Computability and Complexity
COSC 4200

NP-Complete Problems

Polynomial-Time Reductions

Definition

We say that A is *polynomial-time mapping reducible* to B, and write $A \leq_{\mathrm{P}} B$, if there is a polynomial-time computable function $f : \Sigma^* \to \Sigma^*$ such that for all $w \in \Sigma^*$,

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Two implications:

$$\begin{aligned} w \in A &\Rightarrow f(w) \in B \\ f(w) \in B &\Rightarrow w \in A \end{aligned}$$

Or equivalently:

$$\begin{aligned} w \in A &\Rightarrow f(w) \in B \\ w \notin A &\Rightarrow f(w) \notin B \end{aligned}$$

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A problem B is *NP-complete* if

1. B is in NP, and
2. every A in NP is $\leq_{\mathrm{P}}$-reducible to B.

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Proof. 3 $\Rightarrow$ 2 and 2 $\Rightarrow$ 1 are trivial.

To see 1 $\Rightarrow$ 3, assume 1 is true and let B be an NP-complete problem with $B \in \mathrm{P}$.

- Let $A \in \mathrm{NP}$ be arbitrary.
- Then $A \leq_{\mathrm{P}} B$ because B is NP-complete.
- Because $B \in \mathrm{P}$, we have $A \in \mathrm{P}$ as well.

Therefore $\mathrm{P} = \mathrm{NP}$. $\square$

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2. *Every* NP*-complete problem is in* P.
3. $\mathrm{P} = \mathrm{NP}$.

Corollary

The following are equivalent:

1. *Some* NP*-complete problem is not in* P.
2. *Every* NP*-complete problem is not in* P.
3. $\mathrm{P} \neq \mathrm{NP}$.

Satisfiability

A *propositional formula* is a sentence made from variables, operators $\wedge$, $\vee$, and $\neg$, and parentheses. An example is

$$\phi = (x_1 \vee (x_2 \wedge x_3)) \wedge \neg(x_3 \wedge x_4).$$

A *3CNF formula* is a propositional formula consisting of the conjunction of a number of disjunctive clauses with at most 3 literals. An example of this is

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (x_2 \vee x_5) \wedge (\neg x_1 \vee x_4 \vee x_5)$$

Satisfiability

An *assignment* for a formula ϕ is a true/false setting of ϕ's variables.

An assignment *satisfies* ϕ if it makes ϕ evaluate to true in the standard way.

A formula ϕ is *satisfiable* if it has a satisfying assignment.

We define two satisfiability decision problems:

$$\mathrm{SAT} = \{\phi \,|\, \phi \text{ is a satisfiable formula}\}$$

$$\mathrm{3SAT} = \{\phi \,|\, \phi \text{ is a satisfiable 3CNF formula}\}$$

Cook-Levin Theorem (1971, 1973)

$\mathrm{3SAT}$ is NP-complete.

This requires showing that for every $A \in \mathrm{NP}$, $A \leq_{\mathrm{P}} \mathrm{3SAT}$.

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- There is a polynomial-time nondeterministic Turing machine for A.
- The reduction from A to $\mathrm{3SAT}$ expresses the computation of the nondeterministic Turing machine as a formula.
- We will first show $A \leq_{\mathrm{P}} \mathrm{SAT}$ and then we'll modify our proof to get $A \leq_{\mathrm{P}} \mathrm{3SAT}$.

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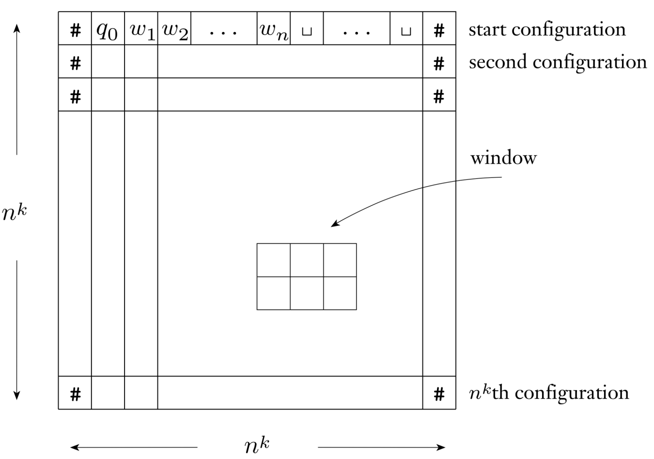
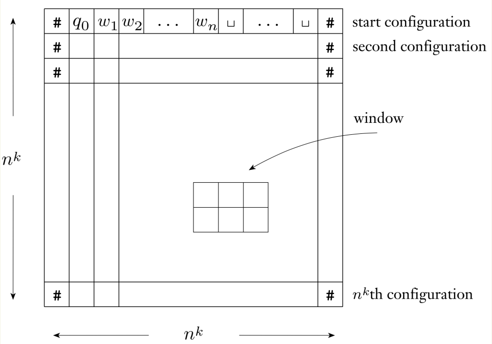
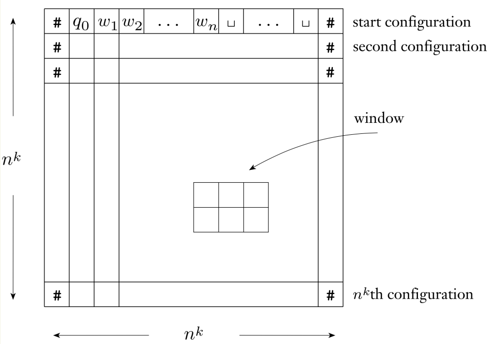
A *tableau* for N on w is an $n^k \times n^k$ table whose rows are the configurations of a branch of the computation of N on input w.





- For convenience later, we assume each configuration starts and ends with $\#$.



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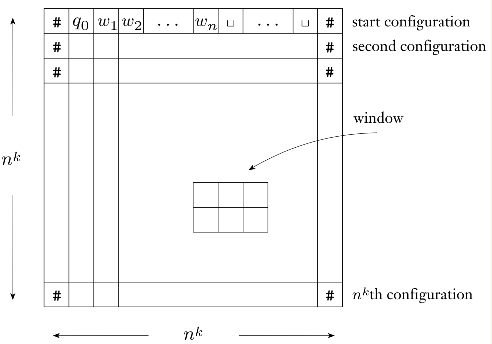
- The first row of the tableau is that start configuation of N on w.



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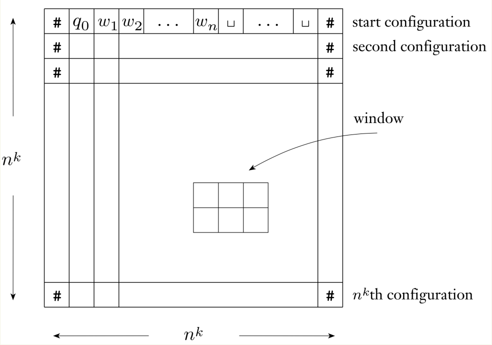
- The first row of the tableau is that start configuation of N on w.
- Each row follows the previous one according to N's transition function.



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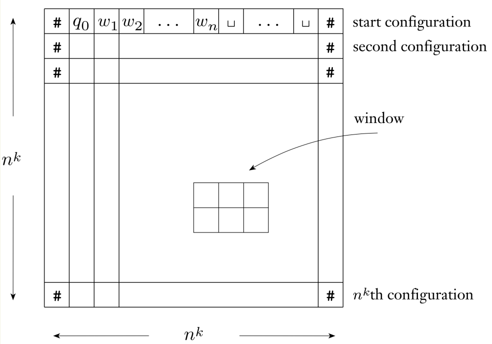
- A tableau is *accepting* if any row is an accepting configuration.



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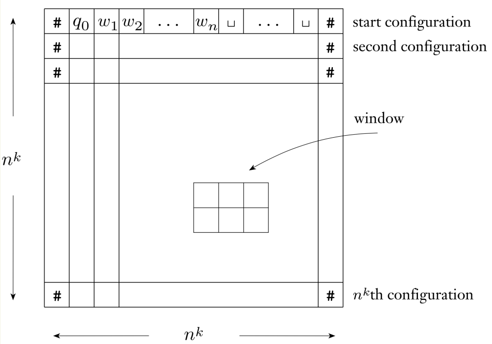
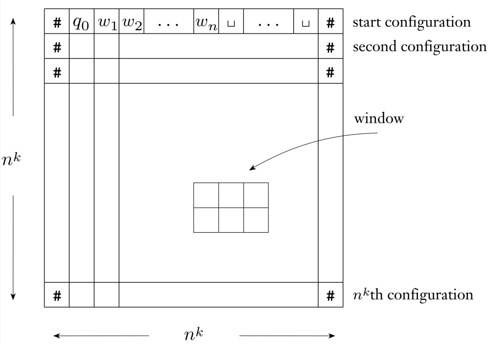
- A tableau is *accepting* if any row is an accepting configuration.
- Every accepting computation path of N corresponds to an accepting tableau.



Our reduction f will take an input w and produce a formula ϕ.

- Let $C = Q \cup \Gamma \cup \{\#\}$, where Q is the set of states and Γ is the tape alphabet.



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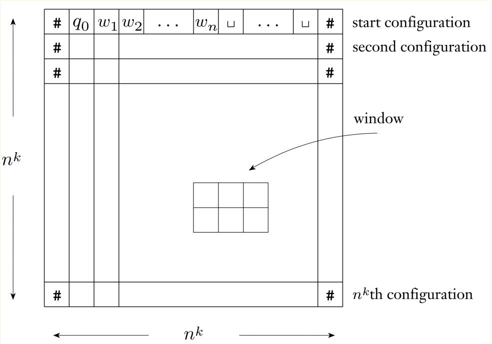

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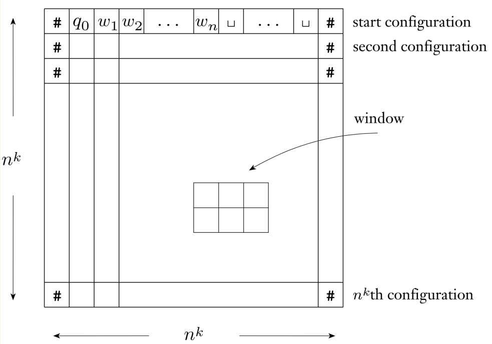
- Each of the $(n^k)^2$ entries of a tableau is called a *cell*.



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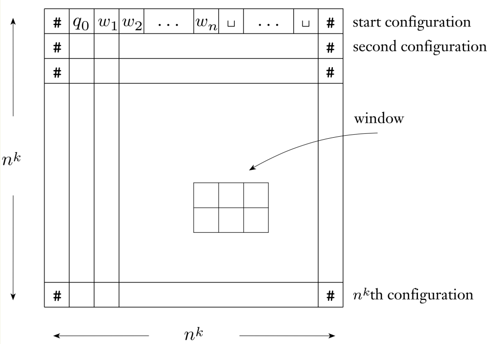
- *cell*$[i, j]$ is the cell in row i and column j.



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- Each of the $(n^k)^2$ entries of a tableau is called a *cell*.
- $cell[i,j]$ is the cell in row i and column j.
- If $x_{i,j,s} = 1$, it means $cell[i,j]$ contains an s.

Our formula ϕ will be the conjunction of four parts:

$$\phi = \phi_{\text{cell}} \wedge \phi_{\text{start}} \wedge \phi_{\text{move}} \wedge \phi_{\text{accept}}$$

We will define these four parts separately.

ϕ_{cell}

Recall the variable $x_{i,j,s}$ corresponds to placing an s in $cell[i,j]$.

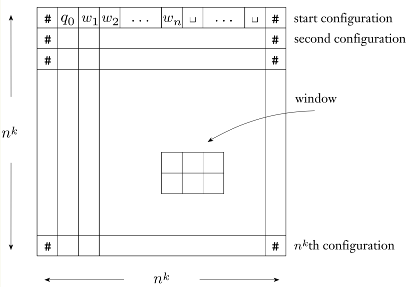
The formula ϕ_{cell} ensures that exactly one variable is turned on for each cell.

$$\phi_{\text{cell}} = \bigwedge_{1 \le i,j \le n^k} \left[\left(\bigvee_{s \in C} x_{i,j,s} \right) \wedge \left(\bigwedge_{\substack{s,t \in C \\ s \neq t}} (\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}}) \right) \right]$$

The first part ensures that some variable is turned on. The second part ensures that only one variable is turned on.

Note that ϕ_{cell} has size $O(n^{2k})$ and may be computed in time $O(n^{2k})$.

ϕ_{start}

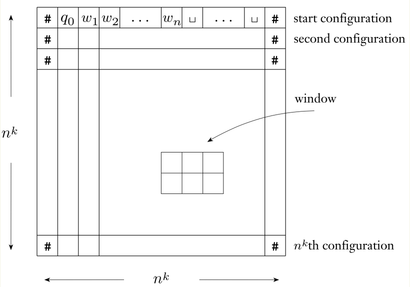


The formula ϕ_{start} ensures the first row of the tableau is the start configuration of N on w.

$$
\begin{aligned}
\phi_{\mathrm{start}} \quad = \quad & x_{1,1,\#} \wedge x_{1,2,q_0} \wedge \\
& x_{1,3,w_1} \wedge x_{1,4,w_2} \wedge \cdots \wedge x_{1,n+2,w_n} \wedge \\
& x_{1,n+3,\sqcup} \wedge \cdots x_{1,n^k-1,\sqcup} \wedge x_{1,n^k,\#}
\end{aligned}
$$

Note that ϕ_{start} has size $O(n^k)$ and may be computed in time $O(n^k)$.

ϕ_{accept}

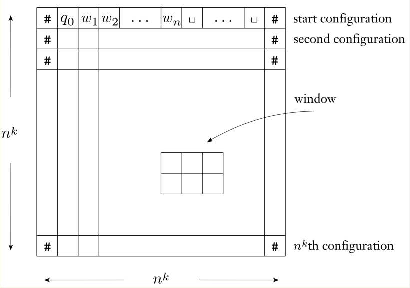


The formula ϕ_{accept} ensures that an accepting computation appears in the tableau. The accept state q_{accept} must appear in the tableau.

$$\phi_{\mathrm{accept}} = \bigvee_{1 \le i,j \le n^k} x_{i,j,q_{\mathrm{accept}}}$$

Note that ϕ_{accept} has size $O(n^{2k})$ and may be computed in time $O(n^{2k})$.

ϕ_{move}



The formula ϕ_{move} ensures that each row of tableau is a configuration that follows the configuration in the previous row by one move of N's transition function. The formula ensures that each 2×3 window of cells is legal.

ϕ_{move}

Suppose $\delta(q_1, a) = \{(q_1, b, R)\}$ and $\delta(q_1, b) = \{(q_2, c, L), (q_2, a, R)\}$.

(a)

a	q_1	b
q_2	a	c

(b)

a	q_1	b
a	a	q_2

(c)

a	a	q_1
a	a	b

(d)

#	b	a
#	b	a

(e)

a	b	a
a	b	q_2

(f)

b	b	b
c	b	b

FIGURE 7.39
Examples of legal windows

(a)

a	b	a
a	a	a

(b)

a	q_1	b
q_2	a	a

(c)

b	q_1	b
q_2	b	q_2

FIGURE 7.40
Examples of illegal windows

ϕ_{move}

The (i,j)-window has $cell[i,j]$ in the upper central position. Let the six cells of the window be $a_1,\ldots,a_6$. Then we define

$$\psi_{i,j} = \bigvee_{\substack{a_1,\ldots,a_6 \\ \text{is a legal window}}} \left(\begin{array}{l} x_{i,j-1,a_1} \wedge x_{i,j,a_2} \wedge x_{i,j+1,a_3} \wedge \\ x_{i+1,j-1,a_1} \wedge x_{i+1,j,a_2} \wedge x_{i+1,j+1,a_3} \end{array} \right)$$

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Finally, we define

$$\phi_{\mathrm{move}} = \bigwedge_{\substack{1 \le i \le n^k \\ 1 < j < n^k}} \psi_{i,j}.$$

Note that ϕ_{move} has size $O(n^{2k})$ and may be computed in time $O(n^{2k})$.

Our reduction f outputs the formula

$$\phi = \phi_{\text{cell}} \wedge \phi_{\text{start}} \wedge \phi_{\text{move}} \wedge \phi_{\text{accept}}.$$

Each of the four parts of ϕ is computable in $O(n^{2k})$ time, so ϕ is computable in $O(n^{2k})$ time. Thus f is polynomial-time computable.

We have

$$\begin{aligned} w \in A \quad &\iff \quad N \text{ accepts } w \\ &\iff \quad \text{there is an accepting computation history of } N \text{ on } w \\ &\iff \quad \text{there is an accepting tableau for } N \text{ on } w \\ &\iff \quad \phi \text{ is satisfiable.} \end{aligned}$$

Therefore $A \leq_{\mathrm{m}} \mathrm{SAT}$ via f.

We now modify the reduction to get $A \leq_{\mathrm{m}} 3\mathrm{SAT}$. In

$$\phi = \phi_{\mathrm{cell}} \wedge \phi_{\mathrm{start}} \wedge \phi_{\mathrm{move}} \wedge \phi_{\mathrm{accept}},$$

the formulas ϕ_{cell}, ϕ_{start}, and ϕ_{accept} are in CNF. We can use the distributive law to put ϕ_{move} into CNF form as well.

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To get 3CNF formulas we use additional variables to split each clause with more than 3 literals into multiple clauses that have 3 literals. For example, the clause $(a_1 \vee a_2 \vee a_3 \vee a_4)$ is replaced by $(a_1 \vee a_2 \vee z) \wedge (\overline{z} \vee a_3 \vee a_4)$, where z is a new variable.

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More generally, a clause

$$(a_1 \vee a_2 \vee \ldots \vee a_l)$$

with l literals is replaced by $l - 2$ clauses:

$$(a_1 \vee a_2 \vee z_1) \wedge (\overline{z_1} \vee a_3 \vee z_2) \wedge (\overline{z_2} \vee a_2 \vee z_3) \wedge \cdots \wedge (\overline{z_{l-3}} \vee a_{l-1} \vee a_l)$$

where the z_i's are new variables. $\Box$

We have shown that 3SAT is NP-complete. We can use the following lemma to build on this to show more problems are NP-complete.

Lemma

$\leq_{\mathrm{P}}$ is transitive. That is, if $A \leq_{\mathrm{P}} B$ and $B \leq_{\mathrm{P}} C$, then $A \leq_{\mathrm{P}} C$.

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Proof. Let f be the reduction from A to B and let g be the reduction from B to C. Then

- $I \in A \Leftrightarrow f(I) \in B$ for all instances I,
- $J \in B \Leftrightarrow g(J) \in C$ for all instances J,
- f is computable in some polynomial $p(n)$ time, and
- g is computable in some polynomial $q(n)$ time.

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Let $n = |I|$. To compute $h(I)$ we need at most

- $p(n)$ time to compute $f(I)$.
- $q(p(n))$ time to compute $g(f(I))$ because $|f(I)| \le p(n)$.

The total time $p(n) + q(p(n))$ is polynomial.

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The total time $p(n) + q(p(n))$ is polynomial.

Therefore $A \leq_{\mathrm{P}} C$ via h. $\square$

Corollary

Let $A, B \in \mathrm{NP}$*. If* A *is* NP*-complete and* $A \leq_{\mathrm{P}} B$*, then* B *is also* NP*-complete.*

Corollary

Let $A, B \in \mathrm{NP}$. If A is NP-complete and $A \leq_{\mathrm{P}} B$, then B is also NP-complete.

Proof. Because A is NP-complete, $C \leq_{\mathrm{P}} A$ for every $C \in \mathrm{NP}$. By transitivity and $A \leq_{\mathrm{P}} B$, we have $C \leq_{\mathrm{P}} B$ for every $B \in \mathrm{NP}$. $\square$

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To prove that a problem $B \in \mathrm{NP}$ is NP-complete, we don't have to reduce every problem in NP to it. We may select any NP-complete problem A and show that $A \leq_{\mathrm{P}} B$.

NP-Completeness Proofs

Here is the method of showing a problem B is NP-complete.

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   Then we need to prove that for every instance I of A,

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   This involves showing two implications: $I \in A \implies I' \in B$ and $I' \in B \implies I \in A$.

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   $$I \in A \iff I' \in B.$$

   This involves showing two implications: $I \in A \implies I' \in B$ and $I' \in B \implies I \in A$.

   - To show that $I \in A \implies I' \in B$, assume that $I \in A$. Then there is some witness w for I that shows I is a member of A. We use this witness to construct a witness w' that shows $I' \in B$.

3. Reduce A to B.

   For this, we need to define a mapping of instances I of A to instances I' of B.

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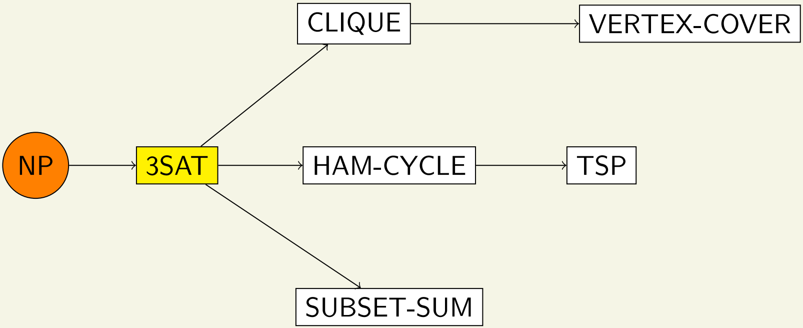
   Finally, we have to verify that the mapping is polynomial-time computable. Usually this doesn't require formal analysis of the algorithm.

Plan for NP-completeness reductions:



Clique

In a graph $G = (V, E)$, a *clique* is a fully interconnected subset of the vertices. That is, a clique is a set $V' \subseteq V$ such that for all $u, v \in V'$, $(u, v) \in E$. The clique decision problem is

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Notice that we can solve this with a brute force search through all subgraphs of size k, which looks like polynomial time $O(n^k)$, but when $k = \Omega(n)$, say $n/3$, the time required is exponential.

However, it is easy to see that $\mathrm{CLIQUE} \in \mathrm{NP}$. The clique itself is a witness. So $\langle G, k\rangle \in \mathrm{CLIQUE} \iff (\exists V')|V'| = k$ and V' is a clique in G.

Theorem

CLIQUE *is* NP*-complete.*

Proof. We will show $\mathrm{3SAT} \leq_{\mathrm{P}} \mathrm{CLIQUE}$. Let $\phi = C_1 \wedge \ldots \wedge C_k$ be a 3CNF formula, where each C_i is a clause with at most 3 literals. We construct a graph G as follows.

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For each clause $C_i = \left(l_1^{(i)} \vee l_2^{(i)} \vee l_3^{(i)}\right)$ we put vertices $v_1^{(i)}$, $v_2^{(i)}$, $v_3^{(i)}$ in G. (If C_i has 1 or 2 literals, we put 1 or 2 vertices in G.)

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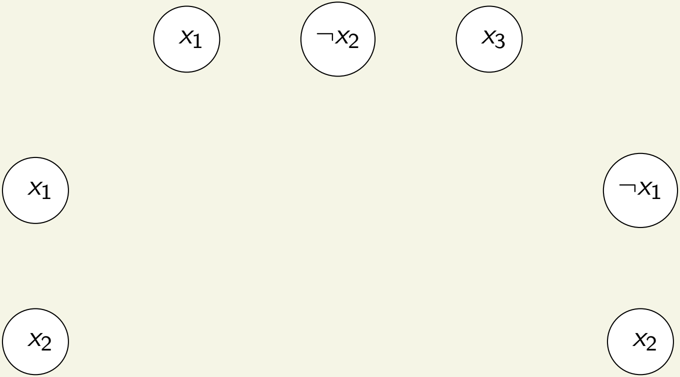
1. $i \neq j$. In other words, $v_a^{(i)}$ and $v_b^{(j)}$ correspond to literals in different clauses.
2. $l_a^{(i)} \neq \neg l_b^{(j)}$. In other words, the corresponding literals are consistent.

For example, consider the formula

$$\phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2) \wedge (\neg x_1 \vee x_2).$$

The graph G appears below. The three vertices at the top correspond to the first clause, the two at the left are for the second clause, and the two at the right are for the third clause.



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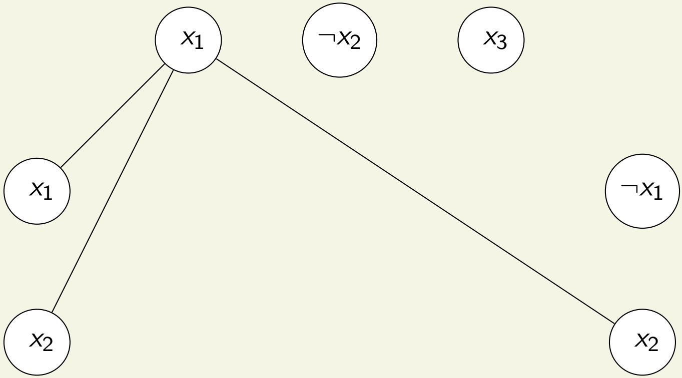

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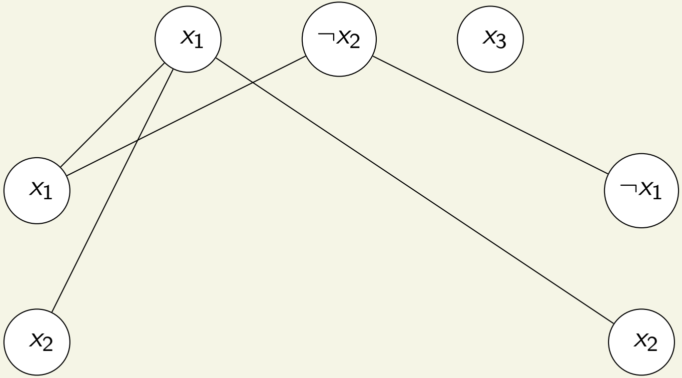

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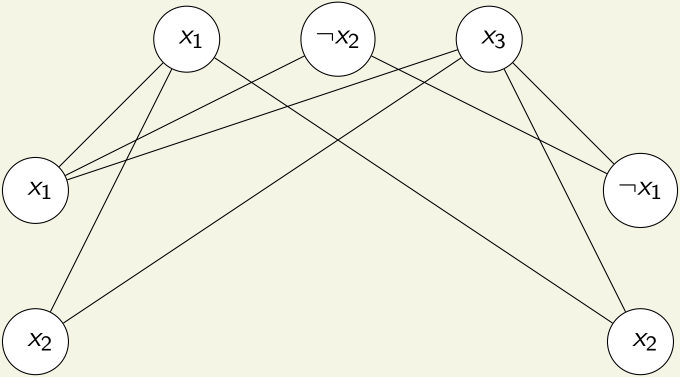

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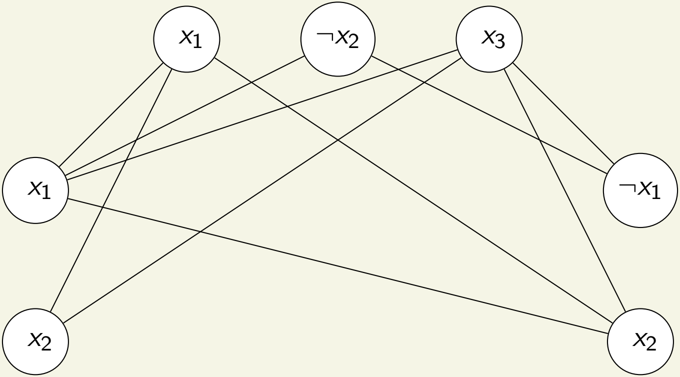

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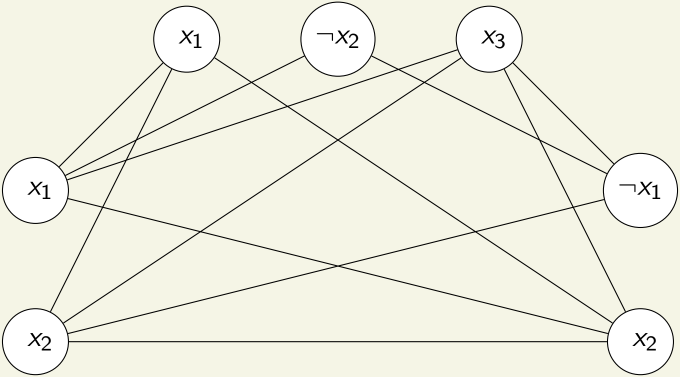

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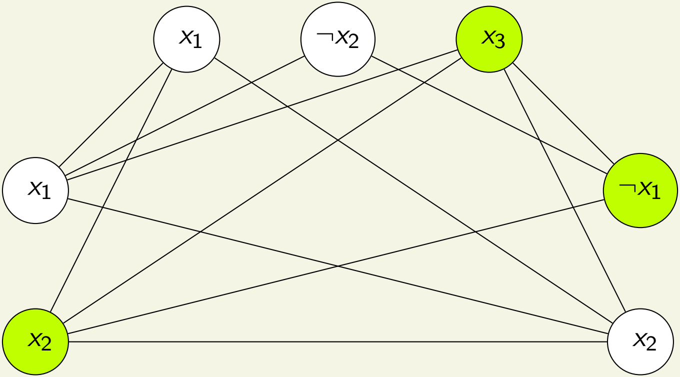

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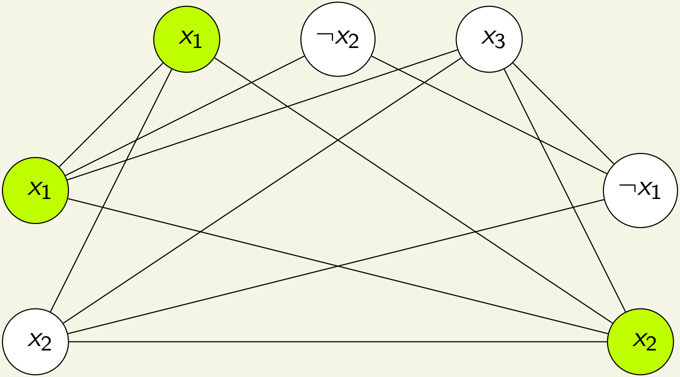

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This shows $\phi \in \mathrm{3SAT} \Rightarrow \langle G, k \rangle \in \mathrm{CLIQUE}$.

Conversely, suppose that G has a clique of size k. Let V' be this clique.

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From V' we obtain a satisfying assignment for ϕ by setting the variables to make each of the corresponding literals true.

- We can do this because there is no edge between a literal and its negation, thus we won't ever try to set a variable to both true and false.
- If a variable does not correspond to a vertex in V', then we can set it arbitrarily.

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This assignment satisfies at least one literal in each clause of ϕ. Therefore, $\langle G, k\rangle \in \mathrm{CLIQUE} \Rightarrow \phi \in \mathrm{3SAT}$.

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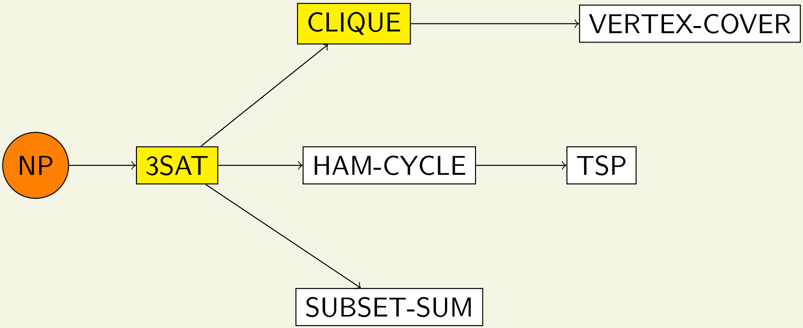
Since the mapping $\phi \mapsto \langle G, k \rangle$ can be computed in polynomial time, we have that $\mathrm{3SAT} \leq_{\mathrm{P}} \mathrm{CLIQUE}$. $\square$

Plan for NP-completeness reductions:



Vertex Cover

A *vertex cover* of a graph $G = (V, E)$ is a subset $V' \subseteq V$ such that if $(u, v) \in E$, then $u \in V'$ or $v \in V'$ (or both). In other words, a vertex cover is a subset of the vertices that touches all edges in the graph. The decision problem is

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Note that V, the set of all vertices in the graph, is always a vertex cover. Also, if V' is a vertex cover and $V'' \supseteq V'$, then V'' is also a vertex cover. The goal is to determine if small vertex covers exist.

Theorem

$\mathrm{VERTEX\text{-}COVER}$ *is* NP*-complete.*

Proof. To see that $\mathrm{VERTEX\text{-}COVER} \in \mathrm{NP}$, note that $\langle G, k\rangle \in \mathrm{VERTEX\text{-}COVER} \iff (\exists V')|V'| \leq k$ and V' is a vertex cover for G. The set V' is the witness and all we have to do is check if V' covers all the edges, which can be done in polynomial time.

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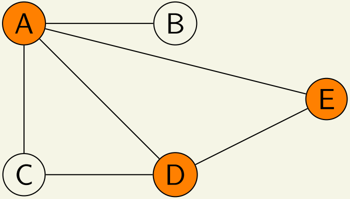
To show completeness, we will reduce CLIQUE to $\mathrm{VERTEX\text{-}COVER}$. Given a graph $G = (V, E)$, we define the complementary graph $\bar{G} = (V, \bar{E})$, where

$$\bar{E} = \{(u, v) \in V \times V \,|(u, v) \notin E \text{ and } u \neq v\}\,.$$

We claim that G has a clique of size k if and only if $\bar{G}$ has a vertex cover of size $|V| - k$.

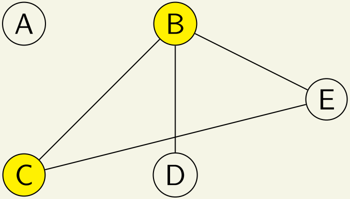
Examples

G:



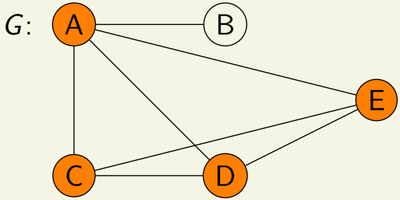
$\{A, D, E\}$ is a clique of size 3

$\overline{G}$:

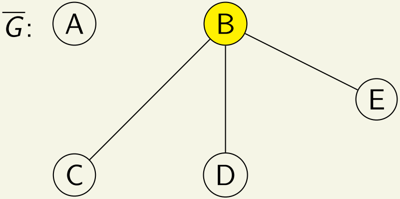


$\{B, C\}$ is a vertex cover of size 2

Examples



$\{A, C, D, E\}$ is a clique of size 4



$\{B\}$ is a vertex cover of size 1

Examples

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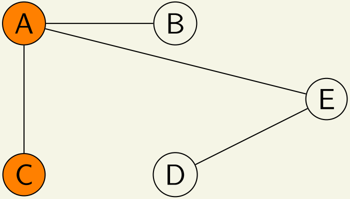

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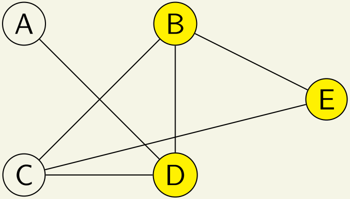

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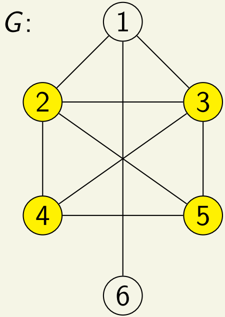
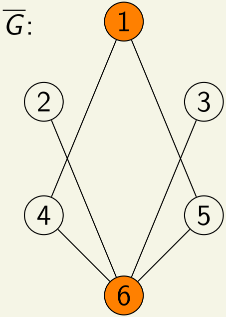


$V' = \{2, 3, 4, 5\}$
is a clique of size 4



$V - V' = \{1, 6\}$
is a vertex cover of size 2

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Since $|V - V'| = |V| - k$, this shows

$$\langle G, k\rangle \in \mathrm{CLIQUE} \Rightarrow \langle \bar{G}, |V| - k\rangle \in \text{VERTEX-COVER}.$$

Now suppose that $\bar{G}$ has a vertex cover V' with $|V'| = |V| - k$. Then for all $u, v \in V$, we have

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$$\langle \bar{G}, |V| - k \rangle \in \text{VERTEX-COVER} \Rightarrow \langle G, k \rangle \in \text{CLIQUE}.$$

We have shown that for any graph $G = (V, E)$ and number k,

$$\langle G, k \rangle \in \mathrm{CLIQUE} \iff \langle \bar{G}, |V| - k \rangle \in \mathrm{VERTEX\text{-}COVER}.$$

Since the mapping $\langle G, k \rangle \mapsto \langle \bar{G}, |V| - k \rangle$ can be computed in polynomial time, we have $\mathrm{CLIQUE} \leq_{\mathrm{P}} \mathrm{VERTEX\text{-}COVER}$. $\square$

Plan for NP-completeness reductions: