SCIM104
AY24/25 SEM 2
skibidi miimi

First-Order DEs

You can solve a first-order differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

by multiplying every term by the **integrating factor**

$$e^{\int P(x)\,dx}.$$

Second-Order Differential Equations

1. The nature of the roots α and β of the *auxiliary equation* determines the **general solution** to **Second-order homogeneous differential equations**

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

The **natures** of the roots α and β of the auxiliary equation

$$am^2 + bm + c = 0$$

- **Case 1:** $b^2 > 4ac$
  two real roots α and β ($\alpha \neq \beta$). The general solution :
  $$y = Ae^{\alpha x} + Be^{\beta x}$$
- **Case 2:** $b^2 = 4ac$
  one repeated root α. The general solution:
  $$y = (A + Bx)e^{\alpha x}$$
- **Case 3:** $b^2 < 4ac$
  two complex conjugate roots α and β equal to $p \pm qi$. The general solution:
  $$y = e^{px}(A\cos qx + B\sin qx)$$

2. To find the general solution to **non-homogeneous differential equation.**

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Form of $f(x)$	Form of particular integral
p	λ
$p + qx$	$\lambda + \mu x$
$p + qx + rx^2$	$\lambda + \mu x + \nu x^2$
pe^{kx}	λe^{kx}
$p\cos\omega x + q\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$

- Choose an appropriate form for the particular integral (P.I.) and substitute into the original equation to find the values of any coefficients.
- The general solution is:
  $$y = \text{C.F.} + \text{P.I.}$$

4. You can use a given substitution to reduce second-order differential equations into differential equations of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Vector Operations

- **Scalar Multiplication:**
  $$c\mathbf{v} = \langle cv_1,\, cv_2\rangle.$$
- **Dot Product:**
  $$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2.$$
- **Cross Product:**
  $$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}.$$

Vector Field

$$\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j} = \langle P(x,y), Q(x,y)\rangle.$$

Gradient Fields

If $f(x,y)$ is a scalar function, its gradient is a vector field:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Directional Derivative

Gradient:

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

Directional Derivative:

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y)\cdot\mathbf{u}$$

where if $\mathbf{u} = \langle a, b\rangle$,

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b.$$

Use unit vector:

$$\hat{\mathbf{u}} = \frac{1}{\sqrt{x^2+y^2+z^2}}(x,y,z) \quad ***$$

The maximum of $D_{\mathbf{u}}f$ is $|\nabla f|$, direction is given by ∇f.
- If $\nabla f = 0$, then f is locally constant.
- If $\mathbf{F} = \nabla f$, f is called a **potential function**.
- Conservative field: $\mathbf{F} = -\nabla f$.

Divergence

The **divergence** of a vector field

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

on $\mathbb{R}^3$ is given by:

$$\text{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

- If $\text{div}\,\mathbf{F} > 0$, the field behaves like a source.
- If $\text{div}\,\mathbf{F} < 0$, the field behaves like a sink.
- If $\text{div}\,\mathbf{F} = 0$, the field is incompressible.

Curl

If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of its components exist, then the curl of $\mathbf{F}$ is given by:

$$\text{curl}\,\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$.

$$\text{curl}\,\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

- If $\text{curl}\,\mathbf{F} = 0$, the field is irrotational (no local rotation).
- If $\text{curl}\,\mathbf{F} \neq 0$, the field has a swirling or rotational effect.

Laplace Operator

For a scalar function $f(x,y,z)$, the Laplacian is defined as:

$$\nabla^2 f = \nabla\cdot(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

commonly used in physics, in Laplace's equation: $\nabla^2 f = 0$.

Line Integrals

$$\int_C f(x,y)\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i,$$

provided the limit exists. Using the arc length formula:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt,$$

we obtain:

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds.$$

Line Integrals in Space

If C is a smooth space curve given by parametric equations

$$x = x(t),\quad y = y(t),\quad z = z(t),\quad a \le t \le b,$$

and f is a function of three variables, then the line integral of f along C is

$$\int_C f(x,y,z)\,ds$$

$$= \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$$

Observe that the integrals can be written in the more compact vector notation:

$$\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|\,dt.$$

Line Integral of Vector Fields

If $\mathbf{F}$ is a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \le t \le b$, then its line integral of $\mathbf{F}$ along C is given by:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt.$$

Double and Surface Integrals

Double Integrals

If f is continuous on the rectangle $R = [a,b]\times[c,d]$, then

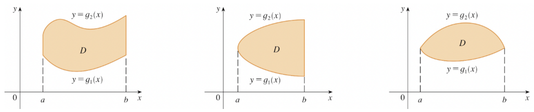
$$\iint_R f(x,y)\,dA = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$$

$$= \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy.$$

Double Integrals over General Regions

Type I

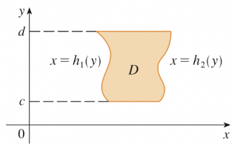
$$\iint_R f(x,y)\,dA = \int_a^b\left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right)dx$$



$$R = \left\{(x,y)\,\middle|\, a \le x \le b,\ g_1(x) \le y \le g_2(x)\right\}.$$

Type II

$$\iint_R f(x,y)\,dA = \int_c^d\left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right)dy$$



$$R = \left\{(x,y)\,\middle|\, c \le y \le d,\ h_1(y) \le x \le h_2(y)\right\}.$$

Surface Integral

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k},\quad (u,v)\in D.$$

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\,\|\mathbf{r}_u\times\mathbf{r}_v\|\,dA.$$

With the formula for a line integral:

$$\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|\,dt.$$

$$\iint_S 1\,dS = \iint_D \|\mathbf{r}_u\times\mathbf{r}_v\|\,dA = A(S).$$

Graphical Surfaces

Any surface S with equation $z = g(x,y)$ can be regarded as a parametric surface with parametric equations:

$$x = x,\quad y = y,\quad z = g(x,y)$$

and so we have

$$\mathbf{r}_x\times\mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_x\times\mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}.$$

Therefore, in this case, the surface integral becomes:

$$\iint_S f(x,y,z)\,dS$$

$$= \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA.$$

Differentiation

$\mathbf{f(x)}$	$\mathbf{f'(x)}$
$\sin x$	$\cos x$
$\cos x$	$sin x$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x\tan x$
$\cot x$	$-\csc^2 x$
$\csc x$	$-\csc x\cot x$
$\dfrac{f(x)}{g(x)}$	$\dfrac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$

Integration (+ constant)

$\mathbf{f(x)}$	$\int f(x)\,\mathrm{d}x$
$\sec^2 kx$	$\frac{1}{k}\tan kx$
$\tan x$	$\ln\lvert\sec x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$
$\csc x$	$-\ln\lvert\csc x+\cot x\rvert, \quad \ln\left\lvert\tan\left(\frac{1}{2}x\right)\right\rvert$
$\sec x$	$\ln\lvert\sec x+\tan x\rvert, \quad \ln\left\lvert\tan\left(\frac{1}{2}x+\frac{\pi}{4}\right)\right\rvert$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad (\lvert x\rvert<a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left(x+\sqrt{x^2-a^2}\right) \quad (x>a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left(x+\sqrt{x^2+a^2}\right)$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert \quad (\lvert x\rvert<a)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv-\int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Trigonometric identities

$$\sin(A\pm B)\equiv\sin A\cos B\pm\cos A\sin B$$
$$\cos(A\pm B)\equiv\cos A\cos B\mp\sin A\sin B$$
$$\tan(A\pm B)\equiv\frac{\tan A\pm\tan B}{1\mp\tan A\tan B} \quad \left(A+B\neq\left(k+\frac{1}{2}\right)\pi\right)$$
$$\sin A+\sin B\equiv 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin A-\sin B\equiv 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos A+\cos B\equiv 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A-\cos B\equiv -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\sin^2(\theta)=\frac{1-\cos(2\theta)}{2}$$
$$\cos^2(\theta)=\frac{1+\cos(2\theta)}{2}$$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0

Maclaurin's and Taylor's Series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots+\frac{x^r}{r!}f^{(r)}(0)+\ldots$$
$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^r}{r!}f^{(r)}(a)$$
$$f(a+x)=f(a)+xf'(a)+\frac{x^2}{2!}f''(a)+\cdots+\frac{x^r}{r!}f^{(r)}(a)$$
$$e^x=\exp(x)=1+x+\frac{x^2}{2!}+\cdots+\frac{x^r}{r!} \quad \text{for all } x$$
$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{r+1}\frac{x^r}{r} \quad (-1<x\leq 1)$$
$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^r\frac{x^{2r+1}}{(2r+1)!} \quad \text{for all } x$$
$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^r\frac{x^{2r}}{(2r)!} \quad \text{for all } x$$
$$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots+(-1)^r\frac{x^{2r+1}}{2r+1} \quad (-1\leq x\leq 1)$$

Power series solution

$$y=c_0+c_1x+c_2x^2+c_3x^3+\cdots$$

of the differential equation

$$y''-2xy'+x^2y=0.$$
$$y'=c_1+2c_2x+3c_3x^2+4c_4x^3+5c_5x^4+6c_6x^5+\cdots,$$
$$y''=2c_2+6c_3x+12c_4x^2+20c_5x^3+30c_6x^4+\cdots.$$

Substitute into the equation:

$$\begin{aligned}y''-2xy'+x^2y=&\left(2c_2+6c_3x+12c_4x^2+20c_5x^3+\cdots\right)\\&-2x\left(c_1+2c_2x+3c_3x^2+4c_4x^3+\cdots\right)\\&+x^2\left(c_0+c_1x+c_2x^2+c_3x^3+\cdots\right)\end{aligned}$$

Simplifying and combining terms:

$$2c_2+(6c_3-2c_1)x+(12c_4-4c_2+c_0)x^2+$$
$$(20c_5-6c_3+c_1)x^3+(30c_6-8c_4+c_2)x^4+\cdots=0.$$

This equation must hold for all x, each coefficient must be zero:

$$2c_2=0 \quad\Rightarrow\quad c_2=0,$$
$$6c_3-2c_1=0 \quad\Rightarrow\quad c_3=\frac{1}{3}c_1,$$
$$12c_4+c_0=0 \quad\Rightarrow\quad c_4=-\frac{1}{12}c_0,$$
$$20c_5-6c_3+c_1=0 \quad\Rightarrow\quad 20c_5-6\cdot\frac{1}{3}c_1+c_1=0$$
$$\Rightarrow\quad 20c_5=c_1 \quad\Rightarrow\quad c_5=\frac{1}{20}c_1.$$

Thus, the solution is:

$$y=c_0\left(1-\frac{1}{12}x^4\right)+c_1\left(x+\frac{1}{3}x^3+\frac{1}{20}x^5\right).$$

Define:

$$y_1(x)=1-\frac{1}{12}x^4, \qquad y_2(x)=x+\frac{1}{3}x^3+\frac{1}{20}x^5.$$

The general solution is:

$$y=c_0y_1(x)+c_1y_2(x),$$

which represents two linearly independent solutions.

Boundary conditions

Find y in terms of x, given that

$$\frac{d^2y}{dx^2}-y=2e^x,$$

with initial conditions

$$\frac{dy}{dx}=0 \quad \text{and} \quad y=0 \quad \text{at} \quad x=0.$$

First, consider the homogeneous equation:

$$\frac{d^2y}{dx^2}-y=0.$$
$$m^2-1=0 \quad\Rightarrow\quad m=\pm 1.$$

So, the complementary function (C.F.) is:

$$y=Ae^x+Be^{-x}.$$

let $y=\lambda xe^x$Then:

$$\frac{dy}{dx}=\lambda xe^x+\lambda e^x,$$
$$\frac{d^2y}{dx^2}=\lambda xe^x+\lambda e^x+\lambda e^x=\lambda xe^x+2\lambda e^x.$$

Substitute into the equation:

$$\frac{d^2y}{dx^2}-y=2e^x,$$
$$\lambda xe^x+2\lambda e^x-\lambda xe^x=2e^x,$$
$$2\lambda e^x=2e^x \quad\Rightarrow\quad \lambda=1.$$

The general solution is:

$$y=Ae^x+Be^{-x}+xe^x.$$

Using the initial conditions:

Since $y=0$ at $x=0$:

$$0=A+B \quad\Rightarrow\quad A+B=0.$$

Differentiate the general solution:

$$\frac{dy}{dx}=Ae^x-Be^{-x}+e^x+xe^x.$$

Substitute $x=0$ into $\frac{dy}{dx}=0$ **Solving the simultaneous equations:**

$$A+B=0,$$
$$A-B=-1.$$
$$A=-\frac{1}{2}, B=\frac{1}{2}.$$
$$y=-\frac{1}{2}e^x+\frac{1}{2}e^{-x}+xe^x.$$