Computability and Complexity
COSC 4200

The Myhill-Nerode Theorem

The Relation $\equiv_M$

Let $A \subseteq \Sigma^*$ be a regular language, and let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA for A with no inaccessible states.

M induces an equivalence relation $\equiv_M$ on Σ^* defined by

$$x \equiv_M y \iff \delta^*(q_0, x) = \delta^*(q_0, y)$$

for all $x, y \in \Sigma^*$.

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$$x \equiv_M y \implies (x \in A \Leftrightarrow y \in A).$$

3. It is of *finite index*: it has only finitely many equivalence classes.

Myhill-Nerode Relations

Definition. Let $A \subseteq \Sigma^*$. An equivalence relation $\equiv$ on Σ^* is called a *Myhill-Nerode relation* for A if it has the three properties from the previous slide: that is, it is a right congruence of finite index that refines A.

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- We saw that every DFA M for A yields a Myhill-Nerode relation $\equiv_M$ for A.
- We will now show that every Myhill-Nerode relation $\equiv$ for A yields a DFA $M_\equiv$ for A.

From Myhill-Nerode Relation $\equiv$ to the DFA $M_{\equiv}$

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Because $\equiv$ is a right congruence, δ is well defined.
Since $\equiv$ refines A, we have $x \in A \Leftrightarrow [x] \in F$ for all $x \in \Sigma^*$.

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Recall $\equiv$ is a Myhill-Nerode relation for A and $M_{\equiv} = (Q, \Sigma, \delta, q_0, F)$ where

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Therefore $L(M_{\equiv}) = A$. $\square$

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1. *If $\equiv$ is a Myhill-Nerode relation for A, and if we apply the construction $\equiv \to M_\equiv$ and then apply the construction $M_\equiv \to \equiv_{M_\equiv}$, the relation $\equiv_{M_\equiv}$ is identical to $\equiv$.*

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$$\equiv \; \rightarrow M_\equiv \rightarrow \; \equiv_{M_\equiv} \; = \; \equiv$$

2. *If M is a DFA for A with no inaccessible states, and if we apply the construction $M \rightarrow \equiv_M$ and then apply the construction $\equiv_M \rightarrow M_{\equiv_M}$, the resulting DFA $M_{\equiv_M}$ is isomorphic to M.*

$$M \rightarrow \; \equiv_M \; \rightarrow M_{\equiv_M} \cong M$$

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- If $\equiv_1$ refines $\equiv_2$, then we say $\equiv_2$ is *coarser* than $\equiv_1$.

Let $A \subseteq \Sigma^*$. We define an equivalence relation $\equiv_A$ on Σ^* by

$$x \equiv_A y \iff (\forall z \in \Sigma^*)\ (xz \in A \Leftrightarrow yz \in A)$$

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To see that $\equiv_A$ refines A, use $z = \epsilon$ in the definition of $\equiv_A$:

$$x \equiv_A y \Rightarrow (x \in A \Leftrightarrow y \in A).$$

Lemma

Let $A \subseteq \Sigma^*$ The relation $\equiv_A$ is a right congruence refining A and is the coarsest such relation on Σ^*.

Proof continued. Let $\equiv$ be any equivalence relation on Σ^* that is a right congruence refining A. We have

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so $\equiv_A$ is coarser than $\equiv$. $\square$

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Proof. 1 $\Rightarrow$ 2: Given a DFA M for A, the construction $M \to \equiv_M$ yields a Myhill-Nerode relation for A.

2 $\Rightarrow$ 3: Let $\equiv$ be a Myhill-Nerode relation for A. Then $\equiv$ is a right congruence that refines A, so $\equiv_A$ is coarser than $\equiv$ by the Lemma. Since $\equiv$ is of finite index, this implies $\equiv_A$ is also of finite index.

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3 $\Rightarrow$ 1: If $\equiv_A$ is of finite index, then it is a Myhill-Nerode relation for A by the Lemma. The construction $\equiv_A \to M_{\equiv_A}$ produces a DFA for A. $\square$

Corollary

For any regular language A, the DFA $M_{\equiv_A}$ is the minimal DFA for A.

Proof. If A is regular, then $\equiv_A$ is a Myhill-Nerode relation for A. Also, it is the coarsest such relation: it has the fewest equivalence classes of any Myhill-Nerode relation for A. Therefore $M_{\equiv_A}$ has the fewest states in a DFA for A. $\square$

Applications

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Therefore $\equiv_A$ has infinitely many equivalence classes, at least one for each 0^k, $k \geq 0$. By the Myhill-Nerode Theorem, A is not regular. $\square$

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- $0^{j^2}0^{2k+1} \notin B$ because $j^2 < j^2 + 2k + 1 < j^2 + 2j + 1 = (j+1)^2$ implies $j^2 + 2k + 1$ is not a square.

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Therefore $\equiv_B$ has infinitely many equivalence classes, so B is not regular by the Myhill-Nerode Theorem. $\square$

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Example. $F = \{a^m b^n c^l \mid m, n, l \geq 0 \text{ and if } m = 1 \text{ then } n = l\}$

Recall that we showed F satisfies the conclusion of the Pumping Lemma. We can use the Myhill-Nerode Theorem to prove F is not regular.

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Proof. For all $i < j$, $ab^i \not\equiv_F ab^j$ since

$$ab^i c^i \in A \text{ and } ab^j c^i \notin F.$$

Therefore $\equiv_F$ has infinitely many equivalence classes, so F is not regular by the Myhill-Nerode Theorem. $\square$

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$$x \equiv_A y \text{ if } (\forall z \in \Sigma^*)\ (xz \in A \Leftrightarrow yz \in A)$$

and

$$x \equiv_M y \text{ if } \delta^*(q_0, x) = \delta^*(q_0, y).$$

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$$x \equiv_A y \text{ if } (\forall z \in \Sigma^*)\ (xz \in A \Leftrightarrow yz \in A)$$

and

$$x \equiv_M y \text{ if } \delta^*(q_0, x) = \delta^*(q_0, y).$$

From the proof of the Myhill-Nerode Theorem, we know that $\equiv_A$ and $\equiv_M$ are the same relation. In other words, the states of M "remember" the equivalence of x with respect to $\equiv_A$:

$$\delta^*(q_0, x) \text{ is essentially } [x]_A.$$

Let $A = \{x00 \mid x \in \{0,1\}^*\}$.
Recall $x \equiv_A y$ if $(\forall z \in \Sigma^*)\ (xz \in A \Leftrightarrow yz \in A)$.

1. What is $\equiv_A$-equivalent to ϵ?

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    - $\epsilon \equiv_A 1 \quad \epsilon \equiv_A 11 \quad \epsilon \equiv_A 01$

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$$\begin{aligned} [\epsilon]_A &= \{\epsilon\} \cup \{x \in \{0,1\}^* \mid x \text{ ends in } 1\} \\ &= \{x \in \{0,1\}^* \mid x \text{ does not end in } 0\} \end{aligned}$$

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2. What is $\equiv_A$-equivalent to 0?
   - $0 \equiv_A 10 \quad 0 \equiv_A 1010 \quad 0 \equiv_A 0010$

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2. What is $\equiv_A$-equivalent to 0?
   - $0 \equiv_A 10 \quad 0 \equiv_A 1010 \quad 0 \equiv_A 0010$
   - $0 \not\equiv_A 100$ ($0 \notin A$ but $100 \in A$)
   - $0 \not\equiv_A 1$ ($00 \in A$ but $10 \notin A$)

$$[0]_A = \{x \in \{0,1\}^* \mid x \text{ ends in exactly one } 0\}$$

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3. What is $\equiv_A$-equivalent to 00?
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   - $00 \not\equiv_A 001$ ($00 \in A$ but $001 \notin A$)

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Recall $x \equiv_A y$ if $(\forall z \in \Sigma^*)\ (xz \in A \Leftrightarrow yz \in A)$.

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$$[0]_A = \{x \in \{0,1\}^* \mid x \text{ ends in exactly one } 0\}$$

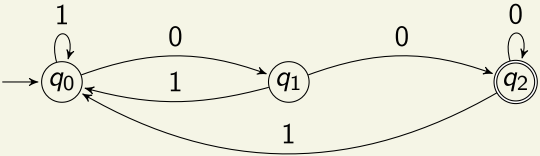
3. What is $\equiv_A$-equivalent to 00?
   - $00 \equiv_A 100 \qquad 00 \equiv_A 10100 \qquad 00 \equiv_A 10000$
   - $00 \not\equiv_A 10$ ($00 \in A$ but $10 \notin A$)
   - $00 \not\equiv_A 001$ ($00 \in A$ but $001 \notin A$)

$$[00]_A = \{x \in \{0,1\}^* \mid x \text{ ends in two or more 0's}\}$$

Example

$$A = \{x00 \mid x \in \{0,1\}^*\}$$

M :



Equivalence classes for $\equiv_M$ and $\equiv_A$:

$[\epsilon]_M =$	$\{x \mid \delta^*(q_0, x) = q_0\} = \epsilon \cup (0 \cup 1)^*1$	$=$	$\left\{ x \;\middle|\; \begin{array}{c} x \text{ does not} \\ \text{end in } 0 \end{array} \right\}$	$= [\epsilon]_A$
$[0]_M =$	$\{x \mid \delta^*(q_0, x) = q_1\} = 0 \cup (0 \cup 1)^*10$	$=$	$\left\{ x \;\middle|\; \begin{array}{c} x \text{ ends in} \\ \text{exactly one } 0 \end{array} \right\}$	$= [0]_A$
$[00]_M =$	$\{x \mid \delta^*(q_0, x) = q_2\} = (0 \cup 1)^*00$	$=$	$\left\{ x \;\middle|\; \begin{array}{c} x \text{ ends in at} \\ \text{least two } 0\text{'s} \end{array} \right\}$	$= [00]_A$

$A = [00]_A$ and $A^c = [\epsilon]_A \cup [0]_A$