Computability and Complexity
COSC 4200

Approximating TSP

Approximation Algorithms

Let X be a minimization problem.
An algorithm $\mathcal{A}$ is an $\alpha(n)$-approximation algorithm if for all n, for all instances of size n, $\mathcal{A}$ returns a solution with value at most $\alpha(n)$ times the value of the optimal solution.

For example:

- a $\log n$-approximation algorithm
- a 2-approximation algorithm

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For example:

- a $\frac{7}{8}$-approximation algorithm
- a $\frac{\log n}{n}$-approximation algorithm

Approximation of TSP

Theorem

If $\mathrm{P} \neq \mathrm{NP}$, then for any polynomial-time computable function $\alpha(n)$, there is no $\alpha(n)$-approximation algorithm for TSP.

Proof. We modify our reduction of $\mathrm{HAM\text{-}CYCLE}$ to TSP. Let $G = (V, E)$ be a graph.

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Define $n = |V|$, $k = n$, and

$$c(i,j) = \begin{cases} 1 & \text{if } (i,j) \in E \\ \alpha(n) \cdot n & \text{if } (i,j) \notin E \end{cases}$$

for all $i, j \in \{1, \ldots, n\}$.

Notice that:

- if $G \in \mathrm{HAM\text{-}CYCLE}$, then there is a TSP tour with cost n.
- if $G \notin \mathrm{HAM\text{-}CYCLE}$, then any TSP tour has cost $> \alpha(n) \cdot n$.

Suppose $\mathcal{A}$ is a polynomial-time $\alpha(n)$-approximation algorithm for TSP. Consider the following Algorithm $\mathcal{B}$.

input: graph G, instance of $\mathrm{HAM\text{-}CYCLE}$

construct the TSP instance from G as described
run $\mathcal{A}$ on the TSP instance
let $c =$ cost of tour output by $\mathcal{A}$

if $c \leq \alpha(n) \cdot n$
    output 'yes'
else
    output 'no'

We claim that Algorithm $\mathcal{B}$ decides $\mathrm{HAM\text{-}CYCLE}$.

Suppose $G \in \mathrm{HAM\text{-}CYCLE}$.

- Then there is a TSP tour with cost n.
- The approximation algorithm $\mathcal{A}$ will find a tour with cost at most $\alpha(n) \cdot n$.
- Therefore algorithm $\mathcal{B}$ will output 'yes.'

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- Then any TSP tour has cost $> \alpha(n) \cdot n$.
- The approximation algorithm $\mathcal{A}$ will return a tour with cost $> \alpha(n) \cdot n$.
- Therefore algorithm $\mathcal{B}$ will output 'no.'

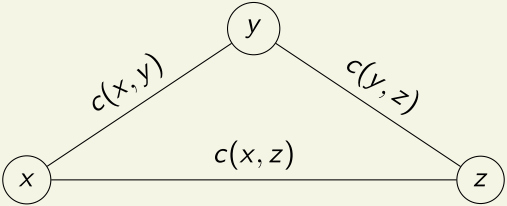
Therefore $\mathcal{B}$ decides $\mathrm{HAM\text{-}CYCLE}$ and $\mathrm{HAM\text{-}CYCLE} \in \mathrm{P}$.
Since $\mathrm{HAM\text{-}CYCLE}$ is NP-complete, $\mathrm{P} = \mathrm{NP}$. $\square$

Metric TSP

We showed there is no hope of approximating general TSP (unless P=NP). However, the cost function in the proof is wild: some costs are low and some are very high.

In the Metric TSP problem, the costs must satisfy the triangle inequality: for all x, y, and z,

$$c(x,z) \le c(x,y) + c(y,z).$$



Approximation Algorithm for Metric TSP

Consider the following algorithm to compute TSP tours:

On input a TSP instance as a weighted graph G:

1. Find a minimum spanning tree, T, of G.

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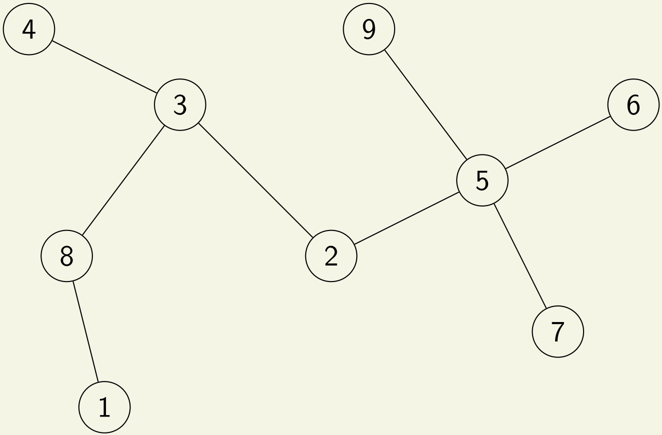
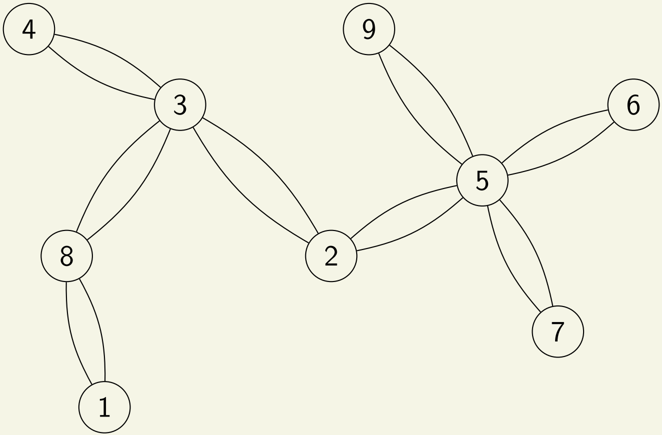
4. Output the tour $\mathcal{C}$ that visits vertices of G in the order of their first appearing in $\mathcal{T}$.

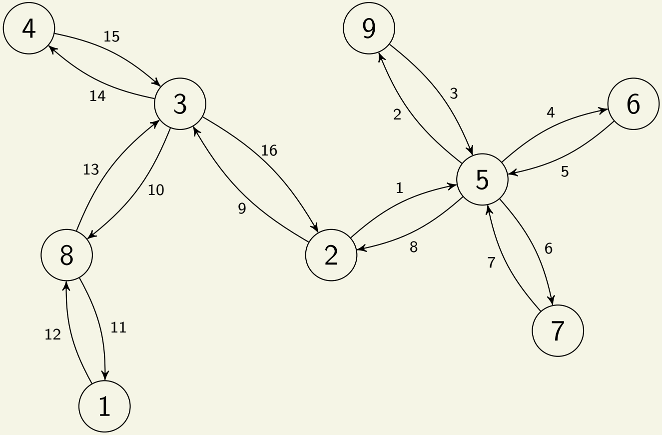
Minimum spanning tree T



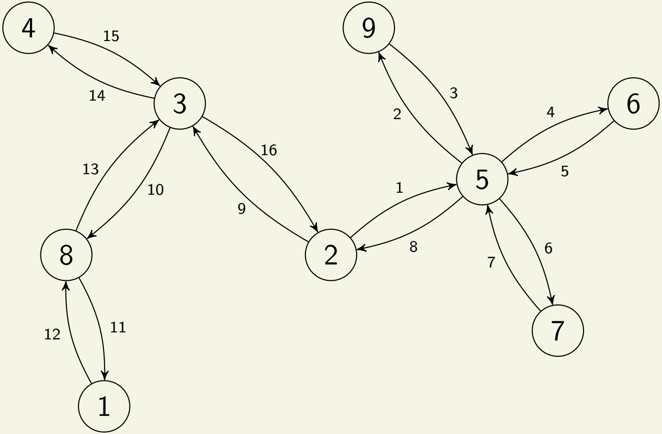
Double every edge to obtain E



Find an Eulerian tour $\mathcal{T}$



Take shortcuts to obtain final tour $\mathcal{C}$



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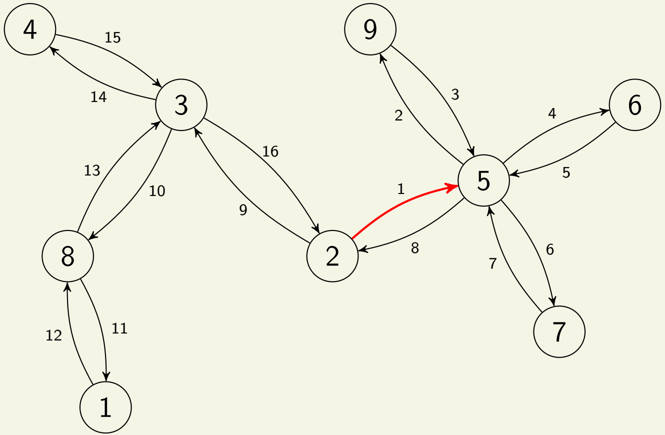

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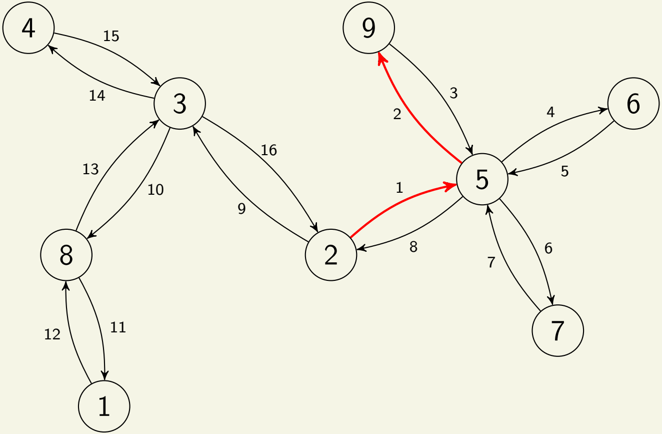

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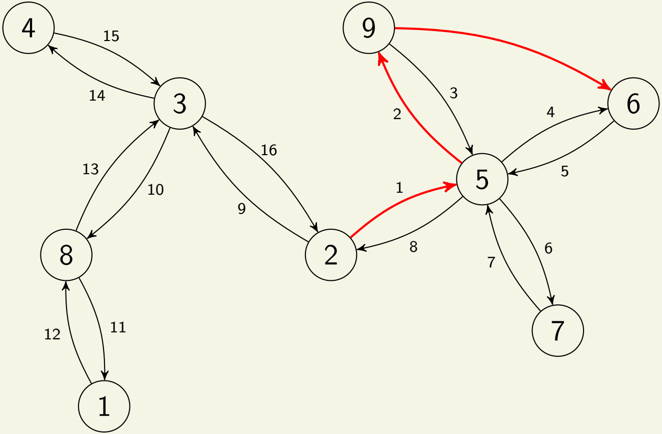

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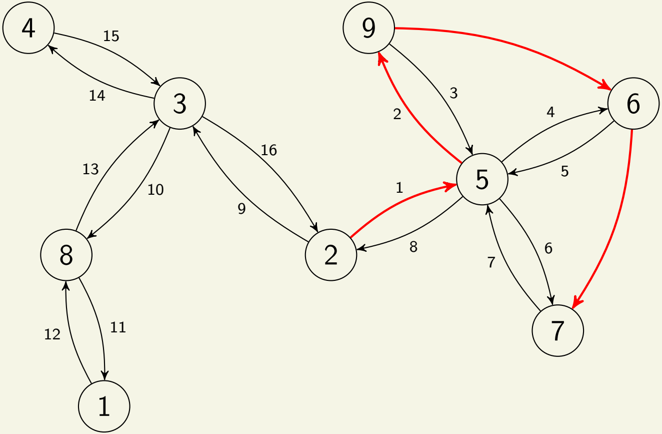

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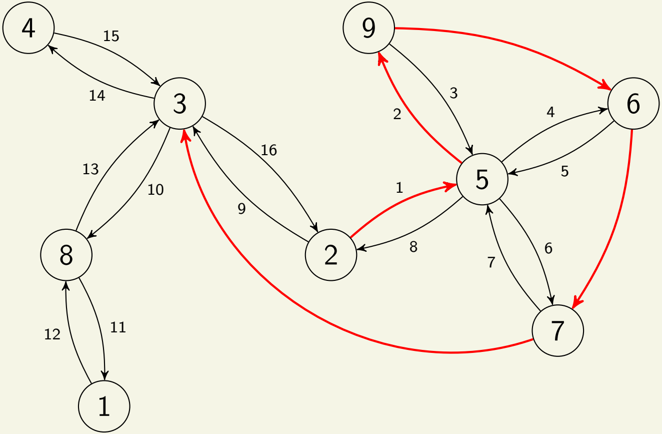

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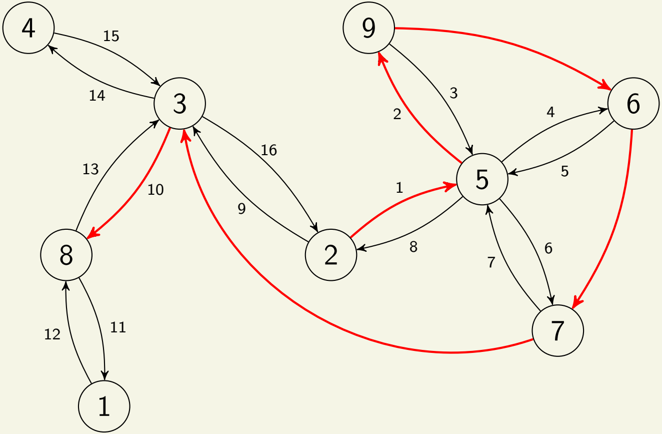

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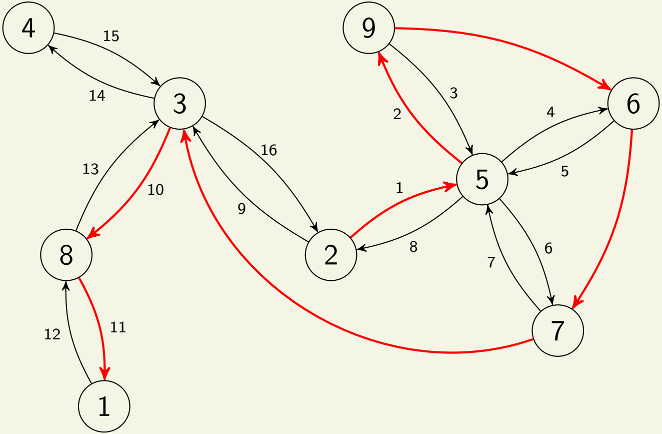

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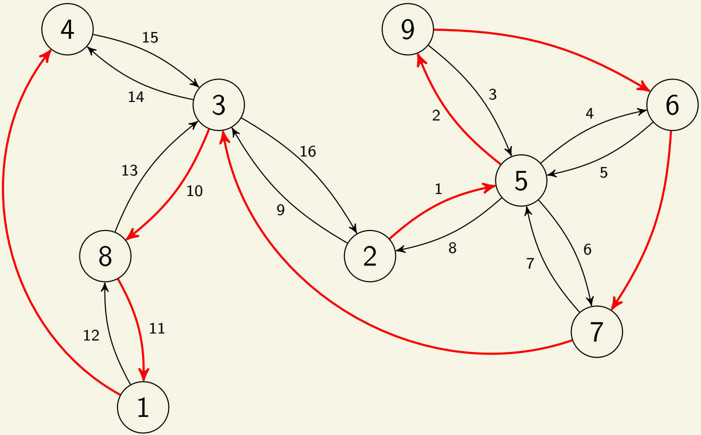

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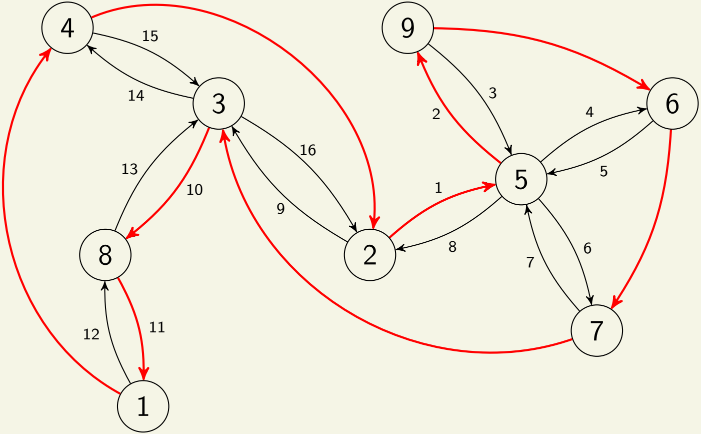

Theorem. This is a 2-appoximation algorithm for Metric TSP.

Proof. Let OPT be the cost of an optimal TSP tour. Observe that:

- $\mathrm{cost}(T) \leq \mathrm{OPT}$. If we take any edge away from an optimal TSP tour, we have a spanning tree U. Thus $\mathrm{cost}(T) \leq \mathrm{cost}(U) \leq \mathrm{OPT}$.

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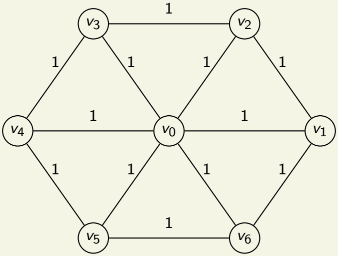
Putting everything together, we have

$$\mathrm{cost}(\mathcal{C}) \leq \mathrm{cost}(\mathcal{T}) = 2 \cdot \mathrm{cost}(T) \leq 2 \cdot \mathrm{OPT}. \quad \square$$

Tight Example for 2-Approximation Algorithm



The distance for all other pairs of vertices is 2.

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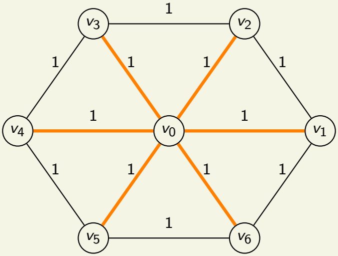

The MST consists of the orange edges and has cost 6.

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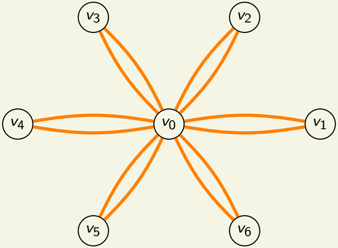

Double the edges of the MST to obtain an Eulerian graph.

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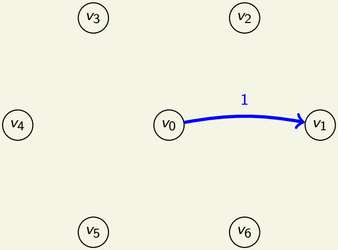

An Eulerian tour. The algorithm could find any of many Eulerian tours, but this one will give worst-case performance.

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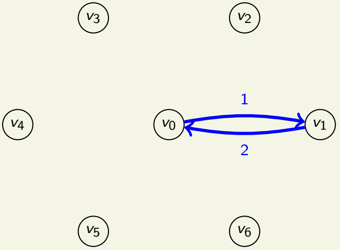

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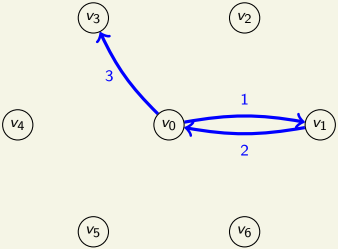

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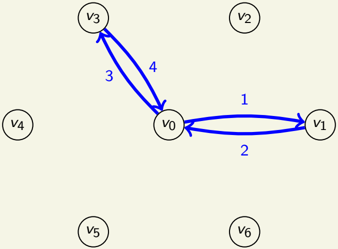

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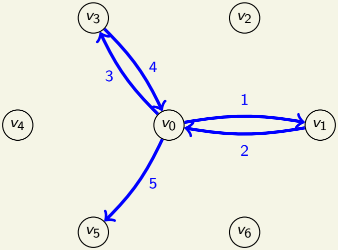

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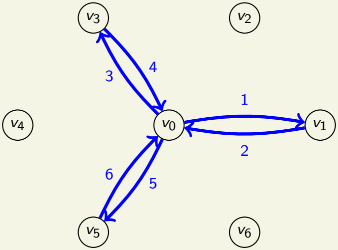

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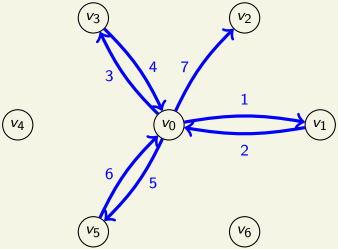

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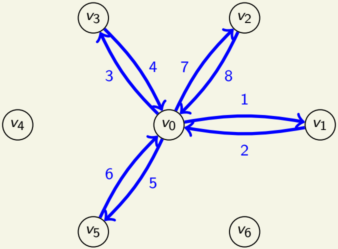

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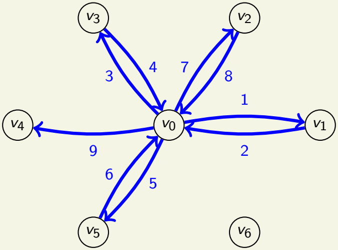

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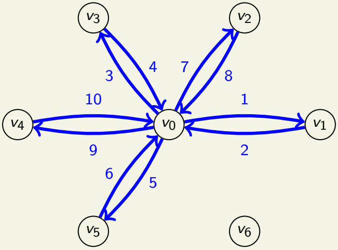

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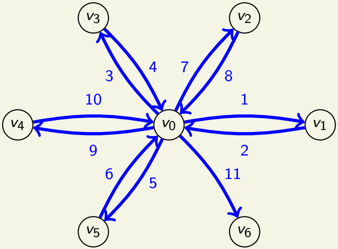

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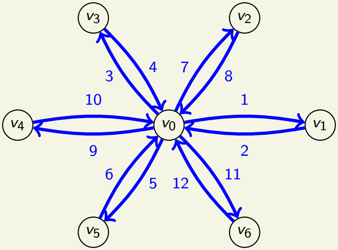

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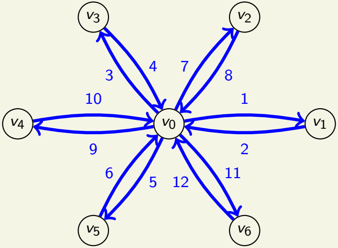

The shortcut tour.

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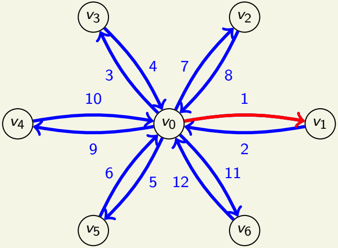

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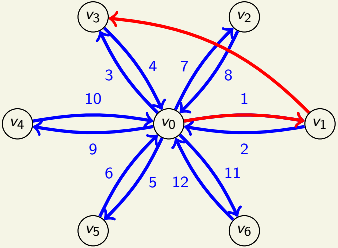

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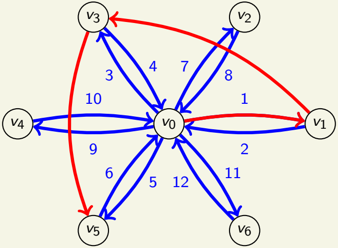

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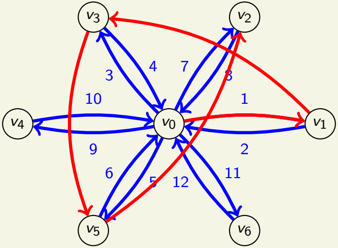

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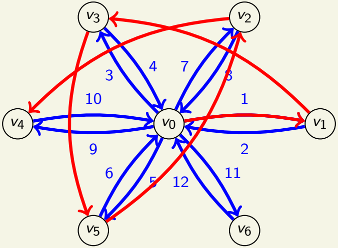

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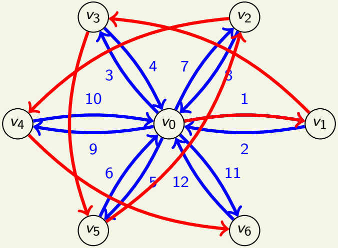

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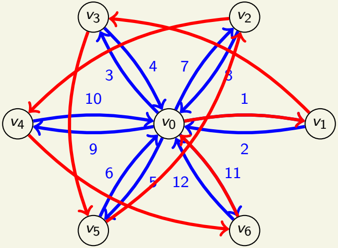

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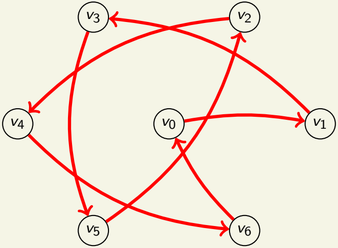

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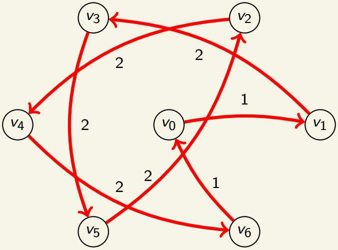

Tour cost: $2(6-1)+2=12$.

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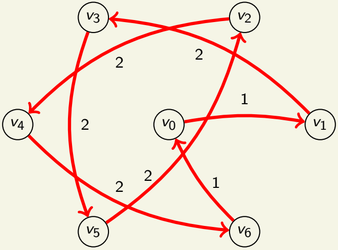
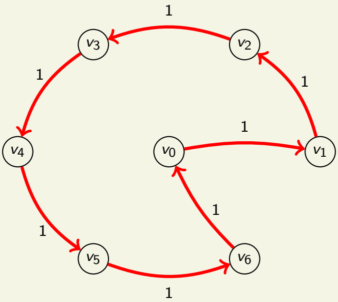


Tour cost: $2(6-1)+2=12$.

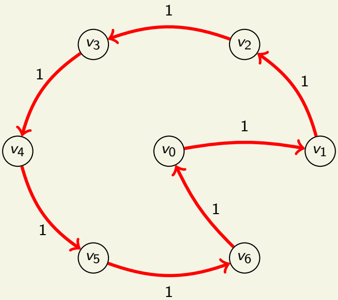
If we change 6 to n and obtain a analogous graph on $n+1$ vertices, the cost is $2(n-1)+2=2n$.

Tight Example for 2-Approximation Algorithm



Optimal tour cost: 7

Tight Example for 2-Approximation Algorithm



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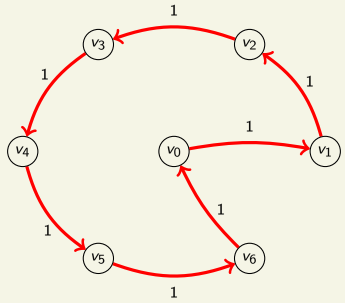

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If we change 6 to n and obtain a analogous graph on $n+1$ vertices, the optimal tour cost is $n+1$.

The performance ratio is $\frac{12}{7} \leq 2$ on the above instance. In general, for the graph on $n+1$ vertices, the performance ratio is:

$$\frac{2n}{n+1}.$$

This approaches 2 as $n \to \infty$.

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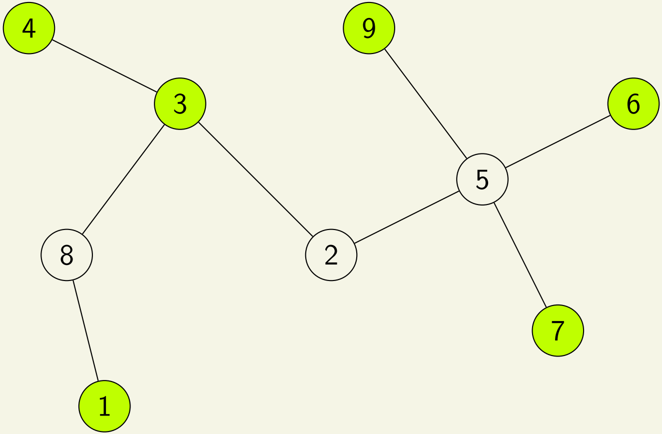
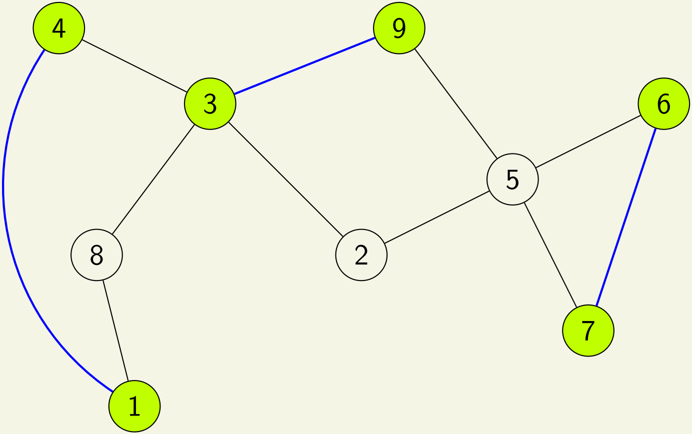
On input a TSP instance as a weighted graph G:

1. Find a minimum spanning tree, T, of G.
2. Compute a minimum cost perfect matching, M, on the set of odd-degree vertices of T. A minimum cost perfect matching is a pairing (matching) of vertices at minimum total edge cost. Add M to T and obtain an Eulerian graph E.
3. Find an Eulerian tour, $\mathcal{T}$, of E.
4. Output the tour $\mathcal{C}$ that visits vertices of G in the order of their first appearing in $\mathcal{T}$.

Minimum spanning tree T, odd degree vertices highlighted



Minimum cost perfect matching added, now Eulerian graph



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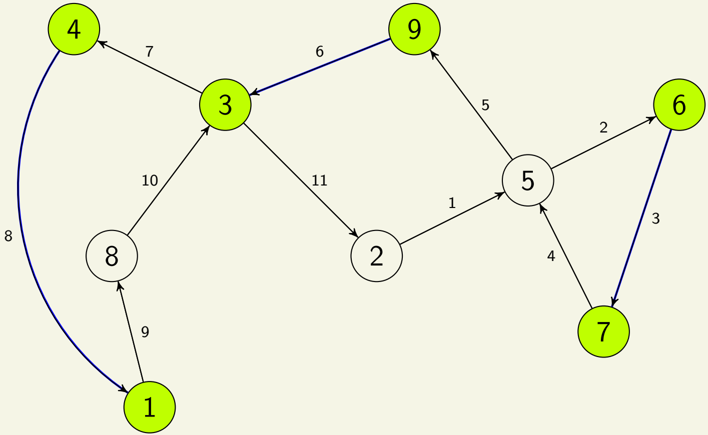
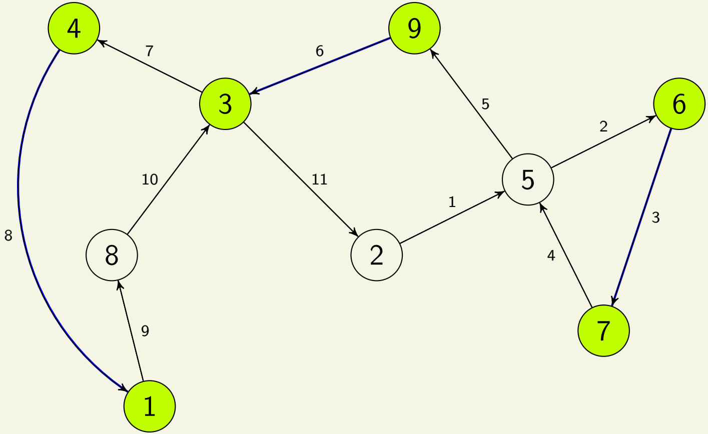


Take shortcuts to obtain final tour $\mathcal{C}$



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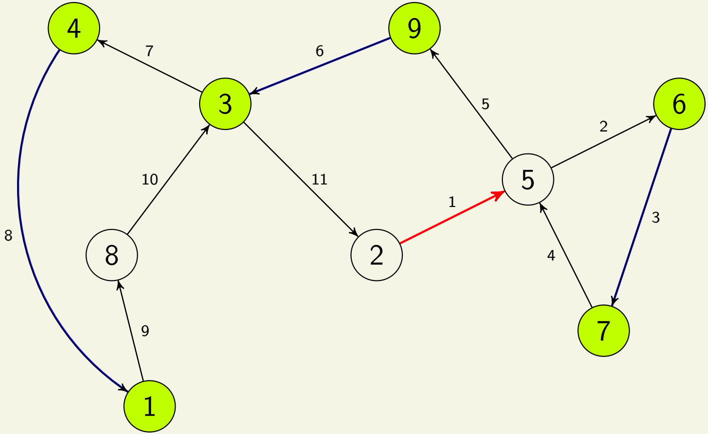

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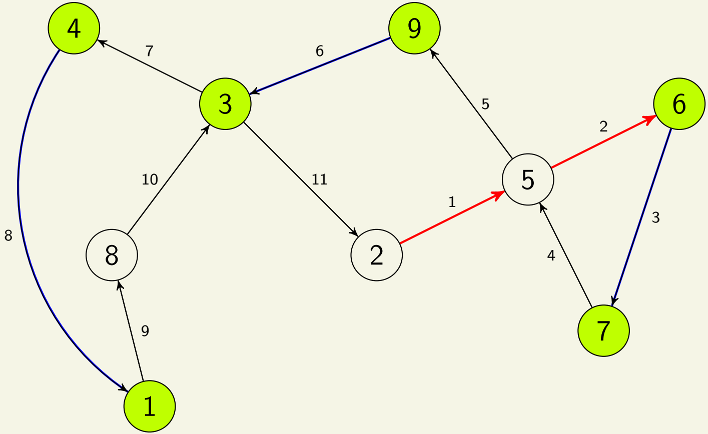

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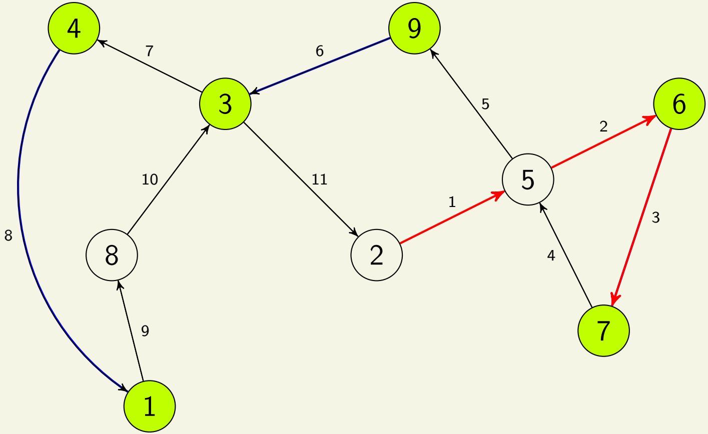

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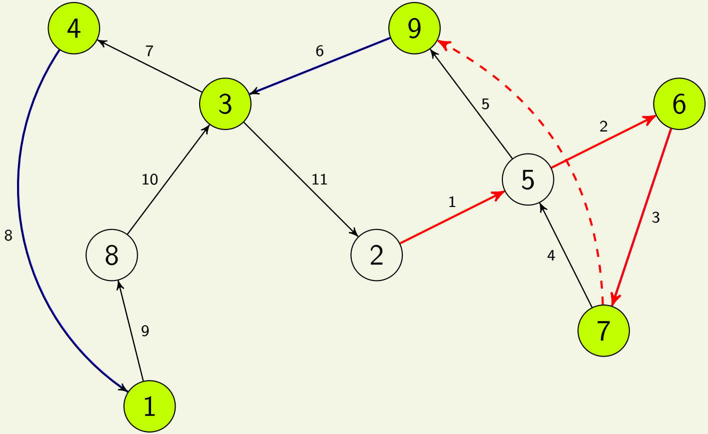

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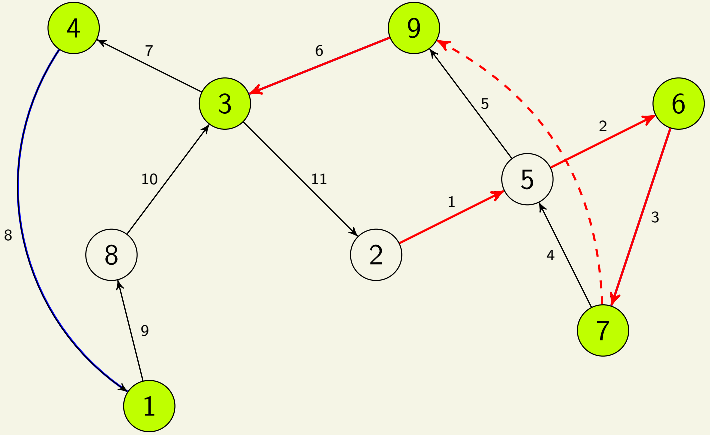

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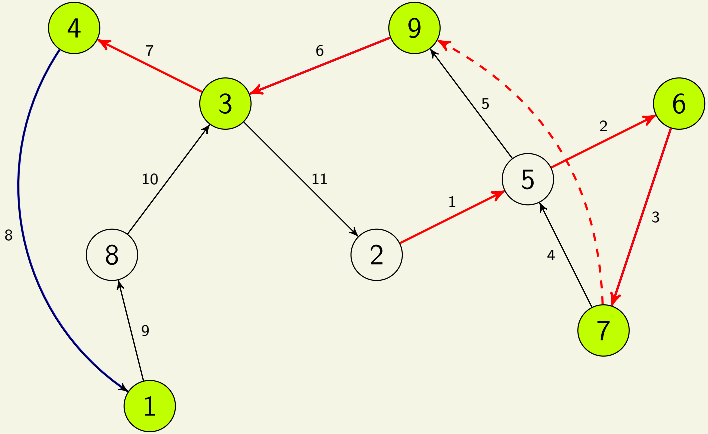

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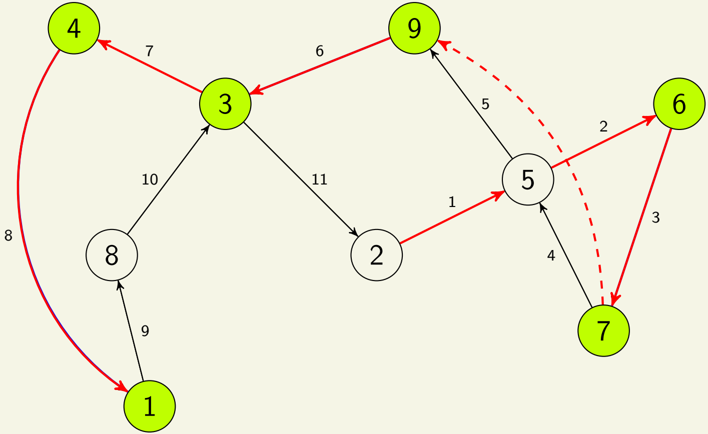

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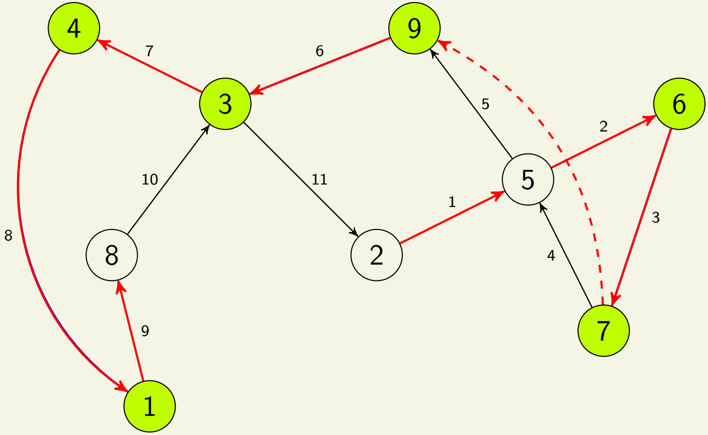

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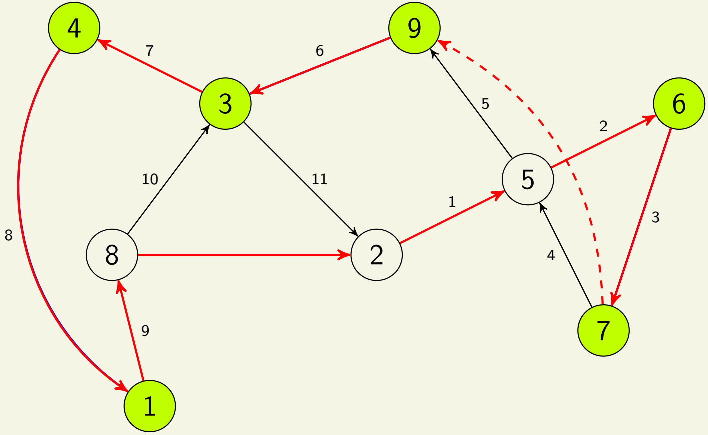

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Therefore the optimal perfect matching has cost $\leq \mathrm{OPT}/2$. $\square$

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Christofides' algorithm (1976) was the best approximation algorithm for Metric TSP until 2020!

In an 86 page paper, Karlin, Klein, and Gharan (2020) gave an algorithm with approximation ratio

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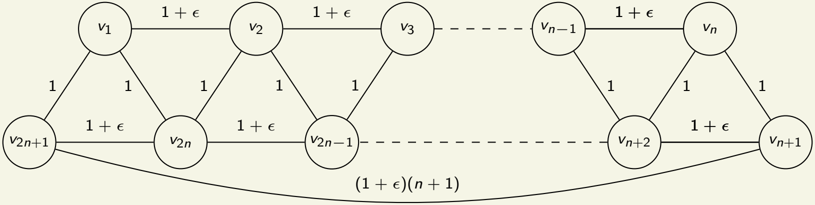
$$\frac{3}{2} - 10^{-36} = 1.499999999999999999999999999999999999.$$

Tight Example for Christofides' Algorithm



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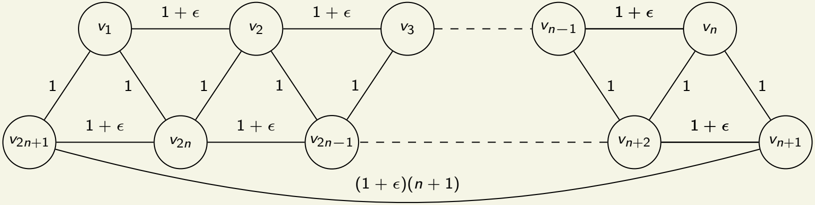
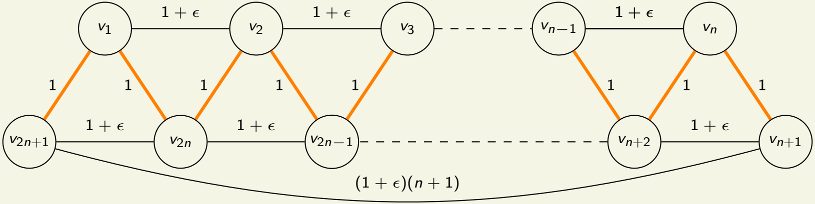


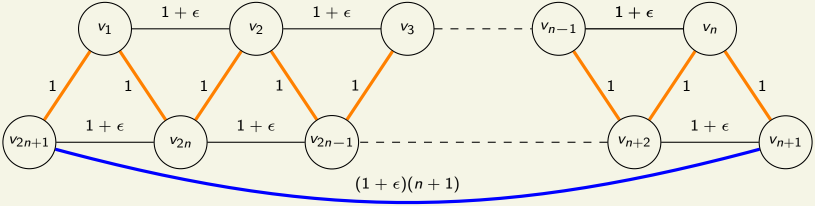
All other edge costs are the cost of the shortest path using the above edges.

Tight Example for Christofides' Algorithm



The MST consists of the orange edges and has cost $2n$.

Tight Example for Christofides' Algorithm



There are two odd degree vertices, v_{n+1} and v_{2n+1}.
They are matched by the minimum cost perfect matching algorithm.

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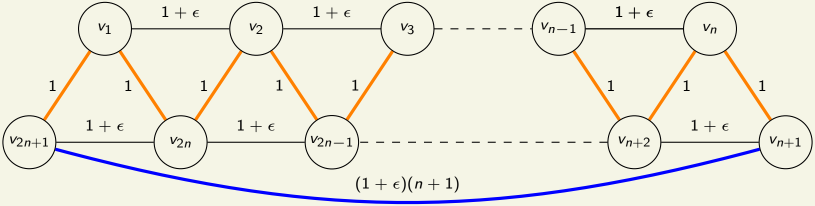

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Tour cost: $2n + (1+\epsilon)(n+1) = (3+\epsilon)n + (1+\epsilon)$

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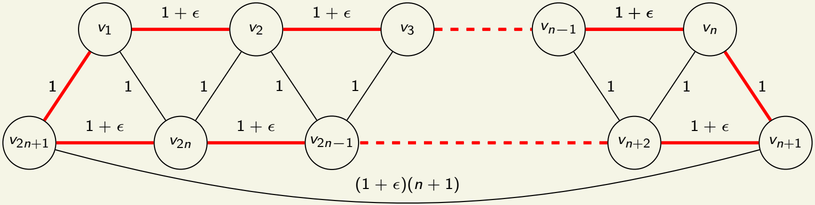

The optimal tour has cost $(2n-1)(1+\epsilon)+2 = 2(1+\epsilon)n+(1-\epsilon)$.

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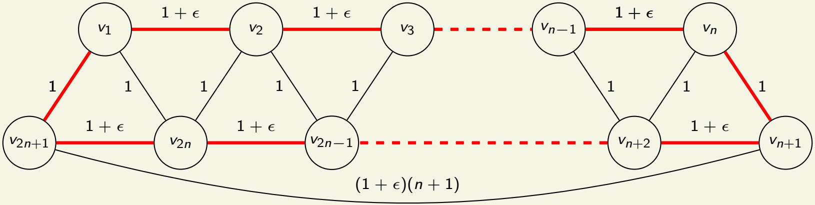

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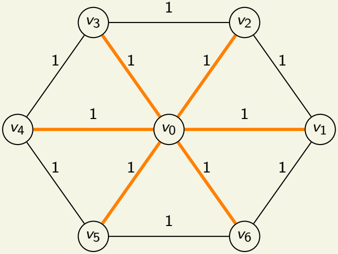
Performance ratio:

$$\frac{(3+\epsilon)n+(1+\epsilon)}{2(1+\epsilon)n+(1-\epsilon)}.$$

This approaches $\frac{3}{2}$ as $n \to \infty$ and $\epsilon \to 0$.

Revisiting Tight Example for 2-Approximation Algorithm

Let's see how Christofides' Algorithm performs on the tight instances for the 2-approximation algorithm.



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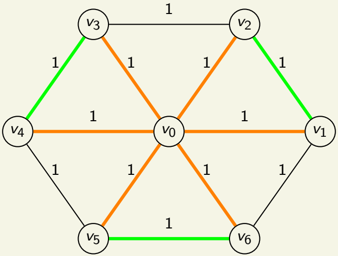
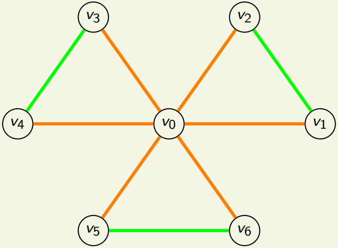


The MST consists of the orange edges.

The green edges are a minimum cost perfect matching of the odd degree vertices in the MST.

Revisiting Tight Example for 2-Approximation Algorithm



An Eulerian tour.

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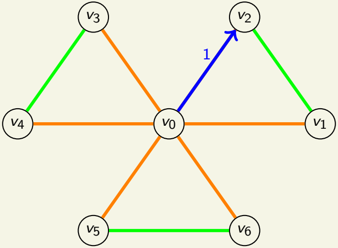

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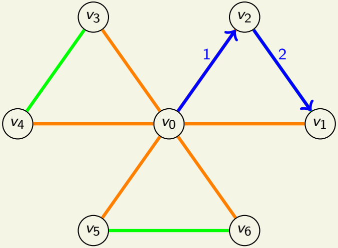

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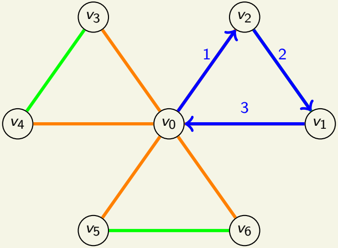

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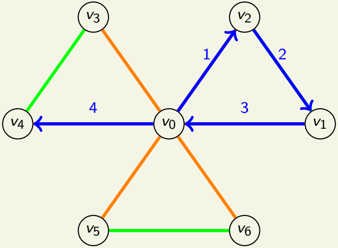

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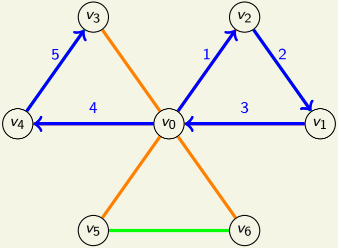

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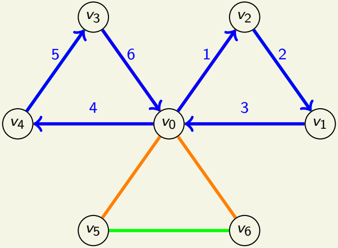

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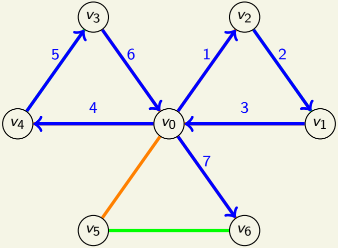

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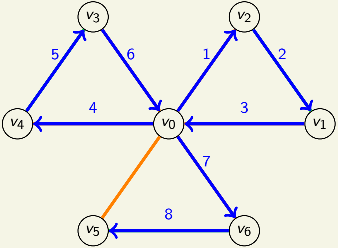

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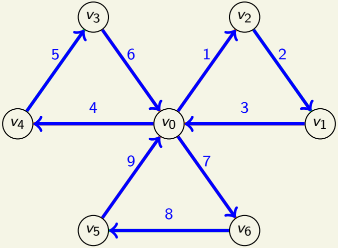

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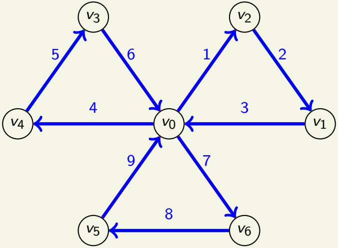

The shortcut tour.

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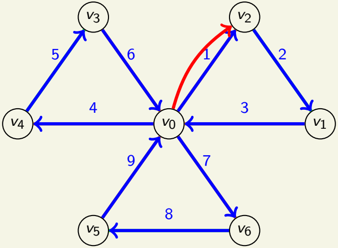

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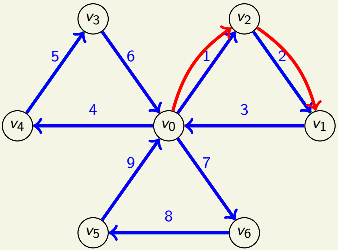

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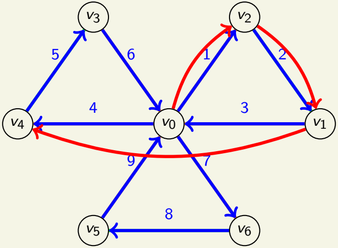

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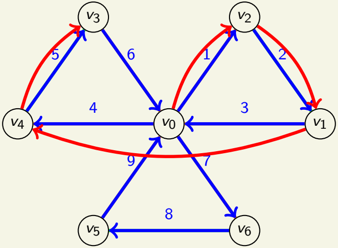

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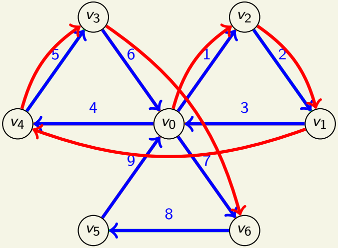

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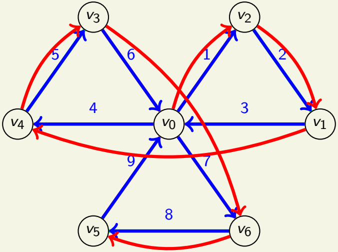

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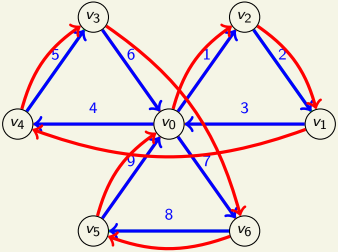

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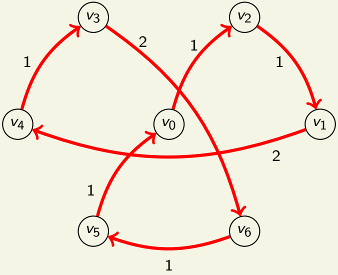
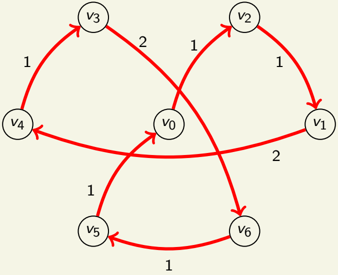


Christofides' algorithm tour cost: 9

Revisiting Tight Example for 2-Approximation Algorithm



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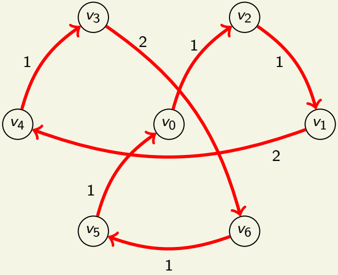


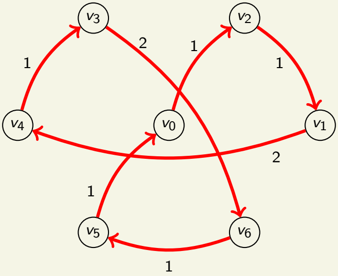
Christofides' algorithm tour cost: 9

Optimal tour cost: 7

Performance ratio: $\frac{9}{7} \leq \frac{3}{2}$

The performance depends on the Eulerian tour. With some Eulerian tours, the algorithm will obtain an optimal TSP tour.

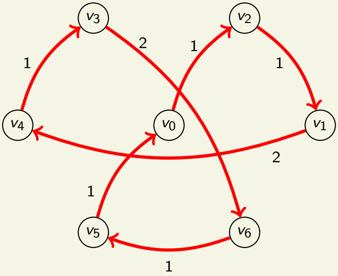
Revisiting Tight Example for 2-Approximation Algorithm



For the analogous graph on $n+1$ vertices with n even, Christofides' algorithm will obtain a tour with cost at most

$$\underbrace{2}_{v_0 \text{ edges}} \cdot 1 + \underbrace{\tfrac{n}{2}}_{\text{edges for pairs}} \cdot 1 + \underbrace{\left(\tfrac{n}{2} - 1\right)}_{\text{edges between pairs}} \cdot 2 = \frac{3n}{2}.$$

Revisiting Tight Example for 2-Approximation Algorithm



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The optimal tour has cost $n+1$, so the performance ratio is

$$\frac{\frac{3n}{2}}{n+1} = \frac{3}{2} \cdot \frac{n}{n+1}.$$