# Game Theory with Computer Science Applications Homework 2

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## Problem 1.

Fisrtly, if $2-(a_1+a_2)<0$, both companies cannot make profit, so they won't do that. Them we consider $2-(a_1+a_2)\geq 0$, To find the best response, we take the derivative with respect to $a_1$ and $a_2$ and set it to zero:

$$\begin{aligned} \frac{\partial u_1}{\partial a_1} &= 2-2\cdot a_1-a_2-1=0 \\ \frac{\partial u_2}{\partial a_2} &= 2-2\cdot a_2-a_1-1=0 \end{aligned} \tag{1}$$

In the Nash equilibrium, each company acts according to its best response function. Thus, in equilibrium, the production quantities of the two companies satisfy:

$$\begin{aligned} &\begin{cases} 1-2a_1-a_2=0 \\ 1-2a_2-a_1=0 \end{cases} \\ \Rightarrow \quad &\begin{cases} a_1=\frac{1}{3} \\ a_2=\frac{1}{3} \end{cases} \end{aligned} \tag{2}$$

Therefore, the Nash equilibrium is: $\left(\frac{1}{3},\frac{1}{3}\right)$.

## Problem 2.

To prove this is the unique Nash equilibrium, consider the following:

1. If one company sets a price higher than its competitor, its payoff is zero because the demand will go to the lower-priced company. To avoid this, the company must set its price below or equal to the competitor's price.

2. If one company sets a slightly lower price than its competitor, it gains all the demand. This creates an incentive to undercut the other company.

3. If both companies set the same price, they split the demand evenly, indicating there's no advantage to setting a higher or lower price.

Furthermore, if $p_i\geq c$, since $f\left(p_i\right)\left(p_i-c\right)\geq f\left(p_i\right)\left(p_i-c\right)/2\geq 0$, therefore, company $i$ will set a lower $p_i$ than it's competitor. But if $p_i<c$, then $f\left(p_i\right)\left(p_i-c\right)<0$, company $i$ will set $p_i=c$ to avoid loss.

Thus, the price $p_1$ should be equal to $p_2$, and $p_1 = p_2 = c$, the unique Nash Equilibrium is $(c, c)$.

## Problem 3.

Let's start by assuming that we have $X_A$ and $X_B$, which $X_A + X_B = r$. The average delay is :

$$X_A \cdot C_A(X_A) + X_B \cdot C_B(X_B) = (r - X_B) \cdot C_B(r) + X_B \cdot C_B(X_B) \tag{3}$$

The $C_B(x)$ is of the form $ax^2 + bx + c$, $a, b, c > 0$. When $x$ is greater than 0, $C_B$ is increasing. So $C_A(x) = C_B(r) \geq C_B(r - x)$, that means $(0, r)$ is **Wardrop Equilibrium(WE)**.

And the delay can be written as:

$$\begin{aligned} &(r - X_B) \cdot \left(ar^2 + br + c\right) + X_B \cdot \left(a{X_B}^2 + bX_B + c\right) \\ &= \left(ar^3 + br^2 + cr\right) + X_B \cdot \left(a{X_B}^2 + bX_B - ar^2 - br\right) \\ &= \left(r^3 - r^2 X_B + {X_B}^3\right) a + \left(r^2 - rX_B + {X_B}^2\right) b + rc \end{aligned} \tag{4}$$

Assume $x$ is the optimal solution to get PoA, then:

$$\text{PoA} = \frac{\text{optimal} \cos \text{t}}{\cos \text{tunderequilibrium}} = \frac{(ar^2 + br + c)\, r}{(r^3 - r^2 x + x^3)\, a + (r^2 - rx + x^2)\, b + rc} \tag{5}$$

**lemma 1** Assume $A, B, C, a, b, c$ and $x, y, z$ are all greater than 0, we proof that:

$$\frac{xa + yb + zc}{xA + yB + zC} \geq \min\left(\frac{a}{A}, \frac{b}{B}, \frac{c}{C}\right) \tag{6}$$

**Proof:**

$$\begin{aligned} &\textit{sine} \quad && xA \cdot \min\left(\frac{a}{A}, \frac{b}{B}, \frac{c}{C}\right) \leq xA \cdot \frac{a}{A} = xa \\ & && yB \cdot \min\left(\frac{a}{A}, \frac{b}{B}, \frac{c}{C}\right) \leq yB \cdot \frac{b}{B} = yb \\ & && zC \cdot \min\left(\frac{a}{A}, \frac{b}{B}, \frac{c}{C}\right) \leq zC \cdot \frac{c}{C} = zc \\ &\textit{thus} && xa + yb + zc \geq (xA + yB + zC) \min\left(\frac{a}{A}, \frac{b}{B}, \frac{c}{C}\right) \\ &\textit{sine} && A, B, C, a, b, c, x, y, z \ > \ 0 \quad \textit{we have} \\ & && \frac{xa + yb + zc}{xA + yB + zC} \geq \min\left(\frac{a}{A}, \frac{b}{B}, \frac{c}{C}\right) \end{aligned} \tag{7}$$

By **lemma 1** and $a, b, c > 0$ we have

$$\frac{1}{\text{PoA}} = \frac{(r^3 - r^2 x + x^3)\, a + (r^2 - rx + x^2)\, b + rc}{(ar^2 + br + c)\, r} \geq \min\left(\frac{r^3 - r^2 x + x^3}{r^3}, \frac{r^2 - rx + x^2}{r^2}, \frac{r}{r}\right) \tag{8}$$

From the question we know $x \in [0, r]$, next we find the minimum value of the three equations on the right side of the equation by derivation:

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{r^3-r^2x+x^3}{r^3}\right)=\frac{-r^2+3x^2}{r^3}=0 \quad \Rightarrow \quad r^2=3x^2 \Rightarrow \frac{r^3-r^2x+x^3}{r^3}\geq\frac{3\sqrt{3}-2}{3\sqrt{3}} \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{r^2-rx+x^2}{r^2}\right)=\frac{-r+2x}{r^2}=0 \quad \Rightarrow \quad r=2x \Rightarrow \frac{r^2-rx+x^2}{r^2}\geq\frac{3}{4}>\frac{3\sqrt{3}-2}{3\sqrt{3}} \\ \frac{r}{r}=1>\frac{3}{4}>\frac{3\sqrt{3}-2}{3\sqrt{3}}\end{gathered} \tag{9}$$

Therefore,

$$\frac{1}{\text{PoA}}\geq\frac{3\sqrt{3}-2}{3\sqrt{3}} \quad \Rightarrow \quad \text{PoA}\leq\frac{3\sqrt{3}-2}{3\sqrt{3}} \tag{10}$$

# Problem 4.

### 1.

From the question we know:

$$\sum_i C_i\left(R_i,R_{-i}\right)=\sum_i\sum_{e\in R_i}\frac{c_e}{f_e\left(R_i,R_{-i}\right)} \tag{11}$$

We swap the order of summation:

$$\sum_i\sum_{e\in R_i}\frac{c_e}{f_e\left(R_i,R_{-i}\right)}=\sum_{e:f_e(R_i,R_{-i})\geq 1}c_e\sum_{i:e\in R_i}\frac{1}{f_e\left(R_i,R_{-i}\right)}=\sum_{e:f_e(R_i,R_{-i})\geq 1}c_e \tag{12}$$

### 2.

Firstly we define the potential function :

$$\Phi(R)=\sum_{e:f_e(R)\geq 1}\sum_{k=1}^{f_e(R)}\frac{c_e}{k}=\sum_{e:f_e(R)\geq 1}c_e\sum_{k=1}^{f_e(R)}\frac{1}{k} \tag{13}$$

In Congestion Game, we have proved that $\Delta\Phi=\Delta U_i$ and the existence of the minimum of $\Phi$ mathmatically equals the existence of NE. Thus, there must exists a Nash Equilibrium $\hat{R}=\left(\hat{R}_1,\ldots,\hat{R}_n\right)$ satisfies $\Phi\left(\hat{R}\right)\leq\Phi\left(R^*\right)$. Then we have:

$$\sum_{e:f_e(\hat{R})\geq 1}c_e\sum_{k=1}^{f_e(\hat{R})}\frac{1}{k}\leq\sum_{e:f_e(R^\star)\geq 1}c_e\sum_{k=1}^{f_e(R^\star)}\frac{1}{k} \tag{14}$$

Since we know that $1\leq f_e(\hat{R})\leq n$, we can infer that

$$\sum_{e:f_e(\hat{R})\geq 1}c_e\leq\sum_{e:f_e(\hat{R})\geq 1}c_e\sum_{k=1}^{f_e(\hat{R})}\frac{1}{k}\leq\sum_{e:f_e(R^\star)\geq 1}c_e\sum_{k=1}^{f_e(R^\star)}\frac{1}{k}\leq\sum_{e:f_e(R^\star)\geq 1}c_e\sum_{k=1}^{n}\frac{1}{k} \tag{15}$$

And now we swap the order of summation, we can get:

$$\sum_{e:f_e(\hat{R})\geq 1}c_e\leq\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\sum_{e:f_e(R^\star)\geq 1}c_e \tag{16}$$

By using the conclusion we have proved in 1. we can prove that:

$$\sum_i C_i\left(\hat{R}_i, \hat{R}_{-i}\right) = \sum_{e:f_e(\hat{R})\geq 1} c_e \leq \left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\sum_i C_i\left(R_i^\star, R_{-i}^\star\right) \tag{17}$$

# Problem 5.

## (1)

Using Rosen's theorem, because $U_1\left(a_1, a_2\right)$ is strictly concave in $a_1$, there must exists a pure strategy Nash Equilibrium. First we assume that there are two different NE, which are $\left(a_1^*, a_2^*\right)$ and $\left(\hat{a}_1, \hat{a}_2\right)$, In the last homework we proved that in the zero-sum game of two players, the $U_1\left(a_1^\star, \hat{a}_2\right), U_1\left(\hat{a}_1, a_2^\star\right)$ are also NE, So we have:

$$U_1\left(a_1^\star, a_2^\star\right) = U_1\left(\hat{a}_1, a_2^\star\right) \tag{18}$$

Due to the concavity of the $U_1\left(a_1, a_2\right)$, we can get

$$U_1\left(\frac{a_1^\star+\hat{a}_1}{2}, a_2^\star\right) > \frac{U_1\left(a_1^\star, a_2^\star\right)+U_1\left(\hat{a}_1, a_2^\star\right)}{2} = U_1\left(a_1^\star, a_2^\star\right) \tag{19}$$

This contradicts the conclusion that $\left(a_1^\star, a_2^\star\right)$ is NE. Therefore, the assumption that there are two different NEs is wrong. There is only one NE.

For any saddle point $\left(a_1^\star, a_2^\star\right)$, we have

$$\begin{gathered} U_1\left(a_1^\star, a_2^\star\right) \geq U_1\left(a_1, a_2^\star\right), \forall a_1 \\ U_1\left(a_1^\star, a_2^\star\right) \leq U_1\left(a_1^\star, a_2\right) \Rightarrow U_2\left(a_1^\star, a_2^\star\right) \geq U_2\left(a_1^\star, a_2\right), \forall a_2 \end{gathered} \tag{20}$$

Therefore, SP is a NE, and because NE is unique, we prove that SP is also unique. Therefore, there exists a unique SP, and it is in pure strategy.

## (2)

The existence of a Nash equilibrium is equivalent to the existence of a maximum value for $L(v, a)$, as defined earlier. This can be proven using Kakutani's fixed-point theorem, which relies on the convexity of $U_1(a_1, a_2)$.