Computability and Complexity
COSC 4200

Reducibility

Mapping Reducibility

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Language A is *mapping reducible* to language B, written $A \leq_{\mathrm{m}} B$, if there is a computable function $f : \Sigma^* \to \Sigma^*$, where for every w,

$$w \in A \Leftrightarrow f(w) \in B.$$

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The reduction maps positive instances to positive instances:

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The reduction converts questions about membership in A to questions about membership in B.

We can combine a reduction with an algorithm that decides B to obtain an algorithm that decides A.

Theorem

If $A \leq_{\mathrm{m}} B$ and B is decidable, then A is decidable.

Proof. Let f be a mapping reduction of A to B and let M be a decider for B. We describe a decider N for A.

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1. Compute $f(w)$.
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    - If M accepts $f(w)$, accept.
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We claim that N decides A.

- If $w \in A$, then $f(w) \in B$ because f reduces A to B. Therefore M accepts $f(w)$ and N accepts w.

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This provides a tool for proving undecidability:

Corollary

If $A \leq_{\mathrm{m}} B$ and A is undecidable, then B is undecidable.

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Corollary

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Proof. Assume the hypothesis and suppose that B is decidable. Then A is decidable by the theorem above, a contradiction. $\square$

Halting Problem for TMs

The *halting problem* for TMs:

$$HALT_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that halts input string } w \}.$$

Theorem

$HALT_{\mathrm{TM}}$ is undecidable.

Second proof. We proved this earlier, but now we give a proof by showing that $A_{\mathrm{TM}} \leq_{\mathrm{m}} HALT_{\mathrm{TM}}$.

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Second proof. We proved this earlier, but now we give a proof by showing that $A_{\mathrm{TM}} \leq_{\mathrm{m}} HALT_{\mathrm{TM}}$.

We must design a computable function f so that for any instance $\langle M, w\rangle$ of A_{TM}, $f(\langle M, w\rangle) = \langle M', w'\rangle$ so that

$$\langle M, w\rangle \in A_{\mathrm{TM}} \Leftrightarrow \langle M', w'\rangle \in HALT_{\mathrm{TM}}.$$

The following Turing machine F computes a reduction f.

F: On input $\langle M, w\rangle$:

1. Construct the following machine M':
   - M': On input x:
     1. Run M on x.
     2. If M accepts, accept.
     3. If M rejects, enter an infinite loop.
2. Output $\langle M', w\rangle$.

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Therefore $\langle M, w\rangle \in A_{\mathrm{TM}} \Rightarrow \langle M', w\rangle \in HALT_{\mathrm{TM}}$.

We have proved

$$\langle M, w\rangle \in A_{\mathrm{TM}} \Leftrightarrow f(M, w) = \langle M', w\rangle \in \mathit{HALT}_{\mathrm{TM}},$$

so $A_{\mathrm{TM}} \leq_{\mathrm{m}} \mathit{HALT}_{\mathrm{TM}}$ via f. Since A_{TM} is undecidable, it follows that $\mathit{HALT}_{\mathrm{TM}}$ is undecidable. $\square$

Regularity Problem for TMs

$$REGULAR_{\mathrm{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is a regular language}\}.$$

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Proof. We will show that $A_{\mathrm{TM}} \leq_{\mathrm{m}} REGULAR_{\mathrm{TM}}$. Given a TM M and a string w, define a TM $M_{(w)}$ as follows:

$M_{(w)}$: On input x:

1. If x has the form $0^n 1^n$, accept.
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- If M does not accept w, then $L(M_{(w)}) = \{0^n 1^n \mid n \geq 0\}$, so $L(M_{(w)})$ is not regular.

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so $A_{\mathrm{TM}} \leq_{\mathrm{m}} \mathit{REGULAR}_{\mathrm{TM}}$ via f.

Since A_{TM} is undecidable, it follows that $\mathit{REGULAR}_{\mathrm{TM}}$ is undecidable. $\square$

Equivalence Problem for TMs

$$EQ_{\mathrm{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\ \}$$

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to EQ_{TM}.

Let R be a TM that immediately rejects all inputs. Then $L(R) = \emptyset$. Our reduction f is defined by

$$f(\langle M \rangle) = \langle M, R \rangle.$$

Then

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so $E_{\mathrm{TM}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$ via f. Since E_{TM} is undecidable, it follows that EQ_{TM} is undecidable. $\square$

In fact, we can even show that EQ_{TM} is not Turing-recognizable or co-Turing-recognizable. For this, we need the following theorem.

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Corollary

If $A \leq_{\mathrm{m}} B$ and A is not co-Turing-recognizable, then B is not co-Turing-recognizable.

Proof. Use the fact $A \leq_{\mathrm{m}} B \iff A^c \leq_{\mathrm{m}} B^c$ and the previous corollary. $\square$

Theorem

EQ_{TM} is not Turing-recognizable.

Proof. We showed that:

- E_{TM} is not Turing-recognizable.
- $E_{\mathrm{TM}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$.

The previous theorem implies that EQ_{TM} is not Turing-recognizable. $\square$

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Proof. We will reduce A_{TM} to EQ_{TM}. Since A_{TM} is not co-Turing-recognizable, this will establish that EQ_{TM} is not co-Turing-recognizable as well.

For any TM M and string w, define a TM $M_{(w)}$ as follows.

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2. If M accepts w, accept x.

Consider any instance $\langle M, w \rangle$ of A_{TM}.

- Suppose $\langle M, w \rangle \in A_{\mathrm{TM}}$. Then M accepts w. Therefore $L(M_{(w)}) = \Sigma^*$.
- Suppose $\langle M, w \rangle \notin A_{\mathrm{TM}}$. Then M does not accept w. Therefore $L(M_{(w)}) = \emptyset$.

Therefore we have

- $\langle M, w \rangle \in A_{\mathrm{TM}} \Rightarrow L(M_{(w)}) = \Sigma^*$
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$$f(\langle M, w\rangle) = \langle M_{(w)}, T\rangle.$$

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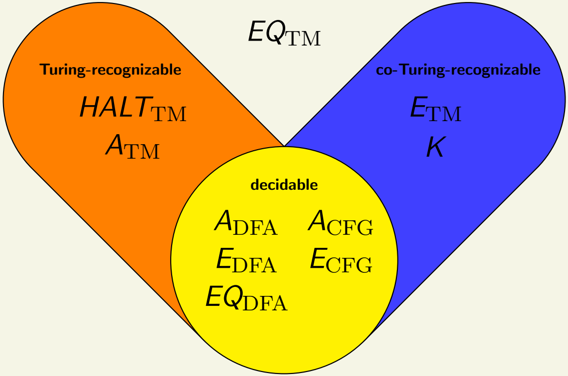
so $A_{\mathrm{TM}} \leq_{\mathrm{m}} EQ_{\mathrm{TM}}$ via f. $\square$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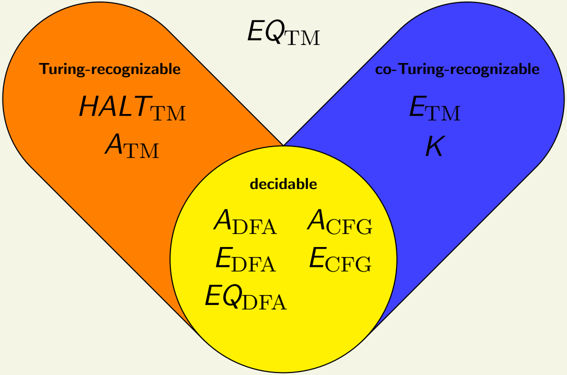

What about EQ_{CFG}?

Define

$$ALL_{\mathrm{DFA}} = \{\langle M \rangle \mid M \text{ is a DFA and } L(M) = \Sigma^*\},$$

$$ALL_{\mathrm{CFG}} = \{\langle G \rangle \mid G \text{ is a CFG and } L(G) = \Sigma^*\}.$$

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Theorem

$\mathrm{ALL}_{\mathrm{DFA}}$ *is decidable.*

Proof. Use similar techniques to E_{DFA} or reduce to EQ_{DFA}. $\square$

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$\mathrm{ALL}_{\mathrm{DFA}}$ *is decidable.*

Proof. Use similar techniques to E_{DFA} or reduce to EQ_{DFA}. □

We will show that ALL_{CFG} is undecidable and as a corollary, that EQ_{CFG} is also undecidable.

$$EQ_{\mathrm{CFG}} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\ \}$$

Definition

Let M be a TM and w be a string.

- An *accepting computation history* of M on w is a sequence of configurations $C_1, \ldots, C_l$, where C_1 is the start configuration of M on w, C_l is an accepting configuration of M, and each C_i follows from C_{i-1} using M's transition function.
- A *rejection computation history* is defined similarly, except that C_l is a rejecting configuration.

Theorem

ALL_{CFG} is undecidable.

Proof. We will show that $A^{c}_{\mathrm{TM}} \leq_{\mathrm{m}} ALL_{\mathrm{CFG}}$. The proof uses computation histories.

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Given a TM and a string w, our goal is to construct a CFG G so that

- M does not accept $w \Rightarrow L(G) = \Sigma^*$,
- M accepts $w \Rightarrow L(G) \neq \Sigma^*$.

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- M accepts $w \Rightarrow L(G) \neq \Sigma^*$.

The idea is to design G sot that G generates all strings that do not encode accepting computation histories of M on w.

The natural way to encode a computation history $C_1, \ldots, C_l$ is as a string

$$\#C_1\#C_2\#C_3\#\cdots\#C_l\#.$$

However, we will encode the computation history as

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The language of our grammar will be

$$L(G) = \left\{ x \in \Sigma^* \;\middle|\; \begin{array}{l} x \text{ is not an accepting computation} \\ \text{history of } M \text{ on } w \end{array} \right\}.$$

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$$x = \#C_1\#C_2^R\#C_3\#C_4^R \cdots \#C_l\#$$

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as well as all x where

4. x is not of the form $\#C_1\#C_2^R\#C_3\#C_4^R \cdots \#C_l\#$.

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- Now D can compare the two configurations, they should match except for the three cells around the head position of C_i.
- D accepts if there is a mismatch. $\square$

The proof that ALL_{CFG} is undecidable reduces $A^c_{\mathrm{TM}} \leq_{\mathrm{m}} ALL_{\mathrm{CFG}}$. Because A^c_{TM} is not Turing-recognizable, we actually have a stronger result:

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ALL_{CFG} is co-Turing-recognizable.

Proof. We need to give a recognition algorithm for ALL^c_{CFG}. Let $s_1, s_2, \ldots$ be an enumeration of Σ^*.

> M: On input $\langle G \rangle$:
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> for $i = 1, 2, \ldots$
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> Run the decision algorithm S for A_{CFG} on input $\langle G, s_i \rangle$.
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Suppose that $\langle G \rangle \notin ALL_{\mathrm{CFG}}$. Then $L(G) \neq \Sigma^*$. Let s_j be the least string that is not in $L(G)$. Then on the j^{th} iteration of the loop, S will reject $\langle G, s_j \rangle$. Therefore M will accept $\langle G \rangle$.

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Now suppose that $\langle G \rangle \in ALL_{\mathrm{CFG}}$. Then $L(G) = \Sigma^*$, so $\langle G, s_i \rangle \in A_{\mathrm{CFG}}$ for all i. Thus S will accept in every iteration and M will run forever on $\langle G \rangle$.

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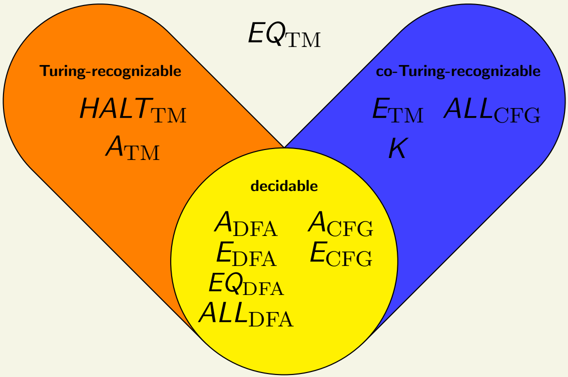
Now suppose that $\langle G \rangle \in ALL_{\mathrm{CFG}}$. Then $L(G) = \Sigma^*$, so $\langle G, s_i \rangle \in A_{\mathrm{CFG}}$ for all i. Thus S will accept in every iteration and M will run forever on $\langle G \rangle$.

Therefore M recognizes ALL^c_{CFG}. $\square$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Recall the *equivalence problem* for CFGs:

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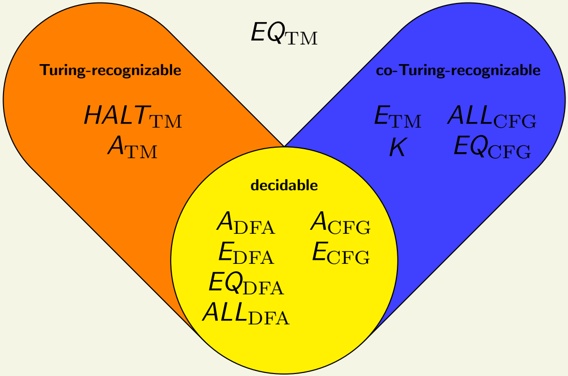
Proof. Similar to ALL_{CFG} is co-Turing-recognizable. □



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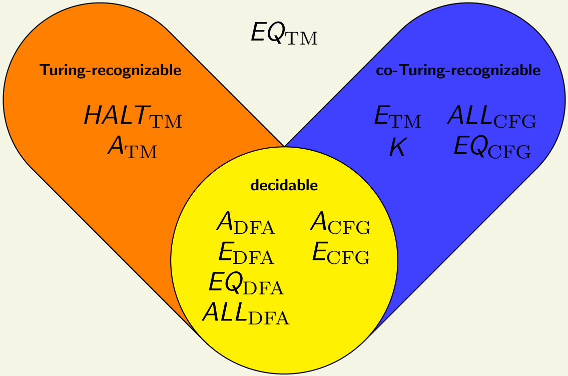

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What about ALL_{TM}?

$$ALL_{\mathrm{TM}} = \{\langle M \rangle \mid G \text{ is a TM and } L(M) = \Sigma^*\}$$

Theorem

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ALL_{TM} is neither Turing-recognizable nor co-Turing-recognizable.

Proof. It suffices to show that $EQ_{\mathrm{TM}} \leq_{\mathrm{m}} ALL_{\mathrm{TM}}$, since EQ_{TM} is not Turing-recognizable and not co-Turing-recognizable.

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  if M_1 accepts w within t steps
    run M_2 on w // for unlimited steps
    if M_2 accepts w, accept x
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M:  On input x:
    if x ∈ 0*
        accept x
    else
        x = 0^t 1w for some w ∈ Σ* and t ≥ 0
        run M1 on w for up to t steps
        run M2 on w for up to t steps
        if M1 accepts w within t steps
            run M2 on w // for unlimited steps
            if M2 accepts w, accept x
            if M2 rejects w, reject x
        else if M2 accepts w within t steps
            run M1 on w // for unlimited steps
            if M1 accepts w, accept x
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We claim that:

$$L(M_1) = L(M_2) \quad \Rightarrow \quad L(M) = \Sigma^*,$$
$$L(M_1) \neq L(M_2) \quad \Rightarrow \quad L(M) \neq \Sigma^*.$$

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In all cases, M accepts x. Therefore $L(M) = \Sigma^*$.

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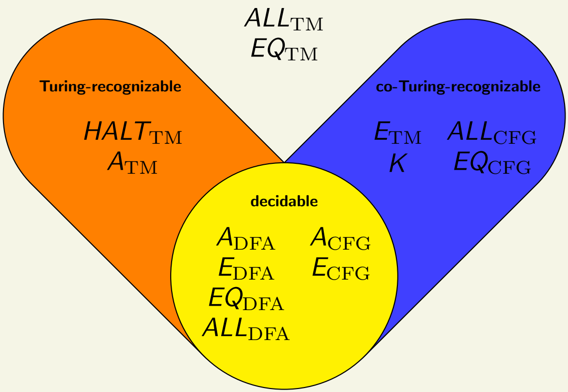
Analogoulsy, if $w \in L(M_2) - L(M_1)$, then $L(M) \neq \emptyset$.

We have shown

$$\begin{aligned} L(M_1) = L(M_2) &\quad\Rightarrow\quad L(M) = \Sigma^*, \\ L(M_1) \neq L(M_2) &\quad\Rightarrow\quad L(M) \neq \Sigma^*. \end{aligned}$$

Since the function $f(\langle M_1, M_2 \rangle) = \langle M \rangle$ is computable, this shows $EQ_{\mathrm{TM}} \leq_{\mathrm{m}} ALL_{\mathrm{TM}}$. Therefore ALL_{TM} is neither Turing-recognizable nor co-Turing-recognizable. $\square$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EQ_{TM}
Turing-recognizable
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K
ALL_{CFG}
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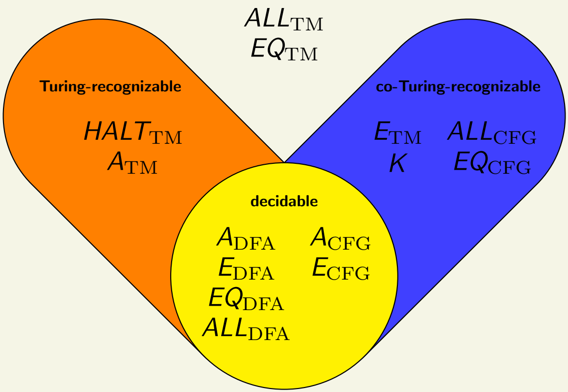

	DFA	CFG	TM
A (acceptance)	decidable	decidable	Turing-recognizable not co-Turing-recognizable
E (emptiness)	decidable	decidable	co-Turing-recognizable not Turing-recognizable
ALL (all strings)	decidable	co-Turing-recognizable not Turing-recognizable	not Turing-recognizable not co-Turing-recognizable
EQ (equivalence)	decidable	co-Turing-recognizable not Turing-recognizable	not Turing-recognizable not co-Turing-recognizable

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Then P is undecidable.

Proof. Let M_{empty} be a TM with $L(M_{\text{empty}}) = \emptyset$. We distinguish two cases.

Case 1: $\langle M_{\text{empty}} \rangle \notin P$. In this case, we show that $A_{\text{TM}} \leq_{\text{m}} P$.

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Case 1: $\langle M_{\text{empty}} \rangle \notin P$. In this case, we show that $A_{\text{TM}} \leq_{\text{m}} P$. Let N be a TM with $\langle N \rangle \in P$. Such an N exists because P is nontrival. Then since $\langle N \rangle \in P$ and $\langle M_{\text{empty}} \rangle \notin P$, we know that $L(N) \neq L(M_{\text{empty}})$, so $L(N) \neq \emptyset$.

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For any TM M and input w, define a new TM $S_{M,w}$ as follows.

$S_{M,w}$: On input x:

1. Simulate M on input w.
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Our reduction f is defined as

$$f(\langle M, w \rangle) = \langle S_{M,w} \rangle.$$

We have

$$\begin{aligned}
\langle M, w\rangle \in A_{\mathrm{TM}} &\Rightarrow L(S_{M,w}) = L(N) \\
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so $A_{\mathrm{TM}} \leq_{\mathrm{m}} P$ via f. Therefore P is undecidable.

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Since $\langle M_{\text{empty}} \rangle \in P$, $\langle M_{\text{empty}} \rangle \notin Q$. Then Case 1 applies to Q and we have $A_{\text{TM}} \leq_{\text{m}} Q$. Therefore Q is undecidable. $\square$

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2. For the second condition, let M_1 accept $\emptyset$ and M_2 accept Σ^*. Then $\langle M_1 \rangle \in E_{\mathrm{TM}}$ and $\langle M_2 \rangle \notin E_{\mathrm{TM}}$.

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