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Further Analysis of an Outcome Set-Based Algorithm for Multiple-Objective Linear Programming[1]

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Abstract. Various difficulties have been encountered in using decision set-based vector maximization methods to solve a multiple-objective linear programming problem (MOLP). Motivated by these difficulties, Benson recently developed a finite, outer-approximation algorithm for generating the set of all efficient extreme points in the outcome set, rather than in the decision set, of problem (MOLP). In this article, we show that the Benson algorithm also generates the set of all weakly efficient points in the outcome set of problem (MOLP). As a result, the usefulness of the algorithm as a decision aid in multiple objective linear programming is further enhanced.

Key Words. Multiple-objective linear programming, efficient sets, weakly efficient sets, outer-approximation algorithms.

1. Introduction

During the past 30 years, operations researchers and many other analysts have become increasingly aware that it is often necessary to explicitly consider several criteria simultaneously in decision making. One of the more popular and practical models that has been used over this time period to aid in decision making with multiple criteria is the multiple objective linear programming model. This model may be written

$$\text{(MOLP)} \quad \text{Vmax}\, Cx, \qquad \text{s.t. } x \in X,$$

where C is a $p \times n$ matrix whose rows c_i, $i = 1, 2, \ldots, p$, are the coefficients of p linear criterion functions, and X is a nonempty, polyhedral set in $\Re^n$.

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Problem (MOLP) and its numerous applications have been studied in literally hundreds of articles, books, and chapters in books; see, for example Refs. 1–25 and references therein.

Various solution concepts for problem (MOLP) have been proposed. Two of the more useful ones are efficiency and weak efficiency. In particular, a point $x^\circ \in \Re^n$ is an efficient or nondominated solution for problem (MOLP) when $x^\circ \in X$ and there exists no $x \in X$ such that $Cx \geq Cx^\circ$ and $Cx \neq Cx^\circ$. A point $\bar{x} \in \Re^n$ is a weakly efficient or weakly nondominated solution for problem (MOLP) when $\bar{x} \in X$ and there exists no $x \in X$ such that $Cx > C\bar{x}$. Let X_E and X_{WE} denote the sets of all efficient solutions and weakly efficient solutions, respectively, of problem (MOLP). Notice that $X_E \subseteq X_{WE}$.

Various approaches that use problem (MOLP) as an aid in multiple-criteria decision making have been proposed. One approach, called the vector maximization approach, has received a good deal of attention. In this approach, all of either X_E or X_{WE} is mathematically generated, or at least some representative portion thereof, without any input from the decision maker (DM). Subsequently, the entire set generated is presented to the DM. The DM, without further aid from the analyst, then chooses a most preferred solution to problem (MOLP) from the generated set. In some cases, all of X_E or X_{WE} is generated; see, e.g., Refs. 1, 2, 8, 12, 17, 21, 24. In other cases, subsets of X_E or X_{WE} are generated, the most popular of which are $(X_E \cap X_{ex})$ and $(X_{WE} \cap X_{ex})$, where X_{ex} denotes the set of all extreme points of X; see, e.g., Refs. 9, 11, 15, 18, 21, 22, 25.

The vector maximization approach has several attractions. At the same time, however, it has two important drawbacks. First, the computational demands of generating all or representative portions of X_E or of X_{WE} grow exponentially with problem size (Refs. 17, 19). Second, the sheer size and nature of the generated set can easily overwhelm the DM (Refs. 17, 18).

Motivated by these difficulties, Benson (Refs. 4, 26) has recently proposed vector maximization algorithms for problem (MOLP) that generate efficient points in the outcome set, rather than in the decision set, of the problem. In particular, in one of these algorithms (Ref. 26), outer approximation is used to generate the set of all efficient extreme points in the outcome set of problem (MOLP). Preliminary experimental applications to randomly-generated problems have demonstrated the computational and practical value of this algorithm (Ref. 26).

In this article, we show that Benson's outer-approximation, outcome set-based algorithm (Ref. 26) also generates the set of all weakly efficient points in the outcome set of problem (MOLP). Since the DM may prefer a nonextreme efficient point in the outcome set to all of the efficient extreme points in this set, this fact further enhances the usefulness of the algorithm of Ref. 26.

In the next section, the steps and key properties of the outcome set-based algorithm of Ref. 26 are summarized. The main result is shown in Section 3, and conclusions are given in the last section.

2. Summary of the Outcome Set-Based Algorithm

Assume henceforth that X is a nonempty, compact polyhedron given by

$$X = \{x \in \Re^n | Ax = b, x \geq 0\},$$

where A is an $m \times n$ matrix and $b \in \Re^m$. Define the sets $Y^=$, $Y_E^=$, $Y_{WE}^=$ by

$$Y^= = \{Cx | x \in X\},$$

$$Y_E^= = \{Cx | x \in X_E\},$$

$$Y_{WE}^= = \{Cx | x \in X_{WE}\}.$$

The sets $Y^=$, $Y_E^=$, $Y_{WE}^=$ are called the outcome set, the efficient outcome set, and the weakly efficient outcome set, of problem (MOLP). It can be shown that the outcome set $Y^=$ is a nonempty compact polyhedron in $\Re^p$ (Ref. 27). It is easy to show that $Y_E^=$ is equal to the set of all efficient outcomes for problem (MOLP), where an efficient or nondominated outcome for problem (MOLP) is a point $y^\circ \in Y^=$ such that there exists no $y \in Y^=$ such that $y \geq y^\circ$ and $y \neq y^\circ$. Similarly, it is easy to show that $Y_{WE}^=$ is equal to the set of all weakly efficient outcomes for problem (MOLP), where a weakly efficient or weakly nondominated outcome for problem (MOLP) is a point $\bar{y} \in Y^=$ such that there exists no $y \in Y^=$ such that $y > \bar{y}$.

The outcome set-based algorithm in Ref. 26 concentrates on generating the set of all efficient extreme points of $Y^=$, i.e., $Y_E^= \cap Y_{ex}^=$, where $Y_{ex}^=$ denotes the set of all extreme points of $Y^=$. Instrumental to the algorithm is the set

$$Y = \{y \in \Re^p | \hat{y} \leq y \leq Cx, \text{ for some } x \in X\},$$

where $\hat{y} \in \Re^p$ is any vector that, for each $i = 1, 2, \ldots, p$, satisfies

$$\hat{y}_i < \min y_i, \qquad \text{s.t. } y \in Y^=.$$

A point $y^\circ \in Y$ is called an efficient or admissible point of Y when no $y \in Y$ exists such that $y \geq y^\circ$ and $y \neq y^\circ$. When $y^\circ \in Y$ and no $y \in Y$ exists such that $y > y^\circ$, then y° is called a weakly efficient or weakly admissible point of Y. Let Y_E and Y_{WE} denote the sets of all efficient and weakly efficient points of Y.

In Ref. 26, it is shown that Y is a nonempty, full-dimensional, compact polyhedron in $\mathfrak{R}^p$ and that $Y_E = Y_E^{=}$. Thus, Y is called an efficiency-equivalent polyhedron for $Y^{=}$.

To identify the set of all efficient extreme points of $Y^{=}$, the algorithm in Ref. 26 uses outer approximation to generate the entire efficiency-equivalent polyhedron Y for the outcome set $Y^{=}$. In the first step of the algorithm, a p-dimensional simplex containing Y is constructed. This simplex S° is given by

$$S^\circ = \{y \in \mathfrak{R}^p | \hat{y} \leq y, \langle e, y \rangle \leq \beta\}, \tag{1}$$

where $e \in \mathfrak{R}^p$ is the vector whose entries are each equal to 1.0 and

$$\beta = \max \langle e, y \rangle, \qquad \text{s.t. } y \in Y.$$

In Ref. 26, it is shown that $S^\circ \supseteq Y$ as required, and a simple method to explicitly enumerate the $p+1$ vertices of S° is given. Notice that S° contains the p faces of Y, $\{y \in Y | y_i = \hat{y}_i\}$, $i = 1, 2, \ldots, p$.

At the beginning of a typical iteration k, $k \geq 0$, of the algorithm, the set of all vertices $V(S^k)$ of the current compact polyhedron S^k containing Y is examined to see whether or not each element of $V(S^k)$ belongs to Y. If so, then $S^k = Y$, and the algorithm stops. Otherwise, a linear inequality is appended to S^k to obtain a new polyhedron S^{k+1} that contains Y and satisfies $S^{k+1} \subset S^k$. The vertex set $V(S^{k+1})$ of S^{k+1} is then explicitly computed, and the algorithm proceeds to the next iteration. In particular, the steps of the outcome set-based algorithm can be summarized as follows.

Algorithm 2.1.

Initialization Step. Compute a point $\bar{p}$ in the interior of Y and construct the p-dimensional simplex S° containing Y given by (1). Find the set $V(S^\circ)$ of all vertices of S°, set $k = 0$, and go to Iteration k.

Iteration k, $k \geq 0$. See Steps $k1$ through $k4$ below.

Step $k1$. If, for each $y \in V(S^k)$, $y \in Y$ is satisfied, then stop: $Y = S^k$. Otherwise, choose any $y^k \in V(S^k)$ such that $y^k \notin Y$ and continue.

Step $k2$. Find the unique value λ_k of λ, $0 < \lambda < 1$, such that $\lambda y^k + (1 - \lambda)\bar{p}$ belongs to the boundary of Y, and set $w^k = \lambda_k y^k + (1 - \lambda_k)\bar{p}$.

Step $k3$. Set $S^{k+1} = S^k \cap \{y \in \mathfrak{R}^p | \langle u^k, y \rangle \leq \langle b, v^k \rangle\}$, where $u^k \in \mathfrak{R}^p$ and $v^k \in \mathfrak{R}^m$ are vectors such that $F^k = \{y \in Y | \langle u^k, y \rangle = \langle b, v^k \rangle\}$ is a face of Y containing w^k.

Step $k4$. Using $V(S^k)$ and the definition of S^{k+1} given in Step $k3$, determine $V(S^{k+1})$. Set $k = k+1$, and to Iteration k.

For each $k \geq 0$, the vectors u^k and v^k in Step $k3$ of the algorithm are found by solving a linear programming problem. For details concerning this linear programming problem and the implementation of the other steps of the outcome set-based algorithm, see Ref. 26.

From Ref. 26, we have the following properties of the outcome set-based algorithm.

Property P1. For each $k \geq 0$, $w^k \in Y_{WE}$.

Property P2. For each $k \geq 0$, $u^k \geq 0$, $u^k \neq 0$, and the face F^k containing w^k given in Step $k3$ is a weakly efficient face of Y.

Property P3. The outcome set-based algorithm is finite and, when it terminates, $S^K = Y$, where $K \geq 0$ is the final iteration number.

3. Main Result

To show that the outcome set-based algorithm generates the entire weakly efficient outcome set $Y_{\bar{W}E}$ of problem (MOLP), we must first state and prove the following two results.

Theorem 3.1. Let $K \geq 0$ denote the final iteration number of the outcome set-based algorithm. Then,

$$Y_{WE} = \bigcup_{k=0}^{K-1} \{y \in Y \mid \langle u^k, y \rangle = \langle b, v^k \rangle\}.$$

Proof. As observed in Section 2, the initial simplex S° of the algorithm contains the p faces of Y,

$$G_i = \{y \in Y \mid y_i = \hat{y}_i\}, \qquad i = 1, 2, \ldots, p.$$

Since $Y \subseteq \mathfrak{R}^p$ is full dimensional, this implies that, for each $i = 1, 2, \ldots, p$, there are no weakly efficient points of Y in the relative interior of G_i. From Step $k3$ of the algorithm, for each $k = 1, 2, \ldots, K$, the polyhedron S^k generated by the algorithm also contains the p faces G_i, $i = 1, 2, \ldots, p$. Therefore, by Property P3 and Step $k3$ of the algorithm, it follows that, at the beginning of iteration K,

$$Y = S^K = \{y \in \mathfrak{R}^p \mid y \geq \hat{y}\} \cap \left[\bigcap_{k=0}^{K-1} H_k\right],$$

where, for each $k = 0, 1, \ldots, K-1$,

$$H_k = \{y \in \mathfrak{R}^p \mid \langle u^k, y \rangle \leq \langle b, v^k \rangle\}.$$

Since Y_{WE} is a union of faces of Y (Ref. 20), and since the faces G_i, $i=1, 2, \ldots, p$, do not belong to Y_{WE}, by Property P2 this implies that

$$Y_{WE} = \bigcup_{k=0}^{K-1} \{y \in Y \mid \langle u^k, y\rangle = \langle b, v^k\rangle\}. \qquad \square$$

Theorem 3.2. The intersection of the set of weakly efficient points of Y with the outcome set of problem (MOLP) is equal to the weakly efficient outcome set of problem (MOLP), i.e., $Y_{WE} \cap Y^{=} = Y^{=}_{WE}$.

Proof. Suppose first that $\bar{y} \in (Y_{WE} \cap Y^{=})$. Then, from Ref. 20, since Y is polyhedral and $\bar{y} \in Y_{WE}$, there exists a point $\bar{u} \in \Re^p$ such that $\bar{u} \geq 0$, $\bar{u} \neq 0$ and $\bar{y}$ is an optimal solution to the problem

$$\max\langle \bar{u}, y\rangle, \qquad \text{s.t. } y \in Y.$$

Because $Y^{=} \subseteq Y$ and $y \in Y^{=}$, this implies that $\bar{y}$ is also an optimal solution to the problem

$$\max\langle \bar{u}, y\rangle, \qquad \text{s.t. } y \in Y^{=}.$$

From Ref. 20, since $\bar{u} \geq 0$ and $\bar{u} \neq 0$, this implies that $\bar{y} \in Y^{=}_{WE}$. Therefore,

$$(Y_{WE} \cap Y^{=}) \subseteq Y^{=}_{WE}.$$

Now suppose that $\bar{y} \in Y^{=}_{WE}$. Then, by definition of $Y^{=}_{WE}$, $\bar{y} \in Y^{=}$. By definition of $Y^{=}$ and the choice of $\hat{y}$, $Y^{=} \subseteq Y$. Therefore, $\bar{y} \in Y$.

Suppose that $\bar{y} \in Y_{WE}$ were not true. From Ref. 20, since $\bar{y} \in Y^{=}_{WE}$ and $Y^{=}$ is a polyhedron, we may choose a point $\bar{u} \in \Re^p$, $\bar{u} \geq 0$, $\bar{u} \neq 0$ such that $\bar{y}$ is an optimal solution to the problem

$$\max\langle \bar{u}, y\rangle, \qquad \text{s.t. } y \in Y^{=}. \tag{2}$$

Because we are assuming that $\bar{y} \notin Y_{WE}$, there exists a point $y^\circ \in Y$ such that $y^\circ > \bar{y}$. Since $y^\circ \in Y$, we know that

$$\hat{y} \leq y^\circ \leq Cx^\circ, \qquad \text{for some } x^\circ \in X.$$

Let

$$y^* = Cx^\circ.$$

Then, $y^* \in Y^{=}$, and since $y^* \geq y^\circ > \bar{y}$,

$$\langle \bar{u}, y^*\rangle \geq \langle \bar{u}, y^\circ\rangle > \langle \bar{u}, \bar{y}\rangle. \tag{3}$$

Since $y^* \in Y^=$, (3) contradicts the fact that $\bar{y}$ is an optimal solution to (2). Therefore, $\bar{y} \in Y_{WE}$ must be true, so that

$$Y^=_{WE} \subseteq (Y_{WE} \cap Y^=),$$

and the theorem is proved. □

Using Theorem 3.1 and its proof and Theorem 3.2, we may now state and prove the main result.

Theorem 3.3. Let $K \geq 0$ denote the final iteration number of the outcome set-based algorithm and, for each $k=0, 1, 2, \ldots, K-1$, let u^k be the vector in $\Re^p$ generated in Step $k3$ of the algorithm. Then, the weakly efficient outcome set $Y^=_{WE}$ of problem (MOLP) is given by

$$Y^=_{WE} = \bigcup_{k=0}^{K-1} Y^*(k),$$

where, for each $k=0, 1, \ldots, K-1$, $Y^*(k)$ denotes the optimal solution set of the linear programming problem

$$\max \langle u^k, y \rangle, \qquad \text{s.t. } y \in Y^=.$$

Proof. From Theorem 3.2,

$$Y^=_{WE} = Y_{WE} \cap Y^=.$$

From Theorem 3.1 and its proof,

$$Y_{WE} = \bigcup_{k=0}^{K-1} \{y \in Y \mid \langle u^k, y \rangle = \langle b, v^k \rangle\},$$

where, for each $y \in Y$ and $k=0, 1, \ldots, K-1$,

$$\langle u^k, y \rangle \leq \langle b, v^k \rangle.$$

Combining these statements yields the desired result. □

Remark 3.1. Notice that, for each $k=0, 1, \ldots, K-1$, the set $Y^*(k)$ in Theorem 3.3 may be generated by solving the linear program $P(u^k)$ given by

$$\begin{aligned} \max \quad & \langle u^k, y \rangle, \\ \text{s.t.} \quad & y - Cx = 0, \\ & Ax = b, \\ & x \geq 0, \end{aligned}$$

and using that

$$Y^*(k) = \{\bar{y} \in \Re^p \mid (\bar{x}, \bar{y}) \in Z^*(k), \text{ for some } \bar{x} \in \Re^n\},$$

where $Z^*(k)$ denotes the optimal solution set to problem $P(u^k)$.

Remark 3.2. In some applications of the vector maximization approach to problem (MOLP), the DM may need to examine not only the set of efficient extreme points in the outcome set, but also the set of efficient points in the outcome set that are non-extreme points. There are two possible reasons for this. First, the DM may have a unique, most preferred solution x^* in the efficient decision set X_E that is not an extreme point of X (Ref. 21). Such a point would be mapped into a point $Cx^* \in Y_E^=$ that is not an extreme point of the outcome set $Y^=$. Second, even if the DM has a most preferred solution $x^* \in X_E$ that is an extreme point of X, it is quite likely that $Cx^* \in Y_E^=$ will not be an extreme point of the outcome set $Y^=$ (Ref. 28). From Theorem 3.3, since $Y_{WE}^= \supseteq Y_E^=$, it follows that the outcome set-based algorithm in Ref. 26 generates all of the efficient outcome set and, therefore, could handle both of these situations.

4. Conclusions

The analysis presented in this article has shown that the outcome set-based algorithm in Ref. 26 for multiple-objective linear programming generates the entire set of weakly efficient points in the outcome set. In addition, from Ref. 26, it is known that this algorithm also generates the set of all efficient extreme points in the outcome set. Taken together, these properties show that the algorithm in Ref. 26 can be used to help a decision maker find a most preferred solution for a multiple-objective linear programming problem regardless of whether or not this solution is associated with an extreme point in the decision set or in the outcome set of the problem.

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