

Theory and Methodology

Constructing the set of efficient objective values in multiple objective linear programs

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Abstract: In earlier work the authors developed an algebraic description of the polyhedron $Y = C[X]$, where C is a $k \times n$ matrix and $X = \{x \in \mathbb{R}^n : Ax \leq b\}$. This algebraic description was then used to determine the Pareto-efficient objective values of the multiple objective linear program: maximize Cx , $x \in X$. In this paper, this approach is modified to obtain a more efficient procedure. In addition, a resulting single-objective nonparametric linear program in \mathbb{R}^{k+1} is developed whose set of optimal basic solutions corresponds to the set of efficient extreme points of Y .

Keywords: Linear optimization, multiple objective, Pareto-efficient point

1. Introduction

A great deal of research has been directed toward solving multiple objective linear programs. Most of this work has been motivated by or involved simplex-like approaches for analyzing the constraint polyhedron. To be more precise, let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$, and C a $k \times n$ matrix. Let the constraint set be

$$X := \{x \in \mathbb{R}^n : Ax \leq b\},$$

and consider the multiple objective linear program

$$\begin{array}{ll} \text{(MOLP)} & \text{'maximize' } Cx \\ & \text{subject to } x \in X. \end{array}$$

Here a set which can be expressed as the intersection of finitely many closed half spaces is called a *polyhedron*. A bounded polyhedron is called a *polytope*. Further the i -th component of a vector v is denoted by $v^{(i)}$, and we write $v_1 \leq v_2$ if $v_1^{(i)} \leq v_2^{(i)}$ for all i , and $v_1 \leq v_2$ if $v_1 \leq v_2$ and $v_1 \neq v_2$. Then

'maximize' in (MOLP) asks for the set

$$M := \{x \in X : \text{there is no } \tilde{x} \in X \text{ such that } Cx \leq C\tilde{x}\},$$

usually called the set of *efficient* or *Pareto-optimal* solutions of (MOLP). Clearly, simplex-type analysis of the constraint set X is a natural approach for analyzing the problem (MOLP) and is effective in many problems (e.g., see [1,2]).

However, from a mathematical point of view it is also natural to consider C as a linear mapping $C: \mathbb{R}^n \rightarrow \mathbb{R}^k$. The problem (MOLP) then transforms to analyzing the set of objective values

$$Y = C[X] = \{y \in \mathbb{R}^k : y = Cx, x \in X\}.$$

In particular, the objective space point of view is to obtain the set

$$E(Y) = \{y \in Y : \text{there is no } \tilde{y} \in Y \text{ such that } y \leq \tilde{y}\}$$

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of Pareto-efficient or nondominated points of Y . Then the set M is the inverse image under C of $E(Y)$.

Disadvantages and limitations of state space approaches for analyzing the constraint set X in applications where, for example, n is large have been discussed in [3–7]. Dauer [5] has shown that in such problems, particularly when k is smaller than n , the objective set Y can be expected to have fewer faces and extreme points than X and, therefore, to be simpler to analyze. Although objective set approaches have been considered in earlier research [6]–[9], such approaches were restricted by the lack of an algebraic description of the convex polyhedron Y . Such a description,

$$Y = \{y \in \mathbb{R}^k : Gy \leq g\},$$

was developed by the authors in [10] and forms the basis of this work.

In this paper we modify the approach in [10] for constructing the description of the set Y and a second set \tilde{Y} for which $E(\tilde{Y}) = E(Y)$. The set \tilde{Y} has the property that all of its extreme points are Pareto efficient. This leads to the development of a polyhedron Z in \mathbb{R}^{k+1} such that the set of all efficient extreme points of the objective set $E(Y)$ corresponds directly to the set of optimal basic solutions of the single objective (nonparametric) linear program

$$\begin{aligned} \text{(LP)} \quad & \text{maximize} \quad z^{(k+1)} \\ & \text{subject to} \quad z \in Z. \end{aligned}$$

These constructions are presented in three parts, which we discuss separately in the following sections. In Section 2 we consider the fact that Y is a subset of the range, $C[\mathbb{R}^n]$, of the mapping C . Thus, we present a construction of the subspace $C[\mathbb{R}^n]$ of \mathbb{R}^k and consider the corresponding Pareto-efficient structure of this subspace.

In Section 3 we extend the approach used earlier [10] and obtain a construction of the polyhedron Y . Essentially, the computations amount to a singular value decomposition of C and the construction of a set of generators of a particular polyhedral cone by finding the set of all optimal extreme point solutions of an appropriate linear program.

A similar technique is used in Section 4 to construct the polyhedron $\tilde{Y} \subset \mathbb{R}^k$ discussed above. The polyhedron Z , from (LP), is then immediate.

2. A representation of the range of C

Suppose C has rank r , and let C_1 denote an $r \times n$ matrix whose row vectors constitute a maximal linearly independent subset of the set of row vectors of C . Let the Moore–Penrose inverse of C_1 be denoted by

$$T_1 := C_1^T (C_1 C_1^T)^{-1}. \quad (1)$$

Then the columns of T_1 constitute a basis for the row-space of C .

If $r = k$, then $C[\mathbb{R}^n] = \mathbb{R}^k$. If $r < k$, then let C_2 denote a $(k - r) \times n$ matrix whose rows are those of C that do not appear in C_1 . Thus, there is a matrix B such that $C_2 = BC_1$ and B is given by

$$B = C_2 T_1. \quad (2)$$

Since the rows of the matrix

$$\bar{C} := \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

are the same as those of C , possibly permuted, the solution set M of (MOLP) is the same as the solution set for ‘maximize’ $\bar{C}x$, subject to $x \in X$. Therefore we assume, without loss of generality, that $Y = \{y \in \mathbb{R}^k : y = \bar{C}x, x \in X\}$ and set $C = \bar{C}$.

Since $Y \subset C[\mathbb{R}^n]$, this subspace of \mathbb{R}^k is of interest. In particular, it has been shown [10, Prop. 3.1] that if $r < k$, then the subspace has the representation

$$C[\mathbb{R}^n] = \{y \in \mathbb{R}^k : (B \mid -I)y = 0\}.$$

In addition, the following results concerning the efficient structure of a subspace of \mathbb{R}^k show that if S is a subspace of \mathbb{R}^k , then either $E(S) = \emptyset$ or $E(S) = S$.

Proposition 2.1. *Suppose S is a subspace of \mathbb{R}^k . Then $E(S) = S$ if and only if $E(S) \neq \emptyset$.*

Proof. The only if part is trivial. Therefore, let $y \in E(S)$ and assume that $\tilde{y} \in S$ and $\tilde{y} \notin E(S)$. Thus, there is $\bar{y} \in S$ such that $\bar{y} - \tilde{y} \geq 0$, and therefore $y + \bar{y} - \tilde{y} \in S$ and $y + \bar{y} - \tilde{y} \geq y$. This contradicts $y \in E(S)$. Hence $E(S) = S$. \square

As a direct consequence of Proposition 2.1 we have the following result.

Corollary 2.2. Suppose S is a subspace of \mathbb{R}^k . Then

- (i) $E(S) = S$ if and only if $0 \in E(S)$.
- (ii) $E(S) = \emptyset$ if and only if $0 \notin E(S)$. \square

For the next proposition and throughout the rest of this paper, e denotes the vector with appropriate dimension and satisfying $e^{(i)} = 1$ for all i .

Proposition 2.3. Suppose $r < k$ and

$$S := \{y \in \mathbb{R}^k : (B| - I)y = 0\}.$$

Then, $E(S) = S$ if and only if the system $Bu \geq 0$, $e^T u = 1$, $u \geq 0$, has no solution.

Proof. By Corollary 2.2, $E(S) = S$ if and only if $0 \in E(S)$. But $0 \in E(S)$ if and only if there is $z > 0$ such that $z^T y \leq 0$ for all $y \in S$ (e.g., see [6, Theorem 2.1]). Therefore, since S is a subspace, $0 \in E(S)$ if and only if there is $z > 0$ such that $z^T y = 0$ for all $y \in S$, i.e., there is $z > 0$ and $z \in S^\perp$, the orthogonal complement of S . Since S^\perp is the subspace spanned by the row vectors of $(B| - I)$, we have $0 \in E(S)$ if and only if there is w such that $z^T = w^T(B| - I)$, i.e. the system $w^T(B| - I) > 0$ has a solution. Therefore, by Gordan's theorem of the alternative (e.g., see [11, p. 31]), $0 \in E(S)$ if and only if the system

$$(B| - I)y = 0, \quad y \geq 0, \quad (3)$$

has no solution. But system (3) has no solution if and only if there is no $u \geq 0$ which solves $Bu \geq 0$, for otherwise

$$y = \begin{pmatrix} u \\ Bu \end{pmatrix}$$

would solve (3). Equivalently, $0 \in E(S)$ if and only if there is no $u \geq 0$ which solves $Bu \geq 0$ and $e^T u = 1$. \square

Now we provide the Algorithm RANGE for computing $C[\mathbb{R}^n]$ and determining its efficient structure.

Step R1: Find C_1 (see Algorithm C_1 of the Appendix), the rank r of C , and the matrix C_2 (C_2 is vacuous if $r = k$).

Step R2: Compute $T_1 = C_1^T(C_1 C_1^T)^{-1}$, e.g. by using the singular value decomposition of C_1 .

Step R3: If $r = k$, then $C[\mathbb{R}^n] = \mathbb{R}^k$ and hence $E(C[\mathbb{R}^n]) = \emptyset$. Go to Step Y1 of Algorithm Y (see Section 3).

Step R4: We have $r < k$, so compute $B = C_2 T_1$; then

$$C[\mathbb{R}^n] = \{y \in \mathbb{R}^k : (B| - I)y = 0\}.$$

Step R5: Apply phase I of the simplex method to the linear program (LP1).

$$\begin{aligned} \text{(LP1)} \quad & \text{maximize} \quad e^T u \\ & \text{subject to} \quad (-B)u \leq 0, \\ & \quad e^T u = 1, \\ & \quad u \geq 0, \quad u \in \mathbb{R}^r. \end{aligned}$$

Step R6: If (LP1) has a basic feasible solution, then $E(C[\mathbb{R}^n]) = \emptyset$. Go to Step Y1 of Algorithm Y.

Step R7: If (LP1) has no basic feasible solution, then $E(C[\mathbb{R}^n]) = C[\mathbb{R}^n]$. Stop.

3. A representation of the set of feasible objective values Y

Since $Y = C[X]$ and X is a polyhedron, it follows that Y is a polyhedron (e.g., see [12, p. 174]), and hence there is a matrix G and a vector g such that

$$Y = \{y \in \mathbb{R}^k : Gy \leq g\}.$$

In this section we construct such G and g . The procedure for finding a representation of Y developed earlier in [10] is based on the fact that there is a one-to-one correspondence between the set of feasible objective values Y and the orthogonal projection of X into the row-space of C . To see this let T_2 be a matrix whose column vectors constitute an orthonormal basis for the orthogonal complement of the row-space of C (T_2 is vacuous if $r = n$). Therefore $T := (T_1 \ T_2)$, with T_1 defined in (1), represents a change of basis for \mathbb{R}^n and satisfies $(C_1 \ T_2^T)^T T = I$. Hence, the representation of the matrix $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with respect to the new basis for \mathbb{R}^n , the columns of T , is defined by the matrix

$$\bar{A} := AT = A(T_1 \ T_2) := (\bar{A}_1 \ \bar{A}_2).$$

The representation of $x \in \mathbb{R}^n$ with respect to the new basis is

$$T^{-1}x = \begin{pmatrix} C_1 \\ T_2^T \end{pmatrix} x = \begin{pmatrix} C_1 x \\ T_2^T x \end{pmatrix} := \begin{pmatrix} u \\ v \end{pmatrix},$$

where $u := C_1 x$ and $v := T_2^T x$. We note that $(u \ 0)^T$ and $(0 \ v)^T$ are the orthogonal projections, with respect to the new basis, of x into the row-space of C and into its orthogonal complement, respectively. Thus, $x \in X$ if and only if

$$ATT^{-1}x \leq b \Leftrightarrow \bar{A}_1 u + \bar{A}_2 v \leq b. \quad (4)$$

By employing (4) one can easily verify the following result [10].

Proposition 3.1. Suppose either (a) $r = n$, or (b) $r < n$ and $\bar{A}_2 = 0$. Then Y has the representation $\{y : Gy \leq g\}$, where

(i) $G = A_1$ and $g = b$, if $r = k$,

$$(ii) \ G = \begin{pmatrix} B & -I \\ -B & I \\ \bar{A}_1 & 0 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$$

if $r < k$. \square

Now assume that $r < n$ and $\bar{A}_2 \neq 0$; then we consider the linear program

$$\begin{aligned} (LP2) \quad & \text{maximize} \quad e^T z \\ & \text{subject to} \quad \bar{A}_2^T z = 0, \\ & \quad e^T z = 1, \\ & \quad z \geq 0, \quad z \in \mathbb{R}^m, \end{aligned}$$

and let P denote the matrix whose row vectors are the optimal basic solutions of (LP2). Equivalently, the rows of P are the vertices of the polytope defined by the constraint set of (LP2). For a survey of the different methods for finding the vertices of convex polyhedral sets, see e.g. [13]. The matrix P is vacuous if (LP2) has no basic feasible solutions, and in this case $Y = C[\mathbb{R}^n]$ [10, Theorems 3.6 and 4.1].

If P is not vacuous, then by definition P has nonnegative entries and $P\bar{A}_2 = 0$. Also, (4) shows that if $(uv)^T$ is viewed as a vector of \mathbb{R}^n with the new basis, then

$$(P\bar{A}_1 \ 0) \begin{pmatrix} u \\ v \end{pmatrix} \leq Pb, \quad v = 0,$$

represents the orthogonal projection of X into the row-space of C , and $P\bar{A}_1 u \leq Pb$ represents Y as a subset of $C[\mathbb{R}^n]$. If $r < n$ and $P\bar{A}_1 = 0$, then either $X = \emptyset$ or $Y = C[\mathbb{R}^n]$ [10, Theorems 3.6 and 4.1]. Otherwise the following result holds.

Proposition 3.2. Suppose $r < n$ and $P\bar{A}_1 \neq 0$. Then Y has the representation $\{y : Gy \leq g\}$, where

(i) $G = P\bar{A}_1$ and $g = Pb$, if $r = k$,

$$(ii) \ G = \begin{pmatrix} B & -I \\ -B & I \\ P\bar{A}_1 & 0 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 0 \\ 0 \\ Pb \end{pmatrix}$$

if $r < k$. \square

This construction of Y from [10] requires the matrix T_2 , whose columns form an orthonormal basis for the orthogonal complement of the row-space of C . Besides the theoretical use of T_2 in this development, its construction is necessary in order to obtain the matrix P via (LP2). We now obtain an alternate description of the constraint set of (LP2) which does not require T_2 .

This alternative approach is motivated by writing $A = A' + A''$, where A' and A'' are matrices whose rows are the projections of the corresponding rows of A into the row-space of C and its orthogonal complement, respectively. It follows that

$$A' = AT_1 C_1, \quad A'' = AT_2 T_2^T. \quad (5)$$

To see this let $x \in \mathbb{R}^n$, then

$$\begin{aligned} x &= TT^{-1}x = (T_1 \ T_2)T^{-1}x \\ &= (T_1 \ T_2) \begin{pmatrix} C_1 \\ T_2^T \end{pmatrix} x = (T_1 C_1 + T_2 T_2^T)x, \end{aligned}$$

and so we have

$$x = T_1 C_1 x + T_2 T_2^T x. \quad (6)$$

Since $(T_1 C_1 x)^T T_2 = 0$ and $(T_2 T_2^T x)^T T_1 = 0$, we have that $T_1 C_1 x$ is in the row space of C and $T_2 T_2^T x$ is in its orthogonal complement. Since $T_1 C_1$ and $T_2 T_2^T$ are both symmetric, (6) yields

$$x^T = x^T T_1 C_1 + x^T T_2 T_2^T,$$

which, when applied to the rows of A , gives

$$A = AT_1 C_1 + AT_2 T_2^T = \bar{A}_1 C_1 + \bar{A}_2 T_2^T.$$

Hence (5) holds. Since

$$A = AT_1C_1 + A(I - T_1C_1),$$

we have the following result.

Proposition 3.3. Let $A'' = \bar{A}_2T_2^T$, then $A'' = A(I - T_1C_1)$ and the constraint set of (LP2) satisfies

$$\begin{aligned} & \{z \in \mathbb{R}^m : \bar{A}_2^T z = 0, e^T z = 1, z \geq 0\} \\ &= \{z \in \mathbb{R}^m : z^T A'' = 0, e^T z = 1, z \geq 0\}. \end{aligned}$$

Proof. From the above discussion we need only show that the constraint set of (LP2) can be given in terms of the matrix A'' . To see this we show that $z^T \bar{A}_2 = 0$ if and only if $z^T A'' = 0$, where $A'' = \bar{A}_2T_2^T$. Suppose $z^T \bar{A}_2 = 0$, then $z^T A'' = z^T \bar{A}_2T_2^T = 0$. On the other hand, suppose that

$$0 = z^T A'' = (z^T \bar{A}_2)T_2^T.$$

Since the rows of T_2^T are linearly independent we have $z^T \bar{A}_2 = 0$. \square

This result shows that (LP2) can be solved by defining the constraint set using

$$A'' = A(I - T_1C_1),$$

which does not require T_2 . It follows that $A'' = 0$ if and only if $\bar{A}_2 = 0$, and so the earlier results [10] are valid with \bar{A}_2 replaced by A'' .

We now provide the Algorithm Y for constructing the set Y .

Step Y1: Compute $\bar{A}_1 = AT_1$.

If $r = n$, then set $H = \bar{A}_1$ and $h = b$. Go to Step Y7.

Step Y2: We have $r < n$, so compute $A'' = A(I - T_1C_1)$.

If $A'' = 0$, then set $H = \bar{A}_1$, $h = b$ and go to Step Y7.

Step Y3: Consider (LP2), with \bar{A}_2 replaced by A'' , and determine P .

If P is vacuous (i.e., (LP2) has no feasible solutions), then go to Step Y6.

Step Y4: Compute $H = P\bar{A}_1$ and $h = Pb$.

If $H \neq 0$, then go to Step Y7.

Step Y5: If $h \geq 0$ does not hold, then $X = \emptyset$. Stop.

Step Y6: We have $Y = C[\mathbb{R}^n]$ and because of the conclusions of Steps R3 and R6. $E(Y) = \emptyset$. Stop.

Step Y7: If $r = k$, then set $G = H$ and $g = h$. And go to Step Y9.

Step Y8: We have $r < k$. Set

$$G = \begin{pmatrix} B & -I \\ -B & I \\ H & 0 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}.$$

Step Y9: Set $Y = \{y : Gy \leq g\}$. Go to Step Y1 of Algorithm \tilde{Y} .

Remarks 3.4. (a) The construction of the matrix G is independent of the vector b .

(b) In the case $r < n$: For any perturbation in the matrix C which yields the same row-space as that of C , the matrices T_2 , \bar{A}_2 , and P remain the same.

(c) In the case $r < k$: The orthogonal projection of Y into the subspace of \mathbb{R}^k spanned by the standard basis e_1, \dots, e_r is given by

$$(H \ 0) \begin{pmatrix} u \\ w \end{pmatrix} \leq 0, \quad w = 0,$$

where $u = [y^{(1)}, \dots, y^{(r)}]^T$ and $w = [y^{(r+1)}, \dots, y^{(k)}]^T$.

(d) In the case $r < k$ and $H \neq 0$: One can easily verify that

(i) If \tilde{u} solves (LP1) and $H\tilde{u} \leq 0$, then $E(Y) = \emptyset$.

(ii) If B has nonnegative entries, then $E(Y) = \{(u_0 \ Bu_0)^T : u_0 \in E(U)\}$ where $U := \{u \in \mathbb{R}^r : Hu \leq h\}$.

The following proposition shows that if the supremum of any of the objective functions over X is not finite, then M , the solution set for (MOLP), is either empty or unbounded.

Proposition 3.5. Suppose $\eta^{(i)} := \sup_{y \in Y} y^{(i)}$ is not finite for some $1 \leq i \leq k$; then M is either empty or unbounded.

Proof. Assume $M \neq \emptyset$, then $E(Y) \neq \emptyset$ and, therefore, $\eta^{(i)} = \infty$. Let $y_0 \in E(Y)$ and $D := \{y \in Y : y^{(i)} \geq y_0^{(i)} + 1\}$. We have $D \neq \emptyset$, since $\eta^{(i)} = \infty$. Since $y_0 \in E(Y)$, there is $z > 0$ such that $z^T y \leq z^T y_0$ for all $y \in Y$. Since D is a polyhedron and $\sup_{y \in D} z^T y \leq z^T y_0$, there is $y_1 \in D$ such that $z^T y \leq z^T y_1$ for all $y \in D$. This implies that y_1 is an efficient point of D . But, by the definition of D , $y^{(i)} < y_1^{(i)}$ for all $y \in \{y : y \in Y, y \notin D\}$, there-

fore $y_1 \in E(Y)$. Thus one can construct a sequence $\{y_0, y_1, \dots\}$ in $E(Y)$ satisfying $y_{j-1}^{(i)} < y_j^{(i)}$ for $j \geq 1$. Hence $E(Y)$ is unbounded and consequently M is unbounded, since M is the inverse image of $E(Y)$ under C . \square

4. A representation of an efficiency-equivalent polyhedron

In this section we construct a polyhedron \tilde{Y} which has the same efficient structure as that of Y , i.e. $E(Y) = E(\tilde{Y})$, and, moreover, which has no extreme points that are not Pareto-efficient. Let $\tilde{Y} := \{y + d : y \in Y, d \leq 0\}$; then [10, Proposition 5.1] shows that $E(Y) = E(\tilde{Y})$. Let Q denote the matrix whose row vectors are the extreme optimal basic solutions of the linear program

$$\begin{aligned} \text{(LP3)} \quad & \text{maximize} \quad e^T z \\ & \text{subject to} \quad (-G^T)z \leq 0, \\ & \quad \quad \quad e^T z = 1, \\ & \quad \quad \quad z \geq 0, \quad z \in \mathbb{R}^m. \end{aligned}$$

Equivalently, the rows of Q are the vertices of the polytope defined by the constraint set of (LP3).

Proposition 4.1. [10, Theorem 5.6]: (i) If Q is vacuous, then $\tilde{Y} = \mathbb{R}^k$ and hence $E(Y) = \emptyset$.

(ii) If Q is nonvacuous, then $E(Y) = E(\tilde{Y})$ where $\tilde{Y} = \{y \in \mathbb{R}^k : QGy \leq Qg\}$. \square

If $QG \neq 0$ then, since QG has nonnegative entries, a direct application of Gale's theorem of the alternative for linear inequalities (e.g., see [11, p. 33]) shows that the system $QGy \leq Qg$ has no solution $y \in \mathbb{R}^k$, i.e. $X = \emptyset$, if and only if there is a row q^T of Q satisfying

$$q^T G = 0 \quad \text{and} \quad q^T g < 0. \quad (7)$$

If none of the rows of Q satisfies (7), we define $\tilde{G} := QG$ and $\tilde{g} := Qg$. Thus, by Proposition 4.1, we have:

(i) If $\tilde{G} = 0$, then $\tilde{Y} = \mathbb{R}^k$ and hence $E(Y) = \emptyset$.

(ii) If $\tilde{G} \neq 0$, then $\tilde{Y} \neq \emptyset$ and $\tilde{Y} = \{y \in \mathbb{R}^k : \tilde{G}y \leq \tilde{g}\}$.

In [10, Proposition 5.7] we proved that all the extreme points of the polyhedron \tilde{Y} are Pareto-efficient. We now prove the following intuitive char-

acteristic for \tilde{Y} . If $\eta^{(i)} := \sup_{y \in Y} y^{(i)}$ is finite, then the representation $\tilde{G}y \leq \tilde{g}$ directly provides the value $\eta^{(i)}$, namely, one of the inequalities of the system $\tilde{G}y \leq \tilde{g}$ has the form $\alpha y^{(i)} \leq \alpha \eta^{(i)}$ for some scalar $\alpha > 0$. We use the notation e_i to denote the vector in \mathbb{R}^k with $e_i^{(j)} = \delta_{ij}$, the Kronecker delta.

Proposition 4.2. Suppose $\eta^{(i)} := \sup_{y \in Y} y^{(i)}$ is finite. Then there is a row vector q^T of Q such that $q^T G = \alpha e_i^T$ and $q^T g = \alpha \eta^{(i)}$ for some $\alpha > 0$.

Proof. By definition of $\eta^{(i)}$, the system

$$Gy \leq g, \quad e_i^T y > \eta^{(i)},$$

has no solution. Therefore, by the nonhomogeneous Farkas' theorem of the alternative (e.g., see [11, p.32]), there is $z \in \mathbb{R}^m$ which satisfies

$$z^T G = e_i^T, \quad z^T g \leq \eta^{(i)}, \quad z \geq 0.$$

Since $z \geq 0$, we have $\beta := 1/(e^T z) > 0$, and therefore $z_o := \beta z$ is an optimal solution for (LP3). Since the set of optimal solutions for (LP3) is a closed bounded convex subset of \mathbb{R}^m , then, by the definition of Q and the Krein–Milman theorem (e.g., see [14, p. 146]), z_o can be expressed as a convex combination of the rows of Q , i.e.,

$$z_o = \sum_{j=1}^p \alpha_j q_j, \quad \sum_{j=1}^p \alpha_j = 1, \quad \alpha_j \geq 0$$

for all $1 \leq j \leq p$,

where $\{q_1^T, \dots, q_p^T\}$ is the set of row vectors of Q . Thus

$$\beta e_i^T = \beta z_o^T G = z_o^T G = \sum_{j=1}^p \alpha_j (q_j^T G). \quad (8)$$

Since $q_j^T G \geq 0$ for all j , (8) implies that for each $1 \leq j \leq p$, $q_j^T G = \beta_j e_i^T$ for some $\beta_j \geq 0$. And also $\beta = \sum_{j=1}^p \beta_j$. Consequently $\beta_{j_o} > 0$ for some $1 \leq j_o \leq p$, since $\beta > 0$. We show that q_{j_o} satisfies the assertion of the theorem. Since $\alpha_{j_o} q_{j_o}^T G = \beta_{j_o} e_i^T$ and $\beta_{j_o} > 0$, it remains to show that $\alpha_{j_o} q_{j_o}^T g = \beta_{j_o} \eta^{(i)}$. Let $\bar{y} \in Y$ such that $\bar{y}^{(i)} = \eta^{(i)}$, we have for all $1 \leq j \leq p$

$$\beta_j \eta^{(i)} = \beta_j e_i^T \bar{y} = \alpha_j q_j^T G \bar{y} \leq \alpha_j q_j^T g. \quad (9)$$

Hence, in particular, $\beta_{j_o} \eta^{(i)} \leq \alpha_{j_o} q_{j_o}^T g$. Also, by em-

playing (9), we have

$$\begin{aligned}\alpha_{j_0} q_{j_0}^T g &= \left(z_0^T - \sum_{j \neq j_0} \alpha_j q_j^T \right) g \\ &\leq \beta \eta^{(i)} - \sum_{j \neq j_0} \alpha_j q_j^T g \\ &\leq \beta_0^{(i)} - \sum_{j \neq j_0} \beta_j \eta^{(i)} = \beta_j \eta^{(i)},\end{aligned}$$

and hence $\alpha_{j_0} q_{j_0}^T g \leq \beta_j \eta^{(i)}$. This completes the proof. \square

In view of Propositions 3.5 and 4.2, we have

Corollary 4.3. *$E(Y)$ is unbounded if and only if $\tilde{G} \neq 0$ and for some i , $1 \leq i \leq k$, none of its rows has the form αe_i for some scalar $\alpha > 0$.* \square

To determine the extreme points of \tilde{Y} , one may apply any of the methods reviewed in [13], or determine the set of optimal basic solutions of

(LP) maximize $z^{(k+1)}$ subject to $z \in Z$,

where the polyhedron Z in \mathbb{R}^{k+1} is defined as

$$\begin{aligned}Z := \{ z = (y \ z^{(k+1)})^T \in \mathbb{R}^{k+1} : \\ (\tilde{G} \ 0)(y \ z^{(k+1)})^T \leq \tilde{g}, \ z^{(k+1)} \leq 0 \}.\end{aligned}$$

We now provide the Algorithm \tilde{Y} for constructing such a polyhedron:

Step $\tilde{Y}1$: Consider (LP3) and determine Q .

If Q is vacuous, then $\tilde{Y} = \mathbb{R}^k$ and hence $E(Y) = \emptyset$. Stop.

Step $\tilde{Y}2$: If any of the rows of Q satisfies (7), then $X = \emptyset$. Stop.

Step $\tilde{Y}3$: Set $\tilde{G} = QG$ and $\tilde{g} = Qg$.

If $\tilde{G} = 0$, then $\tilde{Y} = \mathbb{R}^k$ and hence $E(Y) = \emptyset$. Stop.

Step $\tilde{Y}4$: We have $\tilde{G} \neq 0$, so $\tilde{Y} \neq \emptyset$ and $\tilde{Y} = \{ y \in \mathbb{R}^k : \tilde{G}y \leq \tilde{g} \}$. Stop.

Step $\tilde{Y}5$: If there is i , $1 \leq i \leq k$, such that none of the rows of \tilde{G} has the form αe_i , $\alpha > 0$, then $E(Y)$ is unbounded.

Step $\tilde{Y}6$: Determine optimal basic solutions of (LP).

Step $\tilde{Y}7$: If $z = [z^{(1)}, \dots, z^{(k)}, 0]^T$ is an optimal basic solution of (LP), then $y = [z^{(1)}, \dots, z^{(k)}]^T$ is an extreme point of $E(Y)$.

Remark 4.4. If $y \in E(Y)$, then $\{x \in X : Cx = y\}$ is given by

$$\left\{ x = x_1 + x_2 : x_1 = T_1 [y^{(1)}, \dots, y^{(r)}]^T, \right. \\ \left. Ax_2 \leq b - Ax_1 \right\}.$$

This is true, in general, for any $y \in Y$.

5. Summary and conclusions

Finding an algebraic description for the polyhedron $Y = C[X]$ is of interest in its own right. On the basis of such a description for Y it was natural to think of constructing the polyhedron \tilde{Y} which preserves the efficient structure of Y and whose extreme points are the efficient extreme points of Y .

The linear programs (LP2), (LP3) and (LP) ask for the vertices of their respective constraint sets, and therefore one has the option of selecting the most efficient method for solving these programs. (Again, we refer the reader to [13].) Finally, we note that the procedure given in this paper can be, with no difficulty, adapted for parallel computations. For example, the Algorithm Y could be initiated directly after computing the matrix T_1 , since the matrix B is first needed at Step $Y8$. Also, as Step $\tilde{Y}1$ generates the rows of the matrix Q , Step $\tilde{Y}2$ can be performed in parallel for each such generated row.

Appendix

Suppose S is the subspace of \mathbb{R}^n spanned by the orthonormal vectors s_1, \dots, s_r . If $x \in \mathbb{R}^n$, then the orthogonal projection of x into S , denoted by x_S , is

$$x_S = \sum_{i=1}^r \langle x, s_i \rangle s_i, \quad \|x_S\|^2 = \sum_{i=1}^r (\langle x, s_i \rangle)^2,$$

and $\cos \theta = \|x_S\| / \|x\|$, where θ is the angle the vector x makes with S . Clearly, $\|x_S\| = \|x\|$ if and only if $x \in S$. Also if $x \notin S$, then the vector

$$\tilde{x} := \frac{1}{(\|x\|^2 - \|x_S\|^2)^{1/2}} \left[\left(\sum_{i=1}^r \langle x, s_i \rangle s_i \right) - x \right] \quad (A1)$$

is a unit vector in the orthogonal complement of S and x belongs to the subspace spanned by $\{s_1, \dots, s_r, \tilde{x}\}$.

We write $\mathcal{L}(A)$ to denote the subspace spanned by the vectors in the set A .

Algorithm C_1

The row vectors c_1, \dots, c_k of the matrix C are provided. This algorithm determines $\{c_{\alpha_1}, \dots, c_{\alpha_r}\}$ a maximal linearly independent subset of $\{c_1, \dots, c_k\}$ where $\{\alpha_1, \dots, \alpha_r, \alpha_{r+1}, \dots, \alpha_k\}$ is a permutation of $\{1, 2, \dots, k\}$. It also provides an orthonormal basis $\tilde{c}_{\alpha_1}, \dots, \tilde{c}_{\alpha_r}$ for the row-space of C . At first $\|c_i\|$ is computed for all $i = 1, \dots, k$, $c_{\alpha_1} := c_1$, and $\tilde{c}_{\alpha_1} := c_1 / \|c_1\|$. The j -th iteration of the algorithm provides $A = \{c_{\alpha_1}, \dots, c_{\alpha_j}\}$, $B = \{c_i : c_i \notin A, c_i \in \mathcal{L}(A)\}$, and $\tilde{A} = \{\tilde{c}_{\alpha_1}, \dots, \tilde{c}_{\alpha_j}\}$ where \tilde{A} is an orthonormal basis for $\mathcal{L}(A)$. The $j+1$ iteration designates $c_{\alpha_{j+1}}$ as the row vector $c_i \notin A \cup B$ which has the greatest angle θ with $\mathcal{L}(A)$, provided that $\theta \neq 0$, and determines $\tilde{c}_{\alpha_{j+1}}$ as in (1). We note that (1) shows the computational merit of choosing $c_{\alpha_{j+1}}$ as such. Therefore, it suffices to show the $j+1$ iteration:

Step 1: Compute

$$\delta = \min \left\{ \frac{1}{\|c_i\|^2} \sum_{m=1}^j \langle c_i, \tilde{c}_{\alpha_m} \rangle^2 : c_i \notin A \cup B \right\}$$

and determine $c_{\alpha_{j+1}}$ at which the minimum is achieved.

Step 2:

$$\tilde{c}_{\alpha_{j+1}} = \frac{1}{\|c_{\alpha_{j+1}}\| \sqrt{1 - \delta}} \times \left[\left(\sum_{m=1}^j \langle c_{\alpha_{j+1}}, c_{\alpha_m} \rangle \tilde{c}_{\alpha_m} \right) - c_{\alpha_{j+1}} \right].$$

Step 3: Update A to become $A = \{c_{\alpha_1}, \dots, c_{\alpha_{j+1}}\}$ and \tilde{A} to become $\tilde{A} = \{\tilde{c}_{\alpha_1}, \dots, \tilde{c}_{\alpha_{j+1}}\}$.

Step 4: For each $c_i \notin A \cup B$ evaluate $\delta_i = \sum_{m=1}^{j+1} \langle c_i, \tilde{c}_{\alpha_m} \rangle^2$. If $\delta_i = \|c_i\|^2$, i.e., $c_i \in \mathcal{L}(A)$, then update B to contain c_i .

Step 5: If there is no $c_i \notin A \cup B$, then we have $j+1 = r$. Otherwise, go to Step 1.

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