SHANGHAI JIAO TONG UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING

Game Theory with Computer Science Applications

Homework 1

Due by April 9, 2024

Problem 1. Show that the two-player game illustrated in the following has a unique equilibrium. (Hint: Show that it has a unique pure-strategy equilibrium; then show that player 1, say, cannot put positive weight on both U and M; then show that player 1, say, cannot put positive weight on both U and D, but not on M, for instance.)

$$\begin{pmatrix} & L & M & R \\ U & 1,-2, & -2,1 & 0,0 \\ M & -2,1 & 1,-2 & 0,0 \\ D & 0,0 & 0,0 & 1,1 \end{pmatrix}$$

Problem 2. Let X and Y be subsets of some vector space. Let $f(x, y)$ be a function from $X \times Y$ to $\mathbb{R}$. Show that

$$\sup_{x\in X} \inf_{y\in Y} f(x, y) \le \inf_{y\in Y} \sup_{x\in X} f(x, y).$$

Problem 3. Find a saddle point and the value of the following zero-sum game:

$$\begin{pmatrix} 4 & 3 & 1 & 4 \\ 2 & 5 & 6 & 3 \\ 1 & 0 & 7 & 0 \end{pmatrix}$$

Please show all the steps you used in obtaining the saddle point, such as the relevant LPs. If you used a computer program, please attach a copy of the program.

Problem 4. Repeat Problem 3 for the following matrix:

$$\begin{pmatrix} 0 & 5 & -2 \\ -3 & 0 & 4 \\ 6 & -4 & 0 \end{pmatrix}$$

Problem 5. Find all the NE of the following two-person nonzero-sum game

	b_1	b_2	b_3	b_4
a_1	$(-2,2)$	$(0,-4)$	$(11,-5)$	$(5,-6)$
a_2	$(-4,0)$	$(-1,-1)$	$(11,-2)$	$(4,-3)$
a_3	$(-5,3)$	$(-5,2)$	$(10,0)$	$(3,1)$
a_4	$(-6,2)$	$(-7,1)$	$(1,0)$	$(2,3)$

Problem 6. Consider the following nonzero game. Let (x^*, y^*) and $(\hat{x}, \hat{y})$ be two mixed strategy Nash equilibria of this game. Show that $(x^*, \hat{y})$ and $(\hat{x}, y^*)$ are also Nash equilibria. (Hint: Consider the sum of the payoffs of the two players.)

	L	R
U	$(4,-2)$	$(-3,5)$
D	$(10,-8)$	$(0,2)$

Problem 7. Prove Farka's Lemma. Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^{m\times 1}$. Then exactly one of the following two conditions holds:
(1) $\exists x \in \mathbb{R}^{n\times 1}$ such that $AX = b$, $x \geq 0$;
(2) $\exists y \in \mathbb{R}^{1\times m}$ such that $A^T y \geq 0$, $y^T b < 0$;

Problem 8. Prove Brouwer fixed-point theorem for one-dimensional case. Let $C \in \mathbb{R}$ be a convex, closed and bounded set. Let $f : C \to C$ be a continuous function. Then f has a fixed point, i.e., $\exists x \in C$ such that $x = f(x)$.