Computability and Complexity
COSC 4200

The Pumping Lemma and Nonregular Languages

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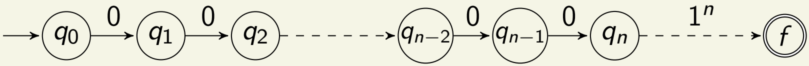

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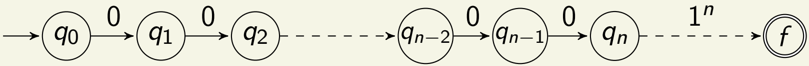

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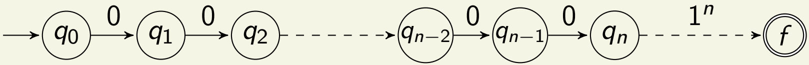
$$\longrightarrow q_0 \xrightarrow{0} q_1 \xrightarrow{0} q_2 \dashrightarrow q_{n-2} \xrightarrow{0} q_{n-1} \xrightarrow{0} q_n \overset{1^n}{\dashrightarrow} f$$

Since $n > p$, M must visit some state twice while reading the 0's.
$\longrightarrow$ For some i, j with $1 \leq i < j \leq n$, we must have $q_i = q_j$.



For some i, j with $1 \leq i < j \leq n$, we must have $q_i = q_j$.

$$\delta^*(q_0, 0^i) = q_i \qquad \delta^*(q_0, 0^j) = q_i$$
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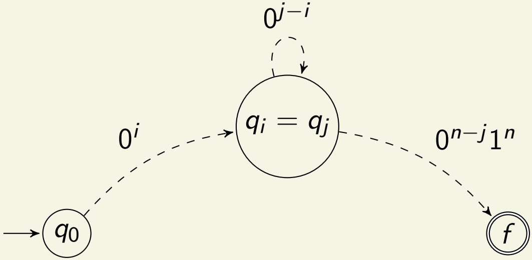

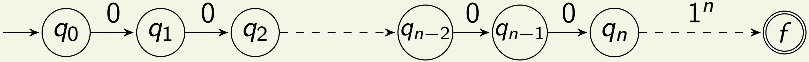

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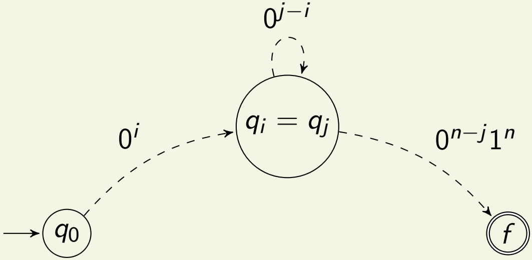

We must also have

$$0^i \left(0^{j-i}\right) \left(0^{j-i}\right) 0^{n-j}1^n = 0^{n+(j-i)}1^n$$

accepted! Therefore M does not accept A.

This technique is formalized as the "Pumping Lemma."

Pumping Lemma

If A is a regular language, then there is a number p (the pumping constant) such that for every $w \in A$ with $|w| \geq p$, w can be divided into three pieces $w = xyz$ satisfying the following conditions:

1. For all $i \geq 0$, $xy^iz \in A$.
2. $|y| > 0$.
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Comments.

- Condition 1 says that xz, xyz, $xyyz$, $xyyyz$, $\ldots$ are all in A.
- Condition 2 says that $y \neq \epsilon$. It is possible for x and z to be ϵ.
- Condition 3 says the combined length of x and y is at most p.

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Let $w \in A$ have length $|w| = n \geq p$. For any $1 \leq i \leq j \leq n$, let $w[i..j]$ be the substring of the i^{th} through j^{th} symbols of w. For each $1 \leq i \leq n$, let $q_i = \delta^*(q_0, w[1..i])$. Then $q_n \in F$ since $w \in A$.

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Since $n + 1 > p$ and there are only p states in Q, some state must be repeated in the sequence

$$q_0, q_1, \ldots, q_n$$

of $n + 1$ states. Let $0 \leq i < j \leq p$ such that $q_i = q_j$.

We now break up w as $w = xyz$ where

$$\begin{aligned} x &= w[1..i] \\ y &= w[i+1..j] \\ z &= w[j+1..n]. \end{aligned}$$

Since $i < j$, $|y| > 0$, and since $j \le p$, $|xy| \le p$. Therefore conditions (2) and (3) are satisfied.

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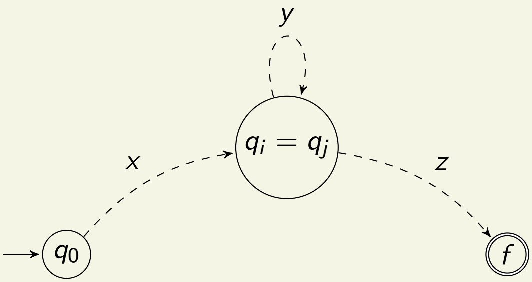
Then

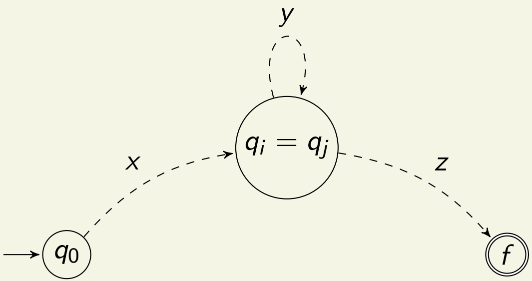
$$\begin{aligned} \delta^*(q_0, x) &= q_i, \\ \delta^*(q_i, y) &= q_j = q_i, \\ \delta^*(q_i, z) &= q_n. \end{aligned}$$

We also have $\delta^*(q_i, y^k) = q_i$ for all k (by induction).



Therefore, for any $k \geq 0$,

$$\delta^*(q_0, xy^k) = \delta^*(\underbrace{\delta^*(q_0, x)}_{q_i}, y^k) = q_i,$$



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$$\delta^*(q_0, xy^k z) = \delta^*(\underbrace{\delta^*(q_0, xy^k)}_{q_i}, z) = q_n \in F$$

and $xy^k z \in A$. $\square$

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Then we know that $x = 0^k$, $y = 0^l$, and $z = 0^{p-k-l}1^p$ for some $k \geq 0$, $l > 0$.

Let $i = 2$. Then

$$\begin{array}{rcl} xy^2z & = & 0^k 0^{2l} 0^{p-k-l} 1^p \\ & = & 0^{p+l} 1^p \end{array}$$

should be in A according to the Pumping Lemma.

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should be in A according to the Pumping Lemma. But since $l > 0$, $p + l \neq p$, so $xy^2z \notin A$. This is a contradiction, so A must not be regular. $\square$

Using the Pumping Lemma to Prove Nonregularity

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5. Conclude that A is not regular.

Note:

- We get to choose w and i in steps 2 and 4.
- We do not get to choose p and x, y, z in steps 1 and 3. We must argue for any possibility of p and x, y, z.

Contrapositive Form of the Pumping Lemma

Original Form:

If A is regular, then
there is some constant p such that
for every $w \in A$ with $|w| \geq p$
there exist x, y, z with $w = xyz$, $|xy| \leq p$, $|y| > 0$ such that
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Contrapositive Form:

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Let $i = 2$. We claim that

$$\begin{aligned} xy^2z &= 0^k 0^{2l} 0^{p^2-k-l} \\ &= 0^{p^2+l} \notin A. \end{aligned}$$

We must show that $p^2 + l$ is not a perfect square.

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and $p^2 + l$ is not a perfect square. Therefore A is not regular by the pumping lemma. $\square$

Example. $E = \{0^i 1^j \mid i \geq j\}$ is not regular.
Proof. Suppose that E is regular, and let p be the pumping constant for E. Let $w = 0^p 1^p$. Let $w = xyz$ where $|xy| \leq p$ and $|y| > 0$.

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Let $i = 0$. Then

$$\begin{aligned} xy^0z &= 0^k 0^{p-k-l} 1^p \\ &= 0^{p-l} 1^p \notin E \end{aligned}$$

because $l > 0$. Therefore E is not regular by the pumping lemma. $\square$

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so B is not regular by the pumping lemma. $\square$

We can also use closure properties to prove nonregularity.

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is also regular. This is a contradiction since we already proved $\{0^n1^n \mid n \geq 0\}$ is not regular. $\square$

Example. $ADD = \left\{ a + b = c \;\middle|\; \begin{array}{l} a, b, c \text{ are binary integers} \\ c \text{ is the sum of } a \text{ and } b \} \end{array} \right\}$

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Then

$$xy^2z \;=\; \text{“}1^{p+l} + 1 = 10^p\text{”} \notin ADD$$

because $1^{p+l} + 1 = 10^{p+l} \neq 10^p$ since $l > 0$. $\square$

The Pumping Lemma Does Not Always Apply

The Pumping Lemma applies in most cases, but there are nonregular languages that satisfy the conclusion of the pumping lemma. Here is one example.

Let $F = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$.

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Let $F = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$.

Suppose F is regular. Then

$$F \cap ab^*c^* = \{ab^n c^n \mid n \geq 0\}$$

is also regular by closure under intersection. This language is not regular by a proof similar to the one we wrote for $\{0^n 1^n \mid n \geq 0\}$. This is a contradiction, so F is not regular.

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- If $m = 1$, then $w = ab^n c^n$. Let $x = \epsilon$, $y = a$, and $z = b^n c^n$. Then for all $i \geq 0$,
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- If $m = 1$, then $w = ab^n c^n$. Let $x = \epsilon$, $y = a$, and $z = b^n c^n$. Then for all $i \geq 0$,
$$xy^i z = a^i b^n c^n \in F.$$

- If $m \geq 2$, then let $x = \epsilon$, $y = aa$, and $z = a^{m-2} b^n c^l$. For all $i \geq 0$,
$$xy^i z = a^{m+2i} b^n c^l \in F.$$

$$F = \{a^m b^n c^l \mid m, n, l \geq 0 \text{ and if } m = 1 \text{ then } n = l\}$$

We now show that F satisfies the conclusion of the Pumping Lemma.

Define $p = 1$. Let $w = a^m b^n c^l \in F$ have $|w| \geq p$. We consider three cases.

- If $m = 1$, then $w = ab^n c^n$. Let $x = \epsilon$, $y = a$, and $z = b^n c^n$. Then for all $i \geq 0$,
$$xy^i z = a^i b^n c^n \in F.$$

- If $m \geq 2$, then let $x = \epsilon$, $y = aa$, and $z = a^{m-2} b^n c^l$. For all $i \geq 0$,
$$xy^i z = a^{m+2i} b^n c^l \in F.$$

- If $m = 0$, then $w = b^n c^l$. There are two subcases:
  - If $j > 0$, then let $x = \epsilon$, $y = b$, and $z = b^{n-1} c^l$.
  - If $j = 0$, then let $x = \epsilon$, $y = c$, and $z = c^{l-1}$.

  In either subcase, $xy^i z \in F$ for all $i \geq 0$.