \dots, n\}$

- si $k \notin I_m(W) \Rightarrow p_W(k) = P(W=k) = 0$
- si $k \in I_m(W)$
  - si $k=0$, $p_W(0) = P(W=0) = P\left(\sum_{i=1}^{n} \underline{X_i} = 0\right)$

$$= P(X_1=0, X_2=0, \dots, X_n=0)$$

$$= p_{\underline{X}}(0,\dots,0) = p^0 (1-p)^{n-0} = (1-p)^n.$$

  - si $k=1$, $p_W(1) = P(W=1) = P\left(\sum_{i=1}^{n} X_i = 1\right) =$

$$= \sum_{\{\underline{x} : \sum_{i=1}^{n} x_i = 1\}} \overline{p_{\underline{X}}(\underline{x})} = \sum_{\{\underline{x} : \sum x_i = 1\}} p(1-p)^{n-1} = \boxed{n} p(1-p)^{n-1}$$

$$= \underbrace{\{\overline{(1,0,\dots,0)}, \overline{(0,1,0,\dots,0)}, \dots, \overline{(0,\dots,0,1)}\}}_{n}$$

$$p_{\underline{X}}(1,0,\dots,0) = p.\underbrace{(1-p)\dots(1-p)}_{n-1} = p(1-p)^{n-1}$$

$\boxed{n}$: # de n-uplas que se pueden formar con una coordenada $=1$ y las $(n-1)$ restantes $=0$ es $\binom{n}{1} = n$

sea $k \in \mathrm{Im}(W)$.

$$P_W(k) = \sum_{\{\underline{x} : \sum x_i = k\}} p(\underline{x}) = \sum_{\{\underline{x} : \sum_{i=1}^n x_i = k\}} \overbrace{p^k (1-p)^{n-k}}^{\text{constante}} = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\{\underline{x} : \sum x_i = k\} \to \{(\underbrace{1,\dots,1}_{k}, \underbrace{0,\dots,0}_{n-k}), \dots\}$$

$$p_{\underline{X}}(\underbrace{1,\dots,1}_{k}, \underbrace{0,\dots,0}_{n-k}) = p^k (1-p)^{n-k}$$

n° de n-uplas que se pueden formar con k coordenadas iguales a 1 y $(n-k)$ coordenadas $= 0$ es $\binom{n}{k}$.

En resumen:

$$p_W(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{si } k \in \{0, 1, \dots, n\} \\ 0 & \text{c.c.} \end{cases}$$

$\therefore\ W \sim B(n, p)$ $\square$

b) X e Y v.a. indep. con $X \sim B(n_1, p)$ e $Y \sim B(n_2, p)$

¿ $\Rightarrow X + Y \sim B(n_1 + n_2, p)$?

Idea: la f.p.m.c de (X,Y) es

$$p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y) = \begin{cases} \overbrace{\binom{n_1}{x} p^x (1-p)^{n_1 - x}}^{p_X(x)} \overbrace{\binom{n_2}{y} p^y (1-p)^{n_2 - y}}^{p_Y(y)} & \text{si } x \in \{0, \dots, n_1\} \text{ e } y \in \{0, \dots, n_2\} \\ 0 & \text{c.c} \end{cases}$$

$$\Rightarrow p(x,y) = \begin{cases} \binom{n_1}{x}\binom{n_2}{y} p^{x+y}(1-p)^{n_1+n_2-(x+y)} & \text{si } \begin{array}{l} x\in\{0,\dots,n_1\} \\ y\in\{0,\dots,n_2\}\end{array} \\ 0 & \text{c.c.} \end{cases}$$

Sea $Z = X+Y$. $\Rightarrow \text{Im}(Z) = \{0,1,\dots,n_1+n_2\}$.

- si $z \notin \text{Im}(Z) \Rightarrow p_Z(z) = P(\underbrace{Z=z}_{=\phi}) = 0$
- si $z \in \text{Im}(Z) \Rightarrow$ ¿?

  - si $z=0 \Rightarrow p_Z(0) = P(X+Y=0) = P(X=0, Y=0)$
  $= p_{X,Y}(0,0) = (1-p)^{n_1+n_2}$

  - $z=1 \Rightarrow p_Z(1) = p_{X,Y}(0,1) + p_{X,Y}(1,0) = \dots \overset{\text{completar}}{=} \left[\binom{n_1}{0}\binom{n_2}{1} + \binom{n_1}{1}\binom{n_2}{0}\right] \cdot p(1-p)^{n_1+n_2-1}$

  $\vdots$

  ¿$= \binom{n_1+n_2}{k}$?

  - $z=k \Rightarrow p_Z(k) \underset{\text{terminar.}}{=} \left[\sum_{i=0}^{k}\binom{n_1}{i}\binom{n_2}{k-i}\right] p^k (1-p)^{n_1+n_2-k}$

$\underline{\mathcal{B}(n_1+n_2, p)}$

Obs: $\displaystyle \sum_{i=0}^{k}\binom{n_1}{i}\binom{n_2}{k-i} = \binom{n_1+n_2}{k} \overbrace{\sum_{i=0}^{k} \frac{\binom{n_1}{i}\binom{n_2}{k-i}}{\binom{n_1+n_2}{k}}}^{=1}$ $\rightarrow$ es la f.p.m de una $\mathcal{H}(k, n_1, n_1+n_2)$

Problema 3: a) X_1, X_2 v.a. indep. con $X_1 \sim P(\lambda_1)$ e $X_2 \sim P(\lambda_2)$

¿ $\Rightarrow X_1 + X_2 \sim P(\lambda_1+\lambda_2)$?

Dem: $X_1 \sim P(\lambda_1) \Rightarrow P_{X_1}(x) = \begin{cases} \dfrac{e^{-\lambda_1}\lambda_1^x}{x!}, & x \in \mathbb{N}\cup\{0\} \\ 0 & c.c. \end{cases}$

$X_2 \sim P(\lambda_2) \Rightarrow P_{X_2}(x) = \begin{cases} \dfrac{e^{-\lambda_2}\lambda_2^x}{x!}, & x \in \mathbb{N}\cup\{0\} \\ 0, & c.c. \end{cases}$

Calculemos $P_{X_1,X_2}(x_1,x_2) = P(X_1 = x_1, X_2 = x_2)$

- si $x_1 \notin \mathbb{N}\cup\{0\}$ o $x_2 \notin \mathbb{N}\cup\{0\} \Rightarrow P_{X_1,X_2}(x_1,x_2) = 0$
- si $x_1, x_2 \in \mathbb{N}\cup\{0\} \Rightarrow \underline{P_{X_1,X_2}(x_1,x_2)} \overset{X_1 \text{ y } X_2 \text{ indep.}}{=} \underline{P_{X_1}}(x_1) \cdot \underline{P_{X_2}}(x_2)$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{\lambda_1^{x_1} \cdot \lambda_2^{x_2}}{x_1!\, x_2!}$$

$W = X + Y \sim$ ¿?, $Im(W) = \mathbb{N}\cup\{0\}$

- si $k \notin Im(W) \Rightarrow P_W(k) = 0$
- si $k \in Im(W) \Rightarrow P_W(k) =$ ¿?

$- \text{ si } k=0, \quad P_W(0) = P(X_1+X_2=0) = P_{X_1,X_2}(0,0)$

$$= e^{-(\lambda_1+\lambda_2)}.$$

$- \text{ si } k=1, \quad P_W(1) = \sum_{\{(x_1,x_2)\,:\,x_1+x_2=1\}} P_{X_1,X_2}(x_1,x_2) = P_{X_1,X_2}(0,1) + P_{X_1,X_2}(1,0)$

$$= e^{-(\lambda_1+\lambda_2)} \left[\frac{\lambda_1^1 \cdot \lambda_2^0}{1!\,0!} + \frac{\lambda_1^0\,\lambda_2^1}{0!\,1!} \right]$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot (\lambda_1+\lambda_2)$$

$\vdots$

$- \text{ si } k \in \mathbb{N} \cup \{0\},$

$$P_W(k) = \sum_{\{(x_1,x_2)\,:\,x_1+x_2=k\}} P_{X_1,X_2}(x_1,x_2) = \sum_{\{(x_1,x_2)\,:\,x_1+x_2=k\}} e^{-(\lambda_1+\lambda_2)} \cdot \frac{\lambda_1^{x_1}\,\lambda_2^{x_2}}{x_1!\,x_2!} =$$

$$= e^{-(\lambda_1+\lambda_2)} \sum_{\{(x_1,x_2)\,:\,x_2=k-x_1\}} \frac{\lambda_1^{x_1} \cdot \lambda_2^{x_2}}{x_1!\,x_2!} = e^{-(\lambda_1+\lambda_2)} \sum_{x_1=0}^{k} \frac{\lambda_1^{x_1} \cdot \lambda_2^{\overset{\geq 0}{k-x_1}}}{x_1!\,\underset{\geq 0}{(k-x_1)}!}$$

si $k-x_1 \geq 0$

$k \geq x_1$

multip. y divido por $k!$

$$\overset{!}{=} \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \sum_{x_1=0}^{k} \overbrace{\frac{k!}{x_1!\,(k-x_1)!}}^{=\binom{k}{x_1}} \cdot \lambda_1^{x_1} \cdot \lambda_2^{k-x_1}$$

Binomio de Newton

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \overbrace{\sum_{x_1=0}^{k} \binom{k}{x_1} \lambda_1^{x_1}\,\lambda_2^{k-x_1}} \overset{!}{=} \frac{e^{-(\lambda_1+\lambda_2)}}{k!} (\lambda_1+\lambda_2)^k \quad \forall\, k \in \mathbb{N}\cup\{0\}$$

$\therefore \; X_1 + X_2 \sim P(\lambda_1 + \lambda_2)$.

Binomio de Newton:

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x}$$

Teorema Central del Límite

Se generaron 1000 muestras de tamaño n

$n=2 \to \left\{ \begin{array}{l} (x_1^1, x_2^1) \\ (x_1^2, x_2^2) \\ \vdots \\ (x_1^{1000}, x_2^{1000}) \end{array} \right\}$ 1000

$n=5 \to \left\{ \begin{array}{l} (x_1^1, \dots, x_5^1) \\ (x_1^2, \dots, x_5^2) \\ \vdots \\ (x_1^{1000}, \dots, x_5^{1000}) \end{array} \right\}$ 1000

$n=30$

$n=200$

P/el primer gráfico:

Si $n=2$, genero:

(x_1^1, x_2^1) con $X_1^1, X_2^1 \sim Be(p=0{,}2) \to \dfrac{x_1^1 + x_2^1}{2}$

(x_1^2, x_2^2) con $x_1^2, x_2^2 \sim Be(p=0{,}2) \to \dfrac{x_1^2 + x_2^2}{2}$

$\vdots$

(x_1^{1000}, x_2^{1000}) con $X_1^{1000}, X_2^{1000} \sim Be(p=0{,}2) \to \dfrac{x_1^{1000} + x_2^{1000}}{2}$

Al final tengo un vector $\begin{pmatrix} \frac{x_1^1 + x_2^1}{2} \\ \vdots \\ \frac{x_1^{1000} + x_2^{1000}}{2} \end{pmatrix}$

Teorema Central del Límite

$X_1, \dots, X_n$ m.a con $E(X_i)=\mu$ y $V(X_i)=\sigma^2$ $\forall i=1 \to n$

entonces para un "n suficientemente grande", $\overline{X}_n$ tiene distribución normal con media $E(\overline{X}_n)=\mu$ y varianza $V(\overline{X})=\dfrac{\sigma^2}{n}$, i.e.:

$$\overline{X} \overset{\cdot}{\approx} N\left(\mu, \frac{\sigma^2}{n}\right) \quad \text{o equivalentemente}$$

$$\to \frac{(\overline{X}-\mu)}{\sigma/\sqrt{n}} \overset{\cdot}{\approx} N(0,1) \quad \text{o} \quad ''$$

$$\to \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\,\sigma} \overset{\cdot}{\approx} N(0,1) \iff \boxed{\sum_{i=1}^{n} X_i \overset{\cdot}{\approx} N(n\mu ; n\sigma^2)}$$

$\square$

Aplicaciones:

$$P(\overline{X} \le x) = P\Big(\underbrace{\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}}_{Z \overset{\cdot}{\approx} N(0,1)} \le \frac{x-\mu}{\sigma/\sqrt{n}}\Big) = \approx \Phi\left(\frac{x-\mu}{\sigma/\sqrt{n}}\right) \quad \forall x \in \mathbb{R}$$

y también

$$P\left(\sum_{i=1}^{n} X_i \le x\right) \approx \Phi\left(\frac{x-n\mu}{\sqrt{n\sigma^2}}\right) \quad \forall x \in \mathbb{R}$$

Regla práctica: "n suficient. grande" si $\underline{\underline{n \geq 30}}$

Problema 4:

X: "tiempo de espera por la mañana" $\sim U[0,4]$

Y: " " " " " la tarde " $\sim U[0,8]$

W: " " " " por la mañana y tarde"

$$W = X+Y$$

$$E(W) = E(X+Y) = E(X)+E(Y) = \frac{4}{2} + \frac{8}{2} = \boxed{6}$$

$X,Y \sim U[a,b]$
$E(X) = \frac{a+b}{2}$

$$V(W) = V(X+Y) \overset{X \text{ e } Y \text{ indep.}}{=} V(X)+V(Y) = \frac{4^2}{12} + \frac{8^2}{12} = \dots = \boxed{\frac{20}{3}} = \sigma^2$$

$X,Y \sim U[a,b]$
$V(X) = \frac{(b-a)^2}{12}$

Sea W_i: "tiempo de espera total en día i" $\forall i=1,\dots,40$

$\Rightarrow W_1, W_2, \dots, W_{40}$ m.a. con $\underset{\mu}{E(W_i)} = 6$ y $V(W_i) = \frac{20}{3} = \sigma^2$

$\sum_{i=1}^{40} W_i$: "tiempo total de espera en 40 días"

3 h. y media $= 3 \times 60 + 30 = 210$ minutos

El enunciado pide:

$$P\left(\sum_{i=1}^{40} W_i \geq 210\right) = ¿? \quad (*)$$

Como $n=40>30$, entonces por TCL: $\forall x \in \mathbb{R}$

$$P\left(\sum_{i=1}^{n} W_i \leq x\right) \approx \Phi\left(\frac{x-n\mu}{\sqrt{n\sigma^2}}\right) = \Phi\left(\frac{x-40\cdot 6}{\sqrt{40\cdot\frac{20}{3}}}\right) =$$

$$= \Phi\left(\frac{x-240}{\sqrt{\frac{800}{3}}}\right)$$

$$\Rightarrow (*) \underset{\sum W_i \text{ es continua}}{=} 1 - P\left(\sum_{i=1}^{40} W_i \leq 210\right) = 1 - \Phi\left(\frac{210-240}{\sqrt{\frac{800}{3}}}\right) = 1-\Phi(-1{,}84)$$

$$= \Phi(1{,}84) \underset{\text{tabla}}{=} 0{,}9671$$

$$\therefore P\left(\sum_{i=1}^{40} W_i \geq 210\right) \approx 0{,}9671$$