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Theory and Methodology

A combined constraint-space, objective-space approach for determining high-dimensional maximal efficient faces of multiple objective linear programs

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Abstract

Characterizations for efficient faces and certain maximal efficient faces of the objective set Y of a linear k-objective minimization problem are presented. These characterizations are used to develop an algorithm for determining high-dimensional maximal efficient faces of Y. The algorithm requires as input an irredundant system of linear inequalities representing the efficiency equivalent polyhedron $\bar{Y} := Y + \mathbb{R}^k_+$. A procedure for obtaining such a representation for $\bar{Y}$ has previously appeared in the literature and is included herein in order to make the paper self-contained. This latter procedure requires, in part, the generation of the efficient extreme points and efficient extreme rays of the constraint polyhedron. Hence, the overall method proposed herein can be viewed as a combined constraint-space, objective-space algorithm. The algorithm is complete for problems with 2 and 3 objectives. An illustrative numerical example is included.

Keywords: Multiple criteria linear programming; Objective space analysis

1. Introduction

In the past two decades various algorithms have been developed for determining the set of efficient solutions of the multiple objective linear program

(MOLP)
Minimize Cx
subject to $x \in X$,

where X is a polyhedral subset $\mathbb{R}^n$ and C is a $k \times n$ matrix [1,2,7,8,12,23]. Although these algorithms vary, they all utilize a simplex-like approach to enumerate the efficient extreme points of X and, together with certain tests and bookkeeping techniques, determine the maximal efficient faces of X. In particular, they all are based on an analysis of the constraint polyhedron X. While a con-

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straint-space analysis of problem (MOLP) is a natural and, in many cases, effective approach for determining the efficient structure of X, it can be complicated if X has a large number of extreme points and faces.

Dauer [3] noted that due to a 'collapsing' effect when the polyhedron X is mapped to $\mathbb{R}^k$ under the linear mapping C, the objective polyhedron $Y := C[X]$ may have significantly fewer extreme points and faces than the polyhedron X. This phenomenon and the fact that there is a one-to-one correspondence between maximal efficient faces of X and maximal efficient faces of Y (see Proposition 2.1) suggest that a promising approach for determining and/or analyzing maximal efficient faces of X is via an analysis of the efficient structure of Y. Of course, the efficient structure of Y is of interest in its own right. Indeed, the selection of a final 'operational' solution from the set of all efficient solutions is primarily based on objective space considerations. Therefore, a good understanding of the efficient structure of Y will help the decision maker to select among the available choices.

The purpose of this paper is to develop theoretical results and computational techniques – in objective space – for identifying maximal efficient faces of Y. Although we do not specify a definitive, computationally efficient technique for identifying all maximal efficient faces of Y in the general case, useful characterisations of certain maximal efficient faces are given. The main theoretical result of the paper is Theorem 3.3 which characterizes maximal efficient faces F having dimension $k - |\mathrm{id}(F)|$, where $\mathrm{id}(F)$ denotes the set of indices of the active constraints defining F. As an immediate corollary (see Corollary 3.5) we obtain a well-known characterization of maximal efficient faces of dimension $k-1$, and in addition, a characterization of maximal efficient faces of dimension $k-2$. These latter characterizations are used in Section 4 to develop an algorithm (Algorithm MEF) which at least generates all maximal efficient faces of dimensions $k-1$ and $k-2$, and may generate other efficient faces as well. (The algorithm is therefore complete for problems with 2 and 3 objectives.)

The characterizations developed in Section 3 are based on having a linear inequality representation either of Y or of a polyhedron $P \subseteq \mathbb{R}^k$ with the same efficient structure as Y. A natural candidate for an efficiency equivalent polyhedron P is the polyhedron

$$\tilde{Y} := Y + \mathbb{R}^k_+$$

(see [5,6,9,10]) which, in addition to having the same efficient structure as Y, has the property that all of its extreme points are efficient. Also, if $Hy \geqslant g$ is a representation for $\tilde{Y}$, then necessarily the matrix H is nonnegative. This latter property of $\tilde{Y}$ is useful since the characterizations given in Section 3 are much simpler when H is nonnegative. It is a representation of $\tilde{Y}$ that is utilized by Algorithm MEF.

Obtaining a linear inequality representation for $\tilde{Y}$ and/or Y has been the focus of several recent research efforts [5,6,9,10,15,17,22]. Although obtaining such a representation is not the focus of this work, in order to make the paper self-contained, in Section 4 we include an algorithm for obtaining a representation of $\tilde{Y}$. The particular algorithm given (Algorithm $\tilde{Y}$) was developed by Gallagher and Saleh [9] and is singled out here due to the fact that, unlike the algorithms specified in [5,6,10,15,17,22], no redundant inequalities are generated. Although most of the characterizations in Section 3 do not depend on having an irredundant system of inequalities, implementing the results is simpler if there are no redundancies.

Section 4 concludes with an example of a 3-objective problem illustrating the use of Algorithm MEF in conjunction with Algorithm $\tilde{Y}$. The entire procedure can be viewed as a combination of constraint-space and objective-space techniques. The constraint-space aspect arises from the fact that Algorithm $\tilde{Y}$ requires the enumeration of the efficient extreme points and efficient extreme rays of the constraint polyhedron X.

We emphasize that the main new contributions of this work are the characterizations of certain maximal efficient faces given in Section 3, the development of Algorithm MEF, and the linking of Algorithm MEF with Algorithm $\tilde{Y}$. A summary and some avenues for future research are given in Section 5.

2. One-to-one correspondence

Since the thrust of this work is on developing techniques to identify maximal efficient faces of Y, it is important to recognize their relationship to the efficient structure of X. Dauer [3] and Dauer and Liu [4] have shown that it is possible for an efficient face F of X to be mapped to an efficient set $C[F]$ of Y that is not an entire face of Y. In this section we show that this cannot happen if F is a maximal efficient face. In fact, in Proposition 2.1, we show that there is a one-to-one correspondence between maximal efficient faces of X and maximal efficient faces of Y.

Before continuing with the development of this one-to-one correspondence we clarify some notation and terminology. Given subsets $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^k$, let

$$C[S] := \{z \in \mathbb{R}^k : z = Cs \text{ for some } s \in S\}$$

and

$$C^{-1}[T] := \{u \in \mathbb{R}^n : Cu \in T\}.$$

Throughout the paper, the term 'efficient' is used in both decision space, $\mathbb{R}^n$, and objective space, $\mathbb{R}^n$. The meaning should be clear from the context. Specifically, a point $x^* \in X$ is said to be an *efficient solution* of problem (MOLP) if there does not exist $x \in X$ satisfying $Cx \gneq Cx^*$; a point $y^* \in Y := C[X]$ is said to be an *efficient objective value* of problem (MOLP) if there does not exist $y \in Y$ satisfying $y \gneq y^*$. A nonempty subset $S \subseteq X$ ($T \subseteq Y$) is said to *efficient* if every point in S (T) is efficient. It is well-known that the set of all efficient solutions (the set of all efficient objective values) is connected and usually nonconvex, and can be expressed as the union of the maximal efficient faces of X (Y). Here, a face F_X (F_Y) of the polyhedron X (Y) is said to be a *maximal efficient face* of X (Y) if it is efficient and if there does not exist an efficient face $\hat{F}_X$ ($\hat{F}_Y$) of X (Y) satisfying $F_X \subsetneqq \hat{F}_X$ ($F_Y \subsetneqq \hat{F}_Y$).

We now state and prove Proposition 2.1 which shows that there is a one-to-one correspondence between maximal efficient faces of X and maximal efficient faces of Y.

Proposition 2.1. *Let X be a polyhedral subset of $\mathbb{R}^n$, let C be a $k \times n$ matrix, and let $Y = C[X]$.*

(a) *If F_X is a maximal efficient face of X, then $C[F_X]$ is a maximal efficient face of Y.*

(b) If *F_Y is a maximal efficient face of Y, then $C^{-1}[F_Y] \cap X$ is a maximal efficient face of X.*

Proof. (a): Let F_X be an efficient face of X, and suppose $C[F_X]$ is not a maximal efficient face of Y. Then there exists an efficient face F_Y of Y such that $C[F_X] \subsetneqq F_Y$. Since F_Y is efficient, there exists $\lambda \in \mathbb{R}^k$, $\lambda > 0$, and $\mu \in \mathbb{R}$ such that

$$\lambda^{\mathrm{T}} z = \mu \geqslant \lambda^{\mathrm{T}} y$$

for all $z \in F_Y$ and all $y \in Y \backslash F_Y$. Now choose $\bar{y} \in F_Y \backslash C[F_X]$ and consider a point $\bar{x} \in C^{-1}(\bar{y}) \cap X$. Then $\bar{x} \in X \backslash F_X$. Moreover, if

$$\tilde{x} \in \operatorname{conv}(F_X \cup \{\bar{x}\})$$

(the convex hull of $F_X \cup \{\bar{x}\}$), then $\lambda^{\mathrm{T}} C\tilde{x} = \mu$. Since $\tilde{x}$ is arbitrary, it follows that $\operatorname{conv}(F_X \cup \{\bar{x}\})$ is efficient. Hence, there exists an efficient face F of X such that $\operatorname{conv}(F_X \cup \{\bar{x}\}) \subseteq F$. Since $F_X \subsetneqq F$, F_X is not a maximal efficient face.

(b): Suppose F_Y is a maximal efficient face of Y. Let

$$F_X = C^{-1}[F_Y] \cap X.$$

Clearly F_X is efficient. Let F be an efficient face of X such that $F_X \subseteq F$. Then, since $F_Y = C[F_X] \subseteq C[F]$, it follows by the maximality of F_Y that $F_Y = C[F]$. Thus,

$$F_X = C^{-1}[F_Y] \cap X = C^{-1}[C[F]] \cap X \supseteq F.$$

Hence, $F_X = F$, so F_X is maximal. □

We note that while the converse of Proposition 2.1 part (b) is true, the converse of part (a) is not. Indeed, one can easily construct a counterexample with, say, $n = 2$ and $k = 1$.

3. Maximal efficient faces in objective space

In this section we present results characterizing when a face F of a polyhedron $P \subseteq \mathbb{R}^k$ is efficient (i.e., minimal with respect to the ordering induced by the nonnegative orthant $\mathbb{R}^k_+$) and characterizing certain maximal efficient faces of P. The latter characterization leads to very simple conditions for determining the maximal efficient faces of dimensions $k-1$ and $k-2$. Throughout, we assume that a linear inequality representation for P is known; in particular, we assume

$$P = \{y \in \mathbb{R}^k : Hy \geqslant g\}.$$

For motivation one may think of P as representing either the objective set $Y = C[X]$ or the efficiency equivalent polyhedron $\tilde{Y} = Y + \mathbb{R}^k_+$. It is important to note that if P represents $\tilde{Y}$, then the matrix H is nonnegative. The results that follow address both the case when H is nonnegative and the case when there are no assumptions on H.

The following notation and terminology are used. Given a nonempty face F of P, let id(F) denote the set of indices of the active constraints defining F. Thus, $j \in \mathrm{id}(F)$ if, and only if,

$$h_j y = g_j \quad \text{for all } y \in F.$$

(Here, and throughout the remainder of the paper, h_j denotes the j-th row of the matrix H, and g_j denotes the j-th component of the vector g.) Partition H and g as

$$H = \begin{bmatrix} H_F^= \\ H_F^+ \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} g_F^= \\ g_F^+ \end{bmatrix},$$

where $H_F^=$ ($g_F^=$) denotes the rows of H (g) indexed by the elements in id(F), and H_F^+ (g_F^+) denotes the remaining rows of H (g). Using this notation, we may write

$$\dim(F) = k - \mathrm{rank}(H_F^=),$$

where dim(F) denotes the dimension of F. A point $y \in F$ is called an *inner point* of F if $H_F^+ y > g_F^+$. It is straightforward to show that every nonempty face has an inner point. (Note that if id(F) contains the index for every row of H, then H_F^+ is vacuous, and consequently, every point in F is an inner point.) Finally, the notation $|J|$ is used to denote the cardinality of a finite set J.

The first result gives a necessary and sufficient condition for a face of P to be efficient.

Proposition 3.1. *Let*

$$P = \{y \in \mathbb{R}^k : Hy \geqslant g\},$$

and let F be a nonempty face of P.

(a) *Then F is efficient if, and only if,*

$$\sum_{j \in \mathrm{id}(F)} \alpha_j h_j > 0$$

for some collection of scalars $\alpha_j \geqslant 0$, $j \in \mathrm{id}(F)$.

(b) *Moreover, if H is nonnegative, then F is efficient if, and only if,*

$$\sum_{j \in \mathrm{id}(F)} h_j > 0.$$

Proof. (a): First assume there exist $\alpha_j \geqslant 0$, $j \in \mathrm{id}(F)$, such that

$$\sum_{j \in \mathrm{id}(F)} \alpha_j h_j > 0.$$

Let $y \in F$ and assume, for the sake of contradiction, that there exists $w \in Y$ with $w \lneqq y$. Then

$$\sum_{j \in \mathrm{id}(F)} \alpha_j h_j w < \sum_{j \in \mathrm{id}(F)} \alpha_j h_j y.$$

Also, since $w \in Y$,

$$\sum_{j \in \mathrm{id}(F)} \alpha_j h_j w \geqslant \sum_{j \in \mathrm{id}(F)} \alpha_j g_j.$$

Finally, since $y \in F$,

$$\sum_{j \in \mathrm{id}(F)} \alpha_j h_j y = \sum_{j \in \mathrm{id}(F)} \alpha_j g_j.$$

Notice that the previous three inequalities imply

$$\sum_{j \in \mathrm{id}(F)} \alpha_j g_j > \sum_{j \in \mathrm{id}(F)} \alpha_j g_j,$$

which yields the contradiction. Hence, no such w can exist; and consequently, y is efficient. Since $y \in F$ was chosen arbitrarily, it follows that F is efficient.

Now suppose there does not exist $\alpha_j \geqslant 0$, $j \in \mathrm{id}(F)$, satisfying

$$\sum_{j \in \mathrm{id}(F)} \alpha_j h_j > 0.$$

Thus, the system

$$\alpha^{\mathrm{T}} H_F^{=} > 0, \quad \alpha \geqslant 0, \; \alpha \in \mathbb{R}^{|\mathrm{id}(F)|},$$

has no solution. Therefore, by Motzkin's Theorem of the Alternative [16, p.28] there exists $u \in \mathbb{R}^k$, $u \gneqq 0$, such that $H_F^{=} u \leqslant 0$. Now, let y be an inner point of F. Then $H_F^{+} y > g_F^{+}$; and consequently, there exists $t \in \mathbb{R}$, $t > 0$, such that $y - tu$ satisfies

$$H_F^{+}(y - tu) \geqslant g_F^{+}.$$

Also,

$$H_F^{=}(y - tu) \geqslant g_F^{=}.$$

Hence, $y - tu \in P$. Since

$$y - tu \lneqq y,$$

it follows that F is not efficient.

(b): Since H is nonnegative,

$$\sum_{j \in \mathrm{id}(F)} \alpha_j h_j > 0$$

for some $\alpha_j > 0$ if, and only if,

$$\sum_{j \in \mathrm{id}(F)} h_j > 0.$$

Hence, (b) follows immediately from (a). □

Note that the nonnegativity of H is needed for the 'only if' direction of Proposition 3.1(b). To see this, consider the polyhedron

$$Y = \{y \in \mathbb{R}^2 : Hy \geqslant 0\},$$

where

$$H = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Then the face $F = \{(0, 0)^{\mathrm{T}}\}$ is efficient, but

$$\sum_{j \in \mathrm{id}(F)} h_j = [1, 0]$$

is not strictly positive. This example also shows that the scaling allowed in part (a) cannot be omitted.

The next proposition refines the results stated in Proposition 3.1 and is important for the proof of Theorem 3.3.

Proposition 3.2. *Let*

$$P = \{y \in \mathbb{R}^k : Hy \geqslant g\},$$

let F be an efficient face of P, and let

$$r = \mathrm{rank}(H_F^{=}).$$

(a) *Then there exists linearly independent rows $h_1, \ldots, h_r$ of $H_F^{=}$ and nonnegative scalars $\beta_1, \ldots, \beta_r$ such that*

$$\sum_{j=1}^{r} \beta_j h_j > 0.$$

(b) *Moreover, if H is nonnegative, then every collection $h_1, \ldots, h_r$* of linearly independent rows of $H_F^{=}$ *satisfies*

$$\sum_{j=1}^{r} h_j > 0.$$

Proof. (a): By Proposition 3.1(a), there exist scalars $\alpha_j > 0$, $j \in \mathrm{id}(F)$, satisfying

$$z := \sum_{j \in \mathrm{id}(F)} \alpha_j h_j > 0.$$

Thus, $z \in \mathrm{cone}\{h_j : j \in \mathrm{id}(F)\}$. But this implies z can be written as a nonnegative linear combination of linearly independent vectors from among $\{h_j : j \in \mathrm{id}(F)\}$ (e.g., see [18, Corollary 17.1.2, p.156]).

(b): Let $h_1, \ldots h_r$ be linearly independent rows of $H_F^{=}$. For the sake of contradiction, suppose the q-th component of $\Sigma_{j=1}^{r} h_j$ is zero. Then by the nonnegativity of H, we have $h_{jq} = 0$ for all $j = 1, \ldots, r$. But Proposition 3.1(b) implies that

$$\sum_{j \in \mathrm{id}(F)} h_j > 0.$$

Hence, there must exist $j_0 \in \mathrm{id}(F) \setminus \{1, \ldots, r\}$ such that h_{j_0} is not a linear combination of $h_1, \ldots, h_r$. This contradicts that $r = \mathrm{rank}(H_F^{=})$. □

To highlight one distinction between parts (a) and (b) of Proposition 3.2, we mention that when H is not nonnegative there may exist $r = \mathrm{rank}(H_F^{=})$ linearly independent rows of $H_F^{=}$ for which no nonnegative linear combination is strictly positive. One can easily see this by considering the polyhedron

$$P = \{y \in R^2 : Hy \geqslant 0\},$$

where

$$H = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

The only efficient face of P is $F = \{(0, 0)^T\}$. Note that $H_F^= = H$; and $h_2 = [1, -1]$ and $h_3 = [1, 0]$ are linearly independent. However, no nonnegative linear combination of h_2 and h_3 is strictly positive. Of course, we do have $h_1 + 2h_3 > 0$.

In Theorem 3.3 we give a characterization of certain maximal efficient faces of P.

Theorem 3.3. *Let*

$$P = \{y \in \mathbb{R}^k : Hy \geqslant g\},$$

and let F be a nonempty face of P.

(a) *Then F is a maximal efficient face of dimension* $k - |\mathrm{id}(F)|$ *if, and only if,*

(1) *there exist scalars* $\alpha_j \geqslant 0$, $j \in \mathrm{id}(F)$, *such that*

$$\sum_{j \in \mathrm{id}(F)} \alpha_j h_j > 0,$$

and

(2) $\Sigma_{j \in J} \beta_j h_j \not> 0$ *for every* $J \subsetneqq \mathrm{id}(F)$ *and every collection of scalars* $\beta_j \geqslant 0$, $j \in J$.

(b) *Moreover, if H is nonnegative, then F is a maximal efficient face of dimension* $k - |\mathrm{id}(F)|$ *if, and only if,*

(3) $\Sigma_{j \in \mathrm{id}(F)} h_j > 0$, *and*

(4) $\Sigma_{j \in J} \mathrm{h}_j \not> 0$ *for every* $J \subsetneqq \mathrm{id}(F)$.

Proof. (a): Suppose conditions (1) and (2) hold. By Proposition 3.1(a), condition (1) implies that F is efficient. To see that

$$\dim(F) = k - |\mathrm{id}(F)|,$$

recall that

$$\dim(F) = k - \mathrm{rank}(H_F^=).$$

Hence, it suffices to show

$$\mathrm{rank}(H_F^=) = |\mathrm{id}(F)|.$$

To this end, let $r = \mathrm{rank}(H_F^=)$. Then by Proposition 3.2(a), there exist linearly independent rows $h_1, \ldots, h_r$ of $H_F^=$ and nonnegative scalars $\alpha_1, \ldots, \alpha_r$ such that

$$\sum_{j=1}^{r} \alpha_j \mathrm{h}_j > 0.$$

Hence, using condition (2), it follows that $r = |\mathrm{id}(F)|$; and so, $\dim(F) = k - |\mathrm{id}(F)|$. Finally, to see that F is a maximal efficient face, note that if $F \subsetneqq F_0$, where F_0 is a face of P, then

$$\mathrm{id}(F_0) \subsetneqq \mathrm{id}(F).$$

Hence, by condition (2) and Proposition 3.1(a), it follows that F_0 is not efficient.

Now suppose F is a maximal efficient face of dimension $k - |\mathrm{id}(F)|$. By Proposition 3.1(a), it follows immediately that condition (1) holds. Suppose, for the sake of contradiction, that condition (2) does not hold. Then there exists $J \subsetneqq \mathrm{id}(F)$ and scalars $\beta_j \geqslant 0$, $j \in J$, such that

$$\sum_{j \in J} \beta_j h_j > 0.$$

Let

$$F_J = \{y \in P : h_j y = g_j \quad \text{for all } j \in J\}.$$

Then clearly $F \subseteq F_J$. We show that F is a proper subset of F_J.

Partition $H_F^=$ and $g_F^=$ as

$$H_F^= = \begin{bmatrix} H_J \\ H_0 \end{bmatrix}$$

and

$$g_F^= = \begin{bmatrix} g_J \\ g_0 \end{bmatrix},$$

where H_J (g_J) consists of the rows $H_F^=$ ($g_F^=$) indexed by $j \in J$, and H_0 consists of the rows $H_F^=$ ($g_F^=$) indexed by $j \in \mathrm{id}(F) \backslash J$. By hypothesis it follows

$$|\mathrm{id}(F)| = \mathrm{rank}(H_F^=),$$

and consequently, the only solution of the system

$$\gamma^{\mathrm{T}} H_J + \delta^{\mathrm{T}} H_0 = 0, \quad \gamma \in \mathbb{R}^{|J|}, \quad \delta \in \mathbb{R}^{|\mathrm{id}(F)| - |J|},$$

is $\gamma = 0$, $\delta = 0$. In particular, there does not exist a solution of the system

$$\gamma^{\mathrm{T}} H_J + \delta^{\mathrm{T}} H_0 = 0, \quad \gamma \in \mathbb{R}^{|J|},$$

$$\delta \in \mathbb{R}^{|\mathrm{id}(F)| - |J|}, \ \delta \gneqq 0.$$

Hence, by Motzkin's Theorem of the Alternative [16, p.28], there exists $u \in \mathbb{R}^k$ such that

$$H_0 u > 0 \text{ and } H_J u = 0.$$

Now, let y be an inner point of F. Then

$$H_J y = g_J,$$

$$H_0 y = g_0,$$

and

$$H_F^+ y > g_F^+.$$

Choose $t > 0$ sufficiently small so that

$$H_F^+(y + tu) > g_F^+.$$

Then we have

$$H_J(y + tu) = g_J,$$

$$H_0(y + tu) > g_0,$$

and

$$H_F^+(y + tu) > g_F^+.$$

Hence, $y + tu \in F_J \backslash F$. Moreover, it follows that $\mathrm{id}(F_J) = J$. Hence,

$$\sum_{j \in \mathrm{id}(F)} \beta_j h_j > 0,$$

and so by Proposition 3.1(a), F_J is efficient. Thus, F_J is an efficient face properly containing F, which contradicts the maximality of F.

(b): If H is nonnegative, then condition (1) is equivalent to condition (3), and condition (2) is equivalent to condition (4). □

Theorem 3.3 gives a useful criteria for determining whether or not a face F of dimension $k - |\mathrm{id}(F)|$ is a maximal efficient face. It is particularly relevant for determining maximal efficient faces of dimensions $k-1$ and $k-2$ due to the following well-known results (e.g., see [11, pp.27,34] or [19, pp.101,105]).

Lemma 3.4. *Let*

$$P = \{y \in \mathbb{R}^k : Hy \geqslant g\},$$

suppose $\dim(P) = k$, *suppose there are no redundant inequalities in the system* $Hy \geqslant g$, *and let* F *be a nonempty face of* P. *Then*

(a) $\dim(F) = k - 1$ *if, and only if,*

$$|\mathrm{id}(F)| = 1,$$

and

(b) $\dim(F) = k - 2$ *if, and only if,*

$$|\mathrm{id}(F)| = 2.$$

Immediate from Theorem 3.3 and Lemma 3.4, we get the following corollary. Part (a) is well-known and could perhaps be considered part of the 'folklore' of MOLP. Although part (b) is stated only for the case when H is nonnegative, a similar characterization could be stated for general H.

Corollary 3.5. *Let*

$$P = \{y \in \mathbb{R}^k : Hy \geqslant g\},$$

suppose $\dim(P) = k$, *suppose there are no redundant inequalities in the system* $Hy \geqslant g$, *and let* F *be a nonempty face of* P.

(a) *Then* F *is a maximal efficient face of dimension* $k - 1$ *if, and only if,*

$$\mathrm{id}(F) = \{i\},$$

where $h_i > 0$.

(b) *If* H *is nonnegative, then* F *is a maximal efficient face of dimension* $k - 2$ *if, and only if,*

$$\mathrm{id}(F) = \{i, j\},$$

where $h_i + h_j > 0$, $h_i \not> 0$, *and* $h_j \not> 0$.

In Section 4 the characterizations given in Corollary 3.5 are utilized in an algorithm for determining high-dimensional maximal efficient faces of the objective set $Y = C[X]$ of an MOLP. Note that the assumption that the system $Hy \geqslant g$ has no redundancies appears to be a restriction in the application of Corollary 3.5 since, in general, the problem of determining if a given system is irredundant and, if necessary, eliminating redundant inequalities is computationally expensive (e.g., see [14,21]). However, our utilization of this result is in conjunction with an algorithm that constructs an irredundant linear inequality representation of the efficiency equivalent polyhedron $\tilde{Y}$; and hence, the characterizations stated in the corollary are immediately applicable.

Certainly Lemma 3.4 does not extend to faces of dimension $k-r$ for $r \geqslant 3$, as one can easily construct an example of a polytope in $\mathbb{R}^3$ with a 0-dimensional face F (i.e., F is a singleton) and with $|\mathrm{id}(F)|$ arbitrarily large. One might conjecture that Lemma 3.4 would extend if the additional assumptions that H is nonnegative and F is a maximal efficient face are imposed. The following example shows that this is not the case. Moreover, it emphasizes the difficulty of extending Theorem 3.3 to obtain a characterization of maximal efficient faces having $|\mathrm{id}(F)| > \mathrm{rank}(H_F^{=})$.

Example 3.6. Let

$P = \mathrm{conv}\{(1,1,1,1)^{\mathrm{T}}, (0,0,2,2)^{\mathrm{T}}\} + \mathbb{R}_+^4.$

Then P has the representation $Hy \geqslant g$, where

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

The only efficient face of P is

$F := \mathrm{conv}\{(1,1,1,1)^{\mathrm{T}}, (0,0,2,2)^{\mathrm{T}}\}.$

Note that

$\mathrm{id}(F) = \{1,2,3,4\}$

and

$\mathrm{rank}(H_F^{=}) = 3.$

Moreover, note that if $J := \{2,3\}$ (or if $J = \{1,4\}$), then

$\sum_{j \in J} h_j > 0$

and

$|J| < \mathrm{rank}(H_F^{=}) < |\mathrm{id}(F)|.$

Finally, we remark that the nonnegativity of H is needed for both directions of Theorem 3.3(b). Indeed, for the 'only if' direction, it is needed for precisely the same reason as in Proposition 3.1(b); namely, to get condition (3). To see that it is needed for the converse, consider the following example.

Example 3.7. Let $P \subseteq \mathbb{R}^3$ be the convex cone generated by $(3,3,0)^{\mathrm{T}}$, $(2,4,1)^{\mathrm{T}}$, $(3,3,2)^{\mathrm{T}}$ and $(4,2,1)^{\mathrm{T}}$. Then $(0,0,0)^{\mathrm{T}}$ is the only efficient point of P, and a linear inequality representation for P is $Hy \geqslant g$, where

$$H = \begin{bmatrix} 5 & -1 & -6 \\ -1 & 5 & -6 \\ -1.75 & 1.75 & 3.5 \\ 1.75 & -1.75 & 3.5 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Letting $F = \{(0,0,0)^{\mathrm{T}}\}$, we have

$\mathrm{id}(F) = \{1,2,3,4\}$

and

$\sum_{j \in \mathrm{id}(F)} h_j = [4,4,1] > [0,0,0].$

Moreover, it is easy to check that if $J \subsetneq \mathrm{id}(F)$, then

$\sum_{j \in J} h_j \ngtr [0,0,0].$

Hence, conditions (3) and (4) are satisfied. However,

$\dim(F) = 0 \neq -1 = 3 - |\mathrm{id}(F)|.$

Example 3.6 also illustrates that one must allow the collection of scalars β_j in (2) of Theorem 3.1(a) to be distinct from the collection α_j in (1).

4. Application to multiple objective linear programming

In this section we utilize the characterizations given in Section 3 to develop techniques for determining, or partially determining, the efficient structure of the multiple objective linear program

(MOLP)

Minimize Cx
subject to $x \in X$,

where X is a polyhedral subset $\mathbb{R}^n$ and C is a $k \times n$ matrix. In particular, an algorithm is given which (at least) lists all maximal efficient faces of dimensions $k-1$ and $k-2$ of the objective polyhedron $Y = C[X]$. Thus, the algorithm is complete for problems with 2 and 3 objectives.

In order to utilize the results in Section 3 one needs a linear inequality representation of a polyhedron $P \subseteq \mathrm{R}^k$ that has the same efficient structure as Y. Two reasonable choices for P are Y, itself, and $\tilde{Y} = Y + \mathbb{R}^k_+$. Two reasons suggest that $\tilde{Y}$ is the better choice. First, if $Hy \geqslant g$ is a representation of $\tilde{Y}$, then necessarily H is nonnegative. Since the characterizations for efficient faces and maximal efficient faces given in Section 3 are much simpler when H is nonnegative, implementing the results for H nonnegative is easier and involves less computation than implementing the corresponding results for general H.

The second reason for choosing $\tilde{Y}$ has to do with obtaining a representative system of inequalities with no redundancies. Although the main characterizations in Section 3 do not depend on having an irredundant system, implementing the results would be cumbersome if the system had many redundant inequalities. Among the available methods for constructing linear inequality representations for Y and $\tilde{Y}$ [5,6,9,10,15,17,22] the only methods that immediately lead to systems with no redundancies are based on the theory of polyhedral polarity (e.g., see [19, Chapter 9]). In these methods one first determines all the extreme points and extreme rays of Y (or $\tilde{Y}$), and then uses these to define a 'polar' polyhedron whose extreme points and/or extreme rays are in one-to-one correspondence with the normals of the hyperplanes defining Y ($\tilde{Y}$). One advantage of applying such a method to $\tilde{Y}$, as opposed to Y, is that all the extreme points of $\tilde{Y}$ are efficient and contribute vital information to the analysis of the problem. In contrast, any effort expended to determine nonefficient extreme points of Y is essentially wasted, since such data contribute no information regarding the efficient structure of Y. A second advantage of applying the theory of polyhedral polarity to $\tilde{Y}$ is that, in general, the polar polyhedron of $\tilde{Y}$ has a simpler structure than the polar polyhedron of Y. In particular, all the extreme points of the polar of $\tilde{Y}$ are nonnegative and every extreme ray is a positive multiple of some extreme point [9]. Hence, in general, it is easier to enumerate the extreme points and extreme rays of the polar of $\tilde{Y}$ than the polar of Y.

Based on the above two reasons, our algorithm utilizes a linear inequality representation for $\tilde{Y}$. In order to make the paper self-contained, we give the following algorithm, due to Gallagher and Saleh [9], for constructing such a system for the polyhedron $\tilde{Y}$. The proof that the algorithm is correct can be found in [9].

Algorithm $\tilde{Y}$

Step 1. Find the efficient extreme points $x_1, \ldots, x_s$ and efficient extreme rays $f_1, \ldots, f_t$ of X. Set

$y_i := Cx_i$ for $i = 1, \ldots, s$, and

$d_j := Cf_j$ for $j = 1, \ldots, t$.

Step 2. Choose $y_0 \in \mathbb{R}^k$ such that

$y_0 < \inf\{y_1, \ldots, y_s\}$.

Form the $k \times s$ matrix

$N := [y_1 - y_0, \cdots, y_s - y_0]$

and the $k \times t$ matrix

$D = [d_1, \cdots, d_t]$,

and define the polar polyhedron

$\Pi := \{\pi \in \mathbb{R}^k_+ : \mathrm{N}^{\mathrm{T}}\pi \geqslant 1, D^{\mathrm{T}}\pi \geqslant 0\}$.

Step 3. Find the vertices $\pi_1, \ldots, \pi_v$ of Π and define the matrix H and the vector g as

$$H := \begin{bmatrix} \pi_1^{\mathrm{T}} \\ \vdots \\ \pi_v^{\mathrm{T}} \end{bmatrix}$$

and

$$g := 1 + Hy_0.$$

The representation for $\tilde{Y}$ is $Hy \geqslant g$. Moreover, no inequality in this representation is redundant.

Note that the first step of Algorithm $\tilde{Y}$ requires the enumeration of the efficient extreme points and efficient extreme rays of X. One could, of course, continue with a constraint-space analysis of the problem to determine all maximal efficient faces of X; and several algorithms have been developed for doing so [1,2,7,8,12,23]. However, all of these algorithms require a considerable amount of computation and memory, especially when X has a large number of efficient extreme points and efficient extreme rays. Therefore, due to the collapsing that can occur when X is mapped to $\mathbb{R}^k$ under C, switching to an objective-space analysis may be advantageous. Although we make no claims that the approach suggested herein is superior in all instances to those based exclusively on an analysis of X, it is reasonable to expect that, at least in some instances, a combination of constraint-space and objective-space techniques will be more effective.

With regard to implementing Algorithm $\tilde{Y}$, we mention that an MOLP software package capable of enumerating the efficient extreme points and efficient extreme rays of X (e.g., ADBASE [20], EFFACET [13]) can also be used to find the vertices of the polyhedron Π. Indeed, one need only solve the single-objective linear program: Minimize 0, subject to $\pi \in \Pi$.

We now give an algorithm, based on the characterizations given in Corollary 3.5, for listing (at least) all maximal efficient faces of dimensions $k-1$ and $k-2$. The inputs for the algorithm are:

1) an irredundant system $Hy \geqslant g$, with H nonnegative, defining the polyhedron $\tilde{Y}$; and

2) a finite set $\mathcal{Y} \subseteq \tilde{Y}$ containing the efficient extreme points of Y and a finite set

$$\mathcal{D} \subseteq \{d \in \mathbb{R}^k : Hd \geqslant 0\}$$

containing the efficient extreme rays of Y.

We use the notation I to denote the set of indices for the rows of H and the notation $\mathcal{P}(I)$ to denote the set of all subsets of I.

Algorithm MEF (Maximal Efficient Faces)

Step 1. Determine the sets

$$\mathcal{Y}_i = \{y \in \mathcal{Y} : h_i y = g_i\}$$

and

$$\mathcal{D}_i = \{d \in \mathcal{D} : h_i d = 0\}.$$

Step 2. Determine the set

$$\mathcal{I}_1 = \{\{i\} \in \mathcal{P}(I) : h_i > 0\},$$

and for each $\{i\} \in \mathcal{I}_1$ output: "$F_{\{i\}} := \operatorname{conv}(\mathcal{Y}_i) + \operatorname{cone}(\mathcal{D}_i)$ is a maximal efficient face of dimension $k-1$."

Step 3. Determine the sets

$$\mathcal{I}_2 = \{\{i,j\} \in \mathcal{P}(I) : h_i + h_j > 0,$$
$$h_i \ngtr 0, h_j \ngtr 0 \text{ and } \mathcal{Y}_i \cap \mathcal{Y}_j \neq \emptyset\}$$

and

$$\hat{\mathcal{I}}_2 = \left\{ J_0 \in \mathcal{I}_2 : \text{ for all } J \in (\mathcal{I}_1 \cup \mathcal{I}_2) \backslash \{J_0\} \text{ either} \right.$$
$$\left. \bigcap_{j \in J_0} \mathcal{Y}_j \not\subseteq \bigcap_{j \in J} \mathcal{Y}_j \text{ or } \bigcap_{j \in J_0} \mathcal{D}_j \not\subseteq \bigcap_{j \in J} \mathcal{D}_j \right\},$$

and for each $J_0 \in \hat{\mathcal{I}}_2$, output: "$F_{J_0} := \operatorname{conv}(\bigcap_{j \in J_0} \mathcal{Y}_j) + \operatorname{cone}(\bigcap_{j \in J_0} \mathcal{D}_j)$ is an efficient face".

Remarks 4.1. (a) The set $\hat{\mathcal{I}}_2$ is defined in Step 3 in order to avoid listing an efficient face properly contained in a larger efficient face identified by the algorithm. It is immediate from Corollary 3.5 that all maximal efficient faces of dimensions $k-1$ and $k-2$ will be listed as output. However, Step 3 may also list efficient faces having dimension smaller than $k-2$. By Theorem 3.3, we know that such faces necessarily have $|\mathrm{id}(F)| > \operatorname{rank}(H_F^{=})$, but unfortunately, we do not know if such a face is necessarily a *maximal* efficient face.

(b) Clearly, since the algorithm finds all maximal efficient faces of dimensions $k-1$ and $k-2$, it is complete for the case when $k=2$. Moreover, it is also complete when $k=3$. To see this, one need only note that since the efficient set of Y is connected, the only time Y will have a maximal efficient face of dimension 0 is when $|\mathcal{Y}|=1$ and $\mathcal{D}=\emptyset$. If this is the case, there is no need to apply Algorithm MEF since one already knows that there is a utopian point.

(c) It is evident that the set $\mathcal{Y}$ must contain all the efficient extreme points of Y, but may contain other points of Y (or even $\tilde{Y}$) as well. Similarly, $\mathcal{D}$ must contain all the efficient extreme rays of Y, but may also contain other rays. We note that if one obtained the extreme points and extreme rays of Y as indicated in Step 1 of Algorithm $\tilde{Y}$, then, due to the collapsing that may occur when X is mapped to $\mathbb{R}^k$ under C, some of the images of extreme points and extreme rays of X may not be extreme for Y.

(d) Clearly, one could continue the process begun in Algorithm MEF by determining sets $\mathcal{J}_\ell$, $\ell \geqslant 3$, containing subsets J of I with $|J|=\ell$, and satisfying

$$\sum_{j\in J} h_j > 0, \quad \sum_{j\in J'} h_j \not> 0$$

for any $J' \subsetneqq J$, and

$$\bigcap_{j\in J} \mathcal{Y}_j \neq \emptyset.$$

However, such an approach seems computationally impractical unless a general theoretical characterization of maximal efficient faces can be determined and incorporated into the search procedure.

(e) Since H is assumed to be nonnegative, the test for efficiency only requires adding vectors and checking whether the sum is strictly positive. If H were not nonnegative, one could use the following linear programming based test to determine whether an index set $J \subseteq I$ satisfies the positivity requirement for efficiency:

$$\begin{aligned} &\text{Maximize} && \rho \\ &\text{subject to} && \sum_{j\in J} \alpha_j h_{ji} - \rho \geqslant 0, \quad i=1,\dots,k, \\ &&& \sum_{j\in J} \alpha_j = 1 \\ &&& \alpha_j \geqslant 0, \quad j\in J, \quad \rho \geqslant 0. \end{aligned}$$

Note that the set J satisfies the positivity requirement if, and only if, the optimal objective value is positive. Of course, one could terminate the test the first time a feasible solution is found having $\rho > 0$.

In Example 4.2 we illustrate the entire procedure of applying both Algorithm $\tilde{Y}$ and Algorithm MEF to a multiple objective linear program.

Example 4.2. Consider the MOLP:

Minimize Cx

subject to $Ax \geqslant b,\ x \geqslant 0$,

where

$$C = \begin{bmatrix} 1 & 4 & 2 & 0 & -1 & -2 \\ 0 & 4 & 0 & -2 & 5 & 3 \\ -1 & -1 & 0 & 3 & 1 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 7 & 45 & 0 & 5 & 62 & 38 \\ -9 & -10 & 0 & 20 & 39 & 19 \\ 1 & 607 & 152 & 21 & 534 & 266 \\ -1 & 3 & 0 & 1 & 6 & 4 \\ -5 & -33 & 0 & 9 & -10 & 0 \\ 3 & -9 & -4 & -7 & -18 & -12 \\ 29 & 9 & 0 & -37 & -94 & -38 \\ 7 & -7 & 0 & -5 & -62 & -38 \end{bmatrix},$$

and

$$b = \begin{bmatrix} 22 \\ 12 \\ 290 \\ 2 \\ -6 \\ -10 \\ -26 \\ -22 \end{bmatrix}.$$

$\tilde{Y}$-Step 1. The efficient extreme points of $X := \{x \in \mathbb{R}^6 : Ax \geqslant b, x \geqslant 0\}$

and the associated efficient objective values of $Y := C[X]$ are

$x_1^{\mathrm{T}} = (0, 0.25, 0, 0.25, 0, 0.50)$,

$y_1^{\mathrm{T}} = (0, 2, 1)$,

$x_2^{\mathrm{T}} = (0, 0, 0.8125, 0, 0.125, 0.375)$,

$y_2^{\mathrm{T}} = (0.75, 1.75, 0.5)$,

$x_3^{\mathrm{T}} = (0.4, 0, 0.5, 0, 0.4, 0)$,

$y_3^{\mathrm{T}} = (1, 2, 0)$,

$x_4^{\mathrm{T}} = (3, 0, 0, 1, 0, 1)$,

$y_4^{\mathrm{T}} = (1, 1, 1)$,

$x_5^{\mathrm{T}} = (2, 0, 0, 1, 0.5, 0)$,

$y_5^{\mathrm{T}} = (1.5, 0.5, 1.5)$,

$x_6^{\mathrm{T}} = (0, 0.4, 0.2, 0.8, 0, 0)$,

$y_6^{\mathrm{T}} = (2, 0, 2)$.

There are no efficient extreme rays.

$\tilde{Y}$-Step 2. Taking $y_0 = (-1, -1, -1)^{\mathrm{T}}$, one obtains

$$N = \begin{bmatrix} 1 & 1.75 & 2 & 2 & 2.5 & 3 \\ 3 & 2.75 & 3 & 2 & 1.5 & 1 \\ 2 & 1.5 & 1 & 2 & 2.5 & 3 \end{bmatrix},$$

and

$$\Pi := \{\pi \in \mathbb{R}^3 : N^{\mathrm{T}}\pi \geqslant 1\}.$$

$\tilde{Y}$-Step 3. The vertices of Π are $(\frac{1}{6}, \frac{1}{6}, \frac{1}{6})^{\mathrm{T}}$, $(\frac{1}{4}, \frac{1}{4}, 0)^{\mathrm{T}}$, $(0, \frac{1}{4}, \frac{1}{4})^{\mathrm{T}}$, $(\frac{1}{3}, 0, \frac{1}{3})^{\mathrm{T}}$, $(1, 0, 0)^{\mathrm{T}}$, $(0, 1, 0)^{\mathrm{T}}$ and $(0, 0, 1)^{\mathrm{T}}$. Hence, an irredundant linear inequality representation for $\tilde{Y} = C[X] + \mathbb{R}^3_+$ is $Hy \geqslant g$, where

$$H = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$g = 1 + Hy_0 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{3} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

MEF-*Step 1.* Utilizing the points $y_1, \dots, y_6$ and the rows of H, one obtains

$\mathcal{Y}_1 = \{y_1, y_2, y_3, y_4\}$,

$\mathcal{Y}_2 = \{y_1, y_4, y_5, y_6\}$,

$\mathcal{Y}_3 = \{y_3, y_4, y_5, y_6\}$,

$\mathcal{Y}_4 = \{y_1, y_3\}$,

$\mathcal{Y}_5 = \{y_1\}$,

$\mathcal{Y}_6 = \{y_6\}$,

$\mathcal{Y}_7 = \{y_3\}$

MEF-*Step 2.* $\mathcal{I}_1 = \{\{1\}\}$, and so the set $F_{\{1\}} := \mathrm{conv}(\mathcal{Y}_1)$ is a maximal efficient face of dimension 2.

MEF-*Step 3.* $\mathcal{I}_2 = \{\{2, 3\}, \{2, 4\}, \{3, 4\}\}$ and $\hat{\mathcal{I}}_2 = \{\{2, 3\}\}$. Hence, the set

$$F_{\{2,3\}} = \mathrm{conv}(\mathcal{Y}_2 \cap \mathcal{Y}_3)$$

is a maximal efficient face of dimension 1.

Since, in this example, $k = 3$, the union of the two maximal efficient faces $F_{\{1\}}$ and $F_{\{2,3\}}$ identified in Algorithm MEF is precisely the set of efficient objective values of the given MOLP. The associated set of efficient feasible solutions is

$$C^{-1}(F_{\{1\}}) \cup C^{-1}(F_{\{2,3\}})$$
$$= \mathrm{conv}\{x_1, x_2, x_3, x_4\} \cup \mathrm{conv}\{x_4, x_5, x_6\}.$$

It is interesting to note that while the points x_2 and x_5 are extreme points of X, the associated objective values y_2 and y_5 are not extreme points of Y. Thus, this example illustrates the collapsing phenomenon mentioned in Remark 4.1(c).

We conclude this section with a brief example illustrating Algorithm MEF when the set $\mathcal{D} \neq \emptyset$.

Example 4.3. Let

$Y = \tilde{Y} = \operatorname{conv}(\mathscr{Y}) + \operatorname{cone}(\mathscr{D}) + \mathbb{R}^3_+,$

where

$\mathscr{Y} = \{(1,1,0)^T, (1,0,1)^T, (0,0,1)^T\}$ and

$\mathscr{D} = \{(2,2,-1)^T\}.$

An irredundant linear inequality representation of Y is $Hy \geqslant g$, where

$$H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Algorithm MEF identifies the 2-dimensional maximal efficient face

$\operatorname{conv}(\mathscr{Y}_1) = \operatorname{conv}(\mathscr{Y})$

in Step 2 and the 1-dimensional maximal efficient face

$$\begin{aligned}&\operatorname{conv}(\mathscr{Y}_7 \cap \mathscr{Y}_8) + \operatorname{cone}(\mathscr{D}_7 \cap \mathscr{D}_8) \\ &\quad = \operatorname{conv}(\{(1,1,0)^T\}) + \operatorname{cone}(\{(2,2,-1)^T\})\end{aligned}$$

in Step 3.

5. Summary and conclusion

Characterizations of efficient faces and certain maximal efficient faces of the objective set Y of a linear k-objective minimization problem have been presented. Based on these characterizations, Algorithm MEF has been proposed for determining high-dimensional maximal efficient faces of Y. Algorithm MEF requires as input an irredundant system of linear inequalities representing the efficiency equivalent polyhedron $\tilde{Y} := Y + \mathbb{R}^k_+$. An algorithm capable of generating such a system of inequalities for $\tilde{Y}$ has previously appeared in the literature [9]. This latter algorithm, referred to herein as Algorithm $\tilde{Y}$, requires, in part, the enumeration of the efficient extreme points and the efficient extreme rays of the constraint polyhedron X. Hence, the linking of Algorithm $\tilde{Y}$ with Algorithm MEF represents a combined constraint-space, objective-space approach for enumerating maximal efficient faces of multiple objective linear programs.

Algorithm MEF, as presented herein, generates all maximal efficient faces of dimensions $k-1$ and $k-2$, and may generate other efficient faces as well. Although the algorithm could be extended to generate all maximal efficient faces of Y (see Remark 4.1(d)), the computational effort to generate lower dimensional faces may be prohibitive. Hence, one avenue that could be pursued to extend the work herein is to seek out reasonable characterizations of lower dimensional maximal efficient faces and incorporate these characterizations into an efficient computational procedure for generating such faces.

Computational experiments related to this work are currently underway. In particular, we are in the initial stages of comparing the approach herein to approaches based entirely on a constraint space analysis (e.g., see [1,2,7,8,12,23]). Also, we intend to compare the performance of these same constraint space algorithms to the performance of Algorithm MEF on the problem of finding the efficient structure of $\tilde{Y}$ given that an irredundant linear inequality representation $Hy \geqslant g$ is known; i.e., the problem Minimize Iy, subject to $Hy \geqslant g$.

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