Game Theory with Computer Science Applications

Lecturer: Zhenzhe Zheng
Scribe: Junhao Dai

Lecture 4
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Main content of this lecture

- Review of NE DSE and CE
- LP Duality
- Zero Sum Game

1 review

NE

$$\begin{aligned} &P^* \ = \ (P_1^*, P_2^*, \cdots \ P_n^*) \ is \ a \ NE \\ &if \ U_i\left(P_i, P_{-i}^*\right) \ \leqslant \ U_\mathrm{i}\left(P_i^*, P_{-i}^*\right) \ \forall \ i, P_i \end{aligned} \tag{1}$$

DSE

$$\begin{aligned} &P^* \ = \ (P_1^*, P_2^*, \cdots \ P_n^*) \ is \ a \ DSE \\ &if \ U_i\left(P_i, P_{-i}^*\right) \ \leqslant \ U_\mathrm{i}\left(P_i^*, P_{-i}\right) \ \forall \ i, P_i, P_{-i} \end{aligned} \tag{2}$$

stronger than NE, may not exist

CE

$$\begin{aligned} &P^*\left(x\right) \quad x \in X_1 \times X_2 \times \cdots \times X_n \quad x = (x_1, x_2, \cdots, x_n) \\ &\sum_{X_{-i}} P^*\left(x_{-i}|x_i\right) U_i\left(x_i^{'}, x_{-i}\right) \ \leqslant \ \sum_{X_{-i}} P^*\left(x_{-i}|x_i\right) U_i\left(x_i, x_{-i}\right) \forall \ x_i, x_i^{'} \end{aligned} \tag{3}$$

2 LP Duality

2.1 Primal Problem

$$\begin{aligned} \max_x \quad & C^T X \\ s.t \quad & AX \leqslant b \\ & X \geqslant 0 \end{aligned} \tag{4}$$

Interpretation

$$\begin{aligned} &x_i \quad amount \ of \ product \ i \ produced \\ &b_i \quad amount \ of \ raw \ material \ of \ type \ i \ available \\ &a_{ij} \quad amount \ of \ raw \ material \ used \ to \ produced \ 1 unit \ of \ j \\ &c_j \quad profit \ from \ 1 unit \ of \ product \ j \end{aligned} \tag{5}$$

2.2 Dual Problem

$$\begin{aligned} \min_{y} \quad & b^T y \\ s.t \quad & A^T y \geqslant C \\ & y \geqslant 0 \end{aligned} \tag{6}$$

$$b^T y^* = C^T X^* \tag{7}$$

y^* and X^* are the solutions to the optimization problem

2.3 Strong Duality of LPs:

If the primal and dual are **feasible**, then both have the same optimal objective value.

Explanation of feasible:

There exists at least one X that satisfies:

$$\begin{aligned} AX &\leqslant b \\ X &\geqslant 0 \end{aligned} \tag{8}$$

There exists at least one y that satisfies:

$$\begin{aligned} A^T y &\geqslant C \\ y &\geqslant 0 \end{aligned} \tag{9}$$

(otherwise seller does not sell)

2.4 Farkas' lemma: (Theorem of alternative)

Exactly one of the following statements is true.

$$\exists x \geqslant 0 \quad s.t\ Ax = b \tag{i}$$

$$\exists y \quad s.t\ y^T A \geqslant 0\ and\ y^T b < 0 \tag{ii}$$

We don't prove this theorem mathematically, but we can explain it intuitively.

Geometric Interpretation of Farkas'

Let a_1, a_2 be the columns of A

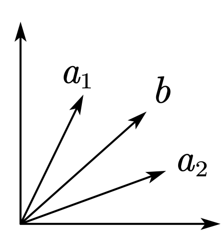

Figure 2: b can be linearly represented by a_1 and a_2

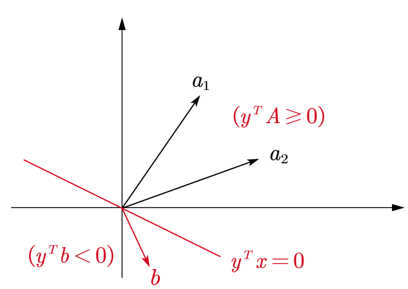

Figure 3: b is not in a half-plane and cannot be expressed

2.5 Proof of Duality

$$\begin{gathered} C^T X \leqslant b^T y \\ C^T X = X^T C \ \leqslant \ X^T A^T y \ \leqslant \ b^T y \\ (sine\ A^T y \ \geqslant \ C \quad X \geqslant 0) \\ (sine\ AX \ \geqslant \ b \quad y \geqslant 0) \end{gathered} \tag{10}$$

Now we will provw that:

$$C^T X^* \geqslant b^T y^* \ for\ optimal\ X^*\ and\ optimal\ y^* \tag{11}$$

Let $C^T X^* \ = \ \Delta$ this implies that

$$\nexists X \qquad s.t \begin{cases} C^T X \geqslant \Delta + \varepsilon \quad (for\ any\ \varepsilon > 0) \\ AX \leqslant b \\ X \geqslant 0 \end{cases} \tag{12}$$

Changing the form we get

$$\begin{cases} C^T X + \ \alpha_0 \ = \ -\Delta - \varepsilon \quad (for\ any\ \varepsilon > 0) \\ AX + \alpha \ = \ b \\ \alpha_0 \alpha X \geqslant 0 \end{cases} \tag{13}$$

$$or \qquad \begin{pmatrix} l - C^T & 1 & 0 \\ A & 0 & I \end{pmatrix} \begin{pmatrix} X \\ \alpha_0 \\ \alpha \end{pmatrix} = \begin{pmatrix} -\Delta - \varepsilon \\ b \end{pmatrix} \tag{14}$$

By Farkas' lemma, $\exists \lambda_0$, λ_1

$$\begin{cases} -C^T \lambda_0 + {\lambda_1}^T A \geqslant 0 \\ \lambda_0 \geqslant 0 \\ \lambda_1 \geqslant 0 \end{cases} \qquad and \quad -(\Delta + \varepsilon)\,\lambda_0 + {\lambda_1}^T b < 0 \tag{15}$$

From the above inequality we can get:

$$\Rightarrow \begin{cases} \frac{{\lambda_1}^T}{\lambda_0} A \geqslant C^T \\ \frac{{\lambda_1}^T}{\lambda_0} \geqslant 0 \\ \frac{{\lambda_1}^T b}{\lambda_0} < \Delta + \varepsilon \end{cases} \tag{16}$$

Let $y = \frac{\lambda_1}{\lambda_0}$

$$s.t \quad \begin{cases} y^T b < \Delta + \varepsilon \\ y^T A \geqslant C^T \\ y \geqslant 0 \end{cases} \tag{17}$$

Thus for each $\varepsilon > 0$, $\exists$ dual fessible y

$$s.t \quad {y^*}^T b \leqslant y^T b < \Delta + \varepsilon \Rightarrow \quad {y^*}^T b \leqslant \Delta = C^T X^* \tag{18}$$

2.6 The Upper Bound and Lower Bound of Primal Problem

$$\begin{aligned} \max_x \quad & C^T X \\ s.t \quad & AX \leqslant b \\ & X \geqslant 0 \end{aligned} \tag{19}$$

It can always be written as:

$$\begin{aligned} \max_x \quad & C^T X \\ s.t \quad & {a_1}^T X \leqslant b_1 \\ & {a_2}^T X \leqslant b_2 \\ & \vdots \\ & {a_n}^T X \leqslant b_n \end{aligned} \tag{20}$$

Suppose we want to get an upper bound of the optimal objective of the primal LP

Let $y_1, y_2, \cdots, y_n \geqslant 0$

Then

$$C^T X \leqslant \left(\sum_i y_i {a_i}^T \right) X \leqslant \sum_i b_i y_i \tag{21}$$

If

$$C^T \leqslant \left(\sum_i y_i {a_i}^T \right) \tag{22}$$

Then $\sum_i b_i y_i$ gives the upper bound.
Thus the highest upper bound is obtained by sovling

$$\begin{aligned} \min \quad & b^T y \\ s.t \quad & \sum_i a_i y_i \geqslant C \ \ \left(or \ A^T y \geqslant C \right) \\ & y \geqslant 0 \end{aligned} \tag{23}$$

This is the dual LP

3 Zero Sum Game

$A(i,j)$	payoff to P_1 when i is $P_1's$ action
$A(i,j)$	payoff to P_2 when j is $P_2's$ action
x	prob distribution over $P_1's$ action
y	prob distribution over $P_2's$ action

Therefore, for P_1 the utility is:

$$U_1(x,y) = \sum_{i,j} A(i,j)\, x_i y_i = x^T A y \tag{24}$$

because of zero sum, we get

$$U_2(x,y) = -U_1(x,y) \tag{25}$$

Thus a NE (x^*, y^*) satisfies

$$\begin{aligned} x^{*T}Ay^* &\geqslant x^TAy^* \quad \forall x \\ and \quad x^{*T}Ay^* &\geqslant x^{*T}Ay \quad \forall y \end{aligned} \tag{26}$$

that means (x^*, y^*) is a saddle point

3.1 Minimax Theorem

① $\min_y \max_x x^TAy = \max_x \min_y x^TAy \quad (This\ is\ called\ the\ value\ of\ the\ game)$

② $x^*\ which\ solves\ \max_x \left(\min_y x^TAy\right) \cdots (A) \quad and \quad y^*\ which\ solves\ \min_y \left(\max_x x^TAy\right) \cdots (B)$

(x^*, y^*) is a NE, $x^{*T}Ay^*$ is the value of the game.

③ $If\ (x^*, y^*)\ is\ a\ NE\ then\ x^{*T}Ay^*\ is\ the\ value\ of\ the\ game.$

$$x^*\ solves\ (A) \qquad y^*\ solves\ (B)$$

④ $Suppose\ P_1\ wants\ to\ maxmize\ its\ worst\ case\ payoff$

$$\max_x \min_y x^TAy \left(sine\ y \geqslant 0 \sum_i y_i = 1\right) = \max_x \min_y \left(x^TA\right)_j$$

3.1.1 Proof of 4

④ $Suppose\ P_1\ wants\ to\ maxmize\ its\ worst\ case\ payoff$

$$\max_x \min_y x^TAy \left(sine\ y \geqslant 0 \sum_i y_i = 1\right) = \max_x \min_y \left(x^TA\right)_j$$

For the LP1 problem, we will transform its form into

$$\begin{aligned} \Rightarrow \qquad & \max_{x,V_1} \quad V_1 \\ s.t \quad & V_1 \leqslant \left(x^TA\right)_j \quad \forall j \\ & x \geqslant 0 \end{aligned}$$

similarly for $P_2's$ problem is

$$\begin{aligned} \Rightarrow \qquad & \min_{y,V_2} \quad V_2 \\ s.t \quad & V_2 \geqslant (Ay)_i \quad \forall i \\ & y \geqslant 0 \end{aligned}$$

Thus by strong duality we have

$$\begin{aligned} & {V_1}^* = {V_2}^* \\ or \qquad & \max_x \min_y \left(x^T A y\right) \; = \; \min_y \max_x \left(x^T A y\right) \end{aligned} \tag{27}$$

3.1.2 Proof of 2

2 $\quad x^*$ *which solves* $\max_x \left(\min_y x^T A y\right) \cdots$ A $\quad$ *and* $\quad y^*$ *which solves* $\min_y \left(\max_x x^T A y\right) \cdots$ B

(x^*, y^*) is a NE, ${x^*}^T A y^*$ is the value of the game.

Let x^* be a solution to $LP1$, from the constraints

$$\begin{aligned} & {V_1}^* \leqslant \left({x^*}^T A\right)_j \quad \forall j \\ \Rightarrow \qquad & \sum_j {V_1}^* y_j^* \leqslant \sum_j \left({x^*}^T A\right)_j {y_j}^* \quad \left(sine \quad \sum_j y_j = 1 \right) \\ \Rightarrow \qquad & {V_1}^* \leqslant {x^*}^T A y^* \\ & where\ y^*\ is\ a\ solution\ to\ LP \end{aligned} \tag{28}$$

similarly we can show that

$$ {V_2}^* \geqslant {x^*}^T A y^* \tag{29}$$

because of ${V_1}^* = {V_2}^*$, we have

$$\begin{aligned} {x^*}^T A y^* \; &= \; \max_x \min_y x^T A y \\ &= \min_y \max_x x^T A y \end{aligned} \tag{30}$$

proof of NE

According to the above proof and conclusion we have:

$$\begin{aligned} \min_y {x^*}^T A y \; = \; \max_x \min_y x^T A y \; = \; \min_y \max_x x^T A y \; = \; \max_x x^T A y^* \; \geqslant \; {x^*}^T A y^* \\ \Rightarrow \qquad \min_y {x^*}^T A y \; \geqslant \; {x^*}^T A y^* \quad \Rightarrow \quad y^*\ solves\ \min_y {x^*}^T A y \end{aligned} \tag{31}$$

similarly we can show that x^* *solves* $\max_x x^T A y^*$
Thus (x^*, y^*) is a NE

3.1.3 Proof of 3

③ *If* (x^*, y^*) *is a NE then* ${x^*}^T A y^*$ *is the value of the game.*

$$x^*\ \textit{solves}\ \text{Ⓐ} \qquad y^*\ \textit{solves}\ \text{Ⓑ}$$

Because (x^*, y^*) *is a NE*, it can be obtained from the definition that

$$\begin{aligned} {x^*}^T A y^* \ &= \ \min_y {x^*}^T A y \ \leqslant \ \max_x \min_y x^T A y \\ \textit{similarly}\ {x^*}^T A y^* \ &= \ \max_x x^T A y^* \ \geqslant \ \min_y \max_x x^T A y \\ \Rightarrow \quad & \min\max \leqslant \ \max\min \\ \textit{and we know}\ \min_y \max_x f(x,y) \ &\geqslant \ \max_x \min_y f(x,y) \end{aligned} \tag{32}$$

Therefore we are proved

$$\min\max = \ \max\min$$

According to the above proof and conclusion

$$\begin{aligned} \min_y {x^*}^T A y \ = \ {x^*}^T A y^* \ &= \ \max_x \min_y x^T A y \ \geqslant \ \min_y x^T A y \\ \Rightarrow \qquad & x^*\ \textit{maxmizes}\ \min_y \left(x^T A y\right) \cdots \text{Ⓐ} \\ & y^*\ \textit{minimizes}\ \max_x \left(x^T A y\right) \cdots \text{Ⓑ} \end{aligned} \tag{33}$$