SHANGHAI JIAO TONG UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING

Game Theory with Computer Science Applications

Homework 2

Due by April 23, 2024

Problem 1. [Cournot Competition] Consider two companies, say company 1 and company 2, which produce identical products. In the Cournot model of competition, companies decide the amount they produce and the market determines a price depending on the total amounts of the products available in the market. The price is higher if the amount of the product is smaller. Let $a_i (i = 1, 2) \in [0, \infty)$ denote the amount of the product produced by company i. Assume that producing one unit of the product costs each company \$1, and the sales price per unit of the product is determined as $[2 - (a_1 + a_2)]^+$. Thus, the payoffs of company 1 and company 2 are given by

$$u_1(a_1, a_2) = a_1[2 - (a_1 + a_2)]^+ - a_1$$

$$u_2(a_1, a_2) = a_2[2 - (a_1 + a_2)]^+ - a_2,$$

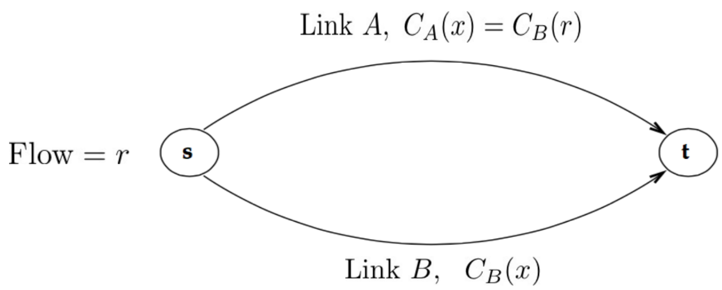
respectively. Fine a pure Nash Equilibrium for this game.

Problem 2. [Bertrand Competition] The Bertrand model is an alternative to the Cournot model of competition. In the Bertrand model, again we consider two companies only, but now each company sets a price and the demand for the product is a function of the lower of the two companies' prices. More precisely, each company i sets a price p_i for the product. The demand for the product is a function of the prices as follows: if company i sets its price lower than that of the other company, i.e., $p_i < p_{-i}$, the demand for the product of company i is given by $f(p_i)$ units, and the demand for the product of the other company is zero. If $p_i = p_{-i}$, then the demand is $f(p_i)/2$ for both companies. Let c_i be the cost for company i to product one unit of the product. Then, the payoff for company i is given by

$$u_i(p_i, p_{-i}) = \begin{cases} f(p_i)(p_i - c_i) & \text{if} \quad p_i < p_{-i}, \\ f(p_i)(p_i - c_i)/2 & \text{if} \quad p_i = p_{-i} \\ 0 & \text{otherwise,} \end{cases}$$

show that when $c_1 = c_2 = c$, $p_1 = p_2 = c$ is the unique NE.

Problem 3. Consider the following Pigou network: show that the price of anarchy (POA)



when $C_B(x)$ is of the form $ax^2 + bx + c, a, b, c > 0$ is upper bounded by $\frac{3\sqrt{3}}{3\sqrt{3}-2}$.

Problem 4. Consider a graph with a set of nodes V and a set of edges E. Let c_e denote the cost of using edge $e \in E$. This graph is accessed by a set of players, where each player chooses to occupy a set of edges. If there are f_e players occupying an edge e, the cost to each such player is $\frac{c_e}{f_e}$. Let R_i be the set of edges occupied by player i. Then its cost is given by $C_i(R_i, R_{-i}) = \sum_{e \in R_i} \frac{c_e}{f_e(R_i, R_{-i})}$. A Nash equilibrium for this problem is a set $(\hat{R}_1, \ldots, \hat{R}_n)$ if there are n players such that $C_i(R_i, \hat{R}_{-i}) \geq C_i(\hat{R}_i, \hat{R}_{-i}), \forall R_i$. Moreover, global optimal solution to this game is a set $(R_1^*, \ldots, R_n^*)$ such that $\sum_i C_i(R_i^*, R_{-i}^*) \leq \sum_i C_i(R_i, R_{-i}), \forall (R_i, R_{-i})$.

1. Show that $\sum_i C_i(R_i, R_{-i}) = \sum_{e: f_e(R_i, R_{-i}) \geq 1} c_e$.

2. Show that there exists a Nash equilibrium $(\hat{R}_1, \ldots, \hat{R}_n)$ such that

$$\sum_i C_i(\hat{R}_i, \hat{R}_{-i}) \leq \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) \sum_i C_i(R_i^*, R_{-i}^*)$$

Hint: Use the potential function used in the congestion game described in the class to classify the Nash equilibrium of this game.

Problem 5. (1) Use Rosen's theorem to prove the following result: consider a zero-sum game, where $U_2(a_1, a_2) = -U_1(a_1, a_2)$. Assume U_1 is strictly concave in a_1, and strictly convex in a_2. Then there exists a unique SP and the SP is in pure strategies (Note: SP= saddle point); (2) During the prove of Rosen's theorem, Why did we have to define the functions $L(v, a)$?