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Outcome Space Partition of the Weight Set in Multiobjective Linear Programming

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Abstract. Approaches for generating the set of efficient extreme points of the decision set of a multiple-objective linear program (P) that are based upon decompositions of the weight set W^0 suffer from one of two special drawbacks. Either the required computations are redundant, or not all of the efficient extreme point set is found. This article shows that the weight set for problem (P) can be decomposed into a partition based upon the outcome set Y of the problem, where the elements of the partition are in one-to-one correspondence with the efficient extreme points of Y. As a result, the drawbacks of the decompositions of W^0 based upon the decision set of problem (P) disappear. The article explains also how this new partition offers the potential to construct algorithms for solving large-scale applications of problem (P) in the outcome space, rather than in the decision space.

Key Words. Multiple-objective linear programming, vector maximization, efficient points, weight set.

1. Introduction and Motivation

Consider the multiple-objective linear programming problem [problem (P)]

$$\begin{aligned} \text{Vmax}\quad & Cx, \\ \text{s.t.}\quad & Ax \le b, \\ & x \ge 0, \end{aligned}$$

where C is a $p \times n$ matrix, A is an $m \times n$ matrix, $b \in R^m$, and x is the vector of decision variables in R^n. Problem (P) has p linear objective functions

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$\langle c_j, x\rangle, j = 1, 2, \ldots, p$, where c_j denotes row j of C. Generally, these objective functions conflict with one another over the decision set

$$X = \{x \in R^n | Ax \leq b, x \geq 0\}$$

of problem (P). As a result, there is rarely a vector $\bar{x} \in X$ such that $\bar{x}$ maximizes $\langle c_j, x\rangle$ over X for each $j = 1, 2, \ldots, p$.

Frequently, a decision maker (DM) with multiple objectives will use problem (P) and the aid of an analyst to find a most-preferred solution, i.e., a solution for problem (P) that lies in X and which the DM prefers over all other solutions that belong to X. To help accomplish this, the concept of efficiency has been found to be very useful. A point $x^0 \in R^n$ is called an efficient solution for problem (P) when $x^0 \in X$ and there exists no point $x \in X$ such that $Cx \geq Cx^0$ and $Cx \neq Cx^0$. Under mild assumptions that hold frequently in practice, it can be shown that any most-preferred solution for problem (P) must also be an efficient solution for the problem (Ref. 1).

To help the DM find a most preferred solution for problem (P), analysts have devised a variety of approaches. One of the well-known approaches is the vector maximum approach. In this approach, the analyst finds either the set X_E of all efficient points for problem (P) or all of the points in X_E that are also elements of the set X_{ex} of extreme points of X. The generated set is presented to the DM. Then, the DM chooose a most-preferred solution from this set. This approach and its ramifications have been studied extensively since the 1950s; see Refs. 1–21 and references therein.

Although the algorithms that use the vector maximum approach to generate X_E or $X_E \cap X_{ex}$ are quite different from one another in certain ways, they all rely upon linear programming techniques including, most notably, simplex method-type procedures and pivots. Most methods rely also upon either decomposing the weight set

$$W^0 \equiv \{w \in R^p | w_j > 0, j = 1, 2, \ldots, p\},$$

finding adjacent efficient bases, or finding adjacent efficient extreme points (Ref. 1).

Regardless of the vector maximum algorithm that one considers for generating X_E or $X_E \cap X_{ex}$, to date neither of these sets can be generated in its entirety for large-scale problems or even moderately-sized problems. There are several reasons for this. Perhaps, most notable are the challenges posed by the sheer size and complexities of the sets X_E and $X_E \cap X_{ex}$.

Most approaches for generating X_E depend upon methods for generating the smaller set $X_E \cap X_{ex}$. But generating even the latter set poses considerable burdens computationally. For instance, Benson (Ref. 22) used the ADBASE algorithm of Steuer (Ref. 23) to generate or attempt to generate

$X_E \cap X_{ex}$ for randomly-generated problems of the form of problem (P) with $p = 4$, m ranging between 25 and 50, and n ranging between 30 and 60. With $m = 25$ and $n = 30$, the average number of efficient extreme points found by ADBASE in a set of ten randomly-generated problems was 7246 points. With $m = 50$ and $n = 50$, this average increased to 83,781 points. With $m = 50$ and $n = 60$, ADBASE was unable to solve any of the ten randomly-generated problems, indicating that the number of elements in $X_E \cap X_{ex}$ in each of these problems exceeded 200,000. For details and other challenges that arise in attempting to generate $X_E \cap X_{ex}$, see Ref. 22.

Motivated by the apparent inability of the vector maximum approach to solve successfully large-scale or medium-scale instances of problem (P), during the 1990s a handful of researchers began investigating the possibility of applying the vector maximum approach to the outcome set

$$Y = \{y \in R^p \mid y = Cx, \text{ for some } x \in X\}$$

of problem (P), rather than to the decision set X of the problem; see Refs. 6, 8, 22, 24–32. This body of research has yielded some important theoretical and practical results.

Theoretical results show that, since p is inevitably much smaller than n, the dimension of the outcome set Y can be expected to be much smaller than the dimension of X, often by multiple orders of magnitude. Generally, the outcome set has far fewer faces and extreme points than the decision set; unlike X, it encapsulates the tradeoffs among the objective functions for the DM in a nonredundant fashion.

In practice, preliminary computational results with algorithms that apply the vector maximum approach to Y have been quite promising. For instance, as expected from the theory, computational evidence to date shows that the set of efficient extreme points in Y is considerably smaller than that in X and can be generated much more easily and quickly (Refs. 22, 24). In addition, complications caused by degeneracy and complex bookkeeping that occur in decision set-based approaches can be avoided in outcome set-based approaches (Refs. 22, 24, 26). Finally, it is well known that the DM prefers to see the efficient solution tradeoffs in the outcome set space R^p, rather than in the decision set space R^n.

However, the best vector maximum approach to apply to Y has yet to be shown definitively. This is because much theoretical, algorithmic, and computational research concerning the application of the vector maximum approach to Y remains to be accomplished. Indeed, this research can be described as being still in its infancy.

Among the many topics yet to be explored is the issue of how useful the idea of decomposing the weight set W^0 would be in helping to generate efficient points in Y. Indeed, this approach has had certain unfortunate limitations when applied to X in an attempt to generate X_E or $X_E \cap X_{ex}$, even for relatively small problems (Refs. 19, 20).

The main purpose of this article is to show that the weight set W^0 for problem (P) can be decomposed into a partition whose elements are in one-to-one correspondence with the efficient extreme points of Y. As we shall see, the elements of this partition have nonempty interiors. Furthermore, as required by the definition of a partition, we will show that, for any two partition elements, if they intersect, the intersection must lie within the boundary of each partition element. We will show by example that, relative to X rather than to Y, the known decompositions of W^0 are not guaranteed to possess these properties.

As a result, we will see yet another advantage of working in the outcome set Y of problem (P), rather than in the decision set. In particular, due to the one-to-one correspondence that we will show between the partition elements of W^0 and the efficient extreme points of Y, there exists the potential to formulate efficient algorithms based upon this partition that generate all of the efficient extreme points of Y without suffering from the limitations of the decomposition approaches of W^0 that have been proposed for generating $X_E \cap X_{ex}$.

In Section 2, we review and illustrate two key weaknesses inherent in approaches for generating $X_E \cap X_{ex}$ based upon the known decompositions of W^0. The main results are given and discussed briefly in Section 3. Conclusions and the potential impact of the main results are presented in the final section.

2. Weaknesses of Decision Set Approaches That Decompose W^0

Two types of decompositions of W^0 have been proposed for helping to generate $X_E \cap X_{ex}$. In addition to the weaknesses inherent in working with the decision set (see Section 1 and Ref. 22), any approach that uses either of these two decompositions to attempt to find $X_E \cap X_{ex}$ has an additional, unique weakness. Both these decompositions of W^0 rely theoretically upon the well-known result that $x^0 \in X_E$ if and only if there exists a vector $w^0 \in W^0$ such that x^0 is an optimal solution to the problem [problem LP(w)]

$$\max (w)^T Cx, \qquad \text{s.t. } x \in X,$$

when $w = w^0$; see e.g., Refs. 1, 14, 20. These approaches and their weaknesses are explained in detail in Refs. 19–20, for example. What follows is a brief summary and illustration of these weaknesses. Henceforth, we assume that the decision set X is bounded and nonempty.

To explain the first decomposition, for each $x \in X$, let

$$W(x) = \{w \in R^p \mid w^T Cx \geq w^T Cx', \text{ for all } x' \in X\}.$$

This approach relies upon the fact that

$$W^0 \subseteq \bigcup_{x \in \bar{X}} W(x), \tag{1}$$

where $\bar{X} = X_E \cap X_{ex}$. For example, the inclusion (1) can be shown by using the Epelman theorem (Ref. 33), Theorem 3.3 in Ref. 34, and the well-known result quoted in the paragraph above. As a result of (1), the weight set W^0 can be decomposed into a union of subsets $[W^0 \cap W(x)]$, $x \in \bar{X}$, of W^0. Since the number of elements in $\bar{X} = X_E \cap X_{ex}$ is finite, this decomposition of W^0 is finite.

A typical method for attempting to generate $\bar{X}$ with the aid of the first decomposition of W^0 could proceed as follows. First, an initial efficient extreme point x^0 for problem (P) is found by using any of a number of known methods; see e.g. Refs. 1 and 4. Next, $W(x^0)$ is generated and $[W^0 \cap W(x^0)]$ is removed from W^0 to create a set $(W^0)^0$. A vector $w^1 \in (W^0)^0$ is then chosen and, with $w = w^1$, any optimal extreme point solution x^1 to problem LP(w) is found. Next, $W(x^1)$ is generated and $[W^0 \cap W(x^1)]$ is removed from $(W^0)^0$ to create a set $(W^0)^1$. The method will continue in this way until, for some $k \geq 0, (W^0)^k$ is empty. At that point, the hope is that

$$X_E \cap X_{ex} = \{x^0, x^1, \ldots, x^k\}.$$

The difficulty that arises in using the first decomposition of W^0 to help generate $X_E \cap X_{ex}$ comes from the fact that there need not be a one-to-one correspondence between the elements x of $\bar{X} = X_E \cap X_{ex}$ and the sets $W^0 \cap W(x)$. In particular, given q distinct elements of $X_E \cap X_{ex}$, where $q \geq 2$, there is a possibility that exactly one set $W^0 \cap W(x)$ corresponds to each of these q elements x. As a result, methods that use the decomposition of W^0 that are based upon the sets $W(x), x \in \bar{X}$, would in such cases generate only one of these q elements of $X_E \cap X_{ex}$. This phenomenon could occur several times during the course of the execution of a method based upon this decomposition of W^0, in which case many elements of $X_E \cap X_{ex}$ would not be found.

The second decomposition of the weight set W^0 is based upon the idea of an efficient basis for problem (P). For the moment, assume that slack variables are introduced in problem (P) to create constraints that are equations, rather than inequalities. For notational convenience, let us assume temporarily and without loss of generality that this yields the problem

$$\begin{aligned} \text{Vmax} \quad & Cx, \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0, \end{aligned}$$

as the form of problem (P), where C, A, b are given as before. Then, X is now temporarily defined by

$$X = \{x \in R^n | Ax = b, x \geq 0\}.$$

Let B be an $m \times m$ feasible basis for the linear program LP(w) for some arbitrary $w \in R^p$. Then, in the second decomposition approach, B is called an efficient basis for problem (P) [and for problem LP(w)] when it is an optimal basis for problem LP(w) for some $w \in W^0$. It can be shown that a feasible basis B for problem LP(w) is an efficient basis for problem (P) if and only if it is a feasible basis for problem (P) and, for some $w \in W^0$,

$$w^T Z \geq 0,$$

where Z is the $p \times (n-m)$ reduced cost matrix for problem (P) corresponding to B. Notice that an efficient basis for problem (P) corresponds to an element of $X_E \cap X_{ex}$.

For a given efficient basis for problem (P), following Zeleny (Ref. 20), let us define the polyhedral cone

$$W(B) = \{w \in R^p | w^T Z \geq 0\},$$

where Z is the reduced cost matrix for problem (P) corresponding to B. From Yu (Ref. 19), it follows that there exist a finite number of efficient bases, $B_1, B_2, \ldots, B_q$, for problem (P) such that

$$W^0 \subseteq \bigcup_{i=1}^{q} W(B_i).$$

As a result, W^0 can be decomposed into a finite union of subsets $[W^0 \cap W(B_i)]$, $i = 1, 2, \ldots, q$, of W^0 such that, for each $x \in X_E \cap X_{ex}$, there exists at least one set $[W^0 \cap W(B_i)]$, $i \in \{1, 2, \ldots, q\}$, such that x corresponds to the basis B_i of problem (P) and, for any $w \in [W^0 \cap W(B_i)]$, x is an optimal solution to problem LP(w). Notice that, due to degeneracy, the number of elements in this decomposition of W^0 can exceed the number of efficient extreme points in X that exist for problem (P).

There are various options available for using the second decomposition of W^0 to help generate $\bar{X} = X_E \cap X_{ex}$. The ADBASE algorithm of Steuer (Ref. 23) embodies an option typically used. In ADBASE, an initial efficient basis for problem (P) is found. Using this basis as a starting point, simplex method-like pivots are used to move iteratively to adjacent efficient bases until all of the efficient bases have been found. Then, it is guaranteed that by finding all elements of X_{ex} that correspond to these bases, $\bar{X} = X_E \cap X_{ex}$ will be found.

The difficulty with the methods that use the second approach for decomposing W^0 to obtain $\bar{X}$ comes from the fact that there is no guarantee that a one-to-one correspondence exists between the elements $[W^0 \cap W(B_i)]$, $i = 1, 2, \ldots, q$, of the decomposition and the elements of $\bar{X}$. In particular, due to degeneracy, many elements of $[W^0 \cap W(B_i)]$, $i = 1, 2, \ldots, q$, may correspond to a given efficient extreme point $\bar{x} \in \bar{X}$. In algorithms like ADBASE, all of these elements would be generated in order to detect $\bar{x}$ and to move on to finding another element of $\bar{X}$, if such an element exists. As a result, although in theory, all of $\bar{X}$ can be found, degeneracies can cause the computations called for by the second approach to be quite redundant and, thus, inefficient.

To illustrate these difficulties, consider the following example.

Example 2.1. In problem (P), let $p = 2$, $n = 3$, $m = 4$, and let

$$C = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 6 \\ 2 \\ 2 \end{bmatrix}.$$

This problem has four efficient extreme points in the decision set. These points are

$$x^1 = (0, 2, 0)^T, \quad x^2 = (0, 2, 2)^T, \quad x^3 = (3, 3, 0)^T, \quad x^4 = (2, 8/3, 2)^T.$$

The nonefficient extreme points in the decision set of this problem are $x^5 = (0, 0, 0)^T$, $x^6 = (0, 0, 2)^T$, $x^7 = (2, 0, 2)^T$, $x^8 = (3, 0, 0)^T$.

It can be shown that

$$\begin{aligned} [W^0 \cap W(x^1)] &= [W^0 \cap W(x^2)] \\ &= \{w \in R^2 \mid 5w_1 - 6w_2 \le 0, w_1, w_2 > 0\}, \\ [W^0 \cap W(x^3)] &= \{w \in R^2 \mid 5w_1 - 6w_2 \ge 0, w_1, w_2 > 0\}, \\ [W^0 \cap W(x^4)] &= \{w \in R^2 \mid 5w_1 - 6w_2 = 0, w_1, w_2 > 0\}. \end{aligned}$$

Thus, although $x^1 \ne x^2$,

$$[W^0 \cap W(x^1)] = [W^0 \cap W(x^2)],$$

in this case. As a result, algorithms for generating $\bar{X}$, that are based upon finding an optimal solution to problem LP(w) as w varies over elements of the decomposition $\{[W^0 \cap W(x^i)],\ i = 1, 2, 3, 4\}$, would fail to detect one of the points x^1 or x^2. Also, notice that $[W^0 \cap W(x^4)]$ has no interior.

If we add slack variables x_4, x_5, x_6, x_7 to rewrite problem (P) in equation form, the resulting constraint matrix, which for ease of notation we will call A, is given by

$$A = \begin{bmatrix} 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In the equation form of problen (P), x^1 is a degenerate extreme point. Corresponding to x^1, there are three degenerate efficient bases given by

$$B^{1,1} = \{x_2, x_4, x_5, x_7\},$$

$$B^{1,2} = \{x_1, x_2, x_4, x_7\},$$

$$B^{1,3} = \{x_2, x_4, x_6, x_7\}.$$

It can shown that

$$[W^0 \cap W(B^{1,1})] = \{w \in R^2 \mid -3w_1 + 2w_2 \geq 0, w_1, w_2 > 0\},$$

$$[W^0 \cap W(B^{1,2})] = \{w \in R^2 \mid 3w_1 - 2w_2 \geq 0,\ -5w_1 + 6w_2 \geq 0, w_1, w_2 > 0\},$$

$$[W^0 \cap W(B^{1,3})] = \{w \in R^2 \mid -5w_1 + 6w_2 \geq 0, w_1, w_2 > 0\}.$$

Notice that, in typical algorithms using this decomposition, all three efficient bases $B^{1,i}$, $i = 1, 2, 3$, would be generated to detect simply the single point $x^1 \in \bar{X}$. Also notice that the elements $[W^0 \cap W(B^{1,i})]$, $i = 1, 2, 3$, of the decomposition of W^0 intersect in their interiors. In particular,

$$[W^0 \cap W(B^{1,3})] = [W^0 \cap W(B^{1,1})] \cup [W^0 \cap W(B^{1,2})].$$

A similar phenomenon occurs at the degenerate efficient extreme point x^2.

Later, we will illustrate via this example that the outcome set-based decomposition of W^0 to be proposed here is a partition of W^0 and does not create situations wherein the use of the decomposition could fail in the ways that the decision set-based decompositions can fail. In particular, the outcome set-based decomposition elements will be shown to be in one-to-one correspondence with the efficient extreme points of Y. Furthermore, problems caused by degeneracy will not occur in the outcome set-based decomposition.

3. Main Results

To develop the main results, we introduce some definitions, additional notation, and preliminary results. All definitions from convex analysis are standard and can be found in Rockafellar (Ref. 35). The following definition is based upon Horst (Ref. 36).

Definition 3.1. Let S be a convex set in R^p with a nonempty interior. A set $\{S^1, S^2, \ldots, S^q\}$ of finitely-many convex subsets S^i, $i = 1, 2, \ldots, q$, of S, each with a nonempty interior, is called a partition of S when S is equal to the union of all of the sets S^i, $i = 1, 2, \ldots, q$, and each pair of sets S^i and S^j, $i \neq j$, intersect at most in their boundaries.

For a convex set S in R^k, the interior and relative interior of S will be denoted by $\operatorname{int} S$ and $\operatorname{ri} S$, respectively. If S is a convex set in R^k, the smallest convex cone containing S is called the convex cone generated by S and is denoted cone(S). If S is a convex set in R^k and $s \in S$, then the normal cone to S at s is

$$\{t \in R^k \,|\, \langle t, s\rangle \geq \langle t, s'\rangle, \text{ for all } s' \in S\}.$$

From Section 1, recall that the outcome set of problem (P) is denoted by Y. Since X is a nonempty, compact polyhedron, it is easy to show from Rockafellar (Ref. 35) that Y is also a nonempty, compact polyhedron. For any point $\bar{y} \in Y$, let

$$Y - \{\bar{y}\} = \{y \in R^p \,|\, y = \hat{y} - \bar{y}, \text{ for some } \hat{y} \in Y\}.$$

Also, for any $y \in Y$, let

$$W(y) = \{w \in R^p \,|\, \langle w, y\rangle \geq \langle w, y'\rangle, \text{ for all } y' \in Y\}.$$

Notice that $W(y)$ is the normal cone to Y at y. It is easy to see that $\hat{x} \in R^n$ is an optimal solution to the linear programming problem LP(w) of Section 2 if and only if $w \in W(\hat{y})$, where $\hat{y} = C\hat{x}$.

Proposition 3.1. Let $Q = \text{cone}(Y - \{y\})$, where $y \in Y$. Then, Q is a polyhedral cone containing the origin and is given by

$$Q = \{\theta(y' - y) \,|\, y' \in Y, \theta \geq 0\}.$$

Proof. From Corollary 19.3.2 in Rockafellar (Ref. 35), since Y is a polyhedron, $Y - \{y\}$ is also a polyhedron. Since $y \in Y$, $Y - \{y\}$ contains the origin. Therefore, from Corollary 19.7.1 of Rockafellar (Ref. 35), cone

$(Y-\{y\})=Q$ is a polyhedral cone containing the origin. Furthermore, by Corollary 2.6.3 of Rockafellar (Ref. 35),

$$Q=\{k(y'-y)\,|\,y'\in Y, k\geq 0\}. \qquad \square$$

For any cone K in R^p, the cone

$$\{z'\in R^p\,|\,\langle z', z\rangle\leq 0, \text{ for all } z\in K\}$$

is called the polar (or dual) cone of K. Let K^0 denote the polar cone of a cone $K\subseteq R^p$.

Proposition 3.2. Let $y\in Y$. Then, the sets $Q=\text{cone}(Y-\{y\})$ and $W(y)$ are polyhedral cones and are polar to one another.

Proof. The set Q is a polyhedral cone by Proposition 3.1. The remainder of the proof follows directly from Theorem 2.1 of Varaiya (Ref. 37), Theorem 14.1 of Rockafellar (Ref. 35), and Corollary 19.2.2 of Rockafellar (Ref. 35). $\square$

Let K be a nonempty, convex set in R^p. Then, the recession cone of K, denoted 0^+K, is given by

$$0^+K=\{d\in R^p\,|\,d+K\subseteq K\}.$$

From Theorem 8.1 of Rockafellar (Ref. 35), when K is a nonempty convex set, 0^+K is a convex cone containing the origin. Recessions cones are used, among other things, to evaluate whether or not convex sets contain unbounded subsets. Notice that, if K is a closed convex cone, then $0^+K=K$. Recall that the lineality of a nonempty convex set K is the dimension of the set $(-0^+K\cap 0^+K)$. Intuitively, the lineality of a nonempty convex set equals the dimension of the largest subspace that, if translated appropriately, could be contained in the set.

Let Y_{ex} denote the (finite) set of extreme points of the outcome set Y for problem (P).

Proposition 3.3. Assume that $y\in Y_{ex}$. Then, the lineality of $Q=\text{cone}(Y-\{y\})$ equals zero.

Proof. Since $y\in Y_{ex}$, it is easy to see that Q is a pointed cone, i.e., that

$$Q\cap -Q=\{0\}.$$

Since Q is a polyhedral cone, it is closed and convex. As we have seen, this implies that

$$O^+Q=Q.$$

As a result,

$$(-O^{+}Q \cap O^{+}Q) = \{0\}.$$

By definition, this means that the lineality of Q equals zero. □

A point $y \in R^p$ is called an admissible (or efficient) point of Y when $y \in Y$ and, if $\hat{y} \in Y$ satisfies $\hat{y} \geq y$, then $\hat{y} = y$. Let Y_E denote the set of all admissible points of Y. Since Y is compact and polyhedral, it is well known that $Y_E \cap Y_{ex}$ is nonempty and has a finite number of elements q. Let $\{y^i | i = 1, 2, \dots, q\}$ denote $Y_E \cap Y_{ex}$. From Section 2, recall that

$$W^0 \subseteq \bigcup_{x \in \bar{X}} [W(x) \cap W^0],$$

where $\bar{X} = X_E \cap X_{ex}$. The first theorem in this section states the counterpart result for Y.

Theorem 3.1. The set W^0 is a subset of $\bigcup_{i=1}^{q} [W(y^i) \cap W^0]$.

Proof. It is easy to see that Y_E equals the set of all efficient points of the multiple objective linear program [problem (PY)] given by

$$\text{Vmax}\, y, \qquad \text{s.t.}\, y \in Y.$$

From this observation, we may apply (1) to problem (PY) to obtain the theorem. □

Theorem 3.1 implies that the weight set

$$W^0 = \{w \in R^p | w > 0\}$$

can be decomposed into a union of the sets $[W^0 \cap W(y^i)]$, $i = 1, 2, \dots, q$. We will call this decomposition of W^0 the outcome set-based decomposition of W^0. It will be our task, in part, to show that this decomposition of W^0 is a partition of W^0. Toward this end, we present the next result.

Theorem 3.2. For each $i \in \{1, 2, \dots, q\}$, $\text{int}\, W(y^i) \neq \emptyset$ and $W^0 \cap \text{int}\, W(y^i) = \text{int}\,(W^0 \cap W(y^i)) \neq \emptyset$.

Proof. Choose $i \in \{1, 2, \dots, q\}$. From Propositions 3.1 and 3.2,

$$Q = \text{cone}(Y - \{y^i\})$$

is a polyhedral cone, and $W(y^i)$ and Q are polar to one another. By Corollary 14.6.1 of Rockafellar (Ref. 35), this implies that the dimension of $W(y^i)$ is $p - j$, where j is the lineality of Q. From Proposition 3.3, $j = 0$.

Therefore, the dimension of $W(y^i) \subseteq R^p$ is p, so that int $W(y^i) \neq \emptyset$. Furthermore, since $y^i \in Y_E$, we know that $W^0 \cap W(y^i) \neq \emptyset$. Since W^0 is an open set with an interior, it follows that

$$W^0 \cap W(y^i) = \operatorname{int} W^0 \cap W(y^i) \neq \emptyset. \tag{2}$$

Since $W^0 = \operatorname{int} W^0$ is open and $W(y^i)$ is a closed, convex set, from (2) and Corollary 6.3.2 of Rockafellar (Ref. 35) it follows that

$$\operatorname{int} W^0 \cap \operatorname{int} W(y^i) \neq \emptyset. \tag{3}$$

From (3), since int $W^0 = W^0$, we see that

$$\operatorname{int} W^0 \cap \operatorname{int} W(y^i) = W^0 \cap \operatorname{int} W(y^i) \neq \emptyset.$$

It is easy to show that

$$\operatorname{int} W^0 \cap \operatorname{int} W(y^i) = \operatorname{int} (W^0 \cap W(y^i)),$$

so that the proof is complete. □

From Proposition 3.2 and Theorems 3.1 and 3.2, to show that $\{(W^0 \cap W(y^i)), i = 1, 2, \ldots, q\}$ is a partition of W^0, it remains to be shown that

$$W^0 \supseteq \bigcup_{i=1}^{q} [W(y^i) \cap W^0] \tag{4}$$

and that, for any $i \neq j$, the sets $[W(y^i) \cap W^0]$ and $[W(y^j) \cap W^0]$ either do not intersect or they intersect only in their boundaries. The inclusion (4) is obvious by inspection. To show the latter property, we will use the following result.

Theorem 3.3. Let $w \in W^0 \cap \operatorname{int} W(y^i)$, for some $i \in \{1, 2, \ldots, q\}$. Then, y^i is the unique optimal solution to the linear program [problem $\mathrm{LP_Y}(w)$]

$$\max \langle w, y \rangle, \qquad \text{s.t. } y \in Y.$$

Proof. Since $w \in W(y^i)$, by definition of $W(y^i)$, y^i is an optimal solution to problem $\mathrm{LP_Y}(w)$. Assume that another optimal solution $\bar{y} \in Y$ for problem $\mathrm{LP_Y}(w)$ exists. We will show that this assumption leads to a contradiction.

From Proposition 3.2, $W(y^i)$ is a polyhedron. Therefore, $W(y^i)$ is a convex set. Since $w \in \operatorname{int} W(y^i)$, there exists a positive scalar M of sufficient magnitude such that, for each $k = 1, 2, \ldots, p$,

$$\begin{aligned} w^k &\equiv w + (w_k/M)(0, 0, \ldots, 0, 1, 0, \ldots, 0)^T \\ &= w + (w_k/M)e^k \end{aligned} \tag{5}$$

also belongs to int $W(y^i)$, where w_k, for each $k=1,2,\ldots,p$, denotes the kth component of w and $e^k=[0,0,0,\ldots,0,1,0,\ldots,0]^T \in R^p$ has the entry 1 in the kth component. Notice that, from (5),

$$\sum_{k=1}^{p} w^k = [p+(1/M)]w,$$

where

$$[p+(1/M)]>0.$$

Therefore, we may express w as

$$w=(1/t)\sum_{k=1}^{p} w^k,$$

where

$$t=[p+(1/M)].$$

This implies that

$$(1/t)\sum_{k=1}^{p} \langle w^k, y^i\rangle = \langle w, y^i\rangle, \tag{6}$$

$$(1/t)\sum_{k=1}^{p} \langle w^k, \bar{y}\rangle = \langle w, \bar{y}\rangle. \tag{7}$$

For each $k\in\{1,2,\ldots,p\}$, since $w^k \in \operatorname{int} W(y^i)$ and $(1/t)>0$, we know that

$$(1/t)\langle w^k, y^i\rangle \geq (1/t)\langle w^k, \bar{y}\rangle. \tag{8}$$

We now claim that, for each $k=1,2,\ldots,p$,

$$\langle w^k, y^i\rangle = \langle w^k, \bar{y}\rangle.$$

To the contrary, suppose that $\bar{k}\in\{1,2,\ldots,p\}$ and

$$\langle w^{\bar{k}}, y^i\rangle \neq \langle w^{\bar{k}}, \bar{y}\rangle.$$

From (8), this implies

$$\langle w^{\bar{k}}, y^i\rangle > \langle w^{\bar{k}}, \bar{y}\rangle.$$

Since $1/t>0$, it follows that

$$(1/t)\langle w^{\bar{k}}, y^i\rangle > (1/t)\langle w^{\bar{k}}, \bar{y}\rangle. \tag{9}$$

From (8) and (9), we obtain

$$(1/t)\sum_{k=1}^{p} \langle w^k, y^i\rangle > (1/t)\sum_{k=1}^{p} \langle w^k, \bar{y}\rangle.$$

By substituting via (6) and (7) in this inequality, we conclude that

$$\langle w, y^i \rangle > \langle w, \bar{y} \rangle.$$

Since $y^i \in Y$, this contradicts that $\bar{y}$ is an optimal solution for problem $\mathrm{LP_Y(w)}$, so that the claim is established.

From the claim established in the previous paragraph and (5), we obtain that, for each $k = 1, 2, \ldots, p$,

$$\langle w + (w_k/M)e^k, y^i \rangle = \langle w + (w_k/M)e^k, \bar{y} \rangle.$$

Since

$$\langle w, y^i \rangle = \langle w, \bar{y} \rangle,$$

this implies that, for each $k = 1, 2, \ldots, p$,

$$(w_k/M)\langle e^k, y^i \rangle = (w_k/M)\langle e^k, \bar{y} \rangle. \tag{10}$$

Recall that $w \in W^0$. Therefore, for $k = 1, 2, \ldots, p$,

$$(w_k/M) > 0.$$

This implies, together with (10),

$$y^i = \bar{y}.$$

This contradicts the assumption that $\bar{y} \neq y^i$. Therefore, no optimal solution for problem $\mathrm{LP_Y(w)}$ exists other than y^i. □

Remark 3.1. In Ref. 38, Benson proved that, given any extreme point $\hat{y}$ of Y_E, there exists a vector $\hat{w} > 0$, $\hat{w} \in R^p$, such that $\hat{y}$ is the unique optimal solution of problem $\mathrm{LP_Y(\hat{w})}$. Theorem 3.3 extends this result by showing that such a vector $\hat{w}$ can be found, in particular, by choosing any $\hat{w} \in W^0 \cap \operatorname{int} W(\hat{y})$.

Given Theorem 3.3, we may now establish the following key result.

Theorem 3.4. Suppose that $i, j \in \{1, 2, \ldots, q\}$ and that $y^i \neq y^j$. Then,

$$[W^0 \cap \operatorname{int} W(y^i)] \cap [W^0 \cap \operatorname{int} W(y^j)] = \emptyset.$$

Proof. Suppose that

$$w \in [W^0 \cap \operatorname{int} W(y^i)] \quad \text{and} \quad w \in [W^0 \cap \operatorname{int} W(y^j)].$$

Then, by Theorem 3.3, y^i is the unique optimal solution to problem $\mathrm{LP_Y(w)}$ and y^j is the unique optimal solution to problem $\mathrm{LP_Y(w)}$ as well. Thus, $y^i = y^j$. But this contradicts the assumption that $y^i \neq y^j$. Hence, no such w exists and the theorem is proven. □

The following result is immediate from Theorems 3.2 and 3.4.

Theorem 3.5. Suppose that $i, j \in \{1, 2, \ldots, q\}$ and that $y^i \neq y^j$. Then, $[W^0 \cap W(y^i)] \neq [W^0 \cap W(y^j)]$.

Remark 3.2. Taken together, Proposition 3.2, Theorems 3.1, 3.2, 3.4, and (4) imply that $\{[W^0 \cap W(y^i)] \,|\, i = 1, 2, \ldots, q\}$ is a partition of W^0. Furthermore, by Theorem 3.5, there is a one-to-one correspondence between the efficient extreme points $\{y^1, y^2, \ldots, y^q\}$ of Y and the weight sets $[W^0 \cap W(y^i)]$, $i = 1, 2, \ldots, q$. On the other hand, by the next result, if $\bar{y}$ and $\hat{y}$ are two nonextreme efficient points of Y that lie in the relative interior of the same efficient face of Y, then

$$[W^0 \cap W(\bar{y})] = [W^0 \cap W(\hat{y})].$$

Theorem 3.6. Let $\bar{y}$ and $\hat{y}$ be two points in $\operatorname{ri} H$, where H is a nonempty face of Y. Then, $W(\bar{y}) = W(\hat{y})$.

Proof. From Rockafellar (Ref. 35), H is a nonempty polyhedron. From Theorem 6.4 of Rockafellar (Ref. 35), since $\bar{y}, \hat{y} \in \operatorname{ri} H$, we may choose $\bar{t}, \hat{t} > 1$ such that

$$\bar{y} + \bar{t}(\hat{y} - \bar{y}) \in H,$$

$$\hat{y} + \hat{t}(\bar{y} - \hat{y}) \in H.$$

By definition, $w \in W(\bar{y})$ if and only if, for all $z \in Y$,

$$\langle w, z - \bar{y} \rangle \leq 0.$$

Equivalently, $w \in W(\bar{y})$ if and only if, for all $z \in Y$,

$$\langle w, z - \hat{y} \rangle + \langle w, \hat{y} - \bar{y} \rangle \leq 0. \tag{11}$$

By setting

$$z = \hat{y} + \hat{t}(\bar{y} - \hat{y})$$

in (11), we see that

$$(\hat{t} - 1)\langle w, \bar{y} - \hat{y} \rangle \leq 0.$$

Since $\hat{t} > 1$, this implies that

$$\langle w, \bar{y} - \hat{y} \rangle \leq 0.$$

By exchanging the roles of $\bar{y}$ and $\hat{y}$ in the previous paragraph, we obtain that

$$\langle w, \hat{y} - \bar{y} \rangle \leq 0.$$

As a result,

$$\langle w, \bar{y} - \hat{y} \rangle = 0,$$

so that

$$\langle w, \hat{y} \rangle = \langle w, \bar{y} \rangle.$$

By definition of $W(\bar{y})$, this implies that $w \in W(\bar{y})$ if and only if $w \in W(\hat{y})$. □

To illustrate these results, consider again Example 2.1.

Example 3.1. Continuation of Example 2.1. The outcome set $Y \subseteq R^p = R^2$ in this case has four extreme points. These are $y^i = Cx^i$, where $i = 1, 3, 5, 8$, and where C and x^i, $i = 1, 3, 5, 8$, are as given earlier. The coordinates of y^i, $i = 1, 3, 5, 8$, are

$$[4, 0]^T, \qquad [9, -6]^T, \qquad [0, 0]^T, \qquad [3, -6]^T,$$

respectively. Since $x^1, x^3 \in X_E$ and $x^5, x^8 \notin X_E$, it follows from Benson (Ref. 34) that

$$Y_E \cap Y_{ex} = \{y^1, y^3\}.$$

It is not difficult to show that

$$[W(y^1) \cap W^0] = \{w \in R^2 \,|\, -5w_1 + 6w_2 \geq 0, w_1, w_2 > 0\},$$

$$[W(y^3) \cap W^0] = \{w \in R^2 \,|\, -5w_1 + 6w_2 \leq 0, w_1, w_2 > 0\}.$$

Notice that, as required by Theorem 3.1 and (4),

$$W^0 = \bigcup_{i=1,3} [W(y^i) \cap W^0].$$

Also, notice that, as called for by Theorem 3.2, for each $i = 1, 3$, $[W(y^i) \cap W^0]$ has a nonempty interior. Furthermore, as guaranteed by Theorem 3.4,

$$[W^0 \cap \operatorname{int} W(y^1)] \cap [W^0 \cap \operatorname{int} W(y^3)] = \emptyset.$$

The intersection of $[W^0 \cap W(y^1)]$ and $[W^0 \cap W(y^3)]$ equals WB, where

$$WB = \{w \in R^2 \,|\, w_1 . w_2 > 0 \text{ and } -5w_1 + 6w_2 = 0\}.$$

Notice that WB lies on the boundary of each of $[W^0 \cap W(y^1)]$ and $[W^0 \cap W(y^3)]$. Since each set $[W^0 \cap W(y^i)]$, $i = 1, 3$, is convex, these observations verify the fact that, by definition,

$$\{[W^0 \cap W(y^1)], [W^0 \cap W(y^3)]\}$$

is a partition of W^0. Notice that the correspondence of y^i with $[W^0 \cap W(y^i)]$, $i = 1, 3$, is one-to-one, as guaranteed by Remark 3.2. Finally, notice that the points

$$\bar{y} = (1/2)y^1 + (1/2)y^3 = (13/2, -3)^T,$$
$$\hat{y} = (4/9)y^1 + (5/9)y^3 = (61/9, -10/3),$$

both lie in the relative interior of the admissible face H of Y, where H is the line segment connecting y^1 and y^3. It is easy to show that, as guaranteed by Theorem 3.6, $W(\bar{y}) = W(\hat{y})$, where

$$W(\bar{y}) = W(\hat{y}) = \{w \in R^2 \mid -5w_1 + 6w_2 = 0, w_1 \geq 0\}.$$

4. Conclusions and Potential Impact

Decompositions of the weight set $W^0 = \{w \in R^p \mid w > 0\}$ based upon the decision set X of the multiple-objective linear programming problem (P) do not provide a good basis for constructing algorithms that generate the set of efficient extreme points in the decision set for the problem. This is mainly because such decompositions form subsets of W^0 that are not in one-to-one correspondence with the efficient extreme points of the decision set of problem (P). Furthermore, these decomposition subsets may lack interiors and may overlap one another in their relative interiors. As a result, algorithms for generating the set of efficient extreme points in X for problem (P) that are based upon these decompositions may call for wasteful, redundant calculations or may fail to generate the entire efficient extreme point set of X.

On the other hand, the results of this paper show how to construct a decomposition of W^0 that is a true partition of W^0 and, consistent with current algorithmic approaches for problem (P), is based upon the outcome set Y, rather than upon the decision set X of the problem. In particular, this partition of W^0 creates subsets of W^0 that are in one-to-one correspondence with the efficient extreme points of Y. Furthermore, the partition elements have interiors and can intersect at most on their boundaries.

The potential impact of these results is that improved outcome set-based approaches may now be investigated for solving large-scale applications of problem (P) in the outcome space R^p. These approaches could use potentially the outcome set-based partition of W^0 given in this article to generate the set of all efficient extreme points of Y in a manner that may resemble the following steps. First, any element w^0 of W^0 is chosen. Next, by solving problem $\mathrm{LP_Y}(\mathrm{w}^0)$, an efficient extreme point y^0 of Y is generated. Following this, the partition element $[W^0 \cap W(y^0)]$ of W^0 is generated and removed from W^0 to create a new weight set $(W^0)^0$. Then, these steps could be repeated with $(W^0)^0$ taking the place of W^0, then $(W^0)^1$ taking the place of $(W^0)^0$, and so on. Eventually, a set $(W^0)^k = \varnothing$ would result, at which point the set of all efficient extreme points $y^0, y^1, \ldots, y^k$ of Y will have been found. We hope in the future that researchers will undertake work to create algorithms using the partition construction of this article that may prove to be capable tangibly for the first time to practically solve and analyze large-scale applications of problem (P).

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