There will be a title – I promise

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Abstract

This paper talks about some stuff.

1 Updated Notation

1. Return to the use of θ rather than the decomposed (ϕ, υ).

2. To make it more clear that the sets previously denoted as $\mathrm{Z}_{\mathcal{B}}$, $\mathrm{H}^i_{\mathcal{B}}\,\forall\, i \in \mathcal{B}$, $\mathrm{E}_{\mathcal{B}}$, $\mathrm{F}_{\mathcal{B}}$, and $\mathrm{D}^i_{\mathcal{B}}\,\forall\, i \in \mathcal{B}$ are each subsets of $\mathcal{B}$ and are available within any Algorithm presented herein for which $\mathcal{B}$ is an input, we modify the notation as follows:

   - $\mathrm{Z}_{\mathcal{B}} \longrightarrow \mathcal{B}.\mathrm{Z}$
   - $\mathrm{H}^i_{\mathcal{B}} \longrightarrow \mathcal{B}.\mathrm{H}^i \quad \forall\, i \in \mathcal{B}$
   - $\mathrm{E}_{\mathcal{B}} \longrightarrow \mathcal{B}.\mathrm{E}$
   - $\mathrm{F}_{\mathcal{B}} \longrightarrow \mathcal{B}.\mathrm{F}$
   - $\mathrm{D}^i_{\mathcal{B}} \longrightarrow \mathcal{B}.\mathrm{D}^i \quad \forall\, i \in \mathcal{B}$

3. We also associate the following scalar information with each f.c.b. $\mathcal{B}$. Again, we assume that each is available within any Algorithm presented herein for which $\mathcal{B}$ is an input.

   - $\mathcal{B}.d$ – intended to represent the dimension of $\mathcal{IR}_{\mathcal{B}}$.

4. For each $i \in \mathcal{B}$, let
$$r^i_{\mathcal{B}}(\theta) = g_{\mathcal{B}}\left(Adj(G(\theta)_{\bullet\mathcal{B}})\right)_{i\bullet} q(\theta). \tag{1}$$

5. For each distinct pair of indices $i, j \in \mathcal{B}$, let
$$l^{i,j}_{\mathcal{B}}(\theta) = g_{\mathcal{B}}\left(Adj(G(\theta)_{\bullet\mathcal{B}})\right)_{i\bullet} G(\theta)_{\bullet\bar{\jmath}}. \tag{2}$$

6. For each $i \in \mathcal{B}$, define
$$\mathcal{B}\,.\mathrm{P}^i := \left\{\ell \in \mathcal{B} : degree((T_{\mathcal{B}}(\theta))_{\ell,\bar{\imath}}) > 0 \text{ or } (T_{\mathcal{B}}(\theta))_{\ell,\bar{\imath}} \text{ is a strictly positive constant}\right\} \tag{3}$$

7. Given a $\theta \in \Theta$ and $\epsilon > 0$, let $B_\epsilon(\theta)$ denote the k-dimensional open ball of radius ϵ centered at θ.

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2 Updated Theory

1. We can sometimes identify elements of $\mathcal{B}$.F when solving NLP_H.

   Theorem 1 *Given a f.c.b. B and distinct $i, j \in \mathcal{B}$, let (λ, θ) be a feasible point of $NLP_H(\mathcal{B}, i, j)$. If $\lambda > 0$ and all inequality constraints of $NLP_H(\mathcal{B}, i, j)$ are satisfied strictly at (λ, θ), then $i \in \mathcal{B}$.F.*

   Proof: Let $\lambda' = \max\{\lambda, \text{ LHS's of inequalities of } NLP_H(\mathcal{B}, i, j) \text{ at } (\lambda, \theta)\}$. Note that $\lambda' > 0$ since all inequality constraints of $NLP_H(\mathcal{B}, i, j)$ are satisfied strictly. Moreover, (λ', θ) is a feasible point to $NLP_F(\mathcal{B}, i)$ and the objective value of $NLP_F(\mathcal{B}, i)$ at this point is $\lambda' > 0$. Hence, the optimal value of $NLP_F(\mathcal{B}, i)$ must be strictly positive showing that $i \in \mathcal{B}$.F by Proposition 4.1 of [1]. ∎

2. We can sometimes determine the dimension of $\mathcal{IR}_{\mathcal{B}}$ upon finding an element i of $\mathcal{B}$.F.

   Theorem 2 *Let a f.c.b. $\mathcal{B}$ and an $i \in \mathcal{B}$.F be given. If $\mathcal{B}.\mathrm{H}^i = \emptyset$ and there exists a point (λ, θ) that is feasible to $NLP_F(\mathcal{B}, i)$ and for which $\lambda > 0$, then $dim(\mathcal{IR}_{\mathcal{B}}) = k$.*

   Proof: Since $\mathcal{B}.\mathrm{H}^i = \emptyset$, from the structure of $NLP_F(\mathcal{B}, i)$ we know that all defining inequalities of $\mathcal{IR}_{\mathcal{B}}$ except the one associate with $i \in \mathcal{B}$ and those whose LHS's are identically zero are satisfied strictly at θ. Hence, there exists $\epsilon > 0$ such that these same defining inequalities of $\mathcal{IR}_{\mathcal{B}}$ are all satisfied strictly at all points in $B_\epsilon(\theta)$. Clearly, the intersection of $B_\epsilon(\theta)$ with the half-space $r^i_{\mathcal{B}}(\theta) \geq 0$ is contained within $\mathcal{IR}_{\mathcal{B}}$ and has dimension k. ∎

3. There exists an alternate NLP to $NLP_A(\mathcal{B}, i, j)$ that can be used to determine the adjacency of $\mathcal{IR}_{\mathcal{B}}$ and $\mathcal{IR}_{\mathcal{B}'}$ along $\mathcal{h}^i_{\mathcal{B}}$.

   Theorem 3 *Let a f.c.b. $\mathcal{B}$ and $i \in \mathcal{B}$ be given such that $dim\left(\mathcal{IR}_{\mathcal{B}}\right) \geq k-1$ and $dim\left(\mathcal{IR}_{\mathcal{B}} \cap \mathcal{h}^i_{\mathcal{B}}\right) = k-1$. For any f.c.b. $\mathcal{B}' \neq \mathcal{B}$ such that $\mid \mathcal{B} \cap \mathcal{B}' \mid \geq h-2$, $\mathcal{IR}_{\mathcal{B}}$ and $\mathcal{IR}_{\mathcal{B}'}$ are adjacent along $\mathcal{h}^i_{\mathcal{B}}$ if and only if one of the following conditions holds:*

   (a) $\mathcal{B}' = (\mathcal{B} \setminus \{i\}) \cup \{\bar{\imath}\}$ and $\left(T_{\mathcal{B}}(\theta)\right)_{i,\bar{\imath}} \not\equiv 0$.

   (b) $\mathcal{B}' = (\mathcal{B} \setminus \{i, j\}) \cup \{\bar{\imath}, \bar{\jmath}\}$, $\left(T_{\mathcal{B}}(\theta)\right)_{i,\bar{\imath}} \equiv 0$, and the following NLP has a strictly positive optimal value:

$$NLP_{A'}(\mathcal{B}, i, j) :=$$

$$\begin{array}{lccr} \max\limits_{\lambda,\theta} & \lambda & & \\ s.t. & l^{j,i}_{\mathcal{B}}(\theta) \geq \lambda & & \\ & r^{\xi}_{\mathcal{B}}(\theta) \geq \lambda & \forall\, \xi \in \left(\mathcal{B} \setminus \left(\mathcal{B}.\mathrm{Z} \cup \mathcal{B}.\mathrm{H}^i \cup \{i\}\right)\right) & (4) \\ & r^{i}_{\mathcal{B}}(\theta) = 0 & & \\ & l^{j,i}_{\mathcal{B}}(\theta) r^{\xi}_{\mathcal{B}}(\theta) - l^{\xi,i}_{\mathcal{B}}(\theta) r^{j}_{\mathcal{B}}(\theta) \geq \lambda & \forall\, \xi \in \left(\mathcal{B}.\mathrm{P}^i \setminus \{j\}\right) & \\ & \theta \in \Theta & & \end{array}$$

   Proof: We focus only on condition (b) as the result is proved for condition (a) in [1].
   $(\Leftarrow)$:
   We establish the desired result by showing that there exists a $(k-1)$-dimensional set $\Theta' \subseteq \Theta$ such that for all $\theta' \in \Theta'$: (I) $\mathcal{C}_{\mathcal{B}}(\theta')$ and $\mathcal{C}_{\mathcal{B}'}(\theta')$ are adjacent along $cone\left(G(\theta')_{\bullet(\mathcal{B}\setminus\{i\})}\right)$, (II) $q(\theta')$ lies in $\mathcal{C}_{\mathcal{B}}(\theta')$, and (III) $q(\theta')$ lies in $\mathcal{C}_{\mathcal{B}'}(\theta')$.

Let (λ^*, θ^*) be a point feasible to $NLP_{A'}(\mathcal{B}, i, j)$ for which $\lambda^* > 0$. Then there must exist an $\epsilon > 0$ such that for all $\theta' \in B_\epsilon(\theta^*)$: (i) $l^{j,i}_{\mathcal{B}}(\theta') > 0$, (ii) $r^{\xi}_{\mathcal{B}}(\theta') > 0$ for all $\xi \in \left(\mathcal{B} \setminus \left(\mathcal{B}.\mathrm{Z} \cup \mathcal{B}.\mathrm{H}^i \cup \{i\}\right)\right)$, and (iii) $l^{j,i}_{\mathcal{B}}(\theta') r^{\xi}_{\mathcal{B}}(\theta') - l^{\xi,i}_{\mathcal{B}}(\theta') r^{j}_{\mathcal{B}}(\theta') > 0$ for all $\xi \in \left(\mathcal{B}.\mathrm{P}^i \setminus \{j\}\right)$. Define

$$\Theta' = B_\epsilon(\theta^*) \cap \hbar^i_{\mathcal{B}} \tag{5}$$

and recognize from (**??**) and (1) that because $r^i_{\mathcal{B}}(\theta') = 0$, we have $\theta^* \in relint\,(\Theta')$. Thus, $dim\,(\Theta') = dim\left(\hbar^i_{\mathcal{B}}\right) = k - 1$.

We now establish claim (I). From (**??**) and (2) we see that because $l^{j,i}_{\mathcal{B}}(\theta') > 0$ for all $\theta' \in B_\epsilon(\theta^*)$, we have that $\left(T_{\mathcal{B}}(\theta')\right)_{j,\bar{\imath}} > 0$ for all $\theta' \in \Theta'$. Therefore, by Proposition 4.4 of [1] we have that $\mathcal{C}_{\mathcal{B}}(\theta')$ and $\mathcal{C}_{\mathcal{B}'}(\theta')$ are adjacent along $cone\left(G(\theta')_{\bullet(\mathcal{B}\setminus\{i\})}\right)$ for all $\theta' \in \Theta'$.

Next, we establish claim (II). From (**??**), (**??**), (1), and (5) we see that because $r^{\xi}_{\mathcal{B}}(\theta') > 0$ for all $\xi \in \left(\mathcal{B} \setminus \left(\mathcal{B}.\mathrm{Z} \cup \mathcal{B}.\mathrm{H}^i \cup \{i\}\right)\right)$ and all $\theta' \in \Theta'$, we have that $\theta' \in relint\left(\mathcal{IR}_{\mathcal{B}} \cap \hbar^i_{\mathcal{B}}\right)$ for all $\theta' \in \Theta'$. Thus, from Observation 2.4 and Definition 2.15 of [1], we have $q(\theta') \in \mathcal{C}_{\mathcal{B}}(\theta')$ for all $\theta' \in \Theta'$.

Finally, we establish claim (III). Recognize that $q(\theta')$ lies in $\mathcal{C}_{\mathcal{B}'}(\theta')$ for all $\theta' \in \Theta'$ if and only if for each $\theta' \in \Theta'$, $q(\theta')$ can be represented as a conic combination of the columns of $G(\theta')_{\bullet\mathcal{B}'}$, i.e., if and only if for each $\theta' \in \Theta'$, there exists $\alpha(\theta') \in \mathbb{R}^h$ such that $\alpha(\theta')_\ell \geq 0$ for all $\ell \in \{1, \ldots, h\}$ and

$$q(\theta') = G(\theta')_{\bullet\mathcal{B}'}\alpha(\theta'). \tag{6}$$

Recognize that because $\mathcal{B}$ is a f.c.b., $\alpha(\theta')$ satisfies (6) if and only if it also satisfies

$$G(\theta')^{-1}_{\bullet\mathcal{B}}q(\theta') = G(\theta')^{-1}_{\bullet\mathcal{B}}G(\theta')_{\bullet\mathcal{B}'}\alpha(\theta'). \tag{7}$$

Recognize that (7) represents a system of h equations. Assuming that the elements of $\alpha(\theta')$ and the individual equations of (7) are indexed by the elements of $\mathcal{B}$, we see that for each $\ell \in \mathcal{B}$, the ℓ^{th} equation of (7) is given by

$$\left(G(\theta')^{-1}_{\bullet\mathcal{B}}q(\theta')\right)_\ell = \sum_{n\in\mathcal{B}}\alpha_n(\theta')\left(G(\theta')^{-1}_{\bullet\mathcal{B}}G(\theta')_{\bullet\mathcal{B}'}\right)_{\ell n}. \tag{8}$$

Since $\mathcal{B}' = (\mathcal{B}\setminus\{i,j\}) \cup \{\bar{\imath},\bar{\jmath}\}$, notice that: (i) when $n = \ell$, we have $\left(G(\theta')^{-1}_{\bullet\mathcal{B}}G(\theta')_{\bullet\mathcal{B}'}\right)_{\ell n} = 1$, and (ii) when $n \neq \ell$ and $n \notin \{i,j\}$, we have $\left(G(\theta')^{-1}_{\bullet\mathcal{B}}G(\theta')_{\bullet\mathcal{B}'}\right)_{\ell n} = 0$. Thus, equation (8) can be expressed as

$$\left(G(\theta')^{-1}_{\bullet\mathcal{B}}q(\theta')\right)_\ell = \alpha_\ell(\theta') + \alpha_i(\theta')\left(G(\theta')^{-1}_{\bullet\mathcal{B}}G(\theta')_{\bullet\mathcal{B}'}\right)_{\ell i} + \alpha_j(\theta')\left(G(\theta')^{-1}_{\bullet\mathcal{B}}G(\theta')_{\bullet\mathcal{B}'}\right)_{\ell j}. \tag{9}$$

Additionally, note that for any $n \in \mathcal{B} \cap \mathcal{B}'$, we have $\left(G(\theta')^{-1}_{\bullet\mathcal{B}}G(\theta')_{\bullet\mathcal{B}'}\right)_{\ell n} = \left(G(\theta')^{-1}_{\bullet\mathcal{B}}G(\theta')\right)_{\ell\bar{n}}$ and, as a result, equation (9) can be written as

$$\left(G(\theta')^{-1}_{\bullet\mathcal{B}}q(\theta')\right)_\ell = \begin{cases} \alpha_i(\theta')\left(G(\theta')^{-1}_{\bullet\mathcal{B}}G(\theta')\right)_{i\bar{\imath}} + \alpha_j(\theta')\left(G(\theta')^{-1}_{\bullet\mathcal{B}}G(\theta')\right)_{i\bar{\jmath}} & \text{if } \ell = i \\ \alpha_i(\theta')\left(G(\theta')^{-1}_{\bullet\mathcal{B}}G(\theta')\right)_{j\bar{\imath}} + \alpha_j(\theta')\left(G(\theta')^{-1}_{\bullet\mathcal{B}}G(\theta')\right)_{j\bar{\jmath}} & \text{if } \ell = j \\ \alpha_\ell(\theta') + \alpha_i(\theta')\left(G(\theta')^{-1}_{\bullet\mathcal{B}}G(\theta')\right)_{\ell\bar{\imath}} + \alpha_j(\theta')\left(G(\theta')^{-1}_{\bullet\mathcal{B}}G(\theta')\right)_{\ell\bar{\jmath}} & \text{otherwise} \end{cases} \tag{10}$$

$$= \begin{cases} \alpha_i(\theta')\left(T_{\mathcal{B}}(\theta')\right)_{i,\bar{\imath}} + \alpha_j(\theta')\left(T_{\mathcal{B}}(\theta')\right)_{i,\bar{\jmath}} & \text{if } \ell = i \\ \alpha_i(\theta')\left(T_{\mathcal{B}}(\theta')\right)_{j,\bar{\imath}} + \alpha_j(\theta')\left(T_{\mathcal{B}}(\theta')\right)_{j,\bar{\jmath}} & \text{if } \ell = j \\ \alpha_\ell(\theta') + \alpha_i(\theta')\left(T_{\mathcal{B}}(\theta')\right)_{\ell,\bar{\imath}} + \alpha_j(\theta')\left(T_{\mathcal{B}}(\theta')\right)_{\ell,\bar{\jmath}} & \text{otherwise} \end{cases} \tag{11}$$

where in (11) follows from (**??**). We now show that for each $\ell \in \mathcal{B}$, $\alpha_\ell(\theta') > 0$ follows from (11) and the fact that each $\theta' \in \Theta'$ satisfies: (i) $l^{j,i}_{\mathcal{B}}(\theta') > 0$, (ii) $r^{\xi}_{\mathcal{B}}(\theta') > 0$ for all $\xi \in \left(\mathcal{B} \setminus \left(\mathcal{B}.\mathrm{Z} \cup \mathcal{B}.\mathrm{H}^i \cup \{i\}\right)\right)$, and (iii) $l^{j,i}_{\mathcal{B}}(\theta') r^{\xi}_{\mathcal{B}}(\theta') - l^{\xi,i}_{\mathcal{B}}(\theta') r^{j}_{\mathcal{B}}(\theta') > 0$ for all $\xi \in \left(\mathcal{B}.\mathrm{P}^i \setminus \{j\}\right)$. To begin, recall that $\left(T_{\mathcal{B}}(\theta)\right)_{i,\bar{\imath}} \equiv 0$.

Next, notice from (??), (??), (1), and (5) that $\left(G(\theta')^{-1}_{\bullet\mathcal{B}}q(\theta')\right)_i = 0$ for all $\theta' \in \Theta'$. Thus, from (11) we have that for every $\theta' \in \Theta'$,

$$\begin{aligned} & \alpha_j(\theta')\left(T_{\mathcal{B}}(\theta')\right)_{i,\bar{\jmath}} = 0 \\ \implies & \alpha_j(\theta') = 0. \end{aligned} \tag{12}$$

Furthermore, equations (11) and (12) show that for each $\theta' \in \Theta'$,

$$\begin{aligned} & \alpha_i(\theta')\left(T_{\mathcal{B}}(\theta')\right)_{j,\bar{\imath}} = \left(G(\theta')^{-1}_{\bullet\mathcal{B}}q(\theta')\right)_j \\ \implies & \alpha_i(\theta') = \frac{\left(G(\theta')^{-1}_{\bullet\mathcal{B}}q(\theta')\right)_j}{\left(T_{\mathcal{B}}(\theta')\right)_{j,\bar{\imath}}}. \end{aligned} \tag{13}$$

As we discussed when we established claim (I), the fact that $l^{j,i}_{\mathcal{B}}(\theta') > 0$ for all $\theta' \in \Theta'$ implies that $\left(T_{\mathcal{B}}(\theta')\right)_{j,\bar{\imath}} > 0$ for all $\theta' \in \Theta'$. Additionally, from (1) and the facts that $j \in \left(\mathcal{B} \setminus \left(\mathcal{B}.\mathrm{Z} \cup \mathcal{B}.\mathrm{H}^i \cup \{i\}\right)\right)$ and $r^{\xi}_{\mathcal{B}}(\theta') > 0$ for all $\xi \in \left(\mathcal{B} \setminus \left(\mathcal{B}.\mathrm{Z} \cup \mathcal{B}.\mathrm{H}^i \cup \{i\}\right)\right)$ and all $\theta' \in \Theta'$, we have that $\left(G(\theta')^{-1}_{\bullet\mathcal{B}}q(\theta')\right)_j > 0$ for all $\theta' \in \Theta'$. Hence, equation (13) shows that $\alpha_i(\theta') > 0$ for all $\theta' \in \Theta'$. Finally, from (12) and (13) we see that for any $\ell \in \mathcal{B} \setminus \{i, j\}$, equation (11) can be written as

$$\begin{aligned} & \alpha_\ell + \frac{\left(G(\theta')^{-1}_{\bullet\mathcal{B}}q(\theta')\right)_j}{\left(T_{\mathcal{B}}(\theta')\right)_{j,\bar{\imath}}}\left(T_{\mathcal{B}}(\theta')\right)_{\ell,\bar{\imath}} = \left(G(\theta')^{-1}_{\bullet\mathcal{B}}q(\theta')\right)_\ell \\ \implies & \alpha_\ell(\theta') = \left(G(\theta')^{-1}_{\bullet\mathcal{B}}q(\theta')\right)_\ell - \left(G(\theta')^{-1}_{\bullet\mathcal{B}}q(\theta')\right)_j \frac{\left(T_{\mathcal{B}}(\theta')\right)_{\ell,\bar{\imath}}}{\left(T_{\mathcal{B}}(\theta')\right)_{j,\bar{\imath}}}. \end{aligned} \tag{14}$$

Now recall that $l^{j,i}_{\mathcal{B}}(\theta')r^{\xi}_{\mathcal{B}}(\theta') - l^{\xi,i}_{\mathcal{B}}(\theta')r^{j}_{\mathcal{B}}(\theta') > 0$ for all $\xi \in \left(\mathcal{B}.\mathrm{P}^i \setminus \{j\}\right)$ and all $\theta' \in \Theta'$. Using the fact that $l^{j,i}_{\mathcal{B}}(\theta') > 0$ for all $\theta' \in \Theta'$, this can be rewritten as $r^{\xi}_{\mathcal{B}}(\theta') - r^{j}_{\mathcal{B}}(\theta')\frac{l^{\xi,i}_{\mathcal{B}}(\theta')}{l^{j,i}_{\mathcal{B}}(\theta')} > 0$ for all $\xi \in \left(\mathcal{B}.\mathrm{P}^i \setminus \{j\}\right)$ and all $\theta' \in \Theta'$. By substituting from (1) and (2) and simplifying, we have

$$\begin{aligned} & g_{\mathcal{B}}\left(Adj(G(\theta')_{\bullet\mathcal{B}})\right)_{\xi\bullet}q(\theta') - g_{\mathcal{B}}\left(Adj(G(\theta')_{\bullet\mathcal{B}})\right)_{j\bullet}q(\theta')\frac{g_{\mathcal{B}}\left(Adj(G(\theta')_{\bullet\mathcal{B}})\right)_{\xi\bullet}G(\theta')_{\bullet\bar{\imath}}}{g_{\mathcal{B}}\left(Adj(G(\theta')_{\bullet\mathcal{B}})\right)_{j\bullet}G(\theta')_{\bullet\bar{\imath}}} > 0 \\ & \qquad \text{for all } \xi \in \left(\mathcal{B}.\mathrm{P}^i \setminus \{j\}\right) \text{ and all } \theta' \in \Theta' \\ \iff & \frac{\left(Adj(G(\theta')_{\bullet\mathcal{B}})\right)_{\xi\bullet}}{det\left(G(\theta')_{\bullet\mathcal{B}}\right)}q(\theta') - \frac{\left(Adj(G(\theta')_{\bullet\mathcal{B}})\right)_{j\bullet}}{det\left(G(\theta')_{\bullet\mathcal{B}}\right)}q(\theta')\frac{\frac{\left(Adj(G(\theta')_{\bullet\mathcal{B}})\right)_{\xi\bullet}}{det\left(G(\theta')_{\bullet\mathcal{B}}\right)}G(\theta')_{\bullet\bar{\imath}}}{\frac{\left(Adj(G(\theta')_{\bullet\mathcal{B}})\right)_{j\bullet}}{det\left(G(\theta')_{\bullet\mathcal{B}}\right)}G(\theta')_{\bullet\bar{\imath}}} > 0 \\ & \qquad \text{for all } \xi \in \left(\mathcal{B}.\mathrm{P}^i \setminus \{j\}\right) \text{ and all } \theta' \in \Theta' \\ \iff & \left(G(\theta')^{-1}_{\bullet\mathcal{B}})\right)_{\xi\bullet}q(\theta') - \left(G(\theta')^{-1}_{\bullet\mathcal{B}})\right)_{j\bullet}q(\theta')\frac{\left(G(\theta')^{-1}_{\bullet\mathcal{B}})\right)_{\xi\bullet}G(\theta')_{\bullet\bar{\imath}}}{\left(G(\theta')^{-1}_{\bullet\mathcal{B}})\right)_{j\bullet}G(\theta')_{\bullet\bar{\imath}}} > 0 \\ & \qquad \text{for all } \xi \in \left(\mathcal{B}.\mathrm{P}^i \setminus \{j\}\right) \text{ and all } \theta' \in \Theta' \\ \iff & \left(G(\theta')^{-1}_{\bullet\mathcal{B}})q(\theta')\right)_{\xi} - \left(G(\theta')^{-1}_{\bullet\mathcal{B}})q(\theta')\right)_{j}\frac{\left(T_{\mathcal{B}}(\theta')\right)_{\xi,\bar{\imath}}}{\left(T_{\mathcal{B}}(\theta')\right)_{j,\bar{\imath}}} > 0 \\ & \qquad \text{for all } \xi \in \left(\mathcal{B}.\mathrm{P}^i \setminus \{j\}\right) \text{ and all } \theta' \in \Theta' \end{aligned} \tag{15}$$

From (14) and (15), it is clear that $\alpha_\ell(\theta') > 0$ for all $\theta' \in \Theta'$ whenever $\ell \in \mathcal{B}.\mathrm{P}^i \setminus \{i, j\}$. Now suppose $\ell \notin \left(\mathcal{B}.\mathrm{P}^i \cup \{i, j\}\right)$. From (3) we see that in this case $\left(T_{\mathcal{B}}(\theta')\right)_{\ell,\bar{\imath}}$ must be a nonpositive constant. Furthermore, we have already established that $(T_{\mathcal{B}}(\theta'))_{j,\bar{\imath}} > 0$ and $\left(G(\theta')^{-1}_{\bullet\mathcal{B}}q(\theta')\right)_j > 0$ for all $\theta' \in \Theta'$. Also notice from (**??**), (**??**), and (1) that for each $\ell \in \mathcal{B} \setminus \{i, j\}$, we have that $\left(G(\theta')^{-1}_{\bullet\mathcal{B}}q(\theta')\right)_\ell \geq 0$ for all $\theta' \in \Theta'$ since $r^{\xi}_{\mathcal{B}}(\theta') > 0$ for all $\xi \in \left(\mathcal{B} \setminus \left(\mathcal{B}.\mathrm{Z} \cup \mathcal{B}.\mathrm{H}^i \cup \{i\}\right)\right)$ and all $\theta' \in \Theta'$. From these facts and (14), we see that $\alpha_\ell(\theta') \geq 0$ even when $\ell \notin \left(\mathcal{B}.\mathrm{P}^i \cup \{i,j\}\right)$. We have now proved that for each $\theta' \in \Theta'$ there exists $\alpha(\theta') \in \mathbb{R}^h$ such that $\alpha(\theta')_\ell \geq 0$ for all $\ell \in \{1, \ldots, h\}$ and $q(\theta') = G(\theta')_{\bullet\mathcal{B}'}\alpha(\theta')$ and hence, claim (III) above is proved.

($\Rightarrow$):

The forward direction of the proof is straightforward as $\mathcal{IR}_{\mathcal{B}}$ and $\mathcal{IR}_{\mathcal{B}'}$ can only be adjacent along $\hbar^i_{\mathcal{B}}$ if $dim\left(\mathcal{IR}_{\mathcal{B}} \cap \mathcal{IR}_{\mathcal{B}'} \cap \hbar^i_{\mathcal{B}}\right) = k-1$. Then, by selecting $\theta' \in relint\left(\mathcal{IR}_{\mathcal{B}} \cap \mathcal{IR}_{\mathcal{B}'} \cap \hbar^i_{\mathcal{B}}\right)$, the logic of the reverse direction of this proof can be reversed to show that the equality constraint of $NLP'_A(\mathcal{B}, i, j)$ is satisfied at θ' and, moreover, all inequality constraints of $NLP'_A(\mathcal{B}, i, j)$ are satisfied strictly at $(\lambda, \theta) = (0, \theta')$. This strict satisfaction of the inequalities of $NLP'_A(\mathcal{B}, i, j)$ when $\lambda = 0$ implies that there must exist an $\epsilon > 0$ such that for all $\lambda' \in B_\epsilon(0)$, all the inequalities of $NLP'_A(\mathcal{B}, i, j)$ are satisfied strictly at (λ', θ'). As $B_\epsilon(0) \cap \{\lambda : \lambda > 0\} \neq \emptyset$, this completes the proof. ■

4. We can sometimes avoid solving NLP_S in Phase 1.

   Observation 4 *Let a f.c.b.* $\mathcal{B}$ *and distinct* $i, j \in \mathcal{B}$ *be given and suppose that there exists a point* (λ, θ, ρ) *that is feasible to either* $NLP^{ph1}_F(\mathcal{B}, i)$ *or* $NLP^{ph1}_H(\mathcal{B}, i, j)$. *Then the point* (θ, ρ) *is feasible to* $NLP_S(\mathcal{B})$.

   As a result of Observation 4, we note that if at any point during Phase 1 we process a f.c.b. $\mathcal{B}$ and discover a point (λ, θ, ρ) during the execution of either BuildF_Ph1($\mathcal{B}$) or BuildZEH_Ph1($\mathcal{B}$) that is feasible to either $NLP^{ph1}_F(\mathcal{B}, i)$ or $NLP^{ph1}_H(\mathcal{B}, i, j)$ for some $i, j \in \mathcal{B}$ and for which $\rho < 0$, then it is unnecessary to solve NLP_S on line 3 of Algorithm **??** and we can move inside the "if" statement given on line 4.

3 Updated Algorithms

Algorithm 1 PartitionΘ($\mathcal{B}_0$) – Partition the parameter space Θ.
Input: An initial f.c.b. $\mathcal{B}_0$ such that $dim(\mathcal{IR}_{\mathcal{B}_0}) = k$.
Output: A partition of $\hat{\Theta}$, denoted $\mathcal{P}$.

```
Let S = {B_0} and P = {IR_{B_0}}.
while S ≠ ∅ do select B from S.
    B.F = BuildF(B)
    for i ∈ F_B do
        Let (S', ℬ) = GetAdjacentRegionsAcross(B, i, ℬ) and set S = S ∪ S'.
        for B' ∈ S' do set P = P ∪ IR_{B'}.
Return P.
```

Algorithm 2 BUILDF($\mathcal{B}$) – Build $\mathcal{B}$.F.

Input: A f.c.b. $\mathcal{B}$ such that $dim(\mathcal{IR}_{\mathcal{B}}) = k$.

Output: The set $\mathcal{B}$.F.

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1: for i ∈ (B \ (B.Z ∪ B.E ∪ B.F)) do solve NLP_F(B, i) to find an optimal solution (λ*, θ*).
2:     if λ* > 0 then
3:         Add (i ∪ B.H^i) to B.F.
4:         if B.d < k and B.H^i = ∅ then set B.d = k.
5: Return B.F and B.d.
```

Algorithm 3 BUILDZEH($\mathcal{B}$) – Build $\mathcal{B}$.Z, $\mathcal{B}$.E, and $\mathcal{B}.\mathrm{H}^i$ for each $i \in \mathcal{B}$. Initialize $\mathcal{B}$.F and $\mathcal{B}.d$.

Input: A f.c.b. $\mathcal{B}$ such that $dim(\mathcal{IR}_{\mathcal{B}}) \geq k - 1$.

Output: The sets $\mathcal{B}$.Z, $\mathcal{B}$.E, $\mathcal{B}$.F, and $\mathcal{B}.\mathrm{H}^i$ for each $i \in \mathcal{B}$.

```
1: Let B.Z = B.E = B.F = ∅, B.H^ℓ = ∅ for each ℓ ∈ B, and B.d = 0.
2: for i ∈ B do
3:     if r^i_B(θ) ≡ 0 then add i to B.Z.
4: for i ∈ (B \ (B.Z ∪ B.E)) do
5:     for j ∈ (B \ (B.Z ∪ B.E ∪ {i})) do
6:         if j ∉ B.H^i then solve NLP_H(B, i, j) to obtain an optimal solution (λ*, θ*).
7:             if λ* = 0 then add (j ∪ B.H^j) to B.H^i.
8:             else if λ* < 0 then add i to B.E and exit the for loop beginning on Line 5.
9:             else if r^ℓ_B(θ*) > 0 for all ℓ ∈ (B \ (B.Z ∪ {i, j})) then add i to B.F.
10:    if B.d < k and B.H^i = ∅ and i ∈ B.F then set B.d = k.
11: Return B.Z, B.E, B.F, B.H^ℓ for each ℓ ∈ B, and B.d.
```

References

[1] Nathan Adelgren. *Advancing Parametric Optimization*. Springer, 01 2021. ISBN 978-3-030-61820-9. doi: 10.1007/978-3-030-61821-6.