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Hybrid Approach for Solving Multiple-Objective Linear Programs in Outcome Space[1]

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Abstract. Various difficulties arise in using decision set-based vector maximization methods to solve a multiple-objective linear programming problem (MOLP). Motivated by these difficulties, some researchers in recent years have begun to develop tools for analyzing and solving problem (MOLP) in outcome space, rather than in decision space. In this article, we present and validate a new hybrid vector maximization approach for solving problem (MOLP) in outcome space. The approach systematically integrates a simplicial partitioning technique into an outer approximation procedure to yield an algorithm that generates the set of all efficient extreme points in the outcome set of problem (MOLP) in a finite number of iterations. Some key potential practical and computational advantages of the approach are indicated.

Key Words. Multiple-objective linear programming, vector maximization, efficient set, outcome set, global optimization.

1. Introduction

A multiple-objective linear programming problem may be written as

$$\text{(MOLP)} \quad \text{Vmax}\, Cx, \qquad \text{s.t. } x \in X,$$

where C is a $p \times n$ matrix with $p \geq 2$ whose rows c_i, $i = 1, 2, \ldots, p$, are the coefficients of p linear criterion functions $\langle c_i, x \rangle$, $i = 1, 2, \ldots, p$, and $X \subseteq \Re^n$ is a nonempty polyhedron. Problem (MOLP) is one of the most popular of the models that are used as aids in decision making with multiple criteria. Numerous studies and applications of problem (MOLP) have been reported

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in the literature in literally hundreds of articles, books, and chapters in books; see for example Refs. 1–27 and references therein.

Many approaches for solving problem (MOLP) use either the concept of efficiency, or the concept of weak efficiency, or both. A point $x^\circ \in \mathfrak{R}^n$ is an efficient (or nondominated) solution for problem (MOLP) when $x^\circ \in X$ and there exists no $x \in X$ such that $Cx \geq Cx^\circ$ and $Cx \neq Cx^\circ$. A point $\bar{x} \in \mathfrak{R}^n$ is a weakly efficient (or weakly nondominated) solution for problem (MOLP) when $\bar{x} \in X$ and there exists no $x \in X$ such that $Cx > C\bar{x}$. Let X_E and X_{WE} denote the set of all efficient solutions and the set of all weakly efficient solutions, respectively, of problem (MOLP). Then, X_E and X_{WE} are called the efficient decision set and the weakly efficient decision set, respectively, of problem (MOLP). Notice that $X_E \subseteq X_{WE}$.

The vector maximization approach is one of the most common of the approaches that use the concept of efficiency or weak efficiency to solve problem (MOLP). In this approach, all of either X_E or X_{WE}, or a representative portion thereof, is mathematically generated without any input from the decision maker (DM). Subsequently, the entire set generated is presented to the DM. The DM, without further aid from the analyst, then chooses a most preferred solution to problem (MOLP) from the generated set.

Let

$$Y^= = \{Cx \mid x \in X\}. \tag{1}$$

The set $Y^=$ is called the outcome set of problem (MOLP). The rationale behind the vector maximization approach is based upon the well-known fact that, as long as the preference function v: $Y^= \to \mathfrak{R}$ of the DM is nondecreasing in its arguments and an efficient solution exists, then a most preferred solution can be found in X_E.

In one class of vector maximization methods for problem (MOLP), the entire efficient decision set X_E or weakly efficient decision set X_{WE} is generated. Some of the most well-known algorithms of this type are presented in Refs. 1–2, 8, 13, 19, 21, 24–25, and 27–28.

In practice, these methods have met with only limited success. There are at least two reasons for this.

First, the computational demands of finding X_E or X_{WE} grow rapidly with problem size, so that only relatively-small problems can be analyzed; see, e.g., Refs. 19, 21, and 29. Second, the sheer size and nature of X_E or X_{WE} have so far precluded the possibility of finding a concrete, practical way of presenting these sets in their entirety in a meaningful way to the DM. In particular, attempts to accomplish this often result in the DM becoming confused or overwhelmed; see, e.g., Refs. 19–20 and 29.

Let X_{ex} denote the set of all extreme points of X. A second class of vector maximization methods for problem (MOLP) involves generating all of $X_E \cap X_{ex}$ or all of $X_{WE} \cap X_{ex}$, that is, all efficient or weakly efficient extreme points of the decision set X; see, e.g., Refs. 9, 11, 14, 16, 21–22, and 24–27. The rationale behind this approach is that, since $X_E \cap X_{ex}$ and $X_{WE} \cap X_{ex}$ are finite, discrete subsets of X_E and X_{WE}, respectively, it ought to be more practical computationally and in practice to generate and present them to the DM, rather than X_E or X_{WE}.

Unfortunately, methods for solving problem (MOLP) that generate $X_E \cap X_{ex}$ or $X_{WE} \cap X_{ex}$ have also achieved only limited success in practice. There are several reasons for this.

First, although $X_E \cap X_{ex}$ and $X_{WE} \cap X_{ex}$ are smaller than X_E and X_{WE}, respectively, it was soon found that the numbers of elements in these sets grow exponentially with problem size. As a result, as the size of problem (MOLP) increases, these sets can quickly become burdensome to generate, and their magnitudes can easily overwhelm the DM; see, e.g., Refs. 21 and 30–35.

Second, most algorithms for generating $X_E \cap X_{ex}$ or $X_{WE} \cap X_{ex}$ require special bookkeeping or backtracking schemes. Although some attempts have been made to streamline these schemes (Ref. 22), the need for them imposes a significant burden on these methods (Refs. 11, 19, 21–22).

Finally, in virtually all of the methods for generating $X_E \cap X_{ex}$ or $X_{WE} \cap X_{ex}$, special care must be taken to handle cases where degenerate extreme points of the decision set X are encountered. These extra steps make implementations of these methods more complicated and slower than they would otherwise be.

Motivated by these difficulties, some researchers in recent years have begun to develop tools for analyzing problem (MOLP) in outcome space rather than in decision space. In particular, ideas for generating all or parts of the efficient outcome set $Y_E^=$ for problem (MOLP) have been given in Refs. 4, 29–31, and 35–38, where

$$Y_E^= = \{Cx \mid x \in X_E\}. \tag{2}$$

The rationale for this approach is threefold.

First, the dimension p of the outcome space is typically much smaller than the dimension n of the decision space. As a result, $Y_E^=$ is invariably much smaller and has a much simpler structure than X_E; see, e.g., Refs. 4, 30–31, and 35–38. Generating all or parts of $Y_E^=$ is therefore expected, in general, to be less computationally demanding than generating all or portions of X_E. In addition, the DM is less likely to be confused or overwhelmed if all or portions of $Y_E^=$ are presented to him or her than if all or portions of X_E were presented.

Second, empirical research has shown that, in practice, the DM prefers searching for a most preferred solution by examining the outcome set rather than the decision set. In particular, the DM prefers to base his or her choice of a most preferred solution on an examination of all or parts of $Y_E^=$, rather than X_E; see, e.g., Refs. 29–31.

Third, it is well known that frequently C maps many points in X_E onto either a single outcome in $Y_E^=$ or onto essentially-equivalent outcomes in $Y_E^=$ (Refs. 39–41). Thus, generating points directly from $Y_E^=$ avoids risking redundant calculations of points in X_E.

In this article, we propose and validate a new hybrid vector maximization approach for solving problem (MOLP) in outcome space. The approach adapts two global optimization, decision set-based methods to outcome space. These methods are a special simplicial partitioning technique of Ban (Refs. 42–43), proposed originally for solving concave minimization problems, and a general outer approximation method that has been used very frequently to help solve a variety of global optimization problems (Refs. 35, 44–45). In particular, the new hybrid algorithm systematically integrates the simplicial partitioning technique into the outer approximation scheme in outcome space so as to generate the set of all efficient extreme points in the outcome set of problem (MOLP) in a finite number of iterations.

The organization of this article is as follows. In the next section, we review the theoretical and algorithmic results that provide the basis for the hybrid approach. In Section 3, the new hybrid algorithm is presented and is shown to generate the set of all efficient extreme points in the outcome set of problem (MOLP) within a finite number of iterations. In Section 4, some key potential practical and computational advantages of the algorithm are indicated. Section 5 gives some conclusions.

2. Theoretical Background

Assume in the remainder of the paper that X, in addition to being a nonempty polyhedron in $\mathfrak{R}^n$, is compact and is given by

$$X=\{x\in\mathfrak{R}^n \mid Ax=b, x\geq 0\},$$

where A is an $m\times n$ matrix and $b\in\mathfrak{R}^m$. In this case, it can be shown that the outcome set $Y^=$ [see (1)] of problem (MOLP) is a nonempty, compact polyhedron in $\mathfrak{R}^p$ (Ref. 46). A point $y^\circ\in\mathfrak{R}^p$ is called an efficient (or nondominated) outcome for problem (MOLP) when $y^\circ\in Y^=$ and there exists no $y\in Y^=$ such that $y\geq y^\circ$ and $y\neq y^\circ$. The set of all efficient outcomes for problem (MOLP) is called the efficient outcome set for problem (MOLP) and is denoted $Y_E^=$. It is an easy exercise to show that $Y_E^=$ may also be defined by (2) and that, since X is nonempty and compact, $Y_E^=$ is nonempty (Refs. 3–4).

Let

$$Y = \{y \in \mathfrak{R}^p \mid \hat{y} \leq y \leq Cx, \text{ for some } x \in X\},$$

where $\hat{y} \in R^p$ is chosen to satisfy

$$\hat{y}_i < \min y_i, \qquad \text{s.t. } y \in Y^=,$$

for each $i = 1, 2, \ldots, p$. A point $\bar{y} \in \mathfrak{R}^p$ is called an efficient point of Y, when $\bar{y} \in Y$ and there exists no $y \in Y$ such that $y \geq \bar{y}$ and $y \neq \bar{y}$. The set of all efficient points of Y is denoted Y_E. From Ref. 35, we have the following result.

Theorem 2.1. The set Y is a nonempty, bounded polyhedron in $\mathfrak{R}^p$ of dimension p, and $Y_E^= = Y_E$.

Due to Theorem 2.1, we will refer to Y as an efficiency-equivalent polyhedron for $Y^=$. Notice from Theorem 2.1 that there exists an $r \times p$ matrix D and a vector $d \in \mathfrak{R}^r$, where $r \geq p+1$, such that

$$Y = \{y \in \mathfrak{R}^p \mid Dy - d \leq 0\}. \tag{3}$$

Thus, if the representation of Y given by (3) were explicitly available, any of a number of decision set-based vector maximization methods (see, e.g., Refs. 1–2, 8, 13, 19, 25, and 28) could be directly applied to the multiple-objective linear program

$$\text{Vmax } y, \qquad \text{s.t. } y \in Y,$$

to find $Y_E = Y_E^=$. Of course, the matrix D and vector d in (3) are not immediately available, so that the practicality of this approach is in question. The hybrid algorithm, as we shall see, takes a more direct approach in its use of Y.

Let $Y_{ex}^=$ and Y_{ex} denote the sets of all extreme points of $Y^=$ and of Y, respectively. The following result is shown in Ref. 35.

Theorem 2.2. Let

$$E = \{y \in Y_{ex} \mid y > \hat{y}\}.$$

Then, $Y_E^= \cap Y_{ex}^= = E$.

From Theorem 2.2, if the set of all extreme points Y_{ex} of the efficiency-equivalent polyhedron is obtained, then the set E of all efficient extreme points in the outcome set $Y^=$ for problem (MOLP) can be easily found. To do so, one simply eliminates from Y_{ex} those points y for which $y_i = \hat{y}_i$ for at least one $i = 1, 2, \ldots, p$. This is the approach taken by the hybrid algorithm.

To generate Y_{ex}, the hybrid algorithm relies in part upon a technique called outer approximation. This technique, which has been employed recently in many global optimization algorithms, comes in a variety of forms (Refs. 44–45). To understand the hybrid algorithm, we now briefly explain the form in which the outer approximation technique is commonly used to minimize a concave function over a polytope.

Let Z be a nonempty, full-dimensional polytope given by

$$Z=\{z\in\mathfrak{R}^q \mid Fz-\theta\leq 0\}, \tag{4}$$

where F is an $h\times q$ matrix and $\theta\in\mathfrak{R}^h$, and suppose that $g: G\rightarrow\mathfrak{R}$ is a concave function, where G is an open convex set in $\mathfrak{R}^q$ containing Z. Then, it is well known that g achieves its minimum over Z at an extreme point of Z (Refs. 44–45). A typical outer approximation procedure for finding such an extreme point may be stated as follows. For any set W, let int W denote the interior of W.

Procedure 2.1. Outer Approximation Procedure.

Initialization Step. Compute a point $\bar{z}\in\operatorname{int} Z$ and construct a nonempty polytope Q° containing Z. Compute the vertex set $V(Q^\circ)$ of Q°; set

$$LB_\circ=\min g(z), \qquad \text{s.t. } z\in V(Q^\circ),$$

and let z° denote any element of $V(Q^\circ)$ for which $g(z^\circ)=LB_\circ$. Set $k=0$, and go to Iteration k.

Iteration k, $k\geq 0$. See Steps $k1$ through $k4$ below.

Step $k1$. If $z^k\in Z$, then stop: z^k minimizes g over Z. Otherwise, continue.

Step $k2$. Find the unique value λ_k of λ, $0<\lambda<1$, for which $\lambda z^k+(1-\lambda)\bar{z}$ belongs to the boundary of Z, and set

$$w^k=\lambda_k z^k+(1-\lambda_k)\bar{z}.$$

Step $k3$. Set

$$Q^{k+1}=Q^k\cap\{z\in\mathfrak{R}^q \mid \langle F_k, z\rangle-\theta_k\leq 0\},$$

where F_k denotes row k of F, and $k\in\{1, 2, \ldots, h\}$ satisfies

$$\langle F_k, w^k\rangle-\theta_k=0.$$

Using $V(Q^k)$ and the definition of Q^{k+1}, compute $V(Q^{k+1})$.

Step $k4$. Set

$$LB_{k+1}=\min g(z), \qquad \text{s.t. } z\in V(Q^{k+1}),$$

and let z^{k+1} denote any element of $V(Q^{k+1})$ for which $g(z^{k+1})=LB_{k+1}$. Set $k=k+1$, and go to Iteration k.

Except for Steps $k2$ and $k3$, this outer approximation procedure can be efficiently implemented using simple algebra and linear programming techniques; see, e.g., Refs. 44–45 and references therein. Step $k2$ can be readily executed by using standard univariate search techniques. Step $k3$ is the most computationally challenging step to execute. In particular, to compute $V(Q^{k+1})$ from $V(Q^k)$ and the definition of Q^{k+1} generally requires using one of the special techniques developed in recent years for the explicit purpose of implementing this step; see, e.g., Refs. 44 and 47 and references therein. Unfortunately, however, even with these techniques, the efficient implementation of this part of Step $k3$ remains a relatively significant challenge with computational demands that seem to increase rather rapidly as the dimension q of Z increases (Refs. 44, 48).

Notice that, in each iteration $k \geq 0$ of the outer approximation procedure, a linear inequality from the representation (4) of Z is generated in Step $k3$. It can be shown that the successive linear inequalities generated in Steps $k3$, $k=0, 1, \dots,$ are distinct from one another (see, e.g., Refs. 44–45 and references therein). Therefore, the outer approximation procedure is finite. In the worst case, it will terminate after h full iterations with $Q^{h+1} = Z$ and LB_{h+1} equal to the minimum of g over Z. Notice in this worst case that, upon termination, $V(Q^{h+1})$ contains a list of all the extreme points of Z.

To generate Y_{ex}, the hybrid algorithm incorporates a special simplicial partitioning technique of Ban (Refs. 42–43) into an outer approximation framework, all in the outcome space, rather than in the decision space, of problem (MOLP). To explain Ban's simplicial partitioning technique, we will adapt and extend the presentation given in Ref. 43. First, we need to recall two well-known definitions (see, e.g., Ref. 44). Let $Z \subseteq \Re^q$ be the nonempty, full-dimensional polytope given by (4).

Definition 2.1. Let I be a finite set of indices. A set $\{Q^i \mid i \in I\}$ of subsets of Z is called a partition of Z when

$$Z = \bigcup_{i \in I} Q^i$$

and

$$Q^i \cap Q^j = \partial Q^i \cap \partial Q^j, \qquad \text{for all } i, j \in I,\ i \neq j,$$

where ∂Q^i denotes the (relative) boundary of Q^i, $i \in I$.

Definition 2.2. Let $M \subseteq \Re^q$ be a full-dimensional simplex (or q-simplex) with vertices $u^1, u^2, \dots, u^{q+1}$. Suppose that

$$v = \lambda u^p + (1 - \lambda) u^t,$$

where p, $t \in \{1, 2, \ldots, q+1\}$, $p \neq t$, and $0 < \lambda < 1$. Let M^1 (M^2, respectively) be the simplex whose vertex set is obtained from M by replacing u^p (u^t respectively) by v. Then, $\{M^1, M^2\}$ is called a bisection of M with respect to v.

Remark 2.1. From Ref. 44, it follows that the bisection $\{M^1, M^2\}$ of M with respect to v in Definition 2.2 forms a partition of M in the sense of Definition 2.1.

As we shall see, the goal of Ban's method is to construct a partition of Z consisting entirely of simplices, each of which lies completely in a face of Z. Central to this process is the following notion.

Definition 2.3. Let $M \subseteq \Re^q$ be a q-simplex with vertices $u^1, u^2, \ldots, u^{q+1}$. Then, M is called trivial with respect to Z when, for each $k = 1, 2, \ldots, h$,

$$(\langle F_k, u^i \rangle - \theta_k)(\langle F_k, u^j \rangle - \theta_k) \geq 0,$$

for all $i, j = 1, 2, \ldots, q+1$.

For any set A, let conv A denote the convex hull of A. From Ref. 43, we have the following result.

Proposition 2.1. Let $M \subseteq \Re^q$ be a q-simplex with vertices $u^1, u^2, \ldots, u^{q+1}$. If M is trivial with respect to Z, then $(M \cap Z)$ is the (subsimplicial) face of M spanned by those vertices of M that lie in Z; i.e.,

$$(M \cap Z) = \operatorname{conv}\{u^i \mid u^i \in Z\}.$$

Ban's simplicial partitioning technique for Z begins by constructing a q-simplex $M^\circ \supseteq Z$. By repeated bisections, M° is partitioned into a set of q-simplices each of which is trivial with respect to Z. This yields a partition of Z into simplices of the form $(M \cap Z)$, where M is a q-simplex in the final partition of M°. The technique may be stated as follows.

Procedure 2.2. Simplicial Partitioning Technique.

Initialization Step. Construct a q-simplex $M^\circ \supseteq Z$. Set $PM^\circ = \{M^\circ\}$, set $k = 1$, and go to Iteration k.

Iteration k, $k \geq 1$. See Steps $k1$ through $k4$ below.

Step $k1$. Remove, if possible, a q-simplex M from PM° such that, for some pair of vertices u^i, u^j of M,

$$(\langle F_k, u^i \rangle - \theta_k)(\langle F_k, u^j \rangle - \theta_k) < 0,$$

and save M, u^i, and u^j. If no such q-simplex M exists, set $k=k+1$, and go to Step $k4$. Otherwise, continue.

Step $k2$. Find the unique value λ_k of λ, $0<\lambda<1$, for which

$$\langle F_k, \lambda u^i+(1-\lambda)u^j\rangle-\theta_k=0,$$

and set

$$v=\lambda_k u^i+(1-\lambda_k)u^j.$$

Step $k3$. Form the bisection $\{M', M''\}$ of M with respect to v. Set $PM^\circ=PM^\circ \cup \{M'\} \cup \{M''\}$, and go to Step $k1$.

Step $k4$. If $k\leq h$, go to Iteration k. Otherwise, stop.

The key properties of the simplicial partitioning technique are summarized in the following theorem.

Theorem 2.3. (a) The simplicial partitioning technique is finite, and it terminates with a partition

$$\{(M\cap Z) \mid M\in\{M^j \mid j\in J\}\}$$

of Z, where J is a finite index set and $\{M^j \mid j\in J\}$ is the set of q-simplices in the partition PM° of M° at the time of termination. Each M^j, $j\in J$, is trivial with respect to Z.

(b) Each extreme point $\bar{z}$ of Z is a vertex of every q-simplex in the set $\{M^j \mid j\in J\}$ that contains $\bar{z}$.

Proof. Part (a) follows, for example, from Ref. 43. To prove part (b), let $\bar{z}\in Z$. Then, by part (a), we may choose a q-simplex $M\in\{M^j \mid j\in J\}$ such that $\bar{z}\in(M\cap Z)$, where M is trivial with respect to Z. Assume that u^i, $i=1, 2, \ldots, q+1$, are the vertices of M. Let I be chosen so that

$$\{u^i \mid i\in I\}=\{u^i, i=1, 2, \ldots, q+1 \mid u^i\in Z\}.$$

Then, by Proposition 2.1,

$$(M\cap Z)=\text{conv}\{u^i \mid i\in I\}. \tag{5}$$

Suppose that, for each $i\in I$, $\bar{z}\neq u^i$. Then, by (5), since $\bar{z}\in(M\cap Z)$, this implies that, for some $\lambda_i\geq 0$, $i\in I$,

$$\bar{z}=\sum_{i\in I}\lambda_i u^i,$$

where $\sum_{i\in I}\lambda_i=1$ and at least two elements of $\{\lambda_i \mid i\in I\}$ are positive. Therefore, $\bar{z}$ is not an extreme point of Z. By the contrapositive, part (b) is proven. □

Remark 2.2. Notice from Theorem 2.3 that the simplicial partitioning technique can be used in a finite method for finding all of the extreme points of Z. To do so, the technique should first be executed until termination to obtain the set $\{M^j | j \in J\}$ of q-simplices mentioned in the theorem. Subsequently, for each $j \in J$, each vertex u of M^j should be examined to see whether or not u is an extreme point of Z. This can be done by using the fact that u is an extreme point of Z if and only if u satisfies the inequalities in (4) defining Z, with q or more of these inequalities holding as equations. From part (b) of the theorem, every extreme point of Z will be found in this way.

In Step $k2$ of the simplicial partitioning technique, notice that λ_k is given by the simple formula

$$\lambda_k = (\theta_k - \langle F_k, u^j \rangle)/(\langle F_k, u^i \rangle - \langle F_k, u^j \rangle).$$

As a result, except for finding an initial q-simplex M° containing Z, the steps of the simplicial partitioning technique can be implemented via elementary vector arithmetic. This is a rather startling observation in view of the ability of the technique to enumerate all extreme points of a polytope and, with minor changes, to solve difficult concave minimization problems. The technique has therefore aroused the interest of leading global optimization researchers (Ref. 43).

The hybrid algorithm for problem (MOLP) applies the outer approximation procedure, with the aid of the simplicial partitioning technique, to the efficiency equivalent polyhedron Y. To help accomplish this, the algorithm constructs an initial simplex in outcome space $\mathfrak{R}^p$ that contains Y. The method for constructing this simplex is based upon the following result, whose proof is in Ref. 35. Let $e \in \mathfrak{R}^p$ denote the vector of ones.

Theorem 2.4. Let $v^\circ = \hat{y}$ and, for each $j = 1, 2, \ldots, p$, define $v^j \in \mathfrak{R}^p$ by

$$v_i^j = \begin{cases} \hat{y}_i, & \text{if } i \neq j, \\ \beta + \hat{y}_j - \langle e, \hat{y} \rangle, & \text{if } i = j, \end{cases}$$

where

$$\beta = \max \langle e, y \rangle, \qquad \text{s.t. } y \in Y.$$

Let

$$S = \operatorname{conv}\{v^j | j = 0, 1, \ldots, p\}.$$

Then, S is a p-simplex in $\mathfrak{R}^p$ with vertex set $\{v^j | j = 0, 1, \ldots, p\}$, and S contains Y. Furthermore, S may also be written as

$$S = \{y \in \mathfrak{R}^p | \langle e, y \rangle - \beta \leq 0, -y_i + \hat{y}_i \leq 0, i = 1, 2, \ldots, p\}. \tag{6}$$

It will also be necessary in the hybrid algorithm to be able to test a given point $w \geq \hat{y}$ for membership in the efficiency equivalent polyhedron Y. Furthermore, given a point $w > \hat{y}$ on the boundary of Y, it will be necessary to determine a linear inequality from among those linear inequalities in outcome space $\Re^p$ that define Y that is binding at w. The next result shows how to accomplish these tasks.

Theorem 2.5. For any $w \in \Re^p$, let $h(w)$ denote the optimal value of the linear programming problem given by

$$
\begin{aligned}
(Q_w) \quad & \max t, \\
\text{s.t.} \quad & Cx - et \geq w, \\
& Ax = b, \\
& x, t \geq 0,
\end{aligned}
$$

where $h(w) = -\infty$ when the feasible region of this problem equals the empty set.

(a) If $w \geq \hat{y}$, then $w \in Y$ if and only if $h(w) \neq -\infty$. In this case, $h(w) \geq 0$.

(b) If $w > \hat{y}$ and w belongs to the boundary of Y, then for all $y \in Y$,

$$\langle u^*, y \rangle - \langle b, v^* \rangle \leq 0,$$

where

$$\langle u^*, w \rangle - \langle b, v^* \rangle = 0,$$

and $(u^{*T}, v^{*T}) \in \Re^{p+m}$ is any optimal solution to the dual linear programming problem to problem (Q_w).

Proof. Part (a) follows easily from the definition of Y. Part (b) can be shown by using logic similar to that used to prove Theorems 2.4 and 2.5 in Ref. 35. □

3. Hybrid Algorithm

The hybrid vector maximization algorithm uses results from Section 2 to generate the set

$$E = (Y_E^= \cap Y_{ex}^=)$$

of all efficient extreme points in the outcome set $Y^=$ of problem (MOLP). To do so, it integrates, in outcome space, the simplicial partitioning technique of Ban into the outer approximation procedure presented in Section 2 in such a way so as to generate the set of all extreme points Y_{ex} of the efficiency equivalent polyhedron Y for $Y^{=}$. From Theorem 2.2, it is then a simple matter to recover E via the identity

$$E=\{y\in Y_{ex} \mid y>\hat{y}\}.$$

The hybrid algorithm may be stated as follows.

Algorithm 3.1. Hybrid Outcome Space Algorithm.

Initialization Step. Compute a point $\bar{p}\in\text{int } Y$, and construct the p-simplex $S=S^{\circ}$ described in Theorem 2.4. Store both the vertex set $V(S^{\circ})$ of $S=S^{\circ}$ and the inequality representation (6) of $S=S^{\circ}$. Set $PM^{\circ}=S^{\circ}$ and $k=0$, and go to Iteration k.

Iteration k, $k\geq 0$. See Steps $k1$ through $k5$ below.

Step $k1$. If, for each $y\in V(S^k)$, $y\in Y$ is satisfied, then go to Step $k5$. Otherwise, choose any $y^k\in V(S^k)$ such that $y^k\notin Y$ and continue.

Step $k2$. Find the unique value λ_k of λ, $0<\lambda<1$, for which $\lambda y^k+(1-\lambda)\bar{p}$ belongs to the boundary of Y, and set

$$w^k=\lambda_k y^k+(1-\lambda_k)\bar{p}.$$

Step $k3$. Set

$$S^{k+1}=S^k\cap\{y\in\Re^p \mid \langle u^k, y\rangle-\langle b, v^k\rangle\leq 0\},$$

where (u^{kT}, v^{kT}) is any optimal solution to the dual linear programming problem to problem (Q_w) with $w=w^k$ (cf. Theorem 2.5).

Step $k4$. With the p-simplex S° playing the role of M° and PM^k used instead of $\{S^{\circ}\}$ in the initialization step as the first simplicial partition of S°, apply the simplicial partitioning technique given in Section 2 to $Z=S^{k+1}$. Let PM^{k+1} denote the simplicial partition of S° at the time of termination of the technique. Using PM^{k+1}, Theorem 2.3, and Remark 2.2, determine $V(S^{k+1})$. Set $k=k+1$ and go to Iteration k.

Step $k5$. Let

$$E=\{y\in V(S^k) \mid y>\hat{y}\}.$$

Then, stop: E is equal to the set of all efficient extreme points of $Y^{=}$.

The point $\bar{p} \in \operatorname{int} Y$ in the initialization step can be computed with the aid of the linear programming problem (Q_w) given in Section 2 in a number of ways. For instance, if we set $w = \bar{y}$ in problem (Q_w), where, for each $i = 1, 2, \ldots, p$,

$$\bar{y}_i = \min y_i, \qquad \text{s.t. } y \in Y,$$

then, for any optimal solution (x^{*T}, t^*) to problem (Q_w), it can be shown that, if $\bar{p}$ is set equal to any strict convex combination of $\bar{y}$ and Cx^*, then $\bar{p} \in \operatorname{int} Y$. This provides a basis for determining various points $\bar{p}$ in int Y.

In Step $k1$, each point $y \in V(S^k)$ can be tested for membership in Y by applying Theorem 2.5(a) in Section 2 with $w = y$. Step $k2$ can be implemented by using simple univariate search to find the unique value λ_k of λ, $0 < \lambda < 1$, such that $h[\lambda y^k + (1 - \lambda)\bar{p}]$ equals 0, where $h: \mathfrak{R}^p \to \mathfrak{R}$ is defined in Theorem 2.5.

Theorem 3.1. The hybrid outcome space algorithm is finite. When it terminates, the set E equals the set of all efficient extreme points of the outcome set $Y^=$ of problem (MOLP).

Proof. First, notice from Theorem 2.3(a) that, for each $k \geq 0$, in Step $k4$ of the algorithm, starting with the simplicial partition PM^k of S°, where each simplex in PM^k is full dimensional and trivial with respect to S^k, a simplicial partition PM^{k+1} of S° is obtained after a finite amount of time. By the same result, PM^{k+1} consists of p-simplices, each of which is trivial with respect to S^{k+1}. By Theorem 2.3(b), every extreme point of S^{k+1} is a vertex of at least one simplex in PM^{k+1}. Therefore, Remark 2.2 can be applied to find $V(S^{k+1})$. Since the procedure in Remark 2.2 is finite, these observations together imply that, for each $k \geq 0$, Step $k4$ determines the vertex set $V(S^{k+1})$ of S^{k+1} in a finite amount of time.

By Theorem 2.4, the p-simplex S° constructed in the initialization step contains Y. From Theorem 2.5(b), for each $k \geq 0$, the vector (u^{k^T}, v^{k^T}) found in Step $k3$ satisfies

$$\langle u^k, y \rangle - \langle b, v^k \rangle \leq 0, \qquad \text{for all } y \in Y,$$

and

$$\langle u^k, w^k \rangle - \langle b, v^k \rangle = 0,$$

where w^k is the point on the boundary of Y found in Step $k2$. Together with the conclusion of the first paragraph of this proof and the statement of the algorithm, these observations imply that the computation of the sets S^i, $i = 0, 1, 2, \ldots$, in the hybrid outcome space algorithm is carried out in precisely the same way as the computation of the sets Q^i, $i = 0, 1, 2, \ldots$, in the outer approximation procedure of Section 2. This implies, by the remarks following the outer approximation procedure, that for some finite number $K \geq 0$, the set S^K generated by the hybrid outcome space algorithm will equal Y. As a result, in Step $K1$, for each $y \in V(S^K)$, $y \in Y$ will be satisfied, and control will shift to Step $K5$, where termination will occur. Since each step of the algorithm can be carried out in finite time, this implies that the algorithm is finite. Furthermore, since $S^K = Y$, it follows from Theorem 2.2 that, upon termination, the set E given in Step $K5$ equals the set of all efficient extreme points of $Y^{=}$. □

4. Potential Advantages

The hybrid outcome space algorithm has a number of potential advantages, both in practice and computationally. Here, we indicate only some key potential advantages, with special attention to those not shared by other vector maximization approaches. These can be summarized as follows.

(a) The hybrid outcome space algorithm is finite and exact. As a result, it is expected to enhance the ability of the DM to find a most preferred solution for Problem (MOLP) that he or she has confidence in.

(b) Since the hybrid outcome space algorithm works in the outcome space $\mathfrak{R}^p$, rather than in the decision space $\mathfrak{R}^n$ of problem (MOLP), it is expected that the number of efficient extreme points that it will need to generate for a given problem will generally be significantly smaller than the number generated by decision space-based vector maximization approaches. As a result, the computational demands of the hybrid outcome space algorithm will potentially be noticeably less than those of these decision space-based approaches.

(c) For the same reason as in (b), it is expected that the output of the hybrid outcome space algorithm is less likely to overwhelm or confuse the DM than the output of typical decision space-based vector maximization methods for problem (MOLP).

(d) A typical DM is expected to prefer viewing the output of the hybrid outcome space algorithm to the output of a vector maximization algorithm for problem (MOLP) that works in the decision space. This is because research has shown that the DM generally prefers to base the choice of a most preferred solution on an examination of points in the efficient outcome set $Y_E^{=}$ rather than in the efficient decision set X_E.

(e) By generating points directly from $Y_E^{=}$ rather than from X_E, it is expected that the hybrid outcome space algorithm will more effectively avoid calculating redundant efficient points for problem (MOLP) than decision space-based vector maximization methods.

(f) The hybrid outcome space algorithm is readily implementable. In particular, the implementation of the algorithm requires the ability to solve linear programming problems, perform simple univariate searches, and execute elementary vector arithmetic. Well-known, efficient procedures are readily available for performing these tasks.

(g) Unlike typical decision space-based algorithms that generate all of the extreme points in X_E, the hybrid outcome space algorithm does not ever call for using any special, complicated bookkeeping, backtracking, or degeneracy schemes. In particular, in the hybrid outcome space algorithm, the bookkeeping is simple, no backtracking is called for, and the only possible degeneracies that can arise will arise in the course of solving linear programming problems. These potential degeneracies can be readily handled by any of a number of simple, well-known rules.

(h) The relatively-difficult task in Step $k4$ of the hybrid outcome space algorithm of finding the vertex set $V(S^{k+1})$ of S^{k+1} is performed by using the simplicial partitioning technique of Ban. Since this technique can be implemented via elementary vector arithmetic, there is a potential of gaining further computational benefits from using Ban's technique in this step.

(i) To further mitigate the calculations required to find $V(S^{k+1})$ in the hybrid outcome space algorithm, Step $k4$ calls for initiating the simplicial partitioning technique of Ban with the initial partition PM^k, rather than S°. Recall that each p-simplex in PM^k is trivial with respect to S^k. As a result, for each $k \geq 0$, in Step $k4$ of the hybrid algorithm, the execution of the first k iterations of the simplicial partitioning technique can be skipped. It follows that for each $k \geq 0$, the simplicial partitioning technique in Step $k4$ of the algorithm will be executed in only one iteration. This is expected to further enhance the computational efficiency of the hybrid outcome space algorithm.

(j) Because of all of the potential advantages listed above, the hybrid outcome space algorithm may prove to be suitable for solving large-scale instances of problem (MOLP) that to date have not been solvable by decision set-based vector maximization methods.

5. Conclusions

In this article, we have presented and validated a new hybrid outcome space approach for solving a multiple objective linear programming problem (MOLP). The approach, which is embodied in an algorithm called the hybrid outcome space algorithm, generates the set of all efficient extreme points in the outcome set, rather than in the decision set, of problem (MOLP). To accomplish this, it uses a hybrid approach in outcome space in which a special simplicial partitioning technique is systematically integrated into an outer approximation procedure.

The hybrid algorithm has a number of potential advantages, both in practice and computationally. Several of these follow from the fact that the algorithm works in outcome space, rather than in decision space. Others follow from the algorithm's use of the hybrid approach and from the relatively-simple mechanics called for in the steps of the algorithm. Because of these potential advantages, the hybrid outcome space algorithm may prove to be suitable for solving large-scale instances of problem (MOLP) that have to date been unsolvable by decision set-based vector maximization methods.

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