Computability and Complexity
COSC 4200

Undecidability

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Some decision problems do not have an algorithmic solutions. Such problems are called *undecidable*. To prove that undecidable problems exist, we will use the technique of *diagonalization*.

First we review the original use of the technique to prove the uncountability of the real numbers.

Countable Sets

The natural numbers is the set $\mathbb{N} = \{0, 1, 2, \ldots\}$.
A function is a bijection if it is both one-to-one and onto.

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A set X is *countable* if there is a bijection $f : \mathbb{N} \to X$.

Then $f(0)$, $f(1)$, $f(2)$,. . . , is a listing of the elements of X.

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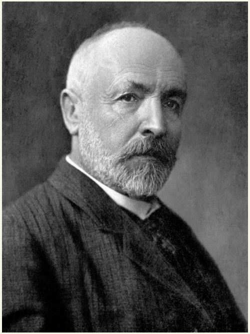
- $L(M)$ for any Turing machine M
- $\{M \mid M \text{ is a Turing machine}\}$

The Real Numbers are Uncountable

Theorem (Cantor, 1874)

The set of real numbers $\mathbb{R}$ is uncountable.



Georg Cantor (1845-1918)

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We define a new number x by $x = 0.e_0 e_1 e_2 e_3 \ldots$, where

$$e_i = \begin{cases} 0 & \text{if } d_i^i \neq 0 \\ 1 & \text{if } d_i^i = 0 \end{cases}$$

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Then for all i, $e_i \neq d_i^i$. Therefore $x \neq f(i)$ for all i, so f is not onto. $\square$

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If M does not halt on w, then U will not halt. This is why U does not decide A_{TM}.

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$$H(\langle M, w\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts } w \\ \text{reject} & \text{if } M \text{ does not accept } w. \end{cases}$$

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Algorithm D: On input $\langle M\rangle$, where M is a TM:

1. Run H on input $\langle M, \langle M\rangle\rangle$.
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For any M,

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Whatever D does, this says that D does the opposite, a contradiction. Therefore neither D or H can exist, so A_{TM} is undecidable. $\square$

To make the diagonalization argument more explicit, we define the *diagonal halting problem*

$$K = \{\langle M \rangle \mid M \text{ does not accept } \langle M \rangle\}.$$

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In either case, M does not decide K correctly on input $\langle M \rangle$. Therefore M does not decide K. $\square$

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Then D decides K:

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But K is undecidable, a contradiction. Therefore A_{TM} is undecidable. $\square$

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If A is co-Turing-recognizable, then A^c is Turing-recognizable. This means there is a Turing machine M such that for all inputs w,

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Equivalently,

- $w \notin A \Rightarrow M$ accepts w.
- $w \in A \Rightarrow M$ does not accept w.

A is *decidable* if there is a TM M such that for all w,

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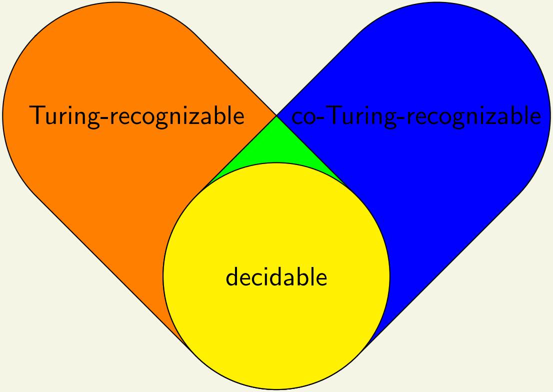

The theorem says the green region is empty.

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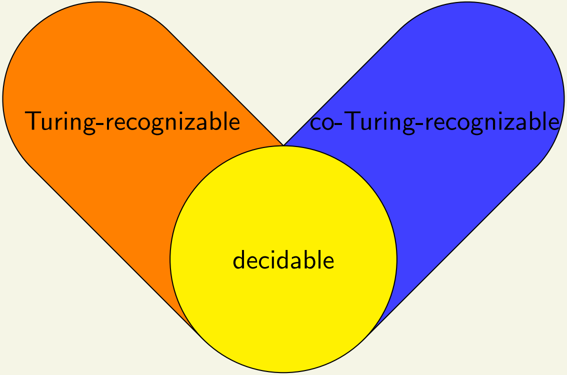

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Proof. Let A be a language.

$(\Rightarrow)$ If A is decidable, then A^c is also decidable.

Since every decidable language is Turing-recognizable, both A and A^c are Turing-recognizable.

($\Leftarrow$) Suppose that A and A^c are Turing-recognizable. Let M_1 be a recognizer for A and let M_2 be a recognizer for A^c.

Then

- $w \in A \Rightarrow M_1$ accepts w.
- $w \notin A \Rightarrow M_1$ does not accept w.

and

- $w \in A \Rightarrow M_2$ does not accept w.
- $w \notin A \Rightarrow M_2$ accepts w.

When M_1 or M_2 does not accept, they may run forever.

Let M be the following TM:

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Running in "parallel" means using two tapes (which can be simulated by a one-tape TM).

- Copy the input from the first tape to the second tape. Move both tape heads to the beginning.
- Run M_1 on the first tape and M_2 on the second tape simultaneously.
- Accept if M_1 accepts. Reject if M_2 accepts.

We have

$$\begin{aligned} w \in A &\Rightarrow M_1 \text{ accepts } w \\ &\Rightarrow M \text{ accepts } w \end{aligned}$$

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$$\begin{aligned} w \notin A &\Rightarrow M_2 \text{ accepts } w \\ &\Rightarrow M \text{ rejects } w. \end{aligned}$$

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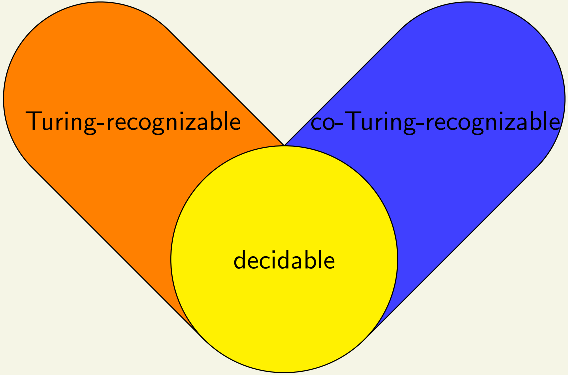
Then M decides A, so A is decidable. $\square$



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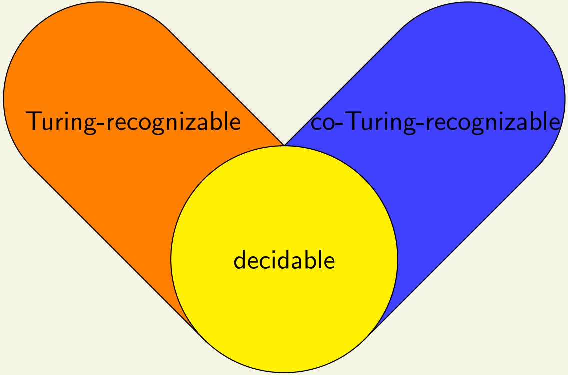

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A language A is decidable if and only if A is Turing-recognizable and co-Turing-recognizable.

Corollary

1. *If A is Turing-recognizable but not decidable, then A is not co-Turing-recognizable.*
2. *If A is co-Turing-recognizable but not decidable, then A is not Turing-recognizable.*

Corollary

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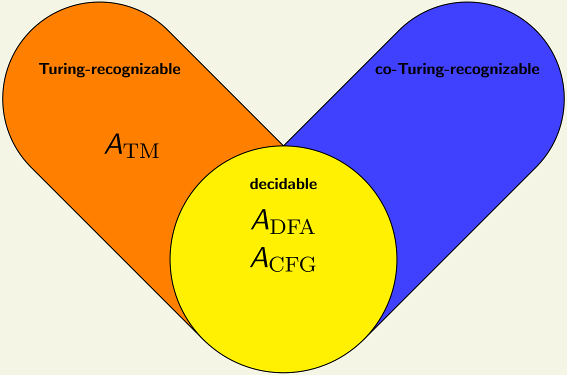
Proof.

We know that A_{TM} is Turing-recognizable. If A_{TM} is also co-Turing-recognizable, then A_{TM} is decidable, but it is not. $\square$


Turing-recognizable
co-Turing-recognizable
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To see that K is co-Turing-recognizable, consider the following TM R:

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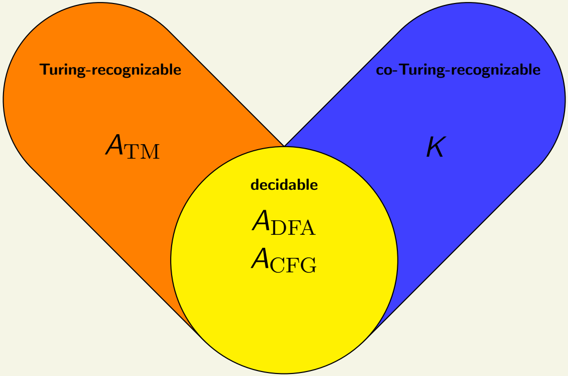
Then R recognizes K^c:

- If $\langle M \rangle \in K$, then M does not accept $\langle M \rangle$, so R will not accept $\langle M \rangle$.
- If $\langle M \rangle \notin K$, then M accepts $\langle M \rangle$, so R will accept $\langle M \rangle$. $\square$


Turing-recognizable
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The *halting problem* for TMs:

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If M does not halt on w, then U will not halt. This is why U does not decide $HALT_{\mathrm{TM}}$.

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1. Run TM R on input $\langle M, w \rangle$.
2. If R rejects, reject.
3. If R accepts, simulate M on w until it halts.
   - If M accepts, accept.
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We now verify that S decides A_{TM}. First suppose $\langle M, w \rangle \in A_{\mathrm{TM}}$:

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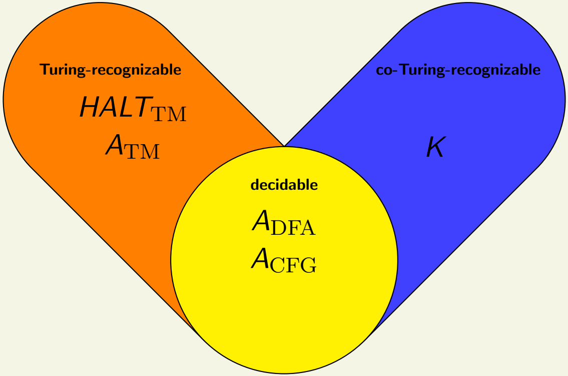
In either case, S rejects $\langle M, w \rangle$.

Since A_{TM} is undecidable, decider S does not exist.

Therefore decider R for $HALT_{\mathrm{TM}}$ does not exist and $HALT_{\mathrm{TM}}$ must be undecidable. $\square$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- If M accepts w, then $L(M_{(w)}) \neq \emptyset$.
- If M does not accept w, then $L(M_{(w)}) = \emptyset$.

Let M be a TM and w be an input for M. Here is the description of $M_{(w)}$. Note that w is hardcoded into $M_{(w)}$.

$M_{(w)}$: on any input x:

1. If $x \neq w$, reject.
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Then if M accepts w, $L(M_{(w)}) = \{w\}$.

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Therefore S decides A_{TM}, a contradiction, so E_{TM} is undecidable. $\square$

We just proved E_{TM} is undecidable. Is it Turing-recognizable?

Theorem

E_{TM} is co-Turing-recognizable.

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Proof. Let $s_1, s_2, \ldots$ be an enumeration of all strings in Σ^*.

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for $i = 1, 2, \ldots$
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Therefore E^c_{TM} is Turing-recognizable. $\square$

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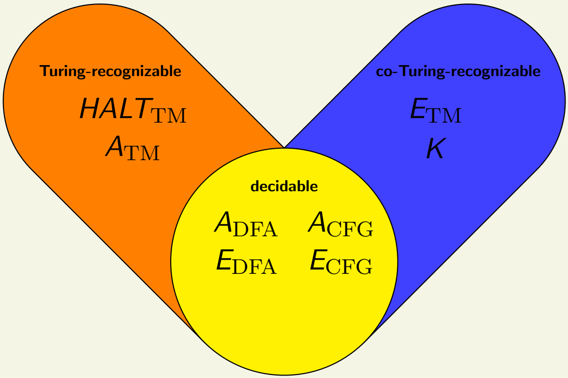
Corollary

E_{TM} is not Turing-recognizable.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