Game Theory with Computer Science Applications Homework 3

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Problem 1.

We assume $W_t = \sum_{i=1}^{n} w_t(i)$, then we get:

$$
\begin{aligned}
W_{t+1} &= \sum_{i=1}^{n} w_t(i)\left(1 - \epsilon c_t(i)\right) \\
&= \sum_{i=1}^{n} w_t(i) - \epsilon \sum_{i=1}^{n} w_t(i) c_t(i) \\
&= W_t - \epsilon W_t \sum_{i=1}^{n} \frac{w_t(i) c_t(i)}{W_t} \\
&= W_t \left(1 - \epsilon \sum_{i=1}^{n} p_t(i) c_t(i)\right)
\end{aligned}
\tag{1}
$$

where $p_t(i) = \frac{w_t(i)}{W_t}$. We take the natural logarithm of the above equation to obtain:

$$
\begin{aligned}
\Rightarrow \quad \ln W_{t+1} = \ln W_t + \ln \left(1 - \epsilon \sum_{i=1}^{n} p_t(i) c_t(i) \right. &\leq \ln W_t - \epsilon \sum_{i=1}^{n} p_t(i) c_t(i) \\
\Rightarrow \quad \epsilon \sum_{i=1}^{n} p_t(i) c_t(i) &\leq \ln W_t - \ln W_{t+1} \\
\Rightarrow \quad \epsilon \sum_{t=1}^{T} \sum_{i=1}^{n} p_t(i) c_t(i) &\leq \ln n - \ln W_{T+1}
\end{aligned}
\tag{2}
$$

We assume $i^\star$ solves the $\min \sum_{t=1}^{T} c_t(i)$, then we have

$$
\begin{aligned}
W_{T+1} \geq w_{T+1}\left(i^\star\right) = \prod_{t=1}^{T} \left(1 - \epsilon c_t\left(i^\star\right)\right) &\geq (1-\epsilon)^{\sum_{t=1}^{T} c_t\left(i^\star\right)} \\
\ln W_{T+1} &\geq \left(\sum_{t=1}^{T} c_t\left(i^\star\right)\right) \ln(1-\epsilon)
\end{aligned}
\tag{3}
$$

Therefore, we proved that the new algorithm also has regret $O\left(\sqrt{T \ln n}\right)$.

Problem 2.

Firstly, we need to find the expected payoff for bidder i with valuation v_i when they bid $\mu_i(v_i) = cv_i + d$. Since v_j is uniformly distributed, we can write this probability that bidder i's bid is higher than the other bidder's bid as:

$$\Pr(\mu_i(v_i) > \mu_j(v_j)) = \Pr(cv_i + d > cv_j + d) = \Pr(v_i > v_j) = v_i \tag{4}$$

We consider the auxiliary function $u(r, v)$ which the expected value for a Player with value v if they decided to bid as if they had value r. The Player will only win if all other players bid less than $b(r)$. Since b is increasing, this is only true if all of the other players value the object less that r. Thus

$$u(r, v) = \Pr(v_j < r)(v - b(r)) \tag{5}$$

For a fixed v we can maximize $u(r, v)$ by setting $\frac{\partial u}{\partial r} = 0$. Of course, $b(r)$ is the optimal bid if we have value v. Thus, we would maximize $u(r, v)$ by setting $r = v$.

$$\begin{aligned} &\frac{\partial u}{\partial r} = (N-1) r^{N-2}(v - b(r)) - r^{N-1} b'(r) = 0 \\ \Rightarrow \quad &(N-1) r^{N-2} v = (N-1) r^{N-2} b(r) + r^{N-1} b'(r) \end{aligned} \tag{6}$$

Substituting $v = r$:

$$\begin{aligned} (N-1) v^{N-2} v &= (N-1) v^{N-2} b(r) + v^{N-1} b'(v) \\ (N-1) v^N &= \frac{\mathrm{d}}{\mathrm{d}v}\left[v^{N-1} b(v)\right] \\ v^{N-1} b(r) &= \int_0^v (N-1) x^N \mathrm{d}x \\ v^{N-1} b(v) &= \frac{N-1}{N} v^N \\ b(v) &= \frac{N-1}{N} v \end{aligned} \tag{7}$$

Thus,

$$b(v) = \frac{N-1}{N} v = \frac{2-1}{2} v = cv + d \tag{8}$$

Therefore, we get $c = \frac{1}{2}$, $d = 0$, $\mu_i(v_i) = \frac{1}{2} v_i$.

Problem 3.

Firstly, we let the potential function be:

$$\Phi(q_i, q_{-i}) = (P(Q) - c) \prod_{i=1}^{n} q_i \tag{9}$$

Then we have:

$$u_i(q_i, q_{-i}) = \frac{\Phi(q_i, q_{-i})}{\prod_{j=1, j \neq i}^{n} q_j} \tag{10}$$

Since q_{-i} is fixed, so

$$\Phi\left(q_i, q_{-i}\right)-\Phi\left(q_i^{\prime}, q_{-i}\right)>0 \Longleftrightarrow u_i\left(q_i, q_{-i}\right)-u_i\left(q_i^{\prime}, q_{-i}\right)>0 \tag{11}$$

Problem 4.

The problem of choosing w^t can be reduced to an approachability problem. The components of an approachability problem are the payoff function $r: \Delta_d \times\ [0,1]^d \rightarrow \mathbb{R}^d$ and a convex set $S \subset \mathbb{R}^d$, define these in terms of ℓ and w as follows:

$$\begin{gathered}
r(w, l)=\left\langle w \cdot \ell-\ell_1, w \cdot \ell-\ell_2, \ldots, w \cdot \ell-\ell_d\right\rangle \\
S=\mathbb{R}_{-}^d:=\left\{v \in \mathbb{R}^d: v_i \leq 0 \forall i\right\} s
\end{gathered} \tag{12}$$

From this definition the average payoff in the approachability problem is:

$$\frac{1}{T} \sum_{t=1}^T r\left(\ell^t, w^t\right)=\left\langle\frac{1}{T} \sum_{t=1}^T \ell^t \cdot w^t-\sum_{t=1}^T \ell_1^t, \ldots, \frac{1}{T} \sum_{t=1}^T \ell^t \cdot w^t-\sum_{t=1}^T \ell_d^t\right\rangle \tag{13}$$

Thus if we can make the average payoff in the approachability problem approach the set S, we get low regret in the online learning problem. To apply Blackwell's Approachability Theorem, we need to verify that $\forall q \exists p$ s.t $r(p, q) \in S$ holds.

By choosing

$$w(\ell)=e_{i^*}, \text { where } i_*=\arg \min_j \ell_j \tag{14}$$

we can get that,

$$r\left(e_{i^*}, \ell\right)=\left\langle\ell_{i^*}-\ell_1, \ldots, \ell_{i^*}-\ell_d\right\rangle \tag{15}$$

which is in $\mathbb{R}_{-}^d$ by definition. So we have proved that $\forall q \exists p$ s.t $r(p, q) \in S$ holds.

The intuition behind this result is that if we observe the loss vector in advance, of course we can beat the best expert. We just choose the minimizer ourselves.Thus, we can apply Blackwell's Approachability Theorem. To conclude, the online learning problem of minimizing the regret can be solved by solving the approachability problem defined by (12).

Problem 5.

All-pay Auction We set $m^A\left(v_i\right)$ as the expected payment of a fixed bidder over v_{-i}, and assuming $m^A(0)=0$. In equilibrium bidder i should bid $b_i\left(v_i\right)$. Suppose he bids some value $s(z)$. We can then derive $m^A\left(v_i\right)$ solving a differential equation as follows. Firsly we let $u^A\left(z, v_i\right)$ be the expected utility of bidder i if he bids $s(z)$ assuming his valuation is v_i. Since the item is allocated to the highest bidder, we can write:

$$\begin{aligned}
u^A\left(z, v_i\right) & =x \cdot \operatorname{Pr}[i \text { winswithbid} s(z)]-m^A(z) \\
& =x \cdot G(z)-m^A(z)
\end{aligned} \tag{16}$$

where G(z) is the c.d.f. and g(z) is the p.d.f. Since bidder i chooses z to maximize his expected utility, it must hold:

$$\frac{\partial}{\partial z} u^A (z, v_i) = xg(z) - \frac{d}{dz} m^A (z) = 0 \tag{17}$$

It is optimal to report $z = v_i$, we obtain:

$$v_i g(v_i) - \frac{d}{dv_i} m^A (v_i) = 0 \tag{18}$$

Finally by solving the differential equation:

$$\begin{aligned} m^A (v_i) &= m^A (0) + \int_0^{v_i} y g(y) \,\mathrm{d}y \\ &= 0 + \int_0^{v_i} y g(y) \,\mathrm{d}y \end{aligned} \tag{19}$$

For All-pay auction, every bidder pays his/her bid for an AP auction we have :

$$\begin{aligned} b_i (v_i) &= \int_0^{v_i} y g(y) \,\mathrm{d}y \\ &= \int_0^{v_i} y (n-1) y^{n-2} \mathrm{d}y = (n-1) \int_0^{v_i} y^{n-1} \mathrm{d}y \\ &= (n-1) \frac{{v_i}^n}{n} = \frac{n-1}{n} {v_i}^n \end{aligned} \tag{20}$$

Therefore, the expected revenue should be $\sum_{i=1}^n E\left(\frac{n-1}{n} {v_i}^n\right)$, that is:

$$\sum_{i=1}^n E\left(\frac{n-1}{n} {v_i}^n\right) = (n-1) \int_0^1 {v_i}^n \mathrm{d}v_i = \frac{n-1}{n+1} \tag{21}$$

Second Price Auction Let $X_1, \ldots, X_n$ denote the independent identically distributed random variables that define the value of the players. We use F and f to denote the cumulative distribution function (cdf) and the density function respectively, and $\frac{\mathrm{d}F}{\mathrm{d}x} = f(x)$. ${Y_k}^{(n)}$ is the random variable that desribes the k’ th largest value among $X_1, \ldots, X_n$.

$$\begin{aligned} & F_1(x) = cdf\left({Y_1}^{(n)}\right) \\ \Rightarrow \quad & F_1(x) = (F(x))^n \\ & F_2(x) = (F(x))^n + n(1 - F(x))(F(x))^{n-1} \\ & \qquad\;\; = x^n + n(1-x) x^{(n-1)} \end{aligned} \tag{22}$$

We have already seen that a dominant strategy in second price auction is to be truthfully bid the true value. Thus $b_i = v_i$, and the expected revenue is simply $E[Y_2(n)]$.

$$\begin{aligned} E[Y_2] &= \int_0^1 x \cdot F_2^{'}(x) \,\mathrm{d}x \\ &= \int_0^1 \left(n x^n + n(n-1) x^{n-1} - n^2 x^n\right) \mathrm{d}x \\ &= \frac{n}{n+1} + (n-1) - \frac{n^2}{n+1} \\ &= \frac{n-1}{n+1} \end{aligned} \tag{23}$$

Therefore, we have proved the revenue equivalence theorem between second-price auction and all-pay auction for N bidders.