SCIM223
AY25/26 SEM 1
nadear

Cylindrical and Spherical coordinate system

Cylindrical $\to$ Rectangular $(r,\theta,z)\mapsto(x,y,z)$

$$x=r\cos\theta,\qquad y=r\sin\theta,\qquad z=z.$$

Rectangular $\to$ Cylindrical $(x,y,z)\mapsto(r,\theta,z)$

$$r=\sqrt{x^2+y^2},\qquad \tan\theta=\frac{y}{x},\qquad z=z.$$

Spherical $\to$ Cylindrical $(\rho,\theta,\phi)\mapsto(r,\theta,z)$

$$r=\rho\sin\phi,\qquad \theta=\theta,\qquad z=\rho\cos\phi.$$

Cylindrical $\to$ Spherical $(r,\theta,z)\mapsto(\rho,\theta,\phi)$

$$\rho=\sqrt{r^2+z^2},\qquad \theta=\theta,\qquad \tan\phi=\frac{r}{z}.$$

Spherical $\to$ Rectangular $(\rho,\theta,\phi)\mapsto(x,y,z)$

$$x=\rho\sin\phi\cos\theta,\qquad y=\rho\sin\phi\sin\theta,\qquad z=\rho\cos\phi.$$

Rectangular $\to$ Spherical $(x,y,z)\mapsto(\rho,\theta,\phi)$

$$\rho=\sqrt{x^2+y^2+z^2},\tan\theta=\frac{y}{x},\cos\phi=\frac{z}{\sqrt{x^2+y^2+z^2}}$$

Restrictions

$$r\ge 0,\quad \rho\ge 0,\quad 0\le\theta<2\pi,\quad 0\le\phi\le\pi.$$

Tangent Plane

Equation of Tangent Plane

Tangent plane at $(a,b,f(a,b))$ Suppose $f(x,y)$ has continuous first partial derivatives at (a,b). A normal vector to the tangent plane is

$$\langle f_x(a,b),\ f_y(a,b),\ -1\rangle.$$

Further, an equation of the tangent plane is given by

$$f_x(a,b)\,(x-a)+f_y(a,b)\,(y-b)-\big(z-f(a,b)\big)=0,$$

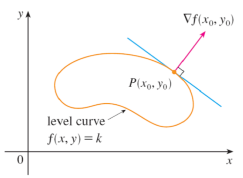
or

$$z\ =\ f(a,b)+f_x(a,b)\,(x-a)+f_y(a,b)\,(y-b).$$

Gradient Vector

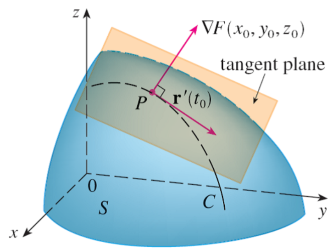
Level Curve vs ∇f

Suppose $f(x,y)$ is differentiable at (x_0,y_0).
Suppose $\nabla f(x_0,y_0)\neq\mathbf{0}$.
Then $\nabla f(x_0,y_0)$ is normal to the level curve $f(x,y)=k$ that contains the point (x_0,y_0).



Level Surface vs ∇f

Suppose $F(x,y,z)$ is a differentiable function of x, y and z at (x_0,y_0,z_0). Suppose $\nabla F(x_0,y_0,z_0)\neq 0$. Then $\nabla F(x_0,y_0,z_0)$ is normal to the level surface $\nabla F(x,y,z)=k$ that contains the point (x_0,y_0,z_0).



Tangent Plane to Level Surface

$\nabla F(x_0,y_0,z_0)\cdot\langle x-x_0,\ y-y_0,\ z-z_0\rangle=0$

Maximizing Rate of Increase/Decrease of f

Suppose f is a differentiable function of two or three variables. Let P denote a given point. Assume $\nabla f(P)\neq\mathbf{0}$. Let $\mathbf{u}$ be a unit vector making an angle θ with ∇f. Then

$$D_{\mathbf{u}}f(P)=\|\nabla f(P)\|\cos\theta.$$

Moreover,

- $\nabla f(P)$ points in the direction of maximum rate of change of f at P (maximum value of $D_{\mathbf{u}}f(P)$ is $\|\nabla f(P)\|$).
- $-\nabla f(P)$ points in the direction of minimum rate of change of f at P (minimum value of $D_{\mathbf{u}}f(P)$ is $-\|\nabla f(P)\|$).

Extrema of Functions of Two Variables

Local and Absolute Maximum

Let $f(x,y):D\to\mathbb{R}$. Then

- f has a local maximum at (a,b) if $f(x,y)\le f(a,b)$ for all points in some disk with center (a,b). The number $f(a,b)$ is called a local maximum value.
- f has an absolute maximum at (a,b) if $f(x,y)\le f(a,b)$ for **all points in the domain** D. The number $f(a,b)$ is called an absolute maximum value.

Local and Absolute Minimum

Let $f(x,y):D\to\mathbb{R}$. Then

- f has a local minimum at (a,b) if $f(x,y)\ge f(a,b)$ for all points in some disk with center (a,b). The number $f(a,b)$ is called a local minimum value.
- f has an absolute minimum at (a,b) if $f(x,y)\ge f(a,b)$ for **all points in the domain** D. The number $f(a,b)$ is called an absolute minimum value.

A necessary condition

If f has a local maximum or minimum at (a,b) and the first–order derivatives of f exist there, then

$$f_x(a,b)=f_y(a,b)=0.$$

Critical Point

Critical or Stationary Point

Let $f(x,y):D\to\mathbb{R}$. Then a point (a,b) is called a critical point of f if

$$f_x(a,b)=0\quad\text{and}\quad f_y(a,b)=0.$$

Saddle Point

Let $f(x,y):D\to\mathbb{R}$. Then a point (a,b) is called a saddle point of f if

- it is a critical point of f, **AND**
- every open disk centered at (a,b) contains points $(x,y)\in D$ for which $f(x,y)<f(a,b)$ and points $(x,y)\in D$ for which $f(x,y)>f(a,b)$.

Lagrange Multiplier

Lagrange Multipliers for Function of Two Variables

Suppose $f(x,y)$ and $g(x,y)$ are differentiable and $\nabla g(x,y)\neq\mathbf{0}$ on the constraint curve $g(x,y)=k$. Suppose the minimum/maximum of $f(x,y)$ subject to the constraint $g(x,y)=k$ occurs at (x_0,y_0). Then

$$\nabla f(x_0,y_0)=\lambda\,\nabla g(x_0,y_0)$$

for some constant λ (a *Lagrange Multiplier*).

Lagrange Multiplier - Three Variables

Suppose $f(x,y,z)$ and $g(x,y,z)$ are differentiable functions such that $\nabla g(x,y,z)\neq\mathbf{0}$ on the constraint surface $g(x,y,z)=k$. Suppose that the minimum/maximum value of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$ occurs at (x_0,y_0,z_0). Then

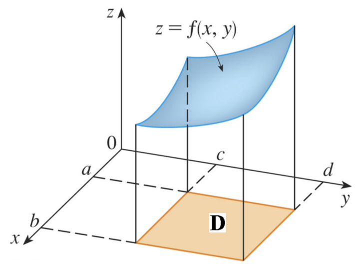
$$\nabla f(x_0,y_0,z_0)=\lambda\,\nabla g(x_0,y_0,z_0)$$

for some constant λ (called a *Lagrange Multiplier*).

Double Integral

Double Integral over Rectangle

Let S be the solid that lies above R and under the graph of f. How can we find the volume of S?



The double integral of f over the rectangle R is

$$\iint_R f(x,y)\,dA=\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}f(x_{ij}^*,y_{ij}^*)\,\Delta A,$$

provided the limit exists and is the same for any choice of the sample points (x_{ij}^*,y_{ij}^*) in R_{ij}, for $1\le i\le m$, $1\le j\le n$.

Double Integral over General Region

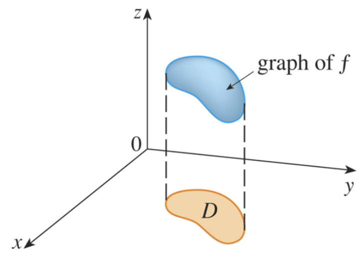
We define the double integral of f over a general region D by

$$\iint_D f(x,y)\,dA\ =\ \iint_R F(x,y)\,dA.$$

Volume as a Double Integral

If $f(x,y)\ge 0$, the volume V of the solid that lies above the region D and below the surface $z=f(x,y)$ is

$$V=\iint_D f(x,y)\,dA.$$



Think of the solid S as the region consisting of points between D and the surface corresponding to f . Then the volume of S is given by the double integral of f over D.

Iterated Double Integral

$$\int_a^b\int_c^d f(x,y)\,dy\,dx$$

means we first integrate with respect to y from c to d (keeping x fixed) and then with respect to x from a to b.

$$\int_c^d\int_a^b f(x,y)\,dx\,dy$$

means we first integrate with respect to x from a to b (keeping y fixed) and then with respect to y from c to d.

Fubini's Theorem

If f is continuous on the rectangle $R=[a,b]\times[c,d]$, then

$$\iint_R f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d\int_a^b f(x,y)\,dx$$

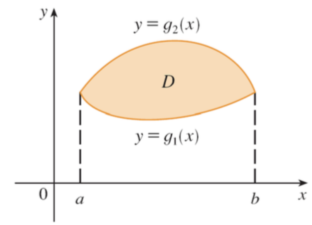
Double Integral over Type I & Type II Regions

Type I Region

A plane region D is said to be of Type I if it lies between the graphs of two continuous functions of x, that is,

$$D=\{(x,y):\ a\le x\le b,\ \ g_1(x)\le y\le g_2(x)\},$$

where $g_1(x)$ and $g_2(x)$ are continuous on $[a,b]$.

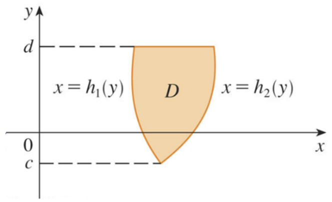


Type II Region

A plane region D is said to be of Type II if it lies between the graphs of two continuous functions of y, that is,

$$D = \{(x, y) : \ c \le y \le d, \ \ h_1(y) \le x \le h_2(y)\},$$

where $h_1(y)$ and $h_2(y)$ are continuous on $[c, d]$.



Double Integral over Type I & Type II Regions

Double Integral over Type I Domain

If f is continuous on a Type I domain D such that

$$D = \{(x, y) : \ a \le x \le b, \ \ g_1(x) \le y \le g_2(x)\},$$

then

$$\iint_D f(x, y)\, dA \ = \ \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

Double Integral over Type II Domain

If f is continuous on a Type II domain D such that

$$D = \{(x, y) : \ c \le y \le d, \ \ h_1(y) \le x \le h_2(y)\},$$

then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

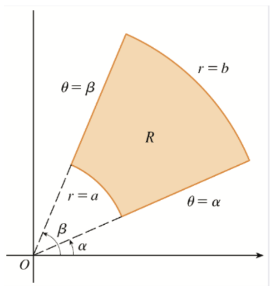
Double Integrals in Polar Coordinates

Relationship between (r, θ) and (x, y)

$$r^2 = x^2 + y^2, \qquad x = r\cos\theta, \qquad y = r\sin\theta.$$

Polar Rectangle

$$R = \{(r, \theta) : \ a \le r \le b, \ \alpha \le \theta \le \beta\}.$$



Change to Polar Coordinates in Double Integral

If f is continuous on a polar rectangle R given by

$$R = \{(r, \theta) : \ 0 \le a \le r \le b, \ \alpha \le \theta \le \beta\},$$
where $0 \le \beta - \alpha \le 2\pi$,

then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, \ r\sin\theta)\ r\, dr\, d\theta.$$

Polar Regions I

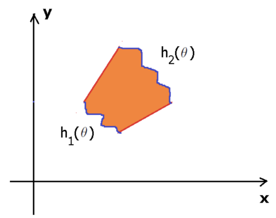
If f is continuous on a polar region D given by

$$D = \{(r, \theta) : \ 0 \le a \le r \le b, \ \ g_1(r) \le \theta \le g_2(r)\},$$

then

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(r)}^{g_2(r)} f(r\cos\theta, \ r\sin\theta)\ r\, d\theta\, dr.$$

$$D = \{(r, \theta) : \ \ \alpha \le \theta \le \beta, \ \ h_1(\theta) \le r \le h_2(\theta)\}.$$



Polar Regions II

If f is continuous on a polar region D given by

$$D = \{(r, \theta) : \ \alpha \le \theta \le \beta, \ \ h_1(\theta) \le r \le h_2(\theta)\},$$

then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, \ r\sin\theta)\ r\, dr\, d\theta.$$

Triple Integral

Fubini's Theorem for Triple Integral

If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz.$$

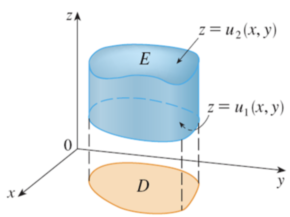
Furthermore, the iterated integral may be evaluated in any order.

Type 1 Region

A solid region E is of Type 1 if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z) : \ (x, y) \in D, \ u_1(x, y) \le z \le u_2(x, y)\},$$

where D is the projection of E onto the xy-plane.



Triple Integral over Type 1 region E

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA.$$

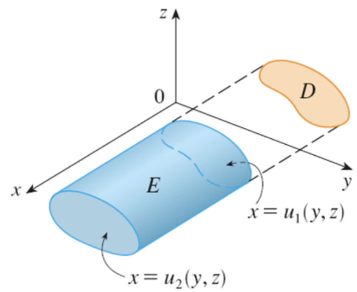
- The most inner integral is with respect to z from $u_1(x, y)$ to $u_2(x, y)$.
- dA is either $dx\, dy$ or $dy\, dx$ depending on how we describe D.

Type 2 Region

A solid region E is of Type 2 if it lies between the graphs of two continuous functions of y and z, that is,

$$E = \{(x, y, z) : \ (y, z) \in D, \ u_1(y, z) \le x \le u_2(y, z)\},$$

where D is the projection of E onto the yz-plane.



Triple Integral over Type 2 region E

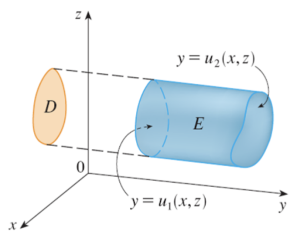
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA.$$

Type 3 Region

A solid region E is of Type 3 if it lies between the graphs of two continuous functions of x and z, that is,

$$E = \{(x, y, z) : \ (x, z) \in D, \ u_1(x, z) \le y \le u_2(x, z)\},$$

where D is the projection of E onto the xz-plane.



Triple Integral over Type 3 region E

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA.$$

Volume of a Solid

Let $f(x, y, z) = 1$ for all $(x, y, z) \in E$ where E is a solid. Then the triple integral of f over E gives the volume V of E:

$$\text{Volume of } E = \iiint_E 1\, dV$$