Computability and Complexity
COSC 4200

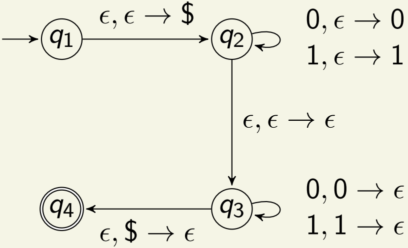
Context-Free Grammars

Context-Free Languages

Context-Free Grammars (CFG)

$$\begin{array}{lcl} S & \rightarrow & LC \mid AR \\ L & \rightarrow & aLb \mid \epsilon \\ R & \rightarrow & bRc \mid \epsilon \\ A & \rightarrow & Aa \mid \epsilon \\ C & \rightarrow & Cc \mid \epsilon \end{array}$$

Pushdown Automata (PDA)



Example of a CFG G:

$$\begin{array}{rcl} A & \to & 0A1 \\ A & \to & B \\ B & \to & \# \end{array}$$

A, B are *variables* (or *nonterminals*)
A is the *start variable*
0, 1, $\#$ are *terminals*
$X \to Y$ is a *substitution rule* (or *production*)

We begin with the start variable and apply substitutions to *derive* strings.

$$A \Rightarrow 0A1$$

We begin with the start variable and apply substitutions to *derive* strings.

$$\begin{aligned} A &\Rightarrow 0A1 \\ &\Rightarrow 00A11 \end{aligned}$$

We begin with the start variable and apply substitutions to *derive* strings.

$$\begin{aligned} A &\Rightarrow 0A1 \\ &\Rightarrow 00A11 \\ &\Rightarrow 000A111 \end{aligned}$$

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$$\begin{aligned} A &\Rightarrow 0A1 \\ &\Rightarrow 00A11 \\ &\Rightarrow 000A111 \\ &\Rightarrow 000B111 \end{aligned}$$

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$$\begin{aligned} A &\Rightarrow 0A1 \\ &\Rightarrow 00A11 \\ &\Rightarrow 000A111 \\ &\Rightarrow 000B111 \\ &\Rightarrow 000\#111 \end{aligned}$$

We begin with the start variable and apply substitutions to *derive* strings.

$$\begin{aligned} A &\Rightarrow 0A1 \\ &\Rightarrow 00A11 \\ &\Rightarrow 000A111 \\ &\Rightarrow 000B111 \\ &\Rightarrow 000\#111 \end{aligned}$$

The *language* $L(G)$ of G is the set of all strings that can be derived from it. In this case,

$$L(G) = \{0^n\#1^n \mid n \geq 0\}.$$

Definition

A *context-free grammar (CFG)* is a 4-tuple $G = (V, \Sigma, R, S)$ where

- V is a finite set called the *variables* (usually V consists of capital letters)
- Σ is a finite set, disjoint from V, called the *terminals*
- R is a finite set of *rules*, with each rule begin of the form

  $$R \to \omega$$

  where $R \in V$ and $\omega \in (V \cup \Sigma)^*$.
  (I.e., R is a variable and ω is a string of variables and terminals.)
- S is the *start variable*.

G:

$$\begin{array}{lcl} A & \rightarrow & 0A1 \\ A & \rightarrow & B \\ B & \rightarrow & \# \end{array}$$

In this example, $G = (V, \Sigma, R, A)$ where

- $V = \{A, B\}$
- $\Sigma = \{0, 1, \#\}$
- $R = \{A \rightarrow 0A1, A \rightarrow B, B \rightarrow \#\}$
- A is the start variable

Let $G = (V, \Sigma, R, S)$ be a CFG. If $u, v, w \in (V \cup \Sigma)^*$ are strings of variables and terminals and $A \to w$ is a rule in the grammar, then we say uAv *yields* uwv, written

$$uAv \Rightarrow uwv.$$

Let $G = (V, \Sigma, R, S)$ be a CFG. If $u, v, w \in (V \cup \Sigma)^*$ are strings of variables and terminals and $A \to w$ is a rule in the grammar, then we say uAv *yields* uwv, written

$$uAv \Rightarrow uwv.$$

We write $u \stackrel{*}{\Rightarrow} v$ if

1. $u = v$ or
2. if a sequence $u_1, \ldots, u_k$ exists for $k \geq 0$ such that

$$u \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \cdots \Rightarrow u_k \Rightarrow v.$$

The *language* of G is

$$L(G) = \{w \in \Sigma^* \mid S \stackrel{*}{\Rightarrow} w\}.$$

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$$L(G) = \{w \in \Sigma^* \mid S \overset{*}{\Rightarrow} w\}.$$

Definition

If $A = L(G)$ for some context-free grammar, then A is a *context-free language*. We let CFL be the class of all context-free languages.

Example. Often we use a compact form for specifying a grammar:

$$S \to 0S1 \mid \epsilon$$

Example. Often we use a compact form for specifying a grammar:

$$S \to 0S1 \mid \epsilon$$

Here the explicit meaning is

- $V = \{S\}$
- $\Sigma = \{0, 1\}$
- $R = \{S \to 0S1, S \to \epsilon\}$
- S is the start variable

The language of this grammar is

$$\{0^n 1^n \mid n \geq 0\}.$$

Example.

$$S \to (S) \mid SS \mid \epsilon$$

Example.

$$S \rightarrow (S) \mid SS \mid \epsilon$$

This grammar generates all strings of properly nested parentheses.

- $S \Rightarrow (S) \Rightarrow ()$
- $S \Rightarrow (S) \Rightarrow (SS) \Rightarrow (()S) \Rightarrow (()())$
- $S \stackrel{*}{\Rightarrow} (()(()()))()$

Example. $S \to (S) \mid SS \mid \epsilon$

$S \stackrel{*}{\Rightarrow} (()(()()))()$ because

$$S$$

Example. $S \to (S) \mid SS \mid \epsilon$

$S \stackrel{*}{\Rightarrow} (()(()()))()$ because

$$S \quad \Rightarrow \quad SS$$

Example. $S \rightarrow (S) \mid SS \mid \epsilon$

$S \overset{*}{\Rightarrow} (()(()()))()$ because

$$\begin{array}{rcl} S & \Rightarrow & SS \\ & \Rightarrow & (S)S \end{array}$$

Example. $S \to (S) \mid SS \mid \epsilon$

$S \overset{*}{\Rightarrow} (()(()()))()$ because

$$\begin{aligned} S &\Rightarrow SS \\ &\Rightarrow (S)S \\ &\Rightarrow (SS)S \end{aligned}$$

Example. $S \to (S) \mid SS \mid \epsilon$

$S \stackrel{*}{\Rightarrow} (()(()()))()$ because

$$\begin{aligned} S &\Rightarrow SS \\ &\Rightarrow (S)S \\ &\Rightarrow (SS)S \\ &\Rightarrow ((S)S)S \end{aligned}$$

Example. $S \to (S) \mid SS \mid \epsilon$

$S \stackrel{*}{\Rightarrow} (()(()()))()$ because

$$\begin{aligned} S &\Rightarrow SS \\ &\Rightarrow (S)S \\ &\Rightarrow (SS)S \\ &\Rightarrow ((S)S)S \\ &\Rightarrow (()S)S \end{aligned}$$

Example. $S \to (S) \mid SS \mid \epsilon$

$S \stackrel{*}{\Rightarrow} (()(()()))()$ because

$$
\begin{aligned}
S &\Rightarrow SS \\
&\Rightarrow (S)S \\
&\Rightarrow (SS)S \\
&\Rightarrow ((S)S)S \\
&\Rightarrow (()S)S \\
&\Rightarrow (()(S))S
\end{aligned}
$$

Example. $S \to (S) \mid SS \mid \epsilon$

$S \stackrel{*}{\Rightarrow} (()(()()))()$ because

$$
\begin{aligned}
S &\Rightarrow SS \\
&\Rightarrow (S)S \\
&\Rightarrow (SS)S \\
&\Rightarrow ((S)S)S \\
&\Rightarrow (()S)S \\
&\Rightarrow (()(S))S \\
&\Rightarrow (()(SS))S
\end{aligned}
$$

Example. $S \to (S) \mid SS \mid \epsilon$

$S \stackrel{*}{\Rightarrow} (()((())))()$ because

$$
\begin{aligned}
S &\Rightarrow SS \\
&\Rightarrow (S)S \\
&\Rightarrow (SS)S \\
&\Rightarrow ((S)S)S \\
&\Rightarrow (()S)S \\
&\Rightarrow (()(S))S \\
&\Rightarrow (()(SS))S \\
&\Rightarrow (()((S)S))S
\end{aligned}
$$

Example. $S \to (S) \mid SS \mid \epsilon$

$S \overset{*}{\Rightarrow} (()(()()))()$ because

$$
\begin{aligned}
S &\Rightarrow SS \\
&\Rightarrow (S)S \\
&\Rightarrow (SS)S \\
&\Rightarrow ((S)S)S \\
&\Rightarrow (()S)S \\
&\Rightarrow (()(S))S \\
&\Rightarrow (()(SS))S \\
&\Rightarrow (()((S)S))S \\
&\Rightarrow (()(()S))S
\end{aligned}
$$

Example. $S \to (S) \mid SS \mid \epsilon$

$S \stackrel{*}{\Rightarrow} (()(()()))()$ because

$$\begin{aligned}
S &\Rightarrow SS \\
&\Rightarrow (S)S \\
&\Rightarrow (SS)S \\
&\Rightarrow ((S)S)S \\
&\Rightarrow (()S)S \\
&\Rightarrow (()(S))S \\
&\Rightarrow (()(SS))S \\
&\Rightarrow (()((S)S))S \\
&\Rightarrow (()(()S))S \\
&\Rightarrow (()(()(S)))S
\end{aligned}$$

Example. $S \to (S) \mid SS \mid \epsilon$

$S \stackrel{*}{\Rightarrow} (()(()()))()$ because

$$
\begin{array}{rcl}
S & \Rightarrow & SS \\
 & \Rightarrow & (S)S \\
 & \Rightarrow & (SS)S \\
 & \Rightarrow & ((S)S)S \\
 & \Rightarrow & (()S)S \\
 & \Rightarrow & (()(S))S \\
 & \Rightarrow & (()(SS))S \\
 & \Rightarrow & (()((S)S))S \\
 & \Rightarrow & (()(()S))S \\
 & \Rightarrow & (()(()(S)))S \\
 & \Rightarrow & (()(()()))S
\end{array}
$$

Example. $S \to (S) \mid SS \mid \epsilon$

$S \overset{*}{\Rightarrow} (()(()()))()$ because

$$
\begin{aligned}
S &\Rightarrow SS \\
&\Rightarrow (S)S \\
&\Rightarrow (SS)S \\
&\Rightarrow ((S)S)S \\
&\Rightarrow (()S)S \\
&\Rightarrow (()(S))S \\
&\Rightarrow (()(SS))S \\
&\Rightarrow (()((S)S))S \\
&\Rightarrow (()(()S))S \\
&\Rightarrow (()(()(S)))S \\
&\Rightarrow (()(()()))S \\
&\Rightarrow (()(()()))(S)
\end{aligned}
$$

Example. $S \to (S) \mid SS \mid \epsilon$

$S \overset{*}{\Rightarrow} (())((())())()$ because

$$
\begin{aligned}
S &\Rightarrow SS \\
&\Rightarrow (S)S \\
&\Rightarrow (SS)S \\
&\Rightarrow ((S)S)S \\
&\Rightarrow (()S)S \\
&\Rightarrow (()(S))S \\
&\Rightarrow (()(SS))S \\
&\Rightarrow (()((S)S))S \\
&\Rightarrow (()(()S))S \\
&\Rightarrow (()(()(S)))S \\
&\Rightarrow (()(()()))S \\
&\Rightarrow (()(()()))(S) \\
&\Rightarrow (()(()()))()
\end{aligned}
$$

Example.

$$\begin{aligned} E &\rightarrow E + T \mid T \\ T &\rightarrow T \times F \mid F \\ F &\rightarrow (E) \mid N \\ N &\rightarrow NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9 \end{aligned}$$

This grammar generates strings like

$$(12 + 8) \times 66$$

$$2020 + (4200 \times 30)$$

$$87 \times ((33 + ((12 + 7) \times 99)) + 47)$$

E

$$
\boxed{
\begin{array}{lcl}
E & \to & E + T \mid T \\
T & \to & T \times F \mid F \\
F & \to & (E) \mid N \\
N & \to & NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{array}
}
$$

$$E \stackrel{*}{\Rightarrow} (12 + 8) \times 66$$

$$E \Rightarrow T$$

$$
\begin{array}{lcl}
E & \rightarrow & E + T \mid T \\
T & \rightarrow & T \times F \mid F \\
F & \rightarrow & (E) \mid N \\
N & \rightarrow & NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{array}
$$

$$E \stackrel{*}{\Rightarrow} (12 + 8) \times 66$$

$$\begin{aligned} E &\Rightarrow T \\ &\Rightarrow T \times F \end{aligned}$$

$$\boxed{\begin{aligned} E &\rightarrow E + T \mid T \\ T &\rightarrow T \times F \mid F \\ F &\rightarrow (E) \mid N \\ N &\rightarrow NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9 \end{aligned}}$$

$$E \stackrel{*}{\Rightarrow} (12 + 8) \times 66$$

$$
\begin{aligned}
E &\Rightarrow T \\
&\Rightarrow T \times F \\
&\Rightarrow F \times F
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
E &\rightarrow E + T \mid T \\
T &\rightarrow T \times F \mid F \\
F &\rightarrow (E) \mid N \\
N &\rightarrow NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{aligned}
}
$$

$$E \stackrel{*}{\Rightarrow} (12 + 8) \times 66$$

$$
\begin{aligned}
E &\Rightarrow T \\
&\Rightarrow T \times F \\
&\Rightarrow F \times F \\
&\Rightarrow (E) \times F
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
E &\rightarrow E + T \mid T \\
T &\rightarrow T \times F \mid F \\
F &\rightarrow (E) \mid N \\
N &\rightarrow NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{aligned}
}
$$

$$E \stackrel{*}{\Rightarrow} (12 + 8) \times 66$$

$$
\begin{aligned}
E &\Rightarrow T \\
&\Rightarrow T \times F \\
&\Rightarrow F \times F \\
&\Rightarrow (E) \times F \\
&\Rightarrow (E + T) \times F
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
E &\rightarrow E + T \mid T \\
T &\rightarrow T \times F \mid F \\
F &\rightarrow (E) \mid N \\
N &\rightarrow NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{aligned}
}
$$

$$E \overset{*}{\Rightarrow} (12 + 8) \times 66$$

$$
\begin{aligned}
E &\rightarrow E + T \mid T \\
T &\rightarrow T \times F \mid F \\
F &\rightarrow (E) \mid N \\
N &\rightarrow NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{aligned}
$$

$$E \stackrel{*}{\Rightarrow} (12 + 8) \times 66$$

$$
\begin{aligned}
E &\Rightarrow T \\
&\Rightarrow T \times F \\
&\Rightarrow F \times F \\
&\Rightarrow (E) \times F \\
&\Rightarrow (E + T) \times F \\
&\Rightarrow (T + T) \times F
\end{aligned}
$$

$$
\begin{array}{rcl}
E & \rightarrow & E + T \mid T \\
T & \rightarrow & T \times F \mid F \\
F & \rightarrow & (E) \mid N \\
N & \rightarrow & NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{array}
$$

$$E \overset{*}{\Rightarrow} (12 + 8) \times 66$$

$$
\begin{array}{rcl}
E & \Rightarrow & T \\
& \Rightarrow & T \times F \\
& \Rightarrow & F \times F \\
& \Rightarrow & (E) \times F \\
& \Rightarrow & (E + T) \times F \\
& \Rightarrow & (T + T) \times F \\
& \Rightarrow & (F + T) \times F
\end{array}
$$

$$
\begin{array}{rcl}
E & \rightarrow & E + T \mid T \\
T & \rightarrow & T \times F \mid F \\
F & \rightarrow & (E) \mid N \\
N & \rightarrow & NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{array}
$$

$$E \stackrel{*}{\Rightarrow} (12 + 8) \times 66$$

$$
\begin{array}{rcl}
E & \Rightarrow & T \\
  & \Rightarrow & T \times F \\
  & \Rightarrow & F \times F \\
  & \Rightarrow & (E) \times F \\
  & \Rightarrow & (E + T) \times F \\
  & \Rightarrow & (T + T) \times F \\
  & \Rightarrow & (F + T) \times F \\
  & \Rightarrow & (F + F) \times F
\end{array}
$$

$$
\begin{array}{rcl}
E & \rightarrow & E + T \mid T \\
T & \rightarrow & T \times F \mid F \\
F & \rightarrow & (E) \mid N \\
N & \rightarrow & NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{array}
$$

$$E \overset{*}{\Rightarrow} (12 + 8) \times 66$$

$$
\begin{array}{rcl}
E & \Rightarrow & T \\
& \Rightarrow & T \times F \\
& \Rightarrow & F \times F \\
& \Rightarrow & (E) \times F \\
& \Rightarrow & (E + T) \times F \\
& \Rightarrow & (T + T) \times F \\
& \Rightarrow & (F + T) \times F \\
& \Rightarrow & (F + F) \times F \\
& \Rightarrow & (N + F) \times F
\end{array}
$$

$$
\begin{array}{lcl}
E & \rightarrow & E + T \mid T \\
T & \rightarrow & T \times F \mid F \\
F & \rightarrow & (E) \mid N \\
N & \rightarrow & NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{array}
$$

$$
E \stackrel{*}{\Rightarrow} (12 + 8) \times 66
$$

$$
\begin{array}{lcl}
E & \Rightarrow & T \\
 & \Rightarrow & T \times F \\
 & \Rightarrow & F \times F \\
 & \Rightarrow & (E) \times F \\
 & \Rightarrow & (E + T) \times F \\
 & \Rightarrow & (T + T) \times F \\
 & \Rightarrow & (F + T) \times F \\
 & \Rightarrow & (F + F) \times F \\
 & \Rightarrow & (N + F) \times F \\
 & \Rightarrow & (NN + F) \times F
\end{array}
$$

$$
\begin{aligned}
E &\rightarrow E + T \mid T \\
T &\rightarrow T \times F \mid F \\
F &\rightarrow (E) \mid N \\
N &\rightarrow NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{aligned}
$$

$$E \overset{*}{\Rightarrow} (12 + 8) \times 66$$

$$
\begin{aligned}
E &\Rightarrow T \\
&\Rightarrow T \times F \\
&\Rightarrow F \times F \\
&\Rightarrow (E) \times F \\
&\Rightarrow (E + T) \times F \\
&\Rightarrow (T + T) \times F \\
&\Rightarrow (F + T) \times F \\
&\Rightarrow (F + F) \times F \\
&\Rightarrow (N + F) \times F \\
&\Rightarrow (NN + F) \times F \\
&\Rightarrow (NN + N) \times F
\end{aligned}
$$

$$
\begin{array}{lcl}
E & \rightarrow & E + T \mid T \\
T & \rightarrow & T \times F \mid F \\
F & \rightarrow & (E) \mid N \\
N & \rightarrow & NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{array}
$$

$$E \stackrel{*}{\Rightarrow} (12 + 8) \times 66$$

$$
\begin{array}{lcl}
E & \Rightarrow & T \\
& \Rightarrow & T \times F \\
& \Rightarrow & F \times F \\
& \Rightarrow & (E) \times F \\
& \Rightarrow & (E + T) \times F \\
& \Rightarrow & (T + T) \times F \\
& \Rightarrow & (F + T) \times F \\
& \Rightarrow & (F + F) \times F \\
& \Rightarrow & (N + F) \times F \\
& \Rightarrow & (NN + F) \times F \\
& \Rightarrow & (NN + N) \times F \\
& \Rightarrow & (NN + N) \times N
\end{array}
$$

$$
\begin{aligned}
E &\rightarrow E + T \mid T \\
T &\rightarrow T \times F \mid F \\
F &\rightarrow (E) \mid N \\
N &\rightarrow NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{aligned}
$$

$$E \overset{*}{\Rightarrow} (12 + 8) \times 66$$

$$
\begin{aligned}
E &\Rightarrow T \\
&\Rightarrow T \times F \\
&\Rightarrow F \times F \\
&\Rightarrow (E) \times F \\
&\Rightarrow (E + T) \times F \\
&\Rightarrow (T + T) \times F \\
&\Rightarrow (F + T) \times F \\
&\Rightarrow (F + F) \times F \\
&\Rightarrow (N + F) \times F \\
&\Rightarrow (NN + F) \times F \\
&\Rightarrow (NN + N) \times F \\
&\Rightarrow (NN + N) \times N \\
&\Rightarrow (NN + N) \times NN
\end{aligned}
$$

$$
\begin{array}{lcl}
E & \rightarrow & E + T \mid T \\
T & \rightarrow & T \times F \mid F \\
F & \rightarrow & (E) \mid N \\
N & \rightarrow & NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{array}
$$

$$E \overset{*}{\Rightarrow} (12 + 8) \times 66$$

$$
\begin{array}{lcl}
E & \Rightarrow & T \\
& \Rightarrow & T \times F \\
& \Rightarrow & F \times F \\
& \Rightarrow & (E) \times F \\
& \Rightarrow & (E + T) \times F \\
& \Rightarrow & (T + T) \times F \\
& \Rightarrow & (F + T) \times F \\
& \Rightarrow & (F + F) \times F \\
& \Rightarrow & (N + F) \times F \\
& \Rightarrow & (NN + F) \times F \\
& \Rightarrow & (NN + N) \times F \\
& \Rightarrow & (NN + N) \times N \\
& \Rightarrow & (NN + N) \times NN \\
& \Rightarrow & (1N + N) \times NN
\end{array}
$$

$$
\boxed{
\begin{aligned}
E &\to E + T \mid T \\
T &\to T \times F \mid F \\
F &\to (E) \mid N \\
N &\to NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{aligned}
}
$$

$$E \overset{*}{\Rightarrow} (12 + 8) \times 66$$

$$
\begin{aligned}
E &\Rightarrow T \\
&\Rightarrow T \times F \\
&\Rightarrow F \times F \\
&\Rightarrow (E) \times F \\
&\Rightarrow (E + T) \times F \\
&\Rightarrow (T + T) \times F \\
&\Rightarrow (F + T) \times F \\
&\Rightarrow (F + F) \times F \\
&\Rightarrow (N + F) \times F \\
&\Rightarrow (NN + F) \times F \\
&\Rightarrow (NN + N) \times F \\
&\Rightarrow (NN + N) \times N \\
&\Rightarrow (NN + N) \times NN \\
&\Rightarrow (1N + N) \times NN \\
&\Rightarrow (12 + N) \times NN
\end{aligned}
$$

$$
\begin{aligned}
E &\rightarrow E + T \mid T \\
T &\rightarrow T \times F \mid F \\
F &\rightarrow (E) \mid N \\
N &\rightarrow NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{aligned}
$$

$$E \stackrel{*}{\Rightarrow} (12 + 8) \times 66$$

$$
\begin{aligned}
E &\Rightarrow T \\
&\Rightarrow T \times F \\
&\Rightarrow F \times F \\
&\Rightarrow (E) \times F \\
&\Rightarrow (E + T) \times F \\
&\Rightarrow (T + T) \times F \\
&\Rightarrow (F + T) \times F \\
&\Rightarrow (F + F) \times F \\
&\Rightarrow (N + F) \times F \\
&\Rightarrow (NN + F) \times F \\
&\Rightarrow (NN + N) \times F \\
&\Rightarrow (NN + N) \times N \\
&\Rightarrow (NN + N) \times NN \\
&\Rightarrow (1N + N) \times NN \\
&\Rightarrow (12 + N) \times NN \\
&\Rightarrow (12 + 8) \times NN
\end{aligned}
$$

$$
\begin{aligned}
E &\rightarrow E + T \mid T \\
T &\rightarrow T \times F \mid F \\
F &\rightarrow (E) \mid N \\
N &\rightarrow NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{aligned}
$$

$$E \overset{*}{\Rightarrow} (12 + 8) \times 66$$

$$
\begin{aligned}
E &\Rightarrow T \\
&\Rightarrow T \times F \\
&\Rightarrow F \times F \\
&\Rightarrow (E) \times F \\
&\Rightarrow (E + T) \times F \\
&\Rightarrow (T + T) \times F \\
&\Rightarrow (F + T) \times F \\
&\Rightarrow (F + F) \times F \\
&\Rightarrow (N + F) \times F \\
&\Rightarrow (NN + F) \times F \\
&\Rightarrow (NN + N) \times F \\
&\Rightarrow (NN + N) \times N \\
&\Rightarrow (NN + N) \times NN \\
&\Rightarrow (1N + N) \times NN \\
&\Rightarrow (12 + N) \times NN \\
&\Rightarrow (12 + 8) \times NN \\
&\Rightarrow (12 + 8) \times 6N
\end{aligned}
$$

$$
\boxed{\begin{array}{lcl}
E & \rightarrow & E + T \mid T \\
T & \rightarrow & T \times F \mid F \\
F & \rightarrow & (E) \mid N \\
N & \rightarrow & NN \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{array}}
$$

$$E \overset{*}{\Rightarrow} (12 + 8) \times 66$$

$$
\begin{array}{lcl}
E & \Rightarrow & T \\
& \Rightarrow & T \times F \\
& \Rightarrow & F \times F \\
& \Rightarrow & (E) \times F \\
& \Rightarrow & (E + T) \times F \\
& \Rightarrow & (T + T) \times F \\
& \Rightarrow & (F + T) \times F \\
& \Rightarrow & (F + F) \times F \\
& \Rightarrow & (N + F) \times F \\
& \Rightarrow & (NN + F) \times F \\
& \Rightarrow & (NN + N) \times F \\
& \Rightarrow & (NN + N) \times N \\
& \Rightarrow & (NN + N) \times NN \\
& \Rightarrow & (1N + N) \times NN \\
& \Rightarrow & (12 + N) \times NN \\
& \Rightarrow & (12 + 8) \times NN \\
& \Rightarrow & (12 + 8) \times 6N \\
& \Rightarrow & (12 + 8) \times 66
\end{array}
$$

Example. $A = \{a^n b^m \mid n < m\}$

$$S \quad \rightarrow \quad aSb \mid Sb \mid b$$

Example. $A = \{a^n b^m \mid n < m\}$

$$S \;\rightarrow\; aSb \mid Sb \mid b$$

$$S \;\Rightarrow\; aSb$$

Example. $A = \{a^n b^m \mid n < m\}$

$$S \quad \rightarrow \quad aSb \mid Sb \mid b$$

$$\begin{aligned} S \quad &\Rightarrow \quad aSb \\ &\Rightarrow \quad aaSbb \end{aligned}$$

Example. $A = \{a^n b^m \mid n < m\}$

$$S \quad \to \quad aSb \mid Sb \mid b$$

$$\begin{aligned} S &\Rightarrow aSb \\ &\Rightarrow aaSbb \\ &\Rightarrow aaaSbbb \end{aligned}$$

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$$\begin{aligned} S &\Rightarrow aSb \\ &\Rightarrow aaSbb \\ &\Rightarrow aaaSbbb \\ &\Rightarrow aaaSbbbb \\ &\Rightarrow aaaSbbbbb \\ &\Rightarrow aaabbbbbb = a^3 b^6 \end{aligned}$$

Example. $B = \{a^n b^m c^l \mid n = m \text{ or } m = l\}$

Example. $B = \{a^n b^m c^l \mid n = m \text{ or } m = l\}$

$$
\begin{array}{rcl}
S & \to & LC \mid AR \\
L & \to & aLb \mid \epsilon \\
R & \to & bRc \mid \epsilon \\
A & \to & Aa \mid \epsilon \\
C & \to & Cc \mid \epsilon
\end{array}
$$

Example. $B = \{a^n b^m c^l \mid n = m \text{ or } m = l\}$

$$\begin{aligned} S &\to LC \mid AR \\ L &\to aLb \mid \epsilon \\ R &\to bRc \mid \epsilon \\ A &\to Aa \mid \epsilon \\ C &\to Cc \mid \epsilon \end{aligned}$$

$$S \Rightarrow LC$$

Example. $B = \{a^n b^m c^l \mid n = m \text{ or } m = l\}$

$$\begin{aligned} S &\rightarrow LC \mid AR \\ L &\rightarrow aLb \mid \epsilon \\ R &\rightarrow bRc \mid \epsilon \\ A &\rightarrow Aa \mid \epsilon \\ C &\rightarrow Cc \mid \epsilon \end{aligned}$$

$$\begin{aligned} S &\Rightarrow LC \\ &\Rightarrow aLbC \end{aligned}$$

Example. $B = \{a^n b^m c^l \mid n = m \text{ or } m = l\}$

$$\begin{aligned} S &\rightarrow LC \mid AR \\ L &\rightarrow aLb \mid \epsilon \\ R &\rightarrow bRc \mid \epsilon \\ A &\rightarrow Aa \mid \epsilon \\ C &\rightarrow Cc \mid \epsilon \end{aligned}$$

$$\begin{aligned} S &\Rightarrow LC \\ &\Rightarrow aLbC \\ &\Rightarrow aaLbbC \end{aligned}$$

Example. $B = \{a^n b^m c^l \mid n = m \text{ or } m = l\}$

$$\begin{aligned} S &\rightarrow LC \mid AR \\ L &\rightarrow aLb \mid \epsilon \\ R &\rightarrow bRc \mid \epsilon \\ A &\rightarrow Aa \mid \epsilon \\ C &\rightarrow Cc \mid \epsilon \end{aligned}$$

$$\begin{aligned} S &\Rightarrow LC \\ &\Rightarrow aLbC \\ &\Rightarrow aaLbbC \\ &\Rightarrow aaaLbbbC \end{aligned}$$

Example. $B = \{a^n b^m c^l \mid n = m \text{ or } m = l\}$

$$\begin{aligned} S &\to LC \mid AR \\ L &\to aLb \mid \epsilon \\ R &\to bRc \mid \epsilon \\ A &\to Aa \mid \epsilon \\ C &\to Cc \mid \epsilon \end{aligned}$$

$$\begin{aligned} S &\Rightarrow LC \\ &\Rightarrow aLbC \\ &\Rightarrow aaLbbC \\ &\Rightarrow aaaLbbbC \\ &\Rightarrow aaabbbC \end{aligned}$$

Example. $B = \{a^n b^m c^l \mid n = m \text{ or } m = l\}$

$$\begin{aligned} S &\rightarrow LC \mid AR \\ L &\rightarrow aLb \mid \epsilon \\ R &\rightarrow bRc \mid \epsilon \\ A &\rightarrow Aa \mid \epsilon \\ C &\rightarrow Cc \mid \epsilon \end{aligned}$$

$$\begin{aligned} S &\Rightarrow LC \\ &\Rightarrow aLbC \\ &\Rightarrow aaLbbC \\ &\Rightarrow aaaLbbbC \\ &\Rightarrow aaabbbC \\ &\Rightarrow aaabbbCc \end{aligned}$$

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$$\begin{aligned} S &\Rightarrow LC \\ &\Rightarrow aLbC \\ &\Rightarrow aaLbbC \\ &\Rightarrow aaaLbbbC \\ &\Rightarrow aaabbbC \\ &\Rightarrow aaabbbCc \\ &\Rightarrow aaabbbCcc \end{aligned}$$

Example. $B = \{a^n b^m c^l \mid n = m \text{ or } m = l\}$

$$
\begin{aligned}
S &\rightarrow LC \mid AR \\
L &\rightarrow aLb \mid \epsilon \\
R &\rightarrow bRc \mid \epsilon \\
A &\rightarrow Aa \mid \epsilon \\
C &\rightarrow Cc \mid \epsilon
\end{aligned}
$$

$$
\begin{aligned}
S &\Rightarrow LC \\
&\Rightarrow aLbC \\
&\Rightarrow aaLbbC \\
&\Rightarrow aaaLbbbC \\
&\Rightarrow aaabbbC \\
&\Rightarrow aaabbbCc \\
&\Rightarrow aaabbbCcc \\
&\Rightarrow aaabbbcc = a^3 b^3 c^2
\end{aligned}
$$

Example. $B = \{a^n b^m c^l \mid n = m \text{ or } m = l\}$

$$\begin{aligned}
S &\to LC \mid AR \\
L &\to aLb \mid \epsilon \\
R &\to bRc \mid \epsilon \\
A &\to Aa \mid \epsilon \\
C &\to Cc \mid \epsilon
\end{aligned}$$

$$S \Rightarrow AR$$

$$\begin{aligned}
S &\Rightarrow LC \\
&\Rightarrow aLbC \\
&\Rightarrow aaLbbC \\
&\Rightarrow aaaLbbbC \\
&\Rightarrow aaabbbC \\
&\Rightarrow aaabbbCc \\
&\Rightarrow aaabbbCcc \\
&\Rightarrow aaabbbcc = a^3b^3c^2
\end{aligned}$$

Example. $B = \{a^n b^m c^l \mid n = m \text{ or } m = l\}$

$$\begin{aligned} S &\rightarrow LC \mid AR \\ L &\rightarrow aLb \mid \epsilon \\ R &\rightarrow bRc \mid \epsilon \\ A &\rightarrow Aa \mid \epsilon \\ C &\rightarrow Cc \mid \epsilon \end{aligned}$$

$$\begin{aligned} S &\Rightarrow AR \\ &\Rightarrow AR \end{aligned}$$

$$\begin{aligned} S &\Rightarrow LC \\ &\Rightarrow aLbC \\ &\Rightarrow aaLbbC \\ &\Rightarrow aaaLbbbC \\ &\Rightarrow aaabbbC \\ &\Rightarrow aaabbbCc \\ &\Rightarrow aaabbbCcc \\ &\Rightarrow aaabbbcc = a^3 b^3 c^2 \end{aligned}$$

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$$\begin{aligned} S &\Rightarrow AR \\ &\Rightarrow AR \\ &\Rightarrow AbRc \end{aligned}$$

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$$
\begin{aligned}
S &\rightarrow LC \mid AR \\
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A &\rightarrow Aa \mid \epsilon \\
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\end{aligned}
$$

$$
\begin{aligned}
S &\Rightarrow AR \\
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$$

$$
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S &\Rightarrow LC \\
&\Rightarrow aLbC \\
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$$\begin{aligned} S &\Rightarrow AR \\ &\Rightarrow AR \\ &\Rightarrow AbRc \\ &\Rightarrow AbbRcc \\ &\Rightarrow Abbcc \end{aligned}$$

$$\begin{aligned} S &\Rightarrow LC \\ &\Rightarrow aLbC \\ &\Rightarrow aaLbbC \\ &\Rightarrow aaaLbbbC \\ &\Rightarrow aaabbbC \\ &\Rightarrow aaabbbCc \\ &\Rightarrow aaabbbCcc \\ &\Rightarrow aaabbbcc = a^3b^3c^2 \end{aligned}$$

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S &\Rightarrow AR \\
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$$\begin{aligned} S &\Rightarrow AR \\ &\Rightarrow AR \\ &\Rightarrow AbRc \\ &\Rightarrow AbbRcc \\ &\Rightarrow Abbcc \\ &\Rightarrow Aabbcc \\ &\Rightarrow Aaabbcc \\ &\Rightarrow Aaaabbcc \\ &\Rightarrow Aaaabbcc \\ &\Rightarrow Aaaaabbcc \\ &\Rightarrow aaaabbcc = a^4b^2c^2 \end{aligned}$$

Theorem

$\mathrm{REG} \subseteq \mathrm{CFL}$. *That is, every regular language is context-free.*

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Proof. Let $A \in \mathrm{REG}$, and let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA for A. For each state $q \in Q$, we make a variable T_q. Formally,

$$V = \{\, T_q \mid q \in Q \}.$$

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The start variable is T_{q_0}. We add a rule

$$T_q \to a T_r$$

if $\delta(q, a) = r$ is a transition in the DFA.

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The start variable is T_{q_0}. We add a rule

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if $\delta(q, a) = r$ is a transition in the DFA. Also, add $T_q \to \epsilon$ for each final state $q \in F$.

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if $\delta(q, a) = r$ is a transition in the DFA. Also, add $T_q \to \epsilon$ for each final state $q \in F$. Formally the set of rules is

$$\begin{aligned} R &= \{ T_q \to aT_{\delta(q,a)} \mid q \in Q \text{ and } a \in \Sigma \} \\ &\quad \cup \{ T_q \to \epsilon \mid q \in F \}. \end{aligned}$$

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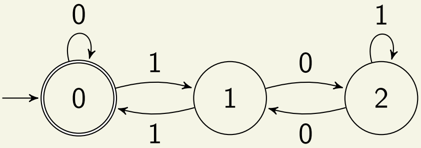
Our grammar is $G = (V, \Sigma, R, T_{q_0})$. Then $L(G) = L(M) = A$. $\square$

Example

$$L(M) = \{x \in \{0,1\}^* \mid \text{bin}(x) \text{ is a multiple of } 3\}$$

M:



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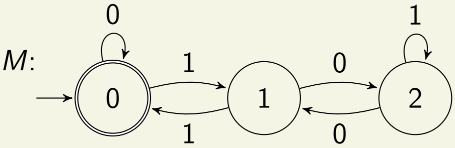

G:

$$\begin{array}{lcl} T_0 & \to & 0\,T_0 \mid 1\,T_1 \mid \epsilon \\ T_1 & \to & 0\,T_2 \mid 1\,T_0 \\ T_2 & \to & 0\,T_1 \mid 1\,T_2 \end{array}$$

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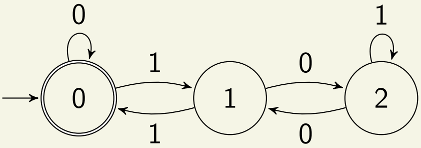

G:

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Let $x = 10101$. Then $\mathrm{bin}(x) = 21$ and $x \in L(M)$.

Example

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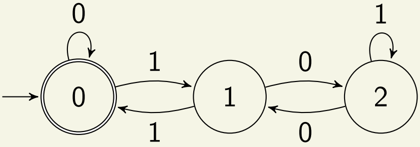

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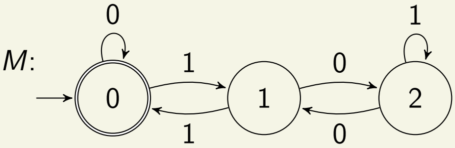
T_0

Example

$$L(M) = \{x \in \{0,1\}^* \mid \text{bin}(x) \text{ is a multiple of } 3\}$$



G:

$$\begin{array}{lcl} T_0 & \to & 0\,T_0 \mid 1\,T_1 \mid \epsilon \\ T_1 & \to & 0\,T_2 \mid 1\,T_0 \\ T_2 & \to & 0\,T_1 \mid 1\,T_2 \end{array}$$

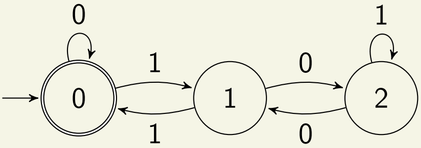
Let $x = 10101$. Then $\text{bin}(x) = 21$ and $x \in L(M)$.

$$T_0 \;\Rightarrow\; 1\,T_1$$

Example

$L(M) = \{x \in \{0,1\}^* \mid \mathrm{bin}(x) \text{ is a multiple of } 3\}$

M:



G:

$$
\begin{array}{lcl}
T_0 & \rightarrow & 0\,T_0 \mid 1\,T_1 \mid \epsilon \\
T_1 & \rightarrow & 0\,T_2 \mid 1\,T_0 \\
T_2 & \rightarrow & 0\,T_1 \mid 1\,T_2
\end{array}
$$

Let $x = 10101$. Then $\mathrm{bin}(x) = 21$ and $x \in L(M)$.

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Example

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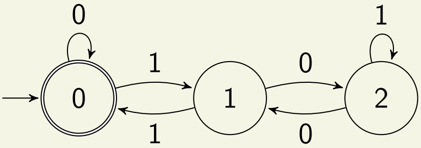

G:

$$\begin{array}{rcl} T_0 & \to & 0\,T_0 \mid 1\,T_1 \mid \epsilon \\ T_1 & \to & 0\,T_2 \mid 1\,T_0 \\ T_2 & \to & 0\,T_1 \mid 1\,T_2 \end{array}$$

Let $x = 10101$. Then $\text{bin}(x) = 21$ and $x \in L(M)$.

$$\begin{array}{rcl} T_0 & \Rightarrow & 1\,T_1 \\ & \Rightarrow & 10\,T_2 \end{array}$$

Example

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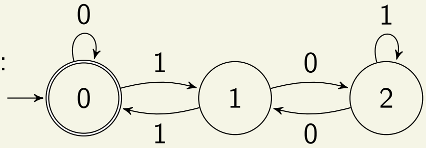


G:

$$\begin{aligned} T_0 &\rightarrow 0\,T_0 \mid 1\,T_1 \mid \epsilon \\ T_1 &\rightarrow 0\,T_2 \mid 1\,T_0 \\ T_2 &\rightarrow 0\,T_1 \mid 1\,T_2 \end{aligned}$$

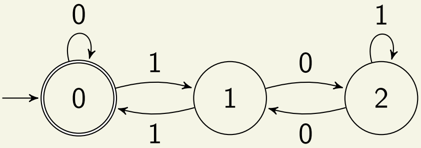
Let $x = 10101$. Then $\mathrm{bin}(x) = 21$ and $x \in L(M)$.

$$\begin{aligned} T_0 &\Rightarrow 1\,T_1 \\ &\Rightarrow 10\,T_2 \end{aligned}$$

Example

$L(M) = \{x \in \{0,1\}^* \mid \mathrm{bin}(x) \text{ is a multiple of } 3\}$

M:



G:

$$
\begin{array}{lcl}
T_0 & \to & 0\,T_0 \mid 1\,T_1 \mid \epsilon \\
T_1 & \to & 0\,T_2 \mid 1\,T_0 \\
T_2 & \to & 0\,T_1 \mid 1\,T_2
\end{array}
$$

Let $x = 10101$. Then $\mathrm{bin}(x) = 21$ and $x \in L(M)$.

$$
\begin{array}{lcl}
T_0 & \Rightarrow & 1\,T_1 \\
 & \Rightarrow & 10\,T_2 \\
 & \Rightarrow & 101\,T_2
\end{array}
$$

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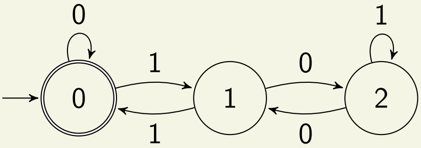

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$$\begin{array}{lcl} T_0 & \rightarrow & 0\,T_0 \mid 1\,T_1 \mid \epsilon \\ T_1 & \rightarrow & 0\,T_2 \mid 1\,T_0 \\ T_2 & \rightarrow & 0\,T_1 \mid 1\,T_2 \end{array}$$

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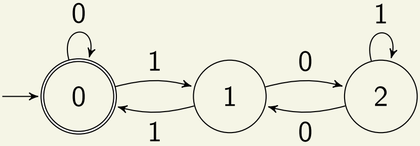

G:

$$\begin{aligned} T_0 &\rightarrow 0\,T_0 \mid 1\,T_1 \mid \epsilon \\ T_1 &\rightarrow 0\,T_2 \mid 1\,T_0 \\ T_2 &\rightarrow 0\,T_1 \mid 1\,T_2 \end{aligned}$$

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Example

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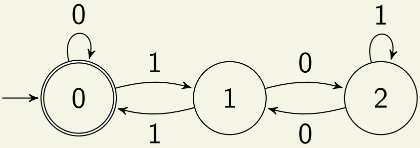
M:



G:

$$
\begin{array}{lcl}
T_0 & \rightarrow & 0\,T_0 \mid 1\,T_1 \mid \epsilon \\
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T_2 & \rightarrow & 0\,T_1 \mid 1\,T_2
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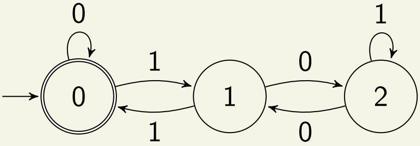

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\begin{aligned}
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\end{aligned}
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$$
\begin{aligned}
T_0 &\Rightarrow 1\,T_1 \\
&\Rightarrow 10\,T_2 \\
&\Rightarrow 101\,T_2 \\
&\Rightarrow 1010\,T_1 \\
&\Rightarrow 10101\,T_0
\end{aligned}
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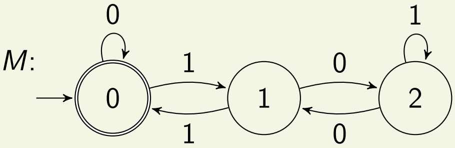
Example

$L(M) = \{x \in \{0,1\}^* \mid \mathrm{bin}(x) \text{ is a multiple of } 3\}$



G:

$$\begin{array}{lcl} T_0 & \to & 0\,T_0 \mid 1\,T_1 \mid \epsilon \\ T_1 & \to & 0\,T_2 \mid 1\,T_0 \\ T_2 & \to & 0\,T_1 \mid 1\,T_2 \end{array}$$

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Example

$$L(M) = \{x \in \{0,1\}^* \mid \mathrm{bin}(x) \text{ is a multiple of } 3\}$$

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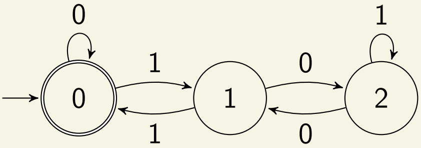

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Let $x = 10101$. Then $\mathrm{bin}(x) = 21$ and $x \in L(M)$.

$$\begin{array}{lcl} T_0 & \Rightarrow & 1\,T_1 \\ & \Rightarrow & 10\,T_2 \\ & \Rightarrow & 101\,T_2 \\ & \Rightarrow & 1010\,T_1 \\ & \Rightarrow & 10101\,T_0 \\ & \Rightarrow & 10101 \end{array}$$

Thus $T_0 \stackrel{*}{\Rightarrow} 10101$, so $10101 \in L(G)$.