The numerical gradient is very simple to compute using the finite difference approximation, but the downside is that it is approximate (since we have to pick a small value of *h*, while the true gradient is defined as the limit as *h* goes to zero), and that it is very computationally expensive to compute. The second way to compute the gradient is analytically using Calculus, which allows us to derive a direct formula for the gradient (no approximations) that is also very fast to compute. However, unlike the numerical gradient it can be more error prone to implement, which is why in practice it is very common to compute the analytic gradient and compare it to the numerical gradient to check the correctness of your implementation. This is called a **gradient check**.

Lets use the example of the SVM loss function for a single datapoint:

$$L_i = \sum_{j \neq y_i} \left[\max(0, w_j^T x_i - w_{y_i}^T x_i + \Delta) \right]$$

We can differentiate the function with respect to the weights. For example, taking the gradient with respect to w_{y_i} we obtain:

$$\nabla_{w_{y_i}} L_i = - \left(\sum_{j \neq y_i} 1(w_j^T x_i - w_{y_i}^T x_i + \Delta > 0) \right) x_i$$

where 1 is the indicator function that is one if the condition inside is true or zero otherwise. While the expression may look scary when it is written out, when you're implementing this in code you'd simply count the number of classes that didn't meet the desired margin (and hence contributed to the loss function) and then the data vector x_i scaled by this number is the gradient. Notice that this is the gradient only with respect to the row of W that corresponds to the correct class. For the other rows where $j \neq y_i$ the gradient is:

$$\nabla_{w_j} L_i = 1(w_j^T x_i - w_{y_i}^T x_i + \Delta > 0) x_i$$

Once you derive the expression for the gradient it is straight-forward to implement the expressions and use them to perform the gradient update.

Gradient Descent

Now that we can compute the gradient of the loss function, the procedure of repeatedly evaluating the gradient and then performing a parameter update is called *Gradient Descent*. Its **vanilla** version looks as follows:

```
# Vanilla Gradient Descent
```

```
while True:
  weights_grad = evaluate_gradient(loss_fun, data, weights)
  weights += - step_size * weights_grad # perform parameter update
```

This simple loop is at the core of all Neural Network libraries. There are other ways of performing the optimization (e.g. LBFGS), but Gradient Descent is currently by far the most common and established way of optimizing Neural Network loss functions. Throughout the class we will put some bells and whistles on the details of this loop (e.g. the exact details of the update equation), but the core idea of following the gradient until we're happy with the results will remain the same.

Mini-batch gradient descent. In large-scale applications (such as the ILSVRC challenge), the training data can have on order of millions of examples. Hence, it seems wasteful to compute the full loss function over the entire training set in order to perform only a single parameter update. A very common approach to addressing this challenge is to compute the gradient over **batches** of the training data. For example, in current state of the art ConvNets, a typical batch contains 256 examples from the entire training set of 1.2 million. This batch is then used to perform a parameter update:

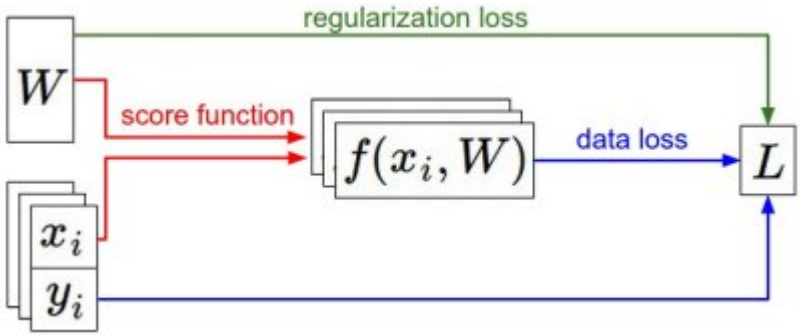
```
# Vanilla Minibatch Gradient Descent

while True:
  data_batch = sample_training_data(data, 256) # sample 256 examples
  weights_grad = evaluate_gradient(loss_fun, data_batch, weights)
  weights += - step_size * weights_grad # perform parameter update
```

The reason this works well is that the examples in the training data are correlated. To see this, consider the extreme case where all 1.2 million images in ILSVRC are in fact made up of exact duplicates of only 1000 unique images (one for each class, or in other words 1200 identical copies of each image). Then it is clear that the gradients we would compute for all 1200 identical copies would all be the same, and when we average the data loss over all 1.2 million images we would get the exact same loss as if we only evaluated on a small subset of 1000. In practice of course, the dataset would not contain duplicate images, the gradient from a mini-batch is a good approximation of the gradient of the full objective. Therefore, much faster convergence can be achieved in practice by evaluating the mini-batch gradients to perform more frequent parameter updates.

The extreme case of this is a setting where the mini-batch contains only a single example. This process is called **Stochastic Gradient Descent (SGD)** (or also sometimes **on-line** gradient descent). This is relatively less common to see because in practice due to vectorized code

optimizations it can be computationally much more efficient to evaluate the gradient for 100 examples, than the gradient for one example 100 times. Even though SGD technically refers to using a single example at a time to evaluate the gradient, you will hear people use the term SGD even when referring to mini-batch gradient descent (i.e. mentions of MGD for "Minibatch Gradient Descent", or BGD for "Batch gradient descent" are rare to see), where it is usually assumed that mini-batches are used. The size of the mini-batch is a hyperparameter but it is not very common to cross-validate it. It is usually based on memory constraints (if any), or set to some value, e.g. 32, 64 or 128. We use powers of 2 in practice because many vectorized operation implementations work faster when their inputs are sized in powers of 2.

Summary



Summary of the information flow. The dataset of pairs of **(x,y)** is given and fixed. The weights start out as random numbers and can change. During the forward pass the score function computes class scores, stored in vector **f**. The loss function contains two components: The data loss computes the compatibility between the scores **f** and the labels **y**. The regularization loss is only a function of the weights. During Gradient Descent, we compute the gradient on the weights (and optionally on data if we wish) and use them to perform a parameter update during Gradient Descent.

In this section,

- We developed the intuition of the loss function as a **high-dimensional optimization landscape** in which we are trying to reach the bottom. The working analogy we developed was that of a blindfolded hiker who wishes to reach the bottom. In particular, we saw that the SVM cost function is piece-wise linear and bowl-shaped.
- We motivated the idea of optimizing the loss function with **iterative refinement**, where we start with a random set of weights and refine them step by step until the loss is minimized.
- We saw that the **gradient** of a function gives the steepest ascent direction and we discussed a simple but inefficient way of computing it numerically using the finite difference approximation (the finite difference being the value of *h* used in computing the numerical gradient).