Methods for another implementation of the Trajectory Planner

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1 Parametrization of agent's trajectory generation

We assume an agent model which has built-in motor circuits for controlling its speed and heading. These circuits allow the agent to run a trajectory between 2 subgoal (candidate) target locations and only need to learn input parameterizations to these circuits that are most consistent with the environment outcome, as explained more in the Bayesian observer section. We suggest that these parameters are 1) The scale,V_m , for a continuous speed function, $V(t)$, that increases as the agent moves away from a starting point and decreases as it approaches the target subgoal. 2) The heading offset, ϕ_0, in the agent's heading function, $\phi(t)$; In particular, we define these generative functions as follows:

$$V_n(t) = V_m sin(\omega t); t \in [0 \leq t \leq \frac{T}{2}] \tag{1}$$

$$\phi_n(t) = mt + \phi_n(0); t \in [0 \leq t \leq \frac{T}{2}] \tag{2}$$

where $\omega = \frac{2\pi}{T}$. $\frac{T}{2}$ is the half period of the speed sinusoid and the time an agent takes to run a segment between 2 anchors.

The integration of Eq.1 and Eq.2 over time, t$\in[0,\frac{T}{2}]$, produces an arc-like segment that connects any given pair of consecutive anchors $\{r_n, \theta_n\}$ and $\{r_{n+1}, \theta_{n+1}\}$.

Given a set of N anchors, the agent will run $(N-1)$ segments between these anchors and will estimate 3 parameters for every segment, namely [V_m,m,and $\phi_n(0)$], which fully specify Eq. 1 and 2. Given an input list of anchors, we will next derive these quantities. First, let us define the angle of the vector connecting a pair of consecutive anchors $\{r_n, \theta_n\}$ and $\{r_{n+1}, \theta_{n+1}\}$ as S_{θ_n}, which is given by:

This equation is in line 23 in $version2/OptimizerClass - smoothness - and - PL.m$

$$S_{\theta_n} = \theta_n + \arctan\left[\frac{r_{n+1}\cdot\sin(\theta_{n+1}-\theta_n)}{r_{n+1}\cdot\cos(\theta_{n+1}-\theta_n)-r_n}\right] \tag{3}$$

The offset $\phi_n(0)$ in Eq.2 determines the agent's starting heading. This offset determines how much the agent deviates from running along the target vector connecting two anchors. We will refer to this deviation as ϕ_{0_n} and measure it relative to S_{θ_n}. Thus, if ϕ_{0_n} is positive, then $\phi_n(0) = S_{\theta_n} - \phi_{0_n}$ and the agent runs between the two anchors in an anticlockwise direction relative to S_{θ_n}, and if ϕ_{0_n} is negative then it is running in a clockwise direction relative to S_{θ_n}.
The value of ϕ_{0_n} determines the curvature of the arc connecting 2 anchors, for example, the agent runs a straight line between 2 anchors if its value is 0. As we will also see this quantity can be optimized, along with the anchors locations, to minimize the agent's total path length while maintaining a smooth trajectory.

So far Eq.2 becomes,

$$\phi_n(t) = mt + S_{\theta_n} - \phi_{0_n}; t \in [0 \le t \le \frac{T}{2}] \tag{4}$$

To determine the slope value in Eq.2, m, we assume that the agent's trace a trajectory with entrance and exit angles symmetric about S_{θ_n}, as follows:

$$\phi_n\left(\frac{T}{2}\right) = S_{\theta_n} + \phi_{0_n} \tag{5}$$

and since,

$$\phi_n\left(\frac{T}{2}\right) = m\frac{T}{2} + \phi_n(0) = m\frac{T}{2} + S_{\theta_n} - \phi_{0_n} \tag{6}$$

Thus, equating Eq.5 to Eq.6 gives m,

$$\begin{aligned} m &= \frac{2\left[S_{\theta_n} + \phi_{0_n} - S_{\theta_n} + \phi_{0_n}\right]}{T} \\ &= \frac{4\phi_{0_n}}{T} \end{aligned} \tag{7}$$

Plugging Eq.7 in Eq.4,we get the final form of the heading function

This equation is in line 92 in $version2/OptimizerClass - smoothness - and - PL.m$

$$\phi_n(t) = \frac{4\phi_{0_n}}{T}t + S_{\theta_n} - \phi_{0_n}; t \in [0 \le t \le \frac{T}{2}] \tag{8}$$

To fully specify the speed function described in Eq.1, we will need to derive the maximum speed, V_m, needed to connect two pairs of consecutive anchors separated by Euclidean distance D. For that we will integrate the x, y positions of the agent up to $t = T/2$ and equate it to the Euclidean distance D between the anchors pair,

This equation is in line 71 in $version2/OptimizerClass-smoothness-and-PL.m$

$$D_n = \sqrt{D_{x_n}^2 + D_{y_n}^2} \tag{9}$$

where D_{x_n} and D_{y_n} are given by

This equation is in line 46 in $version2/OptimizerClass-smoothness-and-PL.m$

$$D_{x_n} = (r_{n+1} \cdot cos(\theta_{n+1})) - (r_n \cdot cos(\theta_n)) \tag{10}$$

This equation is in line 49 in $version2/OptimizerClass-smoothness-and-PL.m$

$$D_{y_n} = (r_{n+1} \cdot sin(\theta_{n+1})) - (r_n \cdot sin(\theta_n)) \tag{11}$$

$$D_n = \sqrt{\left[Y\left(\frac{T}{2}\right) - Y_0\right]^2 + \left[X\left(\frac{T}{2}\right) - X_0\right]^2} \tag{12}$$

where,

$$\begin{aligned}
Y\left(\frac{T}{2}\right) &= \int_0^{\frac{T}{2}} V_m \sin\left(\frac{2\pi}{T}t\right) \sin\left(\frac{4\phi_{0_n}}{T}t + S_{\theta_n} - \phi_{0_n}\right) dt + Y_0 && (13)\\
&= I_y + Y_0 && (14)\\
&= \frac{V_m}{2}\left[\frac{T}{2\pi - 4\phi_{0_n}}\left(\sin(\pi - \phi_{0_n} - S_{\theta_n}) - \sin(-S_{\theta_n} + \phi_{0_n})\right)\right. && (15)\\
&\left. - \frac{T}{2\pi + 4\phi_{0_n}}\left(\sin(\pi + \phi_{0_n} + S_{\theta_n}) - \sin(S_{\theta_n} - \phi_{0_n})\right)\right] + Y_0 && (16)
\end{aligned}$$

Similarily for $X(\frac{T}{2})$,

$$\begin{aligned}
X\left(\frac{T}{2}\right) &= \int_0^{\frac{T}{2}} V_m \sin\left(\frac{2\pi}{T}t\right) \cos\left(\frac{4\phi_{0_n}}{T}t + S_{\theta_n} - \phi_{0_n}\right) dt + X_0 && (17)\\
&= I_x + X_0 && (18)\\
&= \frac{V_m}{2}\left[-\frac{T}{2\pi + 4\phi_{0_n}}\left(\cos(\pi + \phi_{0_n} + S_{\theta_n}) - \cos(S_{\theta_n} - \phi_{0_n})\right)\right. && (19)\\
&\left. - \frac{T}{2\pi - 4\phi_{0_n}}\left(\cos(\pi - \phi_{0_n} - S_{\theta_n}) - \cos(-S_{\theta_n} + \phi_{0_n})\right)\right] + X_0 && (20)
\end{aligned}$$

Substituting Eq.17 and 13 in Eq. 12 gives:

$$D_n = \sqrt{I_y^2 + I_x^2}$$
$$= \sqrt{\left(\frac{V_m \cdot \pi T \cdot (A)}{2(\pi^2 - 4\phi_{0_n}^2)}\right)^2 + \left(\frac{V_m \cdot \pi T \cdot (B)}{2(\pi^2 - 4\phi_{0_n}^2)}\right)^2} \tag{21}$$

where A and B are defined as follows,

$$A = sin(\phi_{0_n} + S_{\theta_n}) + sin(S_{\theta_n} - \phi_{0_n}) \tag{22}$$

$$B = cos(\phi_{0_n} + S_{\theta_n}) + cos(S_{\theta_n} - \phi_{0_n}) \tag{23}$$

Simplifying Eq. 21, we arrive at,

$$D_n = \frac{V_m \pi T \cos \phi_{0_n}}{\pi^2 - 4\phi_{0_n}^2} \tag{24}$$

so V_m becomes,

$$V_m = \frac{D_n.(\pi^2 - 4\phi_{0_n}^2)}{\pi T cos(\phi_{0_n})} \tag{25}$$

As seen from Eq.25 the maximum speed to connect 2 anchors increases with the Euclidean distance, D_n, between them.

We can also see that Eq.25 is undefined at $\phi_{0_n} = \pm\pi/2$, thus we will rewrite it to avoid indeterminacy by redefining ϕ_{0_n} as follows,

This equation is in line 58 and 67 in $version2/OptimizerClass - smoothness - and - PL.m$

$$\epsilon = |\phi_{0_n}| - \frac{\pi}{2} \tag{26}$$

To substitute Eq.26 in Eq.25, $\phi_{0_n} = (\epsilon + \frac{\pi}{2})$ or $\phi_{0_n} = -(\epsilon + \frac{\pi}{2})$. Plugging either solutions in Eq.25 yields the same result since $cos(x)$ is an even function, $cos(-x) = cos(x)$, as well $|x|^2 = x^2$, i.e what matters is the difference between the absolute value of ϕ_{0_n} and $\frac{\pi}{2}$ and not its sign.

Substituting ϕ_{0_n} from Eq.26 into 25 gives,

$$V_m = \frac{D_n(\pi^2 - 4(\epsilon + \frac{\pi}{2})^2)}{\pi T cos(\epsilon + \frac{\pi}{2})} \tag{27}$$

Since, $cos(x + \frac{\pi}{2}) = -sin(x)$,

$$
\begin{aligned}
V_m &= \frac{D_n(\pi^2 - 4(\epsilon^2 + \frac{\pi^2}{4} + \epsilon\pi))}{\pi T cos(\epsilon + \frac{\pi}{2})} \\
&= \frac{D_n(-4\epsilon^2 - 4\epsilon\pi)}{-\pi T sin(\epsilon)} \\
&= \frac{D_n(4\epsilon + 4\pi)}{\pi T sinc(\epsilon)}
\end{aligned} \tag{28}
$$

We can write T in Eq.28 as function of D_n if we assume a linear timing function given the distance between an anchor pair, such that $T_m = \frac{D_m}{\rho}$ where ρ is the distance covered per unit time. Here, the agent takes time of $T/2$ to cover distance D_n, thus we can write T as

This equation is in line 78 in $version2/OptimizerClass - smoothness - and - PL.m$

$$
T = \frac{2D_n}{\rho} \tag{29}
$$

Putting Eq.29 in Eq.28 yields,

This equation is in lines 73-75 in $version2/OptimizerClass - smoothness - and - PL.m$

$$
V_m = \frac{\rho(4\pi + 4\epsilon)}{2\pi sinc(\epsilon)} \tag{30}
$$

As seen from Eq. 30 the value of V_m is no longer undefined at ϕ_{0_n} and is defined over all values of $\phi_{0_n} \in [-\pi, \pi]$.

Finally we can substitute 30 into 1, to get the closed form for the speed function.

This equation is in line 89 and 90 in $version2/OptimizerClass - smoothness - and - PL.m$

$$
v(t) = \frac{\rho(4\pi + 4\epsilon)}{2\pi sinc(\epsilon)} sin\left(\frac{2\pi}{T}t\right) \tag{31}
$$

1.1 Generative functions optimization for path length and smoothness

We want the agent to run smooth but short paths thus we define our loss function to balance path length optimization while minimizing the angular changes at anchors.

The total cost for an n-anchor trajectory is the sum of lengths of $(N-1)$ segments and the sum of angular changes at $(N-2)$ anchors (exit angle from the current anchor - arrival angle from the segment connected to the previous anchor), excluding home anchors since there are no anchors before or after them.

This equation is in line 138 in $version2/OptimizerClass - smoothness - and - PL.m$

$$L = w_1 \sum_{n=1}^{N-1} pl(n) + w_2 \kappa \quad (32)$$

To determine the path length of the n^{th} segment between the anchors (n) and $(n+1)$, we numerically integrate the second norm of changes in the (x_t^n, y_t^n) positions, as follows.

This equation is in line 109 in $version2/OptimizerClass - smoothness - and - PL.m$

$$pl(n) = \sum_{t=1}^{\frac{T}{2}} (\sqrt{(x_t^n - x_{t-1}^n)^2 + (y_t^n - y_{t-1}^n)^2} \quad (33)$$

We determine the angular changes at anchor points as the cost of angular discontinuity and sudden heading change that the agent will pay to minimize its total path length, it is given by

This equation is in line 135 in $version2/OptimizerClass - smoothness - and - PL.m$

$$\kappa = \sum_{n=2}^{N-1} K(n) \quad (34)$$

where N is total number of anchors, and $K(n)$ is,

This equation is in line 35-38 in $version2/OptimizerClass - smoothness - and - PL.m$

$$\begin{aligned} K(n) &= |\phi_n(0) - \phi_{n-1}\left(\frac{T}{2}\right)| \\ &= \left|[S_{\theta_n} - \phi_{0_n}] - [S_{\theta_{n-1}} + \phi_{0_{n-1}}]\right| \end{aligned} \quad (35)$$

We minimize Eq.32 with respect to the anchors radial distances and angles (r_n, θ_n), headings deviations from the vector connecting each anchor pair (ϕ_{0_n}) while satisfying the following non-linear constraints: 1) anchors locations can move around their initial values (r_{0_n}, θ_{0_n}) but within a circle of radius tol_r 2) the anchors do not move outside the arena's bounds.

We compute the first constraint for every anchor C_1^n by calculating the distance between the displaced anchor (r_n, θ_n) and a circle with radius (tol_r) centered around its original location (r_{0_n}, θ_{0_n}). The distances are normalized by the radial scale Rg_r and the angular scale Rg_t to account for different scales, as follows:

This equation is in line 123-130 in $version2/optimize - path - length - and - smoothness.m$

$$C_1^n = \left(\frac{r_n - r_{0_n}}{Rg_r}\right)^2 + \left(\frac{\theta_n - \theta_{0_n}}{Rgt}\right)^2 - tol_r \quad (36)$$