Normal cones to a polyhedral convex set and generating efficient faces in linear multiobjective programming

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NORMAL CONES TO A POLYHEDRAL CONVEX SET AND GENERATING EFFICIENT FACES IN LINEAR MULTIOBJECTIVE PROGRAMMING

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Abstract. In this paper we describe the normal cones to a polyhedral convex set and their polarity with the faces of the set. Then we express optimaltity conditions in terms of negative normal cones and propose a method for generating efficient solution faces of a linear multiobjective programming problem together with some computational examples.

1. Introduction

The concept of normals to a smooth surface was already introduced in classical analysis. It serves as a useful tool to study structure of surfaces and calculus over surfaces. To treat objects without smoothness several generalizations of normals came to light. The first steps were essentially done by Minkowski [10] and later by Fenchel [8] who defined normal cones to a convex set. A systematic study and an extensive exploitation of normal cones were realized by Rockafellar [13]. Further developments in generalizing normal cones to the nonconvex case were set up among others by Clarke [4] who defined normal cones through subdifferentials of Lipschitz functions and also by Morduhovic [11] who used limiting proximal normals to obtain normal cones without convexification (see Rockafellar-Wets [14] for a full description of these developments). Nowadays normal cones are an indispensable device in expressing optimality conditions of nonsmooth optimization problems, existence criteria for variational inequalities, for complementarity problems etc.

The purpose of the present paper is to apply an explicit description of the normal cones to a polyhedral convex set defined by a system of linear inequations to establish the polarity relationship between the faces of the normal cones and the faces of the polyhedral set. Then we introduce the

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notion of negative normal cones and express optimality conditions for a linear multiobjective programming problem in terms of these cones. As an application we propose a quite easy method for generating efficient solution faces and encounter some computational experiences.

2. Preliminaries on polyhedral convex sets

Throughout this paper we denote by P a polyhedral convex set in R^n which is defined by a system of linear inequations

$$\langle a^i, x\rangle \geq b_i, \quad i = 1, ..., p \tag{1}$$

where $a^1, ..., a^p$ are vectors from R^n and $b_1, ..., b_p$ are real numbers. Let us recall that a subset $F \subseteq P$ is said to be a face of P if there is a vector $v \in R^n$ such that

$$F = \{x \in P : \langle v, y - x\rangle \geq 0 \text{ for all } y \in P\}.$$

In other words, F is a face of P if there is a vector $v \in R^n$ such that $F = \arg\min\{\langle v, x\rangle, x \in P\}$, the set of minima of the linear function $\langle v, .\rangle$ on P. The following criterion [13] will be needed in the sequel.

Lemma 2.1. *A nonempty set $F \subseteq P$ is a face if and only if there exists an index set $I_F \subseteq \{1, ..., p\}$ such that F is the solution set to the system*

$$\begin{aligned} &\langle a^i, x\rangle = b_i, \quad i \in I_F, \\ &\langle a^j, x\rangle \geq b_j, \quad j \in \{1, ..., p\}\backslash I_F, \end{aligned} \tag{2}$$

in which case one has $\dim F = n - rank\{a^i : i \in I_F\}$.

Denote by $RecP$ the recession cone of P consisting of all vectors $v \in R^n$ such that $x + tv \in P$ for all $x \in P$ and $t \geq 0$. Then $RecP$ is the solution set to the homogeneous system

$$\langle a^i, x\rangle \geq 0, \quad i = 1, ..., p \tag{3}$$

whenever P is nonempty.

We shall make use of the following notations: For a nonempty subset $X \subseteq R^n$, the positive polar cone of X is denoted by X^0 and is defined by $X^0 = \{v \in R^n : \langle v, x\rangle \geq 0 \text{ for all } x \in X\}$; For a system of vectors $\{v^1, ..., v^k\} \subseteq R^n$, the cone generated by this system, denoted by $cone\{v^1, .., v^k\}$ consists of all positive combinations $\sum_{i=1}^{k} \lambda_i v^i$ with $\lambda_1, ..., \lambda_k \geq 0$. The following version of Farkas' lemma gives us an explicit form of the positive polar cone of the recession cone (Lemma 6.45 [14]).

Lemma 2.2. *Let X be given by the system* (3)*. Then X^0 coincides with $cone\{a^1, ..., a^p\}$.*

3. Normal cones to a polyhedral convex set

Let X be a convex set in R^n and $x_0 \in X$. We recall that the normal cone to X at x_0, denoted by $N_X(x_0)$ consists of the outward normals to the supporting half-spaces to X at x_0, that is

$$N_X(x_0) = \{v \in R^n : \langle v, x - x_0 \rangle \leq 0 \text{ for all } x \in X\}.$$

A detailed study of normal cones to a convex set and their uses can be found in [13] (see also [14] for recent generalizations of normal cones). In this section we consider the case where X is a polyhedral convex set P described in the previous section.

First let us recall a formula to calculate the normal cone to the set P at a point $x_0 \in P$. This formula was given in Theorem 6.46 of [14], however we give here a direct proof for the reader's convenience. For the same reason some other related standard facts that we shall use later, are also given with full proof.

Lemma 3.1. *Let $x_0 \in P$ satisfy the following equations and inequations*

$$\begin{aligned} &\langle a^i, x \rangle = b_i, \quad i \in I(x_0), \\ &\langle a^j, x \rangle > b_j, \quad j \in \{1, ..., p\} \backslash I(x_0), \end{aligned}$$

where $I(x_0)$ is a nonempty index subset of $\{1, ..., p\}$. Then $N_P(x_0) = cone\{-a^i, i \in I(x_0)\}$.

Proof. Let $v \in cone\{-a^i, i \in I(x_0)\}$, that is $v = -\sum_{i \in I(x_0)} \lambda_i a^i$ with $\lambda_i \geq 0$, $i \in I(x_0)$. Then for each $x \in P$ we have

$$\langle v, x - x_0 \rangle = -\sum_{i \in I(x_0)} \lambda_i (\langle a^i, x \rangle - \langle a^i, x_0 \rangle) \leq 0$$

because for each $i \in I(x_0)$ we have $\langle a^i, x_0\rangle = b_i$ and $\langle a^i, x\rangle \geq b_i$ for all $x \in P$. Consequently, $v \in N_P(x_0)$.

Conversely, let $v \in N_P(x_0)$. Then $\langle v, x - x_0\rangle \leq 0$ for all $x \in P$ by definition. Denoting by $cone(P-x_0)$ the cone consisting of vectors $t(x-x_0)$ with $x \in P$ and $t \geq 0$, we see that

$$\langle v, y\rangle \leq 0 \quad \text{for all} \quad y \in cone(P - x_0). \tag{4}$$

We claim that $cone(P - x_0)$ coincides with the set

$$X := \{y \in R^n : \langle a^i, y\rangle \geq 0 \text{ for all } i \in I(x_0)\}$$

which is actually the positive polar cone of the set $\{a^i : i \in I(x_0)\}$. Indeed, let $y \in cone(P - x_0)$, say $y = t(x - x_0)$ for some $x \in P$ and $t \geq 0$. Then for each $i \in I(x_0)$ one has

$$\langle a^i, y\rangle = t\langle a^i, x - x_0\rangle \geq 0.$$

This implies $cone(P - x_0) \subseteq X$. Conversely, let $y \in X$. Let $t > 0$ be sufficiently small such that

$$\langle a^j, ty\rangle \geq -\min\{\langle a^i, x_0\rangle - b_i : i \in \{1, ..., p\}\backslash I(x_0)\}$$

for all $j = 1, ..., p$. Such t exists because $y \in X$ and $\langle a^i, x_0\rangle > b_i$ for $i \notin I(x_0)$. Then we have

$$\langle a^i, ty + x_0\rangle = t\langle a^i, y\rangle + \langle a^i, x_0\rangle \geq b_i, \text{ for } i \in I(x_0),$$

and

$$\langle a^j, ty + x_0\rangle = \langle a^j, ty\rangle + \langle a^j, x_0\rangle \geq b_j, \text{ for } j \in \{1, ..., p\}\backslash I(x_0).$$

These inequalities show that $ty+x_0 \in P$, or equivalently $y \in cone(P-x_0)$. Hence $cone(P - x_0) = X$. This and (4) imply $v \in -X^0$. By Lemma 2.2, $v \in cone\{-a^i : i \in I(x_0)\}$ and the proof is complete. □

Let us denote by N_P the union of all normal cones $N_P(x)$ with $x \in P$. The following proposition establishes the link between N_P and the recession cone of P.

Proposition 3.2. *Assume that P is nonempty. Then we have the relation $N_P = -(recP)^0$.*

Proof. Let $v \in N_P$. There exists $x_0 \in P$ such that $\langle v, x - x_0 \rangle \le 0$ for all $x \in P$. This means that the linear function $\langle v, . \rangle$ attains a maximum at x_0 on P. Consequently $\langle v, u \rangle \le 0$ for all $u \in RecP$, that is $v \in -(RecP)^0$.

Conversely, let $v \in -(recP)^0$. Consider the problem of maximizing $\langle v, . \rangle$ over P. If it has a maximum at some point $x_0 \in P$, then clearly $v \in N_P(x_0) \subseteq N_P$. If not, there must exist a recession direction $u \in RecP$ such that $\langle v, u \rangle > 0$. This is impossible because $v \in -(RecP)^0$. The proof is complete. □

Corollary 3.3. *Assume that P is nonempty. Then N_P is a polyhedral convex cone. Moreover, P is bounded if and only if $N_P = R^n$.*

Proof. Since $RecP$ being defined by the homogeneous system (3) is a polyhedral convex cone, its positive polar cone is polyhedral convex. By Proposition 3.2, N_P is polyhedral convex cone. The second part of the corollary is obtained from Proposition 3.2 and from the fact that P is bounded if and only if $RecP = \{0\}$. □

Observe that if x and y are two relative interior points of a face $F \subseteq P$, then $N_P(x)$ and $N_P(y)$ coincide. For this reason we denote by $N(F)$ the normal cone $N_P(x)$ to P at a relative interior point x of F. Since the number of faces of P is finite, the cone N_P is actually a finite union of polyhedral subcones.

Proposition 3.4. *Assume that F_1 and F_2 are faces of P with $F_1 \subseteq F_2$. Then $N(F_2)$ is a face of $N(F_1)$. Conversely, if N is a face of $N(F_1)$ for some face F_1 of P, then there is a face F of P such that $N(F) = N$ and $F_1 \subseteq F$. In this case $F \ne F_1$ whenever $N \ne N(F_1)$.*

Proof. Let I_{F_1} and I_{F_2} be two index subsets of $\{1, ..., p\}$ determining respectively F_1 and F_2 by (2). Since $F_1 \subseteq F_2$, one has $I_{F_1} \supseteq I_{F_2}$. If $F_1 = F_2$, then $N(F_1) = N(F_2)$ and we are done. So, we may assume $F_1 \ne F_2$. Pick any relative interior points x_1 of F_1 and x_2 of F_2. Then one has for $i = 1, 2$ that

$$\begin{aligned} \langle a^j, x_i \rangle &= b_j, \quad j \in I_{F_i}, \\ \langle a^j, x_i \rangle &> b_j, \quad j \in \{1, ..., p\} \backslash I_{F_i}. \end{aligned}$$

By Lemma 3.1, $N(F_i) = cone\{-a^j : j \in I_{F_i}\}, i = 1, 2$. Consequently $N(F_2) \subseteq N(F_1)$. Assume to the contrary that $N(F_2)$ is not a face of $N(F_1)$. There exists a face $N_0 = cone\{-a^j : j \in I_0\} \subseteq N(F_1)$ which strictly contains $N(F_2)$ and whose relative interior meets $N(F_2)$. Let F_0 be the solution set to the following system

$$\begin{aligned} \langle a^j, x\rangle &= b_j, \quad j \in I_0, \\ \langle a^j, x\rangle &\geq b_j, \quad j \in \{1, ..., p\}\backslash I_0. \end{aligned}$$

We see that $I_0 \subseteq I_1$, hence $F_1 \subseteq F_0 \subseteq F_2$. In particular, $F_0 \neq \emptyset$, and therefore it is a face of P. Let x_0 be a relative interior point of F_0, and as before x_2 a relative interior point of F_2. We wish to show that

$$\langle v, x_2 - x_0\rangle = 0 \text{ for all } v \in N_0. \tag{5}$$

In fact, on one hand for $v \in N_0$ one has $\langle v, x_2 - x_0\rangle \leq 0$. On the other hand, for a relative interior point v_0 of N_0 which also belongs to $N(F_2)$ one has $\langle v_0, x_0 - x_2\rangle \leq 0$, hence $\langle v_0, x_2 - x_0\rangle = 0$. The linear function $v \longmapsto \langle v, x_2 - x_0\rangle$ must then take the value 0 on N_0. In this way (5) is established which implies for each $v \in N_0$, $x \in P$ that

$$\langle v, x - x_2\rangle = \langle v, x - x_0\rangle + \langle v, x_0 - x_2\rangle \leq 0.$$

This shows $v \in N(F_2)$, a contradiction. Hence $N(F_2)$ is a face of $N(F_1)$.

Conversely, let $N(F_1)$ be the normal cone to P at some relative interior point x_1 of a face F_1 of P, and let N be a face of $N(F_1)$. The case $N = N(F_1)$ is trivial, so we consider the case $N \neq N(F_1)$. As in the previous part, $N(F_1) = cone\{-a^i : i \in I_{F_1}\}$. Then $N = cone\{-a^i : i \in I\}$ for some index set $I \subseteq I_{F_1}$. Let F be the solution set to the following system

$$\begin{aligned} \langle a^i, x\rangle &= b_i, \quad i \in I, \\ \langle a^j, x\rangle &\geq b_j, \quad j \in \{1, ..., p\}\backslash I. \end{aligned}$$

Since $I \subseteq I_{F_1}$, we have $F_1 \subseteq F$. In particular, $F \neq \emptyset$ and F is a face of P. Now, we show that F_1 is a proper face of F. Indeed, as N is a proper face of $N(F_1)$, I is a proper subset of I_{F_1} and there exists a nonzero vector $u \in R^n$ such that

$$\begin{aligned} \langle u, a^i\rangle &= 0 \quad \text{for} \quad i \in I, \\ \langle u, a^j\rangle &> 0 \quad \text{for} \quad j \in I_{F_1}\backslash I. \end{aligned} \tag{6}$$

Let us consider the point $x_1 + tu$ with $t > 0$. One has

$$\begin{aligned} \langle a^i, x_1 + tu\rangle &= b_i, \quad i \in I, \\ \langle a^i, x_1 + tu\rangle &= \langle a^i, x_1\rangle + t\langle a^i, u\rangle > b_i, \quad i \in I_{F_1}\backslash I. \end{aligned}$$

and $\langle a^i, x_1 + tu\rangle = \langle a^i, x_1\rangle + t\langle a^i, u\rangle > b_i$ for $i \in \{1, ..., p\}\backslash I_{F_1}$ and for t sufficiently small. Such t exists because $\langle a^i, x_1\rangle > b_i$ for $i \in \{1, ..., p\}\backslash I_{F_1}$. Consequently, $N(F) = N_P(x_1 + tu) = cone\{-a^i : i \in I\} = N$ where t is sufficiently small. It is evident that $F \neq F_1$. The proof is complete. $\square$

For the computation purpose let us introduce the following notion. A nonempty subset $I \subseteq \{1, ..., p\}$ is said to be a normal set if there is some point $x_0 \in P$ such that the normal cone to P at x_0 coincides with the cone generated by $\{-a^i : i \in I\}$. It is obvious that not every subset of $\{1, ..., p\}$ is normal. Below we give a link between the faces of P and the normal subsets of the index set $\{1, ..., p\}$.

Proposition 3.5. *A nonempty convex subset $F \subseteq P$ is a face if and only if there is a normal subset $I \subseteq \{1, ..., p\}$ such that F is defined by the system*

$$\begin{aligned} \langle a^i, x\rangle &= b_i, \quad i \in I, \\ \langle a^j, x\rangle &\geq b_j, \quad j \in \{1, ..., p\}\backslash I, \end{aligned}$$

in which case $\dim F = n - rank\{a^i : i \in I\}$.

Proof. Let F be a face of P. By Lemma 2.1, F is determined by the system (2). Pick any relative interior point x_0 of F. Then $\langle a^j, x_0\rangle > b_j$ for $j \in \{1, ..., p\}\backslash I_F$ and $\langle a^i, x_0\rangle = b_i$ for $i \in I_F$. In view of Lemma 3.1, $N_P(x_0) = cone\{-a^i : i \in I_F\}$ and I_F is a normal subset.

Conversely, let I be a normal subset of the set $\{1, ..., p\}$. Then there is some point $x_0 \in P$ such that $N_P(x_0) = cone\{-a^i : i \in I\}$. Put $a = \sum_{i\in I} a^i$ and consider the problem of minimizing $\langle a, .\rangle$ over P. We wish to show that the set F defined by the system stated in the proposition coincides with the set $\arg\min\{\langle a, y\rangle : y \in P\}$ and by this the proof will be complete. Indeed, let $x \in F$. Then $\langle a, x\rangle = \sum_{i\in I} b_i$. For $y \in P$ one also has $\langle a, y\rangle \geq \sum_{i\in I} b_i$. Consequently $x \in \arg\min\{\langle a, y\rangle : y \in P\}$. Now let x be a minimum of the function $\langle a, .\rangle$ over P. Then

$$\langle a, y - x\rangle \geq 0 \tag{7}$$

for all $y \in P$. We prove that $x \in F$. To this end, we claim that

$$\langle a^i, x_0\rangle = b_i \quad \text{for } i \in I. \tag{8}$$

In fact, if not, there is $i_0 \in I$ such that $\langle a^{i_0}, x_0\rangle > b_{i_0}$. As $-a^{i_0} \in N_P(x_0)$, one obtains

$$\langle a^{i_0}, y\rangle = \langle a^{i_0}, y - x_0\rangle + \langle a^{i_0}, x_0\rangle > b_{i_0}$$

for every $y \in P$ which contradicts the fact that F is nonempty. Taking $y = x_0$ in (7) and taking into account (8) we have

$$\langle a, x_0 \rangle = \sum_{i \in I} b_i \geq \langle a, x \rangle = \sum_{i \in I} \langle a^i, x \rangle$$

which yields $\langle a^i, x \rangle = b_i, i \in I$ because x being in P, $\langle a^i, x \rangle \geq b_i$ for all $i = 1, ..., p$. By this $x \in F$ as requested. $\square$

In the remaining part of this section we assume that the system (1) has no redundant inequalities, i.e there does not exist an index $k \in \{1, ..., p\}$ such that the system (1) is equivalent to

$$\langle a^i, x \rangle \geq b_i, \quad i \in \{1, ..., p\} \setminus \{k\}.$$

Let x_0 be a vertex of P and let $I(x_0)$ denote the set of all active indices at x_0, i.e.

$$I(x_0) = \{i \in \{1, ..., p\} : \langle a^i, x_0 \rangle = b_i\}.$$

The next result tells us when a subset $I \subseteq I(x_0)$ is a normal set.

Corollary 3.6. *Let $I \subseteq I(x_0)$ have $n - 1$ elements. Then it is normal if and only if the system*

$$\begin{aligned} &\langle a^i, x \rangle = b_i, \quad i \in I, \\ &\langle a^j, x \rangle \geq b_j, \quad j \in \{1, ..., p\} \backslash I, \end{aligned} \tag{9}$$

has a solution distinct from x_0.

Proof. If I is normal, then the face F resulted from Proposition 3.5 is the solution set to the system (9) and $\dim F = n - rank\{a^i : i \in I\} \geq n - (n - 1) = 1$. Hence the system has a solution distinct from x_0. Conversely, assume that the system (9) has a solution $x \neq x_0$. Then, since the system (1) has no redundant inequalities, one obtains

$$\langle a^j, x \rangle > b_j \quad \text{for } j \in I(x_0) \setminus I.$$

For a sufficiently small ϵ, one has

$$\begin{aligned} &\langle a^i, x_0 + \epsilon(x - x_0) \rangle = b_i, \quad i \in I, \\ &\langle a^j, x_0 + \epsilon(x - x_0) \rangle > b_j, \quad j \in \{1, ..., p\} \backslash I. \end{aligned}$$

According to Lemma 3.1, $cone\{-a^i : i \in I\} = N_P(x_0 + \epsilon(x - x_0))$, hence I is a normal set. $\square$

Assume that there exist k edges $F_1, ..., F_k$ emanating from the vertex x_0. Then, each of the index sets $I(F_1), ..., I(F_k)$ has at least $(n-1)$ elements. Let $J \subseteq \{1, ..., k\}$ with $|J| = l \leq \min\{k, n-1\}$. Take $x_i \in F_i \setminus \{x_0\}$, $i = 1, ..., k$ and

$$x_J = \frac{x_0}{l+1} + \sum_{j \in J} \frac{x_j}{l+1} \cdot$$

As before, $I(x_J)$ denotes the active index set at the point x_J. Assume that $I(x_J)$ is nonempty. The next result allows us to determine the largest face that contains x_J as a relative interior point.

Proposition 3.7. *The index set above $I(x_J)$ is normal and the face F determined by (9) with $I = I(x_J)$ contains the convex hull of all edges F_j, $j \in \{1, ..., k\}$ satisfying $I(F_j) \supseteq I(x_J)$.*

Proof. It is obvious that $F \neq \emptyset$ and $\langle a^i, x_J \rangle > b_i$ for all $i \in \{1, ..., p\} \setminus I(x_J)$. Therefore, $N_P(x_J) = cone\{-a^i : i \in I(x_J)\}$. In other words, $I(x_J)$ is a normal set. Moreover, for $j \in \{1, ..., k\}$ with $I(F_j) \supseteq I(x_J)$ one has $F_j \subseteq F$. Hence the convex hull of these F_j is contained in F. The proof is complete. $\square$

4. Negative normal cones of a polyhedral convex set

From now on C denotes a $(m \times n)$-matrix with m rows $c^1, ..., c^m$ considered as vectors in R^n. Its transposition is denoted by C^T.

Let $v \in R^n$. We say that v is C-positive if there exist strictly positive numbers $\lambda_1, ..., \lambda_m$ such that $v = \sum_{i=1}^{m} \lambda_i c^i$. If $-v$ is C-positive, we call it C-negative. As usual, R^m_+ denotes the nonnegative orthant of R^m and $intR^m_+$ is its interior. For $z^1 = (z^1_1, ..., z^1_m)$, $z^2 = (z^2_1, ..., z^2_m) \in R^m$, we have the following orders

$z^1 \geq z^2$ if $z^1_i \geq z^2_i$ for $i = 1, ..., m$;
$z^1 > z^2$ if $z^1 \geq z^2$ and $z^1 \neq z^2$;
$z^1 \gg z^2$ if $z^1_i > z^2_i$ for $i = 1, ..., m$.

Below we list some elementary properties of C - positive vectors without proof

a) Let $m = n$ and C the identity matrix. Then $v \in R^n$ is C-positive if and only if $v \gg 0$;

b) The set of C-positive vectors coincides with the relative interior of the cone generated by $c^1, ..., c^m$;

c) If there is a vector which is simultaneously C-positive and C-negative, then $c^1, ..., c^m$ are linearly dependent; The converse is not always true.

The following lemma will be needed.

Lemma 4.1. *Let $x \in R^n$. Then $Cx \geq 0$ (resp. $Cx > 0$) if and only if $\langle v, x\rangle \geq 0$ (resp. $\langle v, x\rangle > 0$) for every C-positive vector $v \in R^n$.*

Proof. It suffices to observe that $z \in R^m, z \geq 0$ (resp. $z > 0$) if and only if $\langle \lambda, z\rangle \geq 0$ (resp. $\langle \lambda, z\rangle > 0$) for all $\lambda \in \operatorname{int} R^m_+$ and that $C^T\lambda$ is C-positive for these λ. □

Let us now return to the normal cones of the polyhedral convex set P defined as in the previous section. We say that the normal cone to P at $x_0 \in P$ is negative if it contains a C-negative vector. Likewise a set $I \subseteq \{1, .., p\}$ is said to be negative if the cone generated by $\{-a^i : i \in I\}$ contains a C-negative vector.

Below is a criterion for I to be negative.

Proposition 4.2. *A subset $I \subseteq \{1, ..., p\}$ is negative if and only if the following system is consistent (has a solution):*

$$\begin{aligned} &\sum_{i\in I} \mu_i a^i = \sum_{j=1}^{m} \lambda_j c^j, \\ &\mu_i \geq 0, \quad i \in I, \\ &\lambda_j > 0, \quad j = 1, ..., m. \end{aligned} \tag{10}$$

Proof. This follows directly from the definition. □

Denote by I_1 the set of indices $i \in \{1, ..., p\}$ with a^i being C-positive; by I_3 the set of indices $i \in \{1, ..., p\}$ with a^i being C-negative and not C-positive. I_2 consists of all remaining indices. Then we have the partition of the index set $\{1, ..., p\}$ by disjoint subsets $I_1 \cup I_2 \cup I_3$. The next result shows how to find negative normal sets outside I_3.

Proposition 4.3. *Assume that $I \subseteq I_1 \cup I_2 \cup I_3$ is a negative normal set such that the cone generated by $\{-a^i, i \in I\}$ is not a linear subspace. Then there exists a negative normal set $I_0 \subseteq I \cap (I_1 \cup I_2)$.*

Proof. Let $I = \{i_1, ..., i_l\} \subseteq I_1 \cup I_2 \cup I_3$ be a negative normal set. We prove the proposition by induction on l. If $l = 1$, then $-a^{i_1}$ is a C-negative because $cone\{-a^{i_1}\}$ is a negative normal cone. Hence, a^{i_1} is a C-positive or $I \subseteq I_1$. Now, let $l > 1$. It is plain that $I \cap (I_1 \cup I_2) \neq \emptyset$. If $I \cap I_3 = \emptyset$, we are done by setting $I_0 = I$. If $I \cap I_3 \neq \emptyset$, say $i_l \in I_3$. Since I is a negative normal set, $cone\{-a^i : i = i_1, ..., i_l\}$ contains a C-negative vector. We state that it does not contain all C-negative vectors in its relative interior. Indeed, if not, as a^{i_l} is C-negative, the $cone\{-a^i : i = i_1, ..., i_l\}$ must contain 0 in its relative interior, hence it is a linear suspace, a contradiction to the hypothesis of the proposition. In this way, there is a C-negative vector v outside the relative interior of $cone\{-a^i : i = i_1, ..., i_l\}$. Joining v with a C-negative vector that $cone\{-a^i : i = i_1, ..., i_l\}$ contains, we find a face of this cone which contains a C-negative vector. The number of vectors generating that face is strictly less than l. By induction it has a negative normal cone generated exclusively by vectors with indices in $(I_1 \cup I_2)$. The index set determining this cone is a negative normal set in $I \cap (I_1 \cup I_2)$ as requested. $\square$

5. Efficient solution faces

Let us consider the following linear multiobjective programming problem denoted by (VP)

$$\min_{x \in M} Cx,$$

where C is an $(r \times n)$-matrix, M is the polyhedral convex set defined by (1) in R^n

We recall that $x_0 \in M$ is an efficient solution of (VP) if there is no other $x \in M$ such that $Cx_0 > Cx$. If every point of a face $F \subseteq M$ is an efficient solution, then F is said to be an efficient (solution) face.

The following scalarization result (Theorem 2.5, Chapter 4 [9]) will be of use (see also Theorem 2.1.5 [16]).

Lemma 5.1. *A point $x_0 \in M$ is an efficient solution of (VP) if and only if there is a positive vector $\lambda \in R^r$ (i.e. $\lambda \gg 0$) such that x_0 is a minimum of the linear function $x \longmapsto \langle \lambda, Cx \rangle$ over M.*

Below is a condition for a point to be an efficient solution in terms of normal cones.

Proposition 5.2. *A point $x_0 \in M$ is an efficient solution of (VP) if and only if the normal cone to M at x_0 is negative, i.e. $N_M(x_0)$ contains a C-negative vector.*

Proof. If $x_0 \in M$ is an efficient solution, then by Lemma 5.1, there is a vector $\lambda \gg 0$ such that $\langle \lambda, Cx - Cx_0 \rangle \geq 0$ for all $x \in M$. Then the vector $v = -C^T\lambda$ is a C-negative vector and belongs to the normal cone to M at x_0. Thus, $N_M(x_0)$ is a negative normal cone. Conversely, if there is a vector $v \in N_M(x_0)$ which is C-negative, then $v = -C^T\lambda$ for some $\lambda \gg 0$. We have also $\langle v, x - x_0 \rangle \leq 0$ for all $x \in M$, or equivalently, x_0 is a minimum of $\langle \lambda, C(.) \rangle$ on M. By Lemma 5.1, x_0 is an efficient solution of (VP). □

Corollary 5.3. *A face $F \subseteq M$ is an efficient solution face if and only if its normal cone is negative.*

Proof. We know from Lemma 5.1 that if a relative interior point of a face F is an efficient solution, then this face is an efficient solution face, that is, every point of F is an efficient solution. Now the corollary is immediate from Proposition 5.2. □

Corollary 5.4. *Let $I(F)$ be the index set determining a face F of M by* (2). *Then F is an efficient solution face if and only if the set $I(F)$ is negative normal.*

Proof. This is another formulation of Corollary 5.3. □

Using the above conditions we can derive some results on the structure of the efficient solution set of (VP).

Corollary 5.5. *The efficient solution set of (VP) is empty if and only if $N_M \cap ri(-cone(c^1, ..., c^r) = \emptyset$. Moreover, if the latter intersection is nonempty, then any of its elements attains a minimum on M which is an efficient solution of (VP).*

Proof. This fact follows from Proposition 5.2 and Proposition 3.2. □

We can also recapture a known connectedness result (Theorems 2.2, 2.3 Chapter 6 [9]).

Corollary 5.6. *The set of efficient solutions of (VP) is pathwise connected, i.e. for every two efficient solutions $x, y \in M$, there exist a finite number of efficient solutions $x_1, ..., x_k$ such that $x_0 = x$, $x_k = y$ and all segments $[x_i, x_{i+1}]$, $i = 0, ..., k-1$ are efficient.*

Proof. Let x and y be two efficient solutions of (VP). By Proposition 5.2, $N_M(x)$ and $N_M(y)$ contain C-negative vectors v_x and v_y respectively.

Then $v_x, v_y \in N_M \cap (ri(-cone(c^1, c^2, .., c^r)))$, where N_M is the union of all normal cones to M. Since N_M and $-cone(c^1, ..., c^r)$ are convex cones, the interval $[v_x, v_y]$ is contained in the intersection $N_M \cap (ri(-cone(c^1, ..., c^r)))$. Let $v_1 = v_x, v_2, ..., v_k = v_y$ be a partition of $[v_x, v_y]$ induced by the partition $\{N_{F_\lambda} : F_\lambda$ are faces of $M,\ \lambda \in \Lambda\}$ of N_M, i.e. $[v_i, v_{i+1}] \subseteq N_{F_{\lambda_i}}$, $i = 1, ..., k-1$. Since v_x, v_y are C - negative, so are $v_1, ..., v_k$. Consequently $N_{F_{\lambda_i}}$, $i = 1, ..., k-1$ are negative normal cones and hence $F_{\lambda_1}, ..., F_{\lambda_{k-1}}$ are efficient faces according to Proposition 5.2. As $v_{i+1} \in N_{F_{\lambda_i}} \cap N_{F_{\lambda_{i+1}}}$, there is an efficient face $F_{i+1} \supseteq F_{\lambda_i}, F_{\lambda_{i+1}}$ with $N_{F_{i+1}} \ni v_{i+1}$. Let $x_i \in F_{\lambda_i}$ such that $N_M(x_i) = N_{F_{\lambda_i}}$, $i = 1, ..., k-1$, $F_1 = F_{\lambda_1}$, $F_k = F_{\lambda_{k-1}}$. Then $[x, x_1] \subseteq F_1$, $[x_1, x_2] \subseteq F_2$,..., $[x_{k-2}, x_{k-1}] \subseteq F_{k-1}$, $[x_{k-1}, y] \subseteq F_k$ and $x, x_1, \ldots, x_{k-1}, y$ form an efficient path joining x and y. The proof is complete. $\square$

Let M be defined by (1) and assume that

(i) $\dim M = n$.

(ii) The system (1) does not contain any redundant inequality

The following result is useful in finding $(n-1)$-dimensional efficient faces.

Corollary 5.7. *Problem (VP) has an $(n-1)$-dimensional efficient face if and only if there is $i_0 \in \{1, ..., p\}$ such that a^{i_0} is a C-positive vector, in which case the face defined by (2) with $I(F) = \{i_0\}$ is an $(n-1)$-dimensional efficient solution face.*

Proof. This is a consequence of Proposition 5.2 and Proposition 3.5. $\square$

We recall that I_3 denotes the set of all indices $i \in \{1, ..., p\}$ such that a^i is C-negative and not C-positive.

Corollary 5.8. *Let $I(F)$ be the index set determining an efficient face F of M by (2) such that the cone generated by $\{a^i : i \in I(F)\}$ is not a linear subspace. If $I(F) \cap I_3 \neq \emptyset$, then F is properly contained in an efficient face F' of M such that $I(F') \cap I_3 = \emptyset$.*

Proof. Invoke Proposition 4.3 and Corollary 5.3. $\square$

6. An application

In this section we shall give a method for numerically solving the problem (VP). The study of normal cones and their relationship with efficient

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faces that we have developed in the previous sections allows us to construct quite simple algorithms to find efficient faces of any dimension.

Throughout this section, without loss of generality we assume that M is an n-dimensional polyhedral set determined by the system (1) and contains no lines, and that there is no redundant inequality in that system.

The general scheme for finding all the efficient solutions of (VP) is standard (see [1], [2], [6] etc.). It consists of three main procedures. The first procedure determines whether the problem (VP) has efficient solutions and to find an initial one if it exists. The second procedure generates all the efficient edges emanating from a given efficient vertex, hence all the efficient vertices and edges of the problem according to the pathwise connectedness of the efficient solution set. The last procedure finds the other efficient faces containing a given efficient vertex when all the efficient edges emanating from it have already been computed.

In our approach the first procedure is based on Corollary 5.5 and seems to be very simple in comparison with the algorithms used in [1], [2], [3] etc. The last two procedures contain two key tests. The first one checks whether an index subset $I \subseteq \{1, ..., p\}$ is negative which indicates whether the solutions to the system (9) with this I are efficient (Section 5). The second test checks whether I is normal, or equivalently whether those solutions form a face of certain dimension (Section 3). In finding efficient faces of dimension larger than one, our algorithm inherits the advantage of [1] that only the information of all the efficient edges containing a given vertex is needed. More importantly, in our program the simplex method is used to solve subsidiary scalar linear problems and is not subject to modification for vector problems as in [1], [2]. Therefore other alternative method for solving scalar linear problems can be applied.

Besides three procedures we also give some "fast" algorithms to deal with particular cases when the number of the variables is low, or when we want to compute merely efficient faces of dimension $(n-1)$.

6.1. Existence of efficient solutions and finding an initial efficient solution for (VP)

According to Corollary 5.5, (VP) has efficient solutions if and only if $N_M \cap ri(-cone(c^1, ..., c^r)) \neq \emptyset$, which is equivalent to $cone(a^1, ..., a^p) \cap ri(cone(c^1, ..., c^r)) \neq \emptyset$. Therefore, to determine whether (VP) has an efficient solution, we solve the system (10) with $I = \{1, ..., p\}$.

Procedure 1.

Step 1. Solve the system (10) with $I = \{1, ..., p\}$.

a) If the system has no solutions, then stop. (VP) has no efficient solutions.

b) Otherwise, go to Step 2.

Step 2. Let $\lambda \gg 0$ be a solution and $v = C^T\lambda$.

a) If $v = 0$, then stop. The set M is efficient.

b) Otherwise, solve the linear programming problem

$$\min\{\langle v, x\rangle, x \in M\}.$$

According to Corollary 5.5, this problem has a solution, say x_0. This x_0 is an initial efficient solution of (VP).

6.2. Determination of Efficient Vertices and Efficient Edges

Let x_0 be an efficient vertex. Recall that $I(x_0)$ denotes the set of active indices at x_0, that is $I(x_0) = \{i \in \{1, ..., p\} : \langle a^i, x_0\rangle = b_i\}$. Any subset $I \subseteq I(x_0)$ with $|I| = n-1$ and $\{a^i, i \in I\}$ linearly independent determines a direction $v \neq 0$ by the following system

$$\langle a^k, v\rangle = 0, \quad k \in I.$$

In view of Corollary 5.4, in order to decide whether the edge emanating from x_0 along the direction v is an efficient edge we have to verify

a) Whether I is negative (equivalently, the cone generated by $\{-a^i : i \in I\}$ contains a C-negative vector);

b) Whether I is normal (equivalently, the system

$$\begin{aligned} \langle a^i, x\rangle &= b_i, \quad i \in I,\\ \langle a^j, x\rangle &\geq b_j, \quad j \in \{1, ..., p\}\backslash I, \end{aligned}$$

is satisfied at a point $(x_0 + tv)$ for some $t \neq 0$ (Corollary 3.6)).

Now we describe a procedure for finding all the efficient solution edges emanating from an efficient vertex x_0.

Procedure 2.

Step 0 (Initialization). Determine the active index set

$$I(x_0) = \{i \in \{1, ..., p\} : \langle a^i, x_0\rangle = b_i\}.$$

a) If $|I(x_0)| = n$, then go to Step 1.

b) Otherwise, go to Step 2.

Step 1. *(x_0 is a nondegenerate vertex)*

Pick $I \subseteq I(x_0)$ with $|I| = n - 1$.

Step 1.1. *(Is I negative ?).* Solve (10) with this I.

a) If it has no solutions, pick another $I \subseteq I(x_0)$ and return to Step 1.1.

b) Otherwise, I is a negative set, go to Step 1.2.

Step 1.2. *(It is sure that I is normal. Find the corresponding efficient solution edge)*

Step 1.2.1. Find a direction v of an edge emanating from x_0 by solving

$$\begin{aligned} \langle a^k, v\rangle &= 1, \quad k \in I(x_0) \setminus I, \\ \langle a^i, v\rangle &= 0, \quad i \in I. \end{aligned}$$

Step 1.2.2. Put

$$t_i = max\{t : \langle a^i, x_0 + tv\rangle \geq b_i, t \geq 0\}, \quad i \in \{1, ..., p\} \setminus I.$$

and $t_0 = \min\{t_i : i \in \{1, ..., p\} \setminus I\}$.

a) If $0 < t_0 < \infty$, then $x_0 + t_0 v$ is an efficient solution vertex and $[x_0, x_0 + t_0 v]$ is an efficient solution edge adjacent to x_0. Store them if they have not been stored before. Pick another $I \subseteq I(x_0)$ and return to Step 1.1.

b) If $t_0 = \infty$, then the ray $\{x_0 + tv : t \geq 0\}$ is an efficient solution ray of the problem. Store the result. Pick another $I \subseteq I(x_0)$ and return to Step 1.1.

Step 2. *(x_0 is a degenerate vertex)*

Pick $I \subseteq I(x_0)$ with $|I| = n - 1$.

Step 2.0. Check whether $rank\{a^i : i \in I\} = n - 1$

a) If "Yes" go to Step 2.1.

b) Otherwise, pick another $I \subseteq I(x_0)$ and return to Step 2.0.

Step 2.1. *(Is I negative ? This step is the same as Step 1.1).* Solve (10) with this I.

a) If it has no solutions, pick another $I \subseteq I(x_0)$ and return to Step 2.0.

b) Otherwise, I is a negative set, go to Step 2.2.

Step 2.2. *(Is I normal? If yes, find the corresponding efficient edge.)*

Step 2.2.1. Find a direction $v \neq 0$ of a possible edge emanating from x_0 by solving

$$\langle a^i, v\rangle = 0, \quad i \in I.$$

Step 2.2.2. Solve the following system

$$\langle a^i, x_0 + tv\rangle \geq b_i, \quad i = 1, ..., p.$$

Let the solution set be $[t_0, 0]$ or $[0, t_0]$ according to $t_0 < 0$ or $t_0 > 0$. The values $t_0 = -\infty$ and $t_0 = \infty$ are possible.

a) If $t_0 = 0$, then no edge of M emanating from x_0 along v. I is not normal. Pick another $I \subseteq I(x_0)$ and go to Step 2.0.

b) If $t_0 \neq 0$ and is finite, then $x_0 + t_0 v$ is an efficient solution vertex and $[x_0, x_0 + t_0 v]$ is an efficient solution edge. Store them if they have not been stored before. Pick another $I \subseteq I(x_0)$ and go to Step 2.0.

c) If t_0 is infinite, say $t_0 = \infty$, then the ray $\{x_0 + tv : t \geq 0\}$ is efficient. Store the result. Pick another $I \subseteq I(x_0)$ and go to Step 2.0.

Remark

i) Since the efficient solution set of (VP) is pathwise connected, by applying the above procedure we are able to generate all the efficient vertices and all the efficient edges of the problem.

ii) In the degenerate cases one can alternatively use methods to first find the subsets of indices corresponding to edges and then check their negativity. It is however advisable to carry out first the test of negativity because it allows to rescind several subsets that do not correspond to efficient edges.

6.3. Determination of other efficient solution faces

Assume that x_0 is an efficient solution vertex of the problem (VP) and $[x_0, x_0 + t_i v_i]$, $i = 1, ..., k$ are efficient edges emanating from x_0 with $t_i > 0$. Here, for the convenience we use $t_i = \infty$ if the ray $\{x_0 + tv_i : t \geq 0\}$ is efficient and $[x_0, x_0 + t_i v_i]$ denotes this ray. Let $I_i \subseteq I(x_0)$, $i = 1, ..., k$ be the negative index sets determining these edges.

Observe first that the largest dimension that an efficient face adjacent to x_0 may have is $min\{k, n-1\}$. For $1 < l \leq \min\{k, n-1\}$, we have the following procedure to find l-dimensional efficient faces adjacent to x_0.

Procedure 3.

Step 0 (Initialization). As Step 0 of Procedure 2. Find $I(x_0)$.

a) If $|I(x_0)| = n$, then go to Step 1.

b) Otherwise, goto Step 2.

Step 1. *(x_0 is nondegenerate vertex)*

Pick $J \subseteq \{1, ..., k\}$, $|J| = l$ and consider $I = \bigcap_{j \in J} I_j$. Since $|I_j| = n - 1$, it is evident that $|I| = n - l$.

Step 1.1. *(Is I negative ?)* Solve (10) with this I.

a) If it has no solutions, pick another J and return to Step 1.1.

b) Otherwise, I is negative, go to Step 1.2.

Step 1.2. *(It is sure that I is normal.)*

The l-dimensional efficient face determined by I contains $conv\{x_0; x_0 + t_i v_i : i \in J\}$ is efficient. Store the result.

Pick another J and return to Step 1.1.

Step 2. *(x_0 is degenerate vertex)*

Pick $J \subseteq \{1, ..., k\}$ with $|J| = l$.

Step 2.0. Consider

$$x_J = \frac{x_0}{l+1} + \sum_{j \in J} \frac{x_0 + \lambda_j v_j}{l+1},$$

where $\lambda_j = t_j$ if t_j is finite and $\lambda_j = 1$ if $t_j = \infty$. Determine the active index set $I(x_J)$ at x_J.

a) If $rank\{a^i : i \in I(x_J)\} < l$, then pick another J and go to Step 2.0.

b) Otherwise, go to Step 2.1.

Step 2.1. *(Is $I(x_J)$ negative ?)* Solve (10) with $I = I(x_J)$.

a) If it has no solutions, then pick another J and go to Step 2.0.

b) Otherwise, $I(x_J)$ is negative, go to Step 2.2.

Step 2.2. *(Find the l-dimensional efficient face containing the edges $[x_0, x_0 + t_j v_j]$, $j \in J$)*

Determine $J_0 = \{j \in \{1, ..., k\} : I_j \supseteq I(x_J)\}$. (It is evident that $J \subseteq J_0$.)

The convex hull of the edges $[x_0, x_0 + t_j v_j], j \in J_0$ is contained in the l-dimensional efficient face we are looking for (Proposition 3.7). Pick another J not containing J_0 with $|J| = l$ and go to Step 2.0.

Remark. Using Procedure 3 one easily obtains an algorithm to determine maximal eficient faces adjacent to a given efficient vertex (an efficient face is maximal if it is not a proper face of a larger efficient face).

6.4. Particular cases

6.4.1. Determination of $(n-1)$ dimensional efficient solution faces

If for some reason we need to find $(n-1)$-dimensional efficient faces only, then the algorithm is very simple (Corollary 5.7) as follows.

Solve (10) with $I = \{i\}, i = 1, ..., p$

a) If it has a solution for some i, then we claim that the face defined by (2) with $I(F) = \{i\}$ is an $(n-1)$-dimensional efficient solution face.

b) Otherwise the problem (VP) has no efficient face of dimension $(n-1)$.

6.4.2. Efficient sets in R^2 and R^3

Sometimes we wish to compute the efficient set of a polyhedron (a bounded polyhedral convex set) which corresponds to the efficient solution set of the problem (VP) with C being the identity matrix. Below we provide an effective and direct algorithm to do this in the case $M \subseteq R^2$ and $M \subseteq R^3$.

a) The case $M \subseteq R^2$

Let us express $a^1, ..., a^p$ in the polar coordinate system

$$a^i = (|a^i|, \theta_i), \quad i = 1, ..., p.$$

By renumbering the indices if necessary, we may assume

$$0 < \theta_1 < \theta_2 < ... < \theta_k < \frac{1}{2}\pi \leq \theta_{k+1} < ... < \theta_p \leq 2\pi.$$

It is evident that $a^1, ..., a^k$ are positive vectors and by Corollay 5.7 each of them determines an efficient edge. Moreover, each of pairs $\{p, 1\}$, $\{1, 2\}$,..., $\{k, k+1\}$ is negative and normal. So by Corollary 5.3 they determine all the $0-$ dimensional efficient faces (vertices) of M. Denote by M_i the intersection point of the lines

$$\langle a^i, x\rangle = b_i$$
$$\langle a^{i+1}, x\rangle = b_{i+1},$$

$i = 0, ..., k$, where $a^0 = a^p$, $b_0 = b_p$. Then the efficient set of M is

$$\bigcup_{i=0}^{k} [M_i, M_{i+1}].$$

b) The case $M \subseteq R^3$

If the dimension of M is three, then it may have efficient faces of dimension 0 or 1 or 2. We recall that a point $x_0 \in M$ is said to be ideal efficient point if $x_0 \leq x$ for all $x \in M$. It is easy to see that M does not possess ideal efficient points if and only if it has efficient faces of dimension 1 or 2. Now we describe an algorithm to determine the set of all efficient points of $M \subseteq R^3$.

With one exceptional case when M has only one efficient point, the efficient set of M can be completely determined if we know all efficient edges.

Step 1. *(To determine whether M possesses an ideal efficient point).*

Solve the linear program

$$\min\{\langle e^i, x\rangle, x \in M\}$$

for $i = 1, 2, 3$, where $e^1 = (1, 0, 0)$, $e^2 = (0, 1, 0)$, $e^3 = (0, 0, 1)$. Let x_1^*, x_2^*, x_3^* be the optimal values of these programs.

a) If $x^* = (x_1^*, x_2^*, x_3^*) \in M$, then x^* is an ideal efficient point of M and it is the unique efficient point of M.

b) Otherwise, go to Step 2.

Step 2. Decompose the index set $\{1, ..., p\}$ into I_1, I_2, I_3, where $I_1 = \{i : a^i \gg 0\}$, $I_3 = \{i : a^i \ll 0\}$, $I_2 = \{1, ..., p\} \setminus (I_1 \cup I_3)$.

a) If $I_1 = \emptyset$, then there are no efficient faces of dimension 2. Go to Step 3 to find efficient faces of smaller dimension.

b) Otherwise, each $a^i, i \in I_1$ determines an efficient face of dimension 2 by the system

$$\begin{aligned} &\langle a^i, x\rangle = b_i, \\ &\langle a^j, x\rangle \geq b_j, \quad j \in \{1, ..., p\}\setminus\{i\}. \end{aligned}$$

Go to Step 3 to find efficient faces of smaller dimension, not included in the above 2-dimensional efficient faces.

Step 3. Choose $\{i, j\} \in I_2$.

Step 3.1. *(Is $\{i, j\}$ negative ?)*

Solve the system

$$ta^i + (1-t)a^j \gg 0,$$

$$0 \leq t \leq 1.$$

a) If it has a solution, then $\{i, j\}$ is negative. Go to Step 3.2.

b) Otherwise, $\{i, j\}$ is not negative, pick other pair $\{i, j\} \in I_2$ and return to Step 3.1.

Step 3.2. *(Is $\{i, j\}$ normal?)*

Determine the set $\Delta_{ij} := \{x \in M : \langle a^i, x\rangle = b_i, \langle a^j, x\rangle = b_j\}$.

a) If $\Delta_{ij} = \emptyset$ or Δ_{ij} is a point, then $\{i, j\}$ is not normal. Pick other pair $i, j \in I_2$ and return to Step 3.1.

b) Otherwise Δ_{ij} is a segment. This segment is an efficient edge. Store it. Pick another $i, j \in I_2$ and return Step 3.1.

Remark. According to Corollary 5.8, Step 2 and Step 3 allow to generate all the efficient set of M because other efficient faces of M are included in those found in these steps.

6.5. Examples

The following examples have been computed in DELPHI 2.0 on PC 486 SX.

Example 1. We begin with the test example given by Yu and Zeleny in [26], (see also [1, 2, 8]). Note that each vertex is nondegenerate.

$$Max \begin{bmatrix} 3 & -7 & 4 & 1 & 0 & -1 & -1 & 8 \\ 2 & 5 & 1 & -1 & 6 & 8 & 3 & -2 \\ 5 & -2 & 5 & 0 & 6 & 7 & 2 & 6 \\ 0 & 4 & -1 & -1 & -3 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_8 \end{bmatrix},$$

$$\text{s.t} \begin{bmatrix} 1 & 3 & -4 & 1 & -1 & 1 & 2 & 4 \\ 5 & 2 & 4 & -1 & 3 & 7 & 2 & 7 \\ 0 & 4 & -1 & -1 & -3 & 0 & 0 & 1 \\ -3 & -4 & 8 & 2 & 3 & -4 & 5 & -1 \\ 12 & 8 & -1 & 4 & 0 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 8 & -12 & -3 & 4 & -1 & 0 & 0 & 0 \\ 15 & -6 & 13 & 1 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_8 \end{bmatrix} \leq \begin{bmatrix} 40 \\ 84 \\ 18 \\ 100 \\ 40 \\ -12 \\ 30 \\ 100 \end{bmatrix}$$

$x_1, ..., x_8 \geq 0.$

We have obtained 29 efficient vertices, 46 efficient edges, 18 efficient faces of dimension 2 and no efficient faces of higher dimension, as expected (see [1] and [2]). These results are included here for sake of completeness.

LIST OF EFFICIENT VERTICES

$v_1 = (0.000, 0.000, 4.837, 8.631, 0.000, 0.000, 10.313, 7.522)$
$v_2 = (0.000, 0.000, 6.834, 10.066, 0.000, 0.000, 6.570, 7.656)$
$v_3 = (0.000, 0.000, 4.157, 10.618, 0.000, 0.000, 1.686, 10.660)$
$v_4 = (0.000, 3.529, 4.539, 0.000, 0.000, 0.000, 16.308, 3.738)$
$v_5 = (0.000, 0.000, 0.000, 7.066, 10.745, 0.000, 11.737, 5.051)$
$v_6 = (0.000, 0.000, 2.213, 4.253, 0.000, 5.801, 19.399, 0.000)$

$v_7 = (0.000, 0.000, 6.022, 11.506, 0.000, 0.000, 0.000, 10.202)$
$v_8 = (0.000, 0.000, 6.375, 11.594, 10.450, 0.000, 0.000, 5.535)$
$v_9 = (0.000, 0.000, 5.000, 11.250, 0.000, 0.000, 0.000, 10.750)$
$v_{10} = (0.000, 0.100, 4.002, 10.801, 0.000, 0.000, 0.000, 11.227)$
$v_{11} = (0.000, 4.137, 3.980, 0.000, 0.000, 0.000, 10.887, 5.433)$
$v_{12} = (0.000, 2.335, 2.889, 0.000, 0.000, 3.870, 20.341, 0.000)$
$v_{13} = (0.000, 0.000, 0.000, 10.000, 27.250, 0.000, 0.000, 1.750)$
$v_{14} = (0.000, 0.000, 0.000, 4.183, 5.982, 4.740, 18.530, 0.000)$
$v_{15} = (0.000, 0.000, 3.874, 10.406, 0.000, 0.000, 0.000, 11.273)$
$v_{16} = (0.000, 3.740, 3.208, 3.323, 0.000, 0.000, 0.000, 9.573)$
$v_{17} = (0.000, 3.647, 0.000, 0.000, 0.000, 6.235, 4.588, 3.412)$
$v_{18} = (0.000, 2.481, 0.000, 0.000, 0.000, 7.747, 12.405, 0.000)$
$v_{19} = (0.000, 2.284, 0.000, 0.000, 7.771, 2.534, 19.192, 0.000)$
$v_{20} = (0.824, 0.000, 0.000, 7.529, 29.137, 0.000, 0.000, 0.000)$
$v_{21} = (0.000, 0.000, 0.000, 9.702, 28.454, 1.191, 0.000, 0.000)$
$v_{22} = (0.000, 0.000, 1.273, 0.000, 0.000, 0.000, 0.000, 11.273)$
$v_{23} = (0.000, 2.462, 2.000, 0.000, 0.000, 0.000, 0.000, 10.154)$
$v_{24} = (0.000, 3.927, 0.000, 0.000, 0.000, 8.585, 0.000, 2.293)$
$v_{25} = (0.000, 3.630, 0.000, 0.000, 0.000, 10.963, 0.000, 0.000)$
$v_{26} = (0.000, 3.467, 0.000, 0.000, 17.511, 0.000, 12.267, 0.000)$
$v_{27} = (0.000, 0.000, 0.000, 5.333, 29.778, 0.000, 0.000, 0.000)$
$v_{28} = (0.000, 1.167, 0.000, 7.667, 29.778, 0.000, 0.000, 0.000)$
$v_{29} = (0.000, 5.000, 0.000, 0.000, 24.667, 0.000, 0.000, 0.000)$

LIST OF EFFICIENT FACES OF DIMENSION 2

$F_1 = [v_1, v_2, v_7, v_9, v_3]$; $F_2 = [v_1, v_3, v_{10}, v_{16}, v_{11}, v_4]$; $F_3 = [v_1, v_5, v_{13}, v_8, v_2]$;
$F_4 = [v_1, v_6, v_{14}, v_5]$; $F_5 = [v_1, v_4, v_{12}, v_6]$; $F_6 = [v_2, v_8, v_7]$;
$F_7 = [v_3, v_9, v_{10}]$; $F_8 = [v_4, v_{11}, v_{17}, v_{18}, v_{12}]$; $F_9 = [v_5, v_{14}, v_{21}, v_{13}]$;
$F_{10} = [v_6, v_{12}, v_{19}, v_{14}]$; $F_{11} = [v_9, v_{15}, v_{10}]$; $F_{12} = [v_{10}, v_{15}, v_{22}, v_{23}, v_{16}]$;
$F_{13} = [v_{12}, v_{18}, v_{19}]$; $F_{14} = [v_{13}, v_{21}, v_{20}]$; $F_{15} = [v_{14}, v_{19}, v_{26}, v_{28}, v_{21}]$;
$F_{16} = [v_{17}, v_{24}, v_{25}, v_{18}]$; $F_{17} = [v_{21}, v_{28}, v_{27}]$; $F_{18} = [v_{26}, v_{29}, v_{28}]$;

Example 2. Solve the problem

$$Min \begin{bmatrix} -x_1 - x_2 - 0.25x_3 \\ x_1 + x_2 + 1.5x_3 \end{bmatrix},$$

s.t $x \in M$,

$M = \{x \in R^3 \mid 2x_1 + x_2 + 2x_3 \geq 2, x_1 + 2x_2 + x_3 \geq 2, -x_1 - x_2 - x_3 \geq -6, x_1, x_2, x_3 \geq 0\}$.

We have obtained 1 two-dimensional efficient face which is determined by $I(F) = \{6\}$. This efficient face contains:

- 5 efficient vertices : $x_1 = (0.67, 0.67, 0)$, $x_2 = (2, 0, 0)$, $x_3 = (0, 2, 0)$, $x_4 = (6, 0, 0)$, $x_5 = (0, 6, 0)$
- 5 efficient edges : $[x_1, x_2]$, $[x_1, x_3]$, $[x_2, x_4]$, $[x_3, x_5]$, $[x_4, x_5]$.

Example 3. Solve the problem

$$Min \begin{bmatrix} -x_1 + 100x_2 + 0x_3 \\ -x_1 - 100x_2 + 0x_3 \\ 0x_1 + 0x_2 - 1x_3 \end{bmatrix},$$

s.t $x \in M$,

$M = \{x \in R^3 \mid x_1 + 2x_2 + 2x_3 \leq 10, 2x_1 + x_2 + 2x_3 \leq 10, 5x_1 + 5x_2 + 6x_3 \leq 30, x_1, x_2, x_3 \geq 0\}$.

We have obtained 3 two-dimensional efficient faces F_1, F_2, F_3 which are determined by $I(F_1) = \{1\}$, $I(F_2) = \{2\}$, $I(F_3) = \{3\}$.

- Face F_1 has 3 vertices : $x_2 = (2, 4, 0)$, $x_4 = (0, 0, 5)$, $x_5 = (0, 5, 0)$ and 3 edges: $[x_2, x_4]$, $[x_2, x_5]$, $[x_4, x_5]$.
- Face F_2 has 3 vertices : $x_1 = (4, 2, 0)$, $x_3 = (5, 0, 0)$, $x_4 = (0, 0, 5)$ and 3 edges: $[x_1, x_3]$, $[x_1, x_4]$, $[x_3, x_4]$.

Face F_3 has 3 vertices : $x_1 = (4, 2, 0)$, $x_2 = (2, 4, 0)$, $x_4 = (0, 0, 5)$ and 3 edges: $[x_1, x_2]$, $[x_1, x_4]$, $[x_2, x_4]$.

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