DePaul University

Department of Mathematical Sciences

MAT 359/459: Simulation Models and Monte Carlo Methods

Project for Spring Quarter, 2021

Purpose: The purpose of this project is to demonstrate the use of Monte Carlo method in financial application.

Problem Description: We have seen some examples of pricing options using Geometric Brownian motion. There are a lot of other models for asset price, for example, combining Vasicek model of varying interest rate with Geometric Brownian motion; combining a deterministic model of volatility with Geometric Brownian motion to create volatility skew and volatility smile; combining the Heston model of a stochastic volatility with Geometric Brownian motion; Merton's jump model that creates jumps of asset price by using Poisson process to count the jumps. Now let's consider a relatively simple situation: combining the Geometric Brownian motion with a model of varying interest rate. **You can either form a group with no more than three people or work individually. Each group/individual can choose a model with a certain weight. Your score of the project will be multiplied by the weight of the model you choose.** Of course, you are encouraged to work out the Jackpot problem with the highest weight!

Notations:

- Initial asset price: $S(0) = 50$
- Asset price: $S(t)$
- Strike price: $K = \$50$
- Initial interest rate: $r(0) = 7\%$
- Interest rate: $r(t)$
- Volatility: $\sigma = 13\%$
- Maturity time: $T = 1$ year

Asset price model:

$$\mathrm{d}S(t) = r(t)S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}B(t), \tag{1}$$

or equivalently (by Ito's lemma),

$$S(t) = S(0)e^{\int_0^t r(s)\mathrm{d}s - \frac{\sigma^2}{2}t + \sigma B(t)}, \tag{2}$$

In simulation:

$$\ln(S(t_j)) = \ln(S(t_{j-1})) + \left(r(t_{j-1}) - \frac{\sigma^2}{2}\right)\Delta + \sigma\sqrt{\Delta}X_j,$$

or equivalently

$$S(t_j) = S(0)e^{\sum_{i=1}^{j} r(t_{i-1})\Delta - \frac{\sigma^2}{2}t_j + \sigma\sqrt{\Delta}(X_1+\cdots+X_j)},$$

$j = 1, ..., d,\ X_j \quad i.i.d \sim N(0, 1)$.

Interest rate models:

1. **Vasicek model (Weight: 1.05)**:

   Mathematical Formulation:

   $$\mathrm{d}r(t) = a(\mu_r - r(t))\mathrm{d}t + \sigma_r \mathrm{d}\tilde{B}(t),$$

   where μ_r is the long term mean level of the interest rate, a is the speed of reversion, σ_r is the instantaneous volatility of the interest rate, and $\tilde{B}(t)$ is a standard Brownian motion that is independent of the Brownian motion $B(t)$ we used in Equation (1).

   In Simulation:

   $$r(t_j) = r(t_{j-1}) + 0.18(0.086 - r(t_{j-1}))\Delta + 0.02\sqrt{\Delta}Z_j,$$

   where $Z_j \quad i.i.d \sim N(0, 1)$.

2. **Rendleman-Bartter model (Weight: 0.95)**:

   Mathematical Formulation: The short-term interest rate follows a Geometric Brownian motion:

   $$\mathrm{d}r(t) = \theta r(t)\mathrm{d}t + \sigma_r r(t)\mathrm{d}\tilde{B}(t).$$

   Or, equivalently,

   $$r(t) = r(0)e^{\left(\theta - \frac{\sigma_r^2}{2}\right)t + \sigma_r \tilde{B}(t)},$$

   where θ is the expected instantaneous rate of change in the interest rate, σ_r is the volatility of interest rate, and $\tilde{B}(t)$ is a standard Brownian motion that is independent of the Brownian motion $B(t)$ we used in Equation (1).

   In Simulation:

   $$r(t_j) = r(0)e^{\left(0.086 - \frac{0.02^2}{2}\right)t_j + 0.02\tilde{B}(t_j)},$$

   or equivalently,

   $$\ln(r(t_j)) = \ln(r(t_{j-1})) + \left(0.086 - \frac{0.02^2}{2}\right)\Delta + 0.02\sqrt{\Delta}Z_j,$$

   where $Z_j \quad i.i.d \sim N(0, 1)$.

3. **Cox-Ingersoll-Ross model (Weight: 1.15)**:

   Mathematical Formulation:

   $$\mathrm{d}r(t) = a(\mu_r - r(t))\mathrm{d}t + \sigma_r \sqrt{r(t)}\mathrm{d}\tilde{B}(t),$$

   which is very similar to Vasicek model, but ensures the interest rate is always positive.

   In Simulation:

   $$r(t_j) = r(t_{j-1}) + 0.23(0.081 - r(t_{j-1}))\Delta + 0.09\sqrt{r(t_{j-1})\Delta}Z_j,$$

   where $Z_j \quad i.i.d \sim N(0, 1)$.

Jackpot model [This is a name I made up.] (Weight: 1.25): In this model, we combine the Vasicek model with the Variance-Gamma model:

$$\ln(S(t_j)) = \ln(S(t_{j-1})) + \left(r(t_{j-1}) + \frac{\ln(1 - 0.15\sigma^2/2)}{0.15} \right) \Delta + \sigma\sqrt{Y_j} X_j,$$

$j = 1, ..., d$, $X_j \quad i.i.d \sim N(0, 1)$, $Y_j \quad i.i.d. \sim Gamma(\Delta/0.15, 0.15)$ (using the shape and scale parameter definition).
And,

$$r(t_j) = r(t_{j-1}) + 0.18(0.086 - r(t_{j-1}))\Delta + 0.02\sqrt{\Delta} Z_j,$$

where $Z_j \quad i.i.d \sim N(0, 1)$.

Solve the following problems:

1. Using a time step of 1 month $d = 12$, calculate the fair price of the European call option with a strike price of \$50 that matures in on year (= 12 months = 52 weeks), with an error tolerance of \$0.05.

2. Repeat the calculation with a time step of 1 week $d = 52$.

Some hints:

- To find the required simulation size, start with $n = 10^4$ for $d = 12$, and $n = 1000$ for $d = 52$.
- The payoff of a European call option depends only on $S(T)$, i.e. $\max(S(T) - K, 0)$.
- The discount factor to discount the payoffs can be estimated as $e^{-\Delta \sum_{j=1}^{d} r(t_j)}$.

Answer the following questions in your written report (not point-by-point, incorporate the answers into your report.):

- What is the sample size required?
- What is your estimated fair price of the European call option?
- Which estimated price is higher? Time step of 1 month or time step of 1 week?
- The payoff of the European option only depends on the price at the maturity time. Should the price of the option depend on the number of time steps used? Why?
- How do these prices compare to the price of the European call option under a Geometric Brownian motion model with $r = 0.07$ and $\sigma = 0.13$?
- Do you have an explanation for the above results?
- What is the computational time of your program? Can you make more efficient?
- Can you try to use some variance reduction method(s) here? [This is not required, but encouraged.]
- Any interesting result? Any doubt? Any suggestion? Additional comments?

Requirement:

Each group consists of no more than 3 people. Assume this is a project given by your client, and you should explain to the audiences the motivation of the project, the method, the tool you use and the conclusion you would like to show. Prepare a double-spaced written report, not to exceed three pages. One group submit one report only. The written report will be due on 6/10/2021, 9:30 PM on d2l.