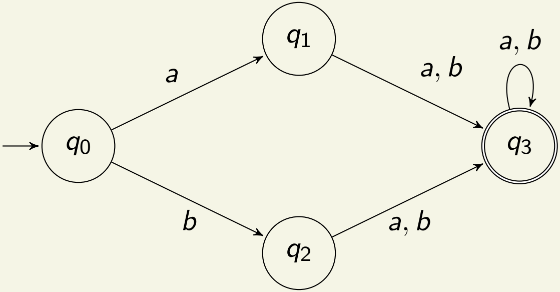
Computability and Complexity
COSC 4200

DFA Minimization

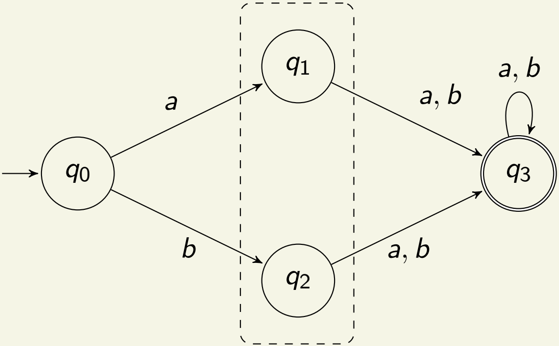
When can we simply a DFA?



q_1 and q_2 are equivalent.

The two states are equivalent because starting from q_1 or q_2, exactly the same strings are accepted.

When can we simply a DFA?



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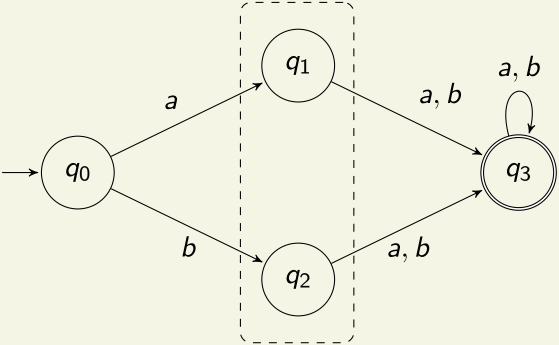
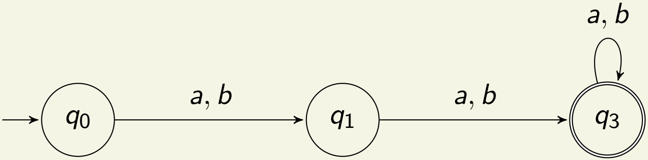
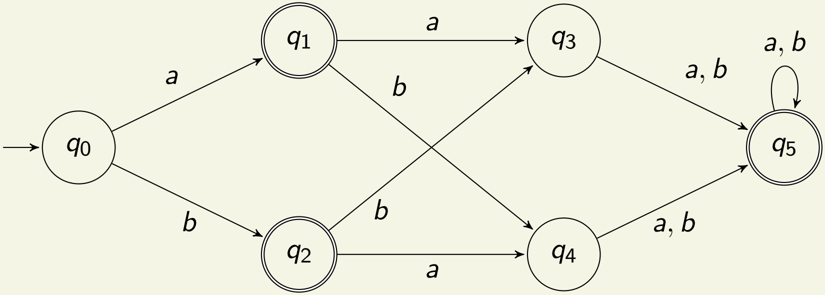


q_1 and q_2 are equivalent. We may collapse these states.

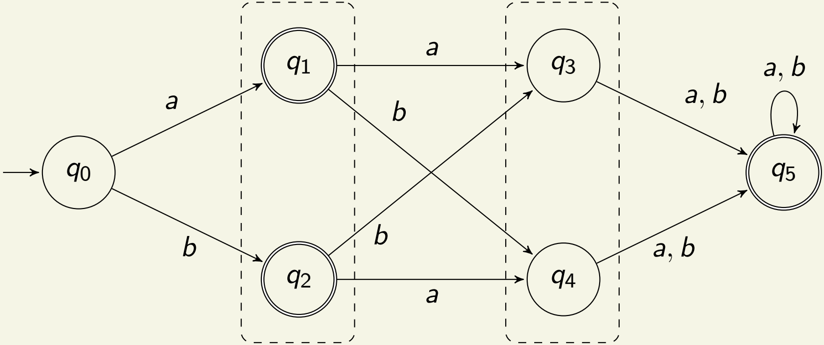


We will develop an algorithm for:

1. Determining which states are equivalent.
2. Building the collapsed automaton.



q_0
q_1
q_2
q_3
q_4
q_5
a
b
a
b
b
a
a, b
a, b
a, b


q_0
q_1
q_2
q_3
q_4
q_5
a
b
a
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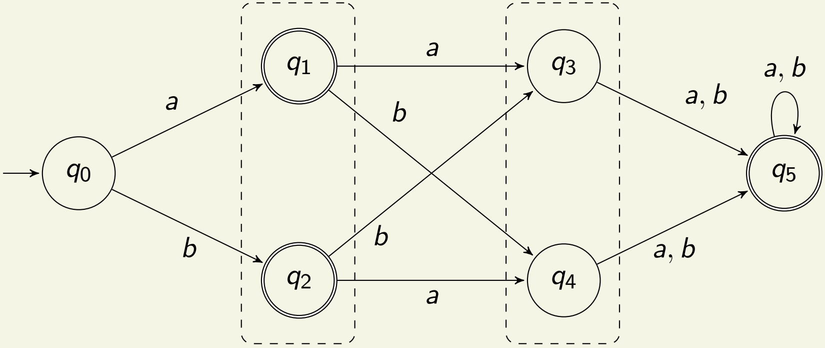
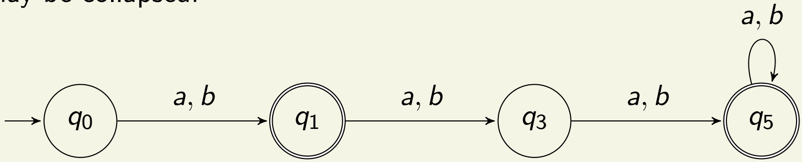
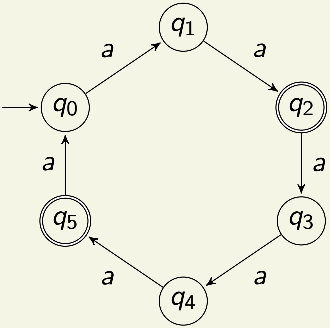
b
a
a, b
a, b
a, b

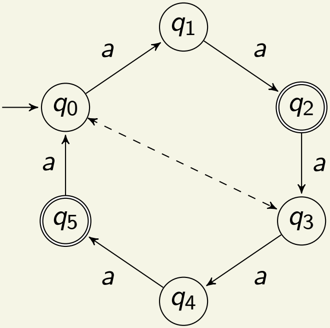


q_1 is equivalent to q_2 and q_3 is equivalent to q_4. These two pairs may be collapsed:





q_0
q_1
q_2
q_3
q_4
q_5
a
a
a
a
a
a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q_3
q_4
q_5
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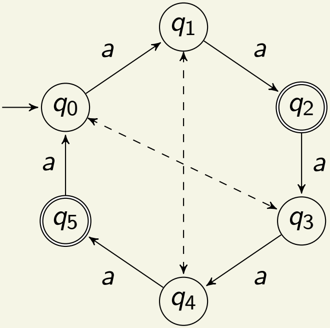
q_0
q_1
q_2
q_3
q_4
q_5
a
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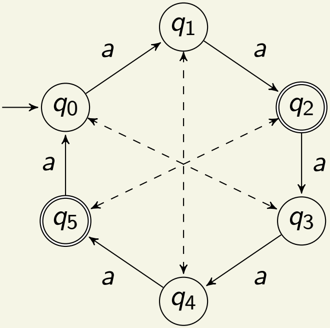
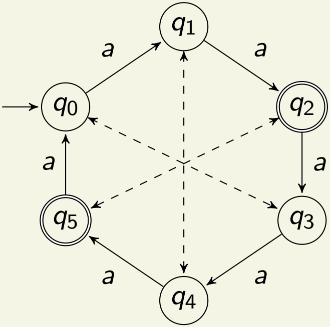
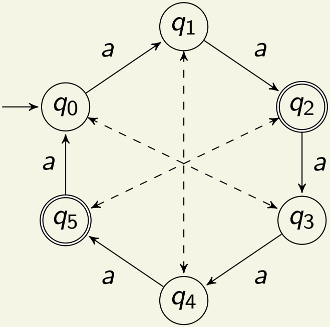

q_1
a
a
q_0
q_2
a
a
q_5
q_3
a
a
q_4



Each state is equivalent with the opposite state in the diagram: q_0 with q_3, q_1 with q_4, q_2 with q_5. Collapsing yields:



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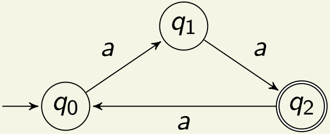

Equivalent States

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA.

Intuitively, two states are equivalent if they "remember" the same information. We can say that two states p and q are equivalent if for every $x \in \Sigma^*$,

$$\delta^*(p, x) \in F \Leftrightarrow \delta^*(q, x) \in F.$$

In other words, p and q are equivalent if exactly the same strings are accepted when we start in p or q. We write $p \approx q$ to denote that p is equivalent to q.

Observe that $\approx$ is an equivalence relation:

- *reflexive:* $p \approx p$ for all p
- *symmetric:* $p \approx q$ implies $q \approx p$
- *transitive:* if $p \approx q$ and $q \approx r$, then $p \approx r$

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For each $p \in Q$, let

$$[p] = \{q \in Q \mid q \approx p\}$$

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These equivalence classes partition Q (so each state belongs to exactly one class), and we have

$$p \approx q \Leftrightarrow [p] = [q].$$

Quotient Automaton

We use $\approx$ to define the *quotient automaton*

$$M/\!\approx\ = (Q', \Sigma, \delta', q_0', F')$$

where

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We claim that $M/\!\approx$ is equivalent to M, that is $L(M/\!\approx) = L(M)$. First, we need to check that the transition function δ' is well-defined.

The transition function δ' is defined by

$$\delta'([p], a) = [\delta(p, a)]$$

for all $[p] \in Q'$, $a \in \Sigma$.

For δ' to be well-defined, it shouldn't matter which equivalence class representative of $[p]$ we use to apply δ. The following lemma says this is the case.

Lemma

If $p \approx q$, then $\delta(p, a) \approx \delta(q, a)$.
Equivalently, if $[p] = [q]$, then $[\delta(p, a)] = [\delta(q, a)]$.

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Equivalently, if $[p] = [q]$, then $[\delta(p,a)] = [\delta(q,a)]$ for all $a \in \Sigma$.

Proof. Suppose $p \approx q$. Then for all $x \in \Sigma^*$,

$$\delta^*(p,x) \in F \iff \delta^*(q,x) \in F.$$

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$$\delta^*(\delta(p, a), y) \in F \quad \iff \quad \delta^*(p, ay) \in F \qquad \text{property of } \delta^*$$

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Therefore $\delta(p, a) \approx \delta(q, a)$. $\square$

The quotient automaton $M/\approx$ has the same language as the original automaton M.

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The proof follows from these two lemmas:

Lemma A

$p \in F \Leftrightarrow [p] \in F'$.

Lemma B

For all $x \in \Sigma^*$, $\delta'^*([p], x) = [\delta^*(p, x)]$.

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Lemma B

For all $x \in \Sigma^*$, $\delta'^*([p], x) = [\delta^*(p, x)]$.

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Assume as our inductive hypothesis the claim

$$\delta'^*([p], x) = [\delta^*(p, x)]$$

holds for some $x \in \Sigma^*$. Let $a \in \Sigma$. For the inductive step, we must show that

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We have

$$\begin{array}{rcll} {\delta'}^*([p], xa) & = & \delta'({\delta'}^*([p], x), a) & \text{definition of } {\delta'}^* \end{array}$$

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$\square$

We now use Lemmas A and B to prove the Theorem.

Theorem

$L(M/{\approx}) = L(M)$.

Proof. For any $x \in \Sigma^*$,

$$x \in L(M/{\approx}) \iff \delta'^*(q'_0, x) \in F' \qquad \text{definition of acceptance}$$

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$\square$

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It turns out that $M/{\approx}$ cannot be collapsed any further. If we apply the quotient construction a second time, the DFA will not change.

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Minimization Algorithm

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3. Repeat the following until there are no more changes:
   for each unmarked $\{p, q\}$
   if $\{\delta(p, a), \delta(q, a)\}$ is marked for some $a \in \Sigma$, then

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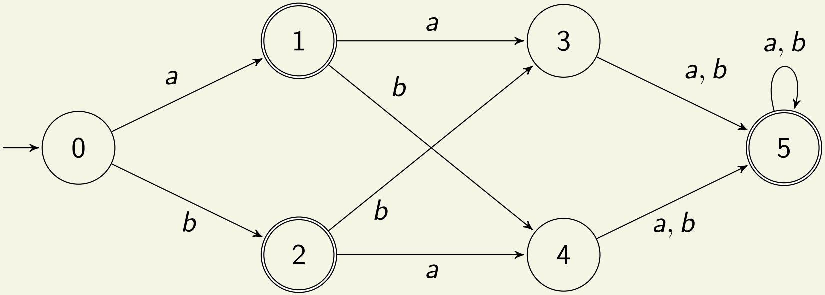
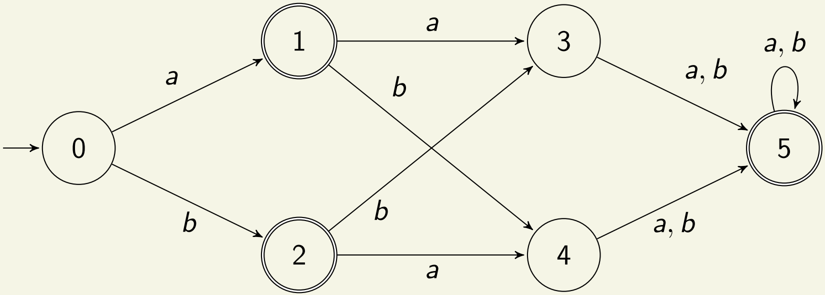
4. Now $p \approx q$ if and only if $\{p, q\}$ is not marked.

Example



	0	1	2	3	4
1					
2					
3					
4					
5					

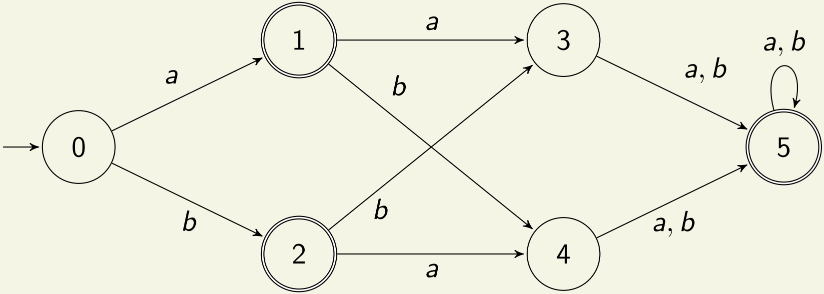
Initialization



Initialization:
Mark $\{p, q\}$ if
$(p \in F$ and $q \notin F)$ or
$(p \notin F$ and $q \in F)$.

	0	1	2	3	4
1	✓				
2	✓				
3		✓	✓		
4		✓	✓		
5	✓			✓	✓

First Pass

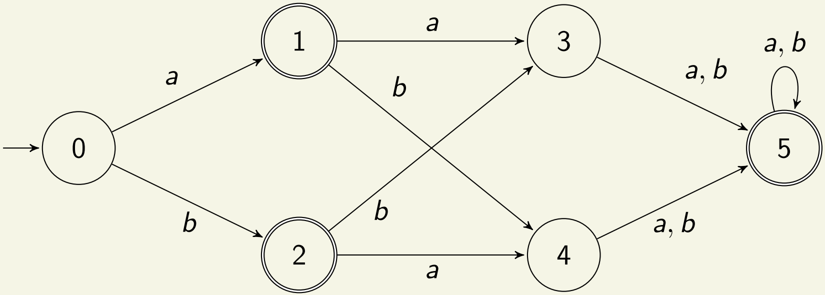


First pass:

- Mark $\{1,5\}$ because $\{1,5\} \xrightarrow{a} \{3,5\}$ is marked
- Mark $\{2,5\}$ because $\{2,5\} \xrightarrow{a} \{4,5\}$ is marked

	0	1	2	3	4
1	✓				
2	✓				
3		✓	✓		
4		✓	✓		
5	✓	✓	✓	✓	✓

Second Pass

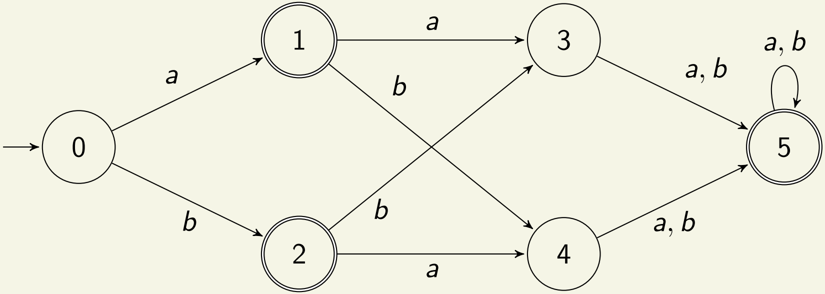


Second pass:

- Mark $\{0,3\}$ because $\{0,3\} \xrightarrow{a} \{1,5\}$ is marked
- Mark $\{0,4\}$ because $\{0,4\} \xrightarrow{b} \{2,5\}$ is marked

	0	1	2	3	4
1	✓				
2	✓				
3	✓	✓	✓		
4	✓	✓	✓		
5	✓	✓	✓	✓	✓

Third Pass



Third pass: no changes

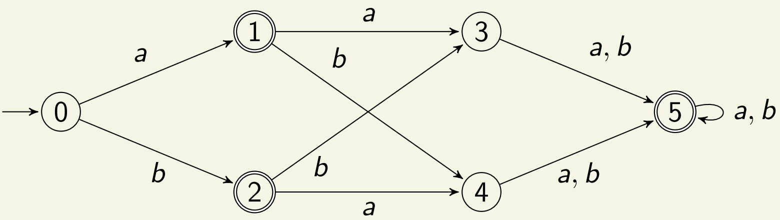
Output of algorithm:

- $1 \approx 2$, $3 \approx 4$
- $[0] = \{0\}$,
  $[1] = \{1, 2\} = [2]$,
  $[3] = \{3, 4\} = [4]$,
  $[5] = \{5\}$

	0	1	2	3	4
1	✓				
2	✓				
3	✓	✓	✓		
4	✓	✓	✓		
5	✓	✓	✓	✓	✓

Quotient Automaton

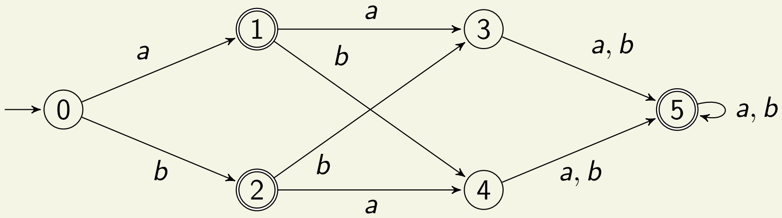
Original DFA:



$$[0] = \{0\},\ [1] = \{1,2\} = [2],\ [3] = \{3,4\} = [4],\ [5] = \{5\}$$

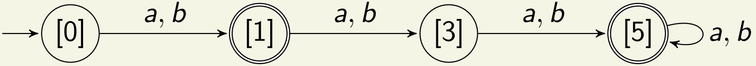
Quotient Automaton

Original DFA:



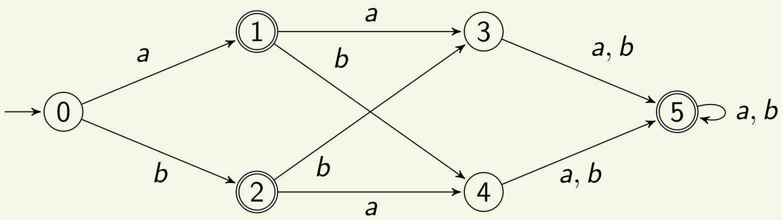
$$[0] = \{0\},\ [1] = \{1, 2\} = [2],\ [3] = \{3, 4\} = [4],\ [5] = \{5\}$$

Quotient Automaton:



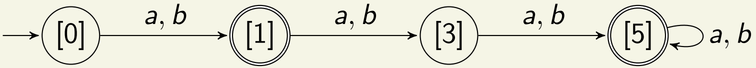
Quotient Automaton

Original DFA:



$$[0] = \{0\},\ [1] = \{1,2\} = [2],\ [3] = \{3,4\} = [4],\ [5] = \{5\}$$

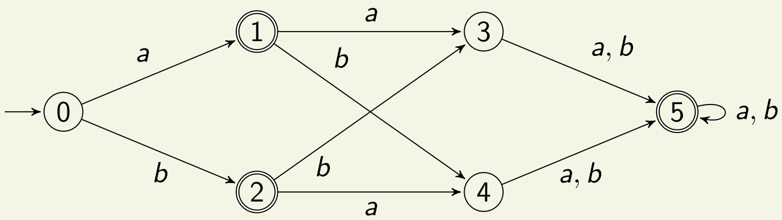
Quotient Automaton:



- $\delta'([0], a) = [\delta(0, a)] = [1]$

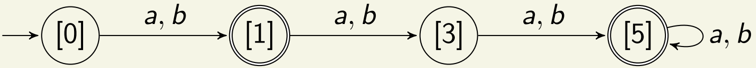
Quotient Automaton

Original DFA:



$$[0] = \{0\},\ [1] = \{1, 2\} = [2],\ [3] = \{3, 4\} = [4],\ [5] = \{5\}$$

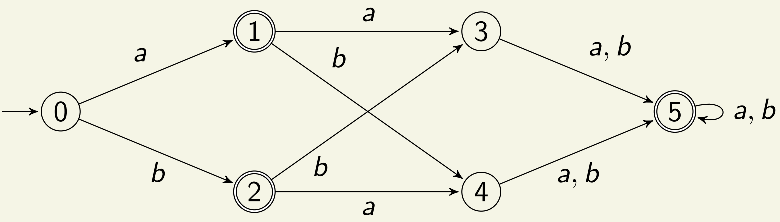
Quotient Automaton:



- $\delta'([0], a) = [\delta(0, a)] = [1]$
- $\delta'([0], b) = [\delta(0, b)] = [2] = [1]$

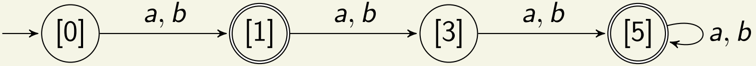
Quotient Automaton

Original DFA:



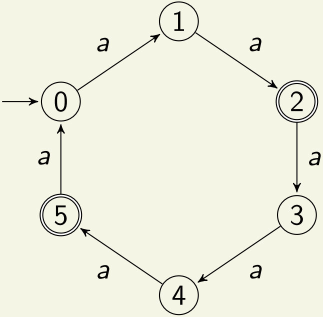
$$[0] = \{0\},\ [1] = \{1, 2\} = [2],\ [3] = \{3, 4\} = [4],\ [5] = \{5\}$$

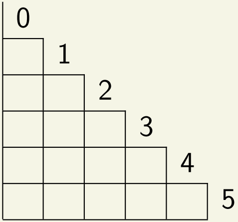
Quotient Automaton:



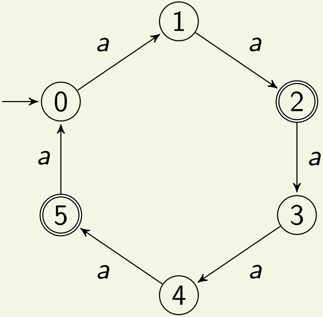
- $\delta'([0], a) = [\delta(0, a)] = [1]$
- $\delta'([0], b) = [\delta(0, b)] = [2] = [1]$
- $\delta'([1], b) = [\delta(1, b)] = [4] = [3]$

Example





Initialization



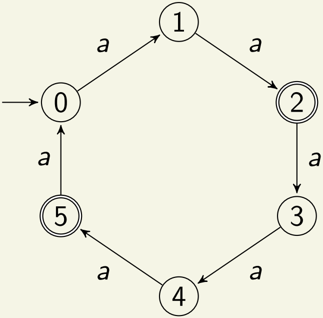
Initialization:
Mark $\{p, q\}$ if
$(p \in F$ and $q \notin F)$ or
$(p \notin F$ and $q \in F)$.

	0	1	2	3	4
1					
2	✓	✓			
3			✓		
4			✓		
5	✓	✓		✓	✓

First Pass

First pass:

- Mark $\{0,1\}$ because $\{0,1\} \xrightarrow{a} \{1,2\}$ is marked
- Mark $\{0,4\}$ because $\{0,4\} \xrightarrow{a} \{1,5\}$ is marked
- Mark $\{1,3\}$ because $\{1,3\} \xrightarrow{a} \{2,4\}$ is marked
- Mark $\{3,4\}$ because $\{3,4\} \xrightarrow{a} \{4,5\}$ is marked



	0	1	2	3	4
1	✓				
2	✓	✓			
3		✓	✓		
4	✓		✓	✓	
5	✓	✓		✓	✓

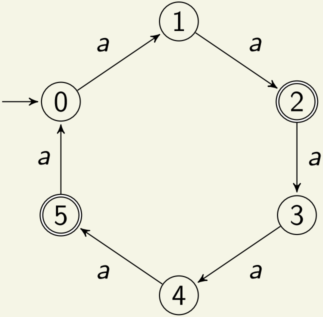
Second Pass

Second pass:

- No changes.

Output of algorithm:

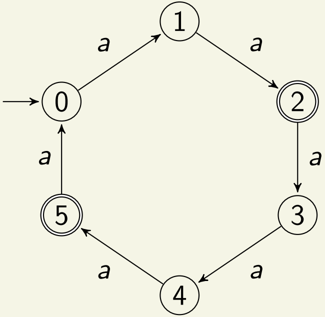
- $0 \approx 3,\ 1 \approx 4,\ 2 \approx 5$
- $[0] = \{0, 3\} = [3]$,
  $[1] = \{1, 4\} = [4]$,
  $[2] = \{2, 5\} = [5]$



	0	1	2	3	4
1	✓				
2	✓	✓			
3		✓	✓		
4	✓		✓	✓	
5	✓	✓		✓	✓

Quotient Automaton

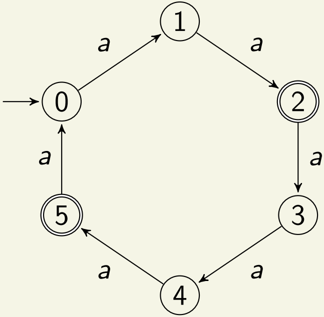
Original DFA:



$$[0] = \{0, 3\} = [3]$$
$$[1] = \{1, 4\} = [4]$$
$$[2] = \{2, 5\} = [5]$$

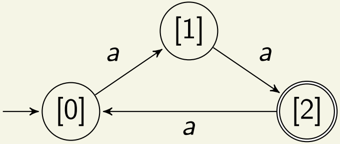
Quotient Automaton

Original DFA:



$$[0] = \{0, 3\} = [3]$$
$$[1] = \{1, 4\} = [4]$$
$$[2] = \{2, 5\} = [5]$$

Quotient Automaton:



$$\delta'([0], a) = [\delta(0, a)] = [1]$$
$$\delta'([1], a) = [\delta(1, a)] = [2]$$
$$\delta'([2], a) = [\delta(2, a)] = [3] = [0]$$