Computability and Complexity
COSC 4200

NP vs. coNP

The Tautology Problem

A propositional formula ϕ is a *tautology* if every assignment to its variables is a satisfying assignment. Let

$$\mathrm{TAUT} = \{\phi \,|\, \phi \text{ is a tautology}\}\,.$$

In our framework of complexity classes, where can we place this problem?

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We can easily solve this problem in polynomial space–by looping over all assignments and checking whether all of them satisfy ϕ. Thus $\mathrm{TAUT} \in \mathrm{PSPACE}$.

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On the other hand, there are witnesses that formulas are *not* tautologies – an assignment that makes a formula ϕ evaluate to false proves that $\phi \notin \mathrm{TAUT}$.

$$\begin{aligned} \phi \notin \mathrm{TAUT} \quad &\iff \quad \phi \text{ is not a tautology} \\ &\iff \quad (\exists \tau)\ \tau \text{ does not satisfy } \phi. \end{aligned}$$

co-classes

For a language $A \subseteq \{0,1\}^*$, the *complement* of A is

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Definition

For a class $\mathcal{C}$ of languages, the *co-class* of $\mathcal{C}$ is

$$\mathrm{co}\mathcal{C} = \{A^c \,|A \in \mathcal{C}\}.$$

In other words, $\mathrm{co}\mathcal{C}$ is the class of all languages whose complements are in $\mathcal{C}$. Note that $\mathrm{co}\mathcal{C}$ is *not* the same as the complement $\mathcal{C}^c$.

From our discussion, we have $\mathrm{TAUT} \in \mathrm{coNP}$.

Closure Under Complementation

Definition

A class $\mathcal{C}$ is *closed under complement* if $\mathcal{C} = \mathrm{co}\mathcal{C}$.

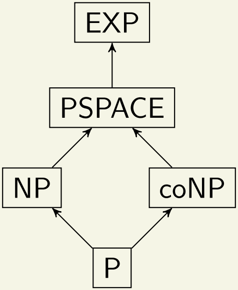
We have studied several classes that have this property, such as P, PSPACE, E, and EXP.

In fact, all deterministic time and space classes are closed under complement. This is because any deterministic algorithm for a language A can easily be modified to get an algorithm for A^c with the same time and space bounds – just flip the accepting and rejecting decisions.

The NP vs. coNP Question

Whether NP is closed under complement, that is whether $\mathrm{NP} = \mathrm{coNP}$, is a major open problem in computational complexity.

If we prove $\mathrm{NP} \neq \mathrm{coNP}$, then we also have that $\mathrm{P} \neq \mathrm{NP}$ and $\mathrm{NP} \neq \mathrm{EXP}$.



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Proof. For 1, we have

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Quantifiers and Predicates

Recall we may define NP using a polynomial-time predicate with polynomial-size witnesses and an existential quantifer:

Theorem

$A \in \mathrm{NP}$ if and only if there is a $D \in \mathrm{P}$ and a polynomial p such that for all $x \in \Sigma^*$,

$$x \in A \iff (\exists w \in \{0,1\}^{\leq p(n)})\ \langle x, w \rangle \in D.$$

For coNP, we use a universal quantifier:

Theorem

$A \in \mathrm{coNP}$ if and only if there is a $D \in \mathrm{P}$ and a polynomial p such that for all $x \in \Sigma^*$,

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Just as NP is analogous to Σ_1, coNP is analogous to Π_1.

coNP-Completeness and the Unsatisfiability Problem

Definition

A language B is coNP-complete if

1. B is in coNP, and
2. every A in coNP is polynomial-time reducible to B.

We will soon prove that TAUT is coNP-complete. To do that, we will first prove a very similar problem is coNP-complete.

A formula ϕ is a *unsatisfiable* if there is no assignment τ that satisfies ϕ. Then we define

$$\mathrm{UNSAT} = \{\phi \,|\, \phi \text{ is a unsatisfiable}\}$$

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$$x \in A^c \iff f(x) \in \mathrm{SAT}.$$

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Therefore $A \leq_{\mathrm{P}} \mathrm{UNSAT}$. Since $A \in \mathrm{coNP}$ was arbitrary, this shows that UNSAT is coNP-complete. $\square$

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Closure Under $\leq_{\mathrm{P}}$-Reductions

Definition

A complexity class $\mathcal{C}$ is closed under $\leq_{\mathrm{P}}$ reductions if $B \in \mathcal{C}$ and $A \leq_{\mathrm{P}} B$ implies $A \in \mathcal{C}$.

Lemma

P, NP, coNP, PSPACE, *and* EXP *are closed under* $\leq_{\mathrm{P}}$*-reductions.*

Proof. We already proved this for P: if $A \leq_{\mathrm{P}} B$ and $B \in \mathrm{P}$, then $A \in \mathrm{P}$. The proofs for PSPACE and EXP are similar.

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V': on input $\langle x, w \rangle$:
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so V' is a verifier for A. Therefore $A \in \mathrm{NP}$.

Now suppose $A \leq_{\mathrm{P}} B$ and $B \in \mathrm{coNP}$. Then $A^c \leq_{\mathrm{P}} B^c$ via the same reduction and $B^c \in \mathrm{NP}$, so $A^c \in \mathrm{NP}$ by the above proof. Therefore $A \in \mathrm{coNP}$. $\square$

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Therefore $\mathrm{NP} = \mathrm{coNP}$. $\square$

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The other direction is similar. $\square$

coNP-Complete Examples

For a graph G, let $\omega(G)$ denote the size of a largest clique in G. We can restate the $\textsc{Clique}$ problem as

$$\textsc{Clique} = \{\langle G, k\rangle \mid \omega(G) \geq k\}.$$

The complementary problem

$$\{\langle G, k\rangle \mid \omega(G) < k\}$$

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$$\begin{aligned}&\{\langle G, k\rangle \mid \omega(G) < k\}\\ &= \{\langle G, k\rangle \mid \text{every clique in } G \text{ has size} < k\}\end{aligned}$$

is coNP-complete.

coNP-Complete Examples

Similarly, the complementary problem of

$$\mathrm{VERTEX\text{-}COVER} = \{\langle G, k\rangle \,|\, G \text{ has a vertex cover of size } k\}$$

is coNP-complete:

$$\{\langle G, k\rangle \mid \text{every vertex cover of } G \text{ has size} > k\}.$$

coNP-Complete Examples

The complementary Hamiltonian cycle problem:

$$\{G \mid G \text{ does not have a Hamiltonian cycle}\}$$

The complementary TSP problem:

$$\left\{ \langle n, c, k \rangle \;\middle|\; \begin{array}{l} c \text{ is a cost function and every} \\ \text{tour has cost at least } k \end{array} \right\}$$

The Complexity Class $\mathrm{NP} \cap \mathrm{coNP}$

$$\begin{tikzcd} \boxed{\mathrm{NP}} & & \boxed{\mathrm{coNP}} \\ & \boxed{\mathrm{NP} \cap \mathrm{coNP}} \arrow[ul] \arrow[ur] & \\ & \boxed{\mathsf{P}} \arrow[u] & \end{tikzcd}$$

The intersection of the classes NP and coNP gives the complexity class $\mathrm{NP} \cap \mathrm{coNP}$.

- For a problem A to be in $\mathrm{NP} \cap \mathrm{coNP}$, we must have $A \in \mathrm{NP}$ and $A \in \mathrm{coNP}$.

- Equivalently, $A \in \mathrm{NP} \cap \mathrm{coNP}$ if A and A^c are both in NP. That is, there are polynomial-time verifiable witnesses for membership in A and in A^c. For any string x, there is a proof that either $x \in A$ or that $x \notin A$.

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- The computation is accepting if all outputs are in $\{1, ?\}$.
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Polynomial-time strong nondeterministic TMs accept exactly the $\mathrm{NP} \cap \mathrm{coNP}$ problems:

$$\mathrm{NP} \cap \mathrm{coNP} = \left\{ L(N) \,\middle|\, \begin{array}{l} N \text{ is a polynomial-time} \\ \text{strong nondeterministic TM} \end{array} \right\}.$$

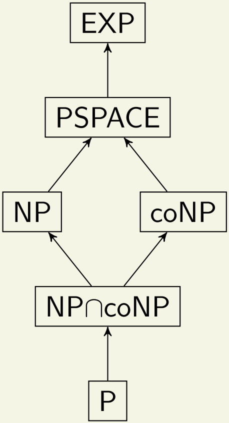
Problems in $\mathrm{NP} \cap \mathrm{coNP}$

Problems in $\mathrm{NP} \cap \mathrm{coNP}$:

- Factoring
- Discrete Logarithm
- Lattice Problems
- Parity Games
- Stochastic Games

Under a derandomization hypothesis, Graph Isomorphism and other isomorphism problems are in $\mathrm{NP} \cap \mathrm{coNP}$.

Open Problems



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