Ex1

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1 Prove / Disprove:

1. $2^{\sqrt{\log(n)}} \in \Theta(n)$

   Disproof:

   $$\lim_{n\to\infty} \frac{2^{\sqrt{\log n}}}{n} = \lim_{n\to\infty} \frac{2^{\sqrt{\log n}}}{2^{\log n}} = \lim_{n\to\infty} 2^{\sqrt{\log n}-\log n}$$

   But we know that $\lim_{n\to\infty} \sqrt{\log n} - \log n = \lim_{n\to\infty} \sqrt{\log n} \cdot \left(1 - \sqrt{\log n}\right) \longrightarrow -\infty$ (because $\sqrt{\log n} \longrightarrow \infty$ and $\left(1 - \sqrt{\log n}\right) \longrightarrow -\infty$)

   So overall we got $\lim_{n\to\infty} 2^{\sqrt{\log n}-\log n} = 0$ therefore $\lim_{n\to\infty} \frac{2^{\sqrt{\log n}}}{n} = 0$

   And because of that (according to the lecture), $2^{\sqrt{\log(n)}} \notin \Theta(n)$

   ■

2. $2^{\sqrt{\log(n)}} \in \omega\left(\log^{10} n\right)$

   Proof:

   $$\lim_{n\to\infty} \frac{2^{\sqrt{\log n}}}{\log^{10} n} = \lim_{n\to\infty} \frac{2^{\sqrt{\log n}}}{2^{\log(\log^{10} n)}} = \lim_{n\to\infty} 2^{\sqrt{\log n}-\log(\log^{10} n)} \lim_{n\to\infty} 2^{\sqrt{\log n}-10\log(\log n)}$$

   But we also know that $\lim_{n\to\infty} \sqrt{\log n} - 10\log(\log n) = \lim_{n\to\infty} \sqrt{\log n}\left(1 - \frac{10\log(\log n)}{\sqrt{\log n}}\right)$

   We can also see that $\lim_{n\to\infty} \frac{10\log(\log n)}{\sqrt{\log n}} \underset{\text{L'Hôpital's rule}}{=} \lim_{n\to\infty} \frac{10 \cdot \frac{1}{n\ln 2} \cdot \frac{1}{\log n}}{\frac{1}{n\ln 2} \cdot \frac{1}{2\sqrt{\log n}}} =$ $\lim_{n\to\infty} \frac{20\sqrt{\log n}}{\log n} = \lim_{n\to\infty} \frac{20}{\sqrt{\log n}} = 0$

Therefore $\lim_{n\to\infty} \frac{2^{\sqrt{\log n}}}{\log^{10} n} = \lim_{n\to\infty} \sqrt{\log n}\left(1 - \frac{10\log(\log n)}{\sqrt{\log n}}\right) =$ $\lim_{n\to\infty} \sqrt{\log n} = \infty$ And according to the Tirgul, we can say that $2^{\sqrt{\log(n)}} \in \omega\left(\log^{10} n\right)$

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3. $\frac{n}{2}\log\frac{n}{2} \in \Omega(n\log n)$

   Proof:

   $\lim_{n\to\infty} \frac{\frac{n}{2}\log\frac{n}{2}}{n\log n} = \frac{1}{2}\lim_{n\to\infty}\frac{\log\frac{n}{2}}{\log n} \underset{\text{(L'Hôpital's rule)}}{=} \frac{1}{2}\lim_{n\to\infty}\frac{\frac{1}{2}\cdot\frac{2}{n}}{\frac{1}{n}} = \frac{1}{2}\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n}} =$ $\lim_{n\to\infty}\frac{1}{2} = \frac{1}{2} > 0$. So overall (according to the lecture) we got $\frac{n}{2}\log\frac{n}{2} \in \Theta(n\log n)$ Thus in particular $\frac{n}{2}\log\frac{n}{2} \in \Omega(n\log n)$

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2 Let f, g be two positive functions. Prove/Disprove:

1. if f and g are monotonic, then $f \in O(g)$ or $g \in O(f)$
   Disproof:

   Let f, g be the following:

   $$f(n) = \begin{cases} n^n, & \text{if n is even} \\ (n-1)^{n-1}, & \text{if n is odd} \end{cases} \quad g(n) = \begin{cases} n^n, & \text{if n is odd} \\ (n-1)^{n-1}, & \text{if n is even} \end{cases}$$

   We will start by proving they are monotonic.
   let j be an even integer, let m be an odd integer.
   $f(j) = j^j, f(j+1) = j^j = f(j)$.
   $f(m) = (m-1)^{m-1} > f(m+1) = (m+1)^{m+1}$. So overall $f(n)$ is decreasing (weak term) for every n.
   $g(j) = (j-1)^{j-1} < (j+1)^{j+1} = g(j+1).g(m) = m^m = g(m+1)$ So overall $g(n)$ is increasing (weak term) for every n.

   We will now prove that $f \notin O(g)$. Suppose there are $c \in \mathbb{R}^+, N_0 \in \mathbb{N}$ which for every $n > N_0$, $f(n) \leq\ c \cdot g(n)$. Let $n_1 > 0$ be an even integer. $f(n) = n_1^{n_1} \leq c \cdot g(n_1) = c \cdot (n_1 - 1)^{n_1 - 1}$. Contradiction! there is not a real positive number c which applies for the above statements for every even integer that is bigger than N_0.

We will now prove that $g \notin O(f)$. Suppose there are $\tilde{c} \in \mathbb{R}, \tilde{N_0} \in \mathbb{N}$ which for every $n > \tilde{N_0}$, $g(n) \leq \tilde{c} \cdot f(n)$. Let $n_0 > \tilde{N_0}$ be an odd integer. $g(n_0) = n_0^{n_0} \leq c \cdot f(n_0) = (n_0 - 1)^{n_0 - 1}$ Contradiction! As we just saw, there is not a real number c which is suitable for that.

$\blacksquare$

2. $\Theta(max(f(n), g(n))) = \Theta(f(n) + g(n))$
   Proof:

   $\subseteq$
   Let $\varphi(n)$ be in $\Theta(max(f(n), g(n)))$.
   That means that there are $c_1, c_2, N_0 \geq 0$ that for every $n > N_0$:
   $c_1 \cdot max(f(n), g(n)) \leq \varphi(n) \leq c_2 \cdot max(f(n), g(n))$ And because of that we see that $\varphi(n) \leq c_2 \cdot (f(n) + g(n))$. So that is why $\varphi(n) \in O(f(n) + g(n))$. But we also know that $c_1 \cdot max(f(n), g(n)) \leq \varphi(n)$ for every integer from a certain point. So $\frac{c_1}{2} \cdot 2 \cdot max(f(n), g(n)) \leq \varphi(n)$. That is why $\frac{c_1}{2} \cdot (f(n) + g(n)) \leq \varphi(n)$ and $\varphi(n) \in \Omega(f(n) + g(n))$. Overall we got $\varphi(n) \in \Theta(f(n) + g(n))$.$\square$

   $\supseteq$
   Let $\psi(n)$ be in $\Theta(f(n) + g(n))$. That means that there are $c_3, c_4, N_1 > 0$ That make for every $n > N_1, c_3(f(n) + g(n)) \leq \psi(n) \leq c_4(f(n) + g(n))$. hence $\psi(n) \leq c_4(f(n)+g(n)) \leq c_4(max(f(n), g(n))+max(f(n), g(n))) \Rightarrow \psi(n) \leq \frac{c_4}{2} \cdot max(f(n), g(n))$ That is why $\varphi(n) \in O(max(f(n), g(n)))$. Let's look at $c_3(f(n)+g(n)) \leq \psi(n)$.We can say that $\psi(n) \geq c_3(f(n)+g(n)) \geq c_3 \cdot max(f(n), g(n))$. So we also got $\psi(n) \in \Omega(max(f(n), g(n)))$. Overall, $\psi(n) \in \Theta(max(f(n), g(n)))$.

   $\blacksquare$

3. If $f(n) \in \Theta(g(n))$ So $\omega(f(n)) = \omega(g(n))$
   Proof:

   Let $\psi(n)$ be in $\omega(f(n))$.
   That means that for every $c \in \mathbb{R} \geq 0$ there is $N_0 \in \mathbb{N}$ which makes for every $n > N_0$, $\psi(n) \geq c \cdot f(n)$. But we know that $f(n) \in \omega(g(n))$. That means that there is $c_1 \in \mathbb{R} \geq 0$ such that $f(n) \geq c_1 \cdot g(n)$. Hence we get $\psi(n) \geq c \cdot c_1 \cdot g(n)$. But c can be every single positive real number, so $c \cdot c_1$ can be every single positive real number, and that proves $\psi(n) \in \omega(g(n))$ i.e $\omega(f(n)) \subseteq \omega(g(n))$.
   Because $f(n) \in \Theta(g(n))$, we can also say $g(n) \in \Theta(f(n))$(By moving the constants to the other side of the equation).

Without any loss of generality, we can state that $\omega(g(n)) \subseteq \omega(f(n))$.
By two-directional inclusion, $\omega(g(n)) = \omega(f(n))$.

$\blacksquare$

3 Determine the runtime of the code, find Θ:

1. The outside loop runs $n-1$ times. The inside one runs $\log_i n$ for every iteration of the outside loop. Overall the runtime is $T(n) = \sum_{i=2}^{n} \log_i n$. $T(n) \leq$
$\sum_{i=2}^{\sqrt{n}} \log_i n + \sum_{i=\sqrt{n}+1}^{n} \log_i n \underset{\text{If the base of the log is lower, the outcome is bigger}}{\leq}$
$\sqrt{n} \cdot \log_2 n + (n - \sqrt{n}) \cdot \log_{\sqrt{n}} n \underset{\log_2 n<\sqrt{n} \text{ for every n from a certain point}}{<}$
$\sqrt{n} \cdot \sqrt{n} + \left(n - \sqrt{n}\right) \cdot 2 = n + 2n - 2\sqrt{n} < 3n.$

   On the other hand, $T(n) \underset{\text{If the base of the log is lower, the outcome is bigger}}{\geq}$
$(n-1) \cdot \log_n n = n - 1 \underset{\text{for every } n>1}{\geq} \frac{1}{2}n.$

   To conclude, we know that $\frac{1}{2}n \leq T(n) \leq 3n$ for every n from a certain point, therefore $T(n) \in \Theta(n)$

$\blacksquare$

2. $T(n) = \sum_{i=1}^{n^2} \sum_{j=1}^{i} \sum_{k=1}^{j} 1 = \sum_{i=1}^{n^2} \sum_{j=1}^{i} j = \sum_{i=1}^{n^2} \frac{i(i+1)}{2} = \frac{1}{2} \sum_{i=1}^{n^2} i^2 + \frac{1}{2} \sum_{i=1}^{n^2} i = \frac{1}{2} \cdot$
$\frac{n^2\left(n^2+1\right)\left(2n^2+1\right)}{6} + \frac{1}{2} \cdot \frac{n^2\left(n^2+1\right)}{2} = \frac{n^6+3n^4+2n^2}{6} \leq \frac{n^6+3n^6+2n^6}{6}$
$= n^6$

   But also $\frac{n^6+3n^4+2n^2}{6} \geq \frac{n^6}{6}$
So overall we got $\frac{n^6}{6} \leq T(n) \leq n^6$

   Therefore by definition, $T(n) \in \Theta(n^6)$

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