SHANGHAI JIAO TONG UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING

Game Theory with Computer Science Applications

Homework 3

Due by May 21, 2024

Problem 1. Suppose we modify the multiplicative weights algorithm for the best expert problem given in the lecture notes so that the weight update step is $\omega_{t+1}(i) := \omega_t(i)(1-\epsilon c_t(i))$ (rather than $\omega_{t+1}(i) := \omega_t(i)(1-\epsilon)^{c_t(i)}$). Show that this algorithm also has regret $O(\sqrt{T \ln n})$.

Problem 2. Consider a first-price auction with two bidders, whose valuations are i.i.d. with uniform distribution, i.e., $v_i \sim \text{Uniform}[0,1]$. Let $\mu_i(v_i)$ be the bid of bidder i when its valuation is v_i. Assume that the bidders use only affine μ_i, i.e., $\mu_i(v_i) = cv_i + d$. Find c, and d such that $\{\mu_i, i = 1,2\}$ form a Bayesian-Nash equilibrium for this game.

Problem 3. Consider the following Cournot competition with I firms. For each firm i, the strategy is to choose a quantity $q_i \in (0,\infty)$, and the payoff function is $u_i(q_i, q_{-i}) = q_i(P(Q)-c)$, where $P(Q)$ with $Q = \sum_{i=1}^{I} q_i$ denotes the inverse demand (price) function. Show that this game is an ordinal potential game. The definition of ordinal potential game is as follows: An ordinal potential game exists if there is a potential function $\Phi : S \to \mathbb{R}$ such that for all agents i with strategy s_i,

$$\Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) > 0 \text{ iff } u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) > 0.$$

Problem 4. Consider an online learning setting where loss vectors $\ell^1, \ell^2, \ldots \in [0,1]^d$ are observed. Prove that we could always choose weights $w^1, w^2, \ldots \in \Delta^d$ (probability simplex) so that

$$\forall \epsilon > 0,\ \exists T \text{ s.t. } \frac{1}{T}\left(\sum_{t=1}^{T} \ell^t \cdot w^t - \sum_{t=1}^{T} \ell_i^t\right) \le \epsilon, \quad \forall i.$$

(*Hint: Reduce the above problem to apply the Blackwell Approachability Theorem. Utilize the equivalence between the following two conditions: (1)* $\forall q \exists p$ *s.t.* $r(p,q) \in S$*; and (2) For all half-spaces* H *containing* S, $\exists p$ *s.t.* $\forall q, r(p,q) \in H$.)

Problem 5. Prove the revenue equivalence theorem between second-price auction and all-pay auction for N bidders with i.i.d. uniform distribution $v_i \sim \text{Uniform}[0,1]$ on single item. In all-pay auction, each bidder pay his/her bid, regardless of whether being allocated, and the bidder with the highest bid is allocated the item. (*Hint: First prove the equilibrium bidding function is* $b_i(v_i) = \frac{N-1}{N} v_i^N$.)