Game Theory with Computer Science Applications

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Main content of this lecture

Revenue Optimal Mechanism

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Problem Setting

- 1 seller 1 item
- N buyer private valuation $\theta_i\ f_i$, where $\theta_i \in [0, \theta_{i,\max}]$, each θ_i is independent.

Mechanism Design

- Allocation

$$\pi_i\left(b_i, b_{-i}\right) \qquad s.t. \pi_i \geq 0 \qquad \sum_{i=1}^{N} \pi_i \leq 1$$

- Payment $q_i\left(b_i, b_{-i}\right)$

The expected allocation and payment:

$$\begin{aligned} \alpha_i\left(\theta_i\right) &= E_{\theta_{-i}}\left[\pi_i\left(\theta_i, \theta_{-i}\right)\left|\theta_i\right]\right. \\ m_i\left(\theta_i\right) &= E_{\theta_{-i}}\left[q_i\left(\theta_i, \theta_{-i}\right)\left|\theta_i\right]\right. \end{aligned}$$

- Payoff

$$\theta_i \alpha_i\left(\hat{\theta}_i\right) - m_i\left(\hat{\theta}_i\right)$$

$$\max \quad \sum_{i=1}^{N} E\left[m_i\left(\theta_i\right)\right]$$

We need to solve for $\{\pi_i\}$ and $\{q_i\}$, and then

$$\begin{aligned} & s.t. \qquad (IC)\ \ and\ \ (IR) \\ (IC): \quad & \theta_{\mathrm{i}} \alpha_{\mathrm{i}}\left(\theta_i\right) - m_i\left(\theta_i\right) \ \geq\ \theta_i \alpha_i\left(\hat{\theta}_i\right) - m_i\left(\hat{\theta}_i\right) \qquad \forall i, \theta_i, \hat{\theta}_i \\ (IR): \quad & \theta_{\mathrm{i}} \alpha_{\mathrm{i}}\left(\theta_i\right) - m_i\left(\theta_i\right) \ \geq\ 0 \qquad \forall i, \theta_i \end{aligned}$$

Translate the conditions of IC and IR into three equivalent constraints:

$$\begin{aligned} & (IC)\,(IR) \Longleftrightarrow \\ & (i) \quad m_i\left(\theta_i\right) = m_i\left(0\right) + \theta_i \alpha_i\left(\theta_i\right) - \int_0^{\theta_i} \alpha_i\left(\theta\right) \mathrm{d}\theta \\ & (ii) \quad \alpha_i\ is\ a\ non-decrea\sin g\ function \\ & (iii) \quad m_i\left(0\right) \leq 0 \end{aligned}$$

Now we can remove the conditions of IC and IR and solve the new optimization problem. Our objective for parameter optimization is:

$$E(m_i(\theta_i)) \; = \; m_i(0) \; + \; \int_0^{\theta_{i,\max}} \alpha_i(\theta_i)\left(\underbrace{\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}}_{\Delta\varphi_i(\theta_i)}\right)\mathrm{d}\theta_i$$

$$\downarrow$$

$$\int \pi_i(\theta_i,\theta_{-i})\underbrace{f_{-i}(\theta_{-i})}_{f_1(\theta_1)f_2(\theta_2)\cdots f_{i-1}(\theta_{i-1})f_{i+1}(\theta_{i+1})\cdots f_N(\theta_N)}\mathrm{d}\theta_{-i}$$

Therefore, our optimization objective becomes:

$$\Rightarrow \sum_{i=1}^{N} E\left[m_i\left(\theta_i\right)\right] = \sum_{i=1}^{N} m_i(0) + \sum_{i=1}^{N} \int \pi_i\left(\theta_i, \theta_{-i}\right)\varphi_i\left(\theta_i\right) f(\vec{\theta}) d\vec{\theta}$$

Optimal Allocation rule is

$$t_i\left(\theta_i, \theta_{-v}\right) = 1 \quad \text{iff} \quad \varphi_i\left(\theta_i\right) = \max_j \varphi_j\left(\theta_j\right) \quad \text{and} \quad \varphi_i\left(\theta_i\right) \geqslant 0$$

Payment rule is :

$$\begin{cases} q_i\left(\theta_i, \theta_{-i}\right) = \partial_i \pi_i\left(\theta_i, \theta_{-i}\right) - \int^{\alpha_i} \pi_i\left(\theta_i; \theta_{-i}\right) d\theta_i. \\ q_v\left(\theta_v, \theta_{-i}\right) = 0 \Rightarrow m_v(0) = 0 \end{cases}$$

Assumption φ_i is a strictly increasing function $\rightarrow$ regular
A sufficient condition for $\varphi_v\left(\theta_i\right)$ to be non-decreasing.

$$\frac{1-F_i\left(\theta_i\right)}{f_r\left(\theta_i\right)} \text{ is non} - \text{decrea}\sin\text{g}$$

$$\frac{f_r\left(\partial_i\right)}{1-F_i\left(\theta_i\right)} \nearrow \quad f(\theta) = \lambda e^{-\lambda\theta} \quad \theta \geqslant 0 \text{ for some } \lambda > 0.$$

Claim If φ_i is strictly increasing then (ii) is satisfied by our allocation rule

$$\begin{aligned}
&\text{if} \quad \pi_i\left(\theta_i, \theta_{-i}\right) > 0 \quad \text{then} \bar{a}_i\left(\hat{\theta}_i, \theta_{-i}\right) = 1 \quad \text{if} \quad \hat{\theta}_i > \theta_i \\
&\Rightarrow \quad \pi_v\left(\hat{\theta}_i, \partial_{-i}\right) \; \geqslant \; \pi_i\left(\theta_i, \theta_i\right) \\
&\text{if} \quad \pi_r\left(\theta_i, \theta_{-2}\right) = 0, \quad \text{then} \\
&\Rightarrow \quad \bar{a}_i\left(\hat{\theta}_i, \theta_{-1}\right) \geqslant \pi_i\left(\theta_i; \theta_{-i}\right) \quad \forall \hat{\theta}_i \geqslant \theta_{-i} \\
&\qquad \alpha_i\left(\theta_i\right) = \mathrm{E}_{\theta_{-\mathrm{i}}}\left[\pi_i\left(\theta_i, \theta_{-i}\right)\right] \quad \text{is} \quad \text{non} - \text{decrea}\sin\text{g}
\end{aligned}$$

Revenue

We take the maximum value among each phi and 0.

$$E\left[\max\left(\varphi_1\left(\theta_1\right)\varphi_2\left(\theta_2\right)\cdots\varphi_N\left(\theta_M\right),0\right)\right]$$

$$r_i\left(\theta_{-i}\right)=\left\{z_i:\varphi_i\left(z_i\right)\geqslant 0\text{and}\varphi_i\left(z_i\right)\geqslant\max_{j\neq i}\varphi_j\left(\theta_j\right)\right\}$$

r_i is the smallest valuation of the bidder i such that she is guaranteed to win the item
Under this situation, we can write the expressions for π_i and q as follows:

$$\pi_{ij}\left(\theta_i,\theta_{-i}\right)=\begin{cases}1 & \text{if}\theta_i>r_i\left(\theta_{-i}\right)\\ 0 & \text{if}\theta_i<r_i\left(\theta_{-i}\right)\end{cases}$$
$$\begin{aligned}q_i\left(\theta_i,\theta_{-\mathrm{i}}\right)&=\theta_i\pi_r\left(\theta_1,\theta_{-\mathrm{i}}\right)-\int_0^{\theta_i}\pi_i\left(\theta_v,\theta_{-i}\right)\mathrm{d}\theta\\&=\theta_iI_{\theta_i>r_i}-\int_0^{\theta_i}I_{\theta>r_i}\mathrm{d}\theta\end{aligned}$$

Next, we will discuss the situation case by case.
If $\theta_i>r_i\left(\theta_i\right)$ then.

$$q_i\left(\theta_i,\theta_{-i}\right)=\theta_i-\left(\theta r-r_i\right)=r_i\left(\theta_i\right)$$

If $\theta_i<r_i\left(\theta_i\right)$

$$q_i\left(\theta_i,\theta_{-i}\right)=0$$

Revenue - optimal Mechanism

Next, we will provide a re-explanation for Revenue-Optimal Mechanism.
(1) The bidder with the largest virtual valuation $\varphi_i\left(\theta_i\right)$ wins
If $\max\limits_j\varphi_j\left(\theta_j\right)\geqslant 0$
If $\max_j\varphi_j\left(\theta_j\right)<0$ the seller does not sell the item.
(2) The wihner (If there is one) pays a amount equal to the smallest value that would stiil win
Example If f_i are all identical

$$\varphi\left(\theta_i\right)=\theta_i-\frac{1-F\left(\theta_i\right)}{f\left(\theta_i\right)}$$

$$\begin{aligned}r_i\left(\theta_i\right)&=\min\left\{z_i:\varphi\left(z_i\right)\geqslant 0.\varphi\left(z_i\right)\geqslant\max_{j\neq i}\varphi\left(\theta_j\right)\right\}\quad\textit{sine non}-\textit{decreasing}\\&=\min\left\{z_i:z_i\geqslant\varphi^{-1}(0).\quad z_i\geqslant\max_{j\neq i}\theta_j\right\}\end{aligned}$$

This is like a second price auction with a reserve price of $\varphi^{-1}(0)$.
A: $\varphi\left(\theta_i\right)<0$
B: $\varphi\left(\theta_1\right)>0$, $\varphi\left(\theta_i\right)<0,i\neq 1$