18.01.2022г. Контролно 2 по СЕМ-СИ

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Вариант 1

Зад. 1 Хвърляме 3 зара

X - е броя на четните в/у трите

Y - е броя на 5-иците в/у трите

а) съвм. разпр. на X и Y = ?

б) разпр. на $Z = \max\{X, Y\}$ = ? $E(Z \mid Y = 1) = ?$

Р-е: а) Нека $p_{k,l} := \mathbb{P}(X = k, Y = l)$

$q_k := \mathbb{P}(X = k)$

$r_l := \mathbb{P}(Y = l)$

$s_m := \mathbb{P}(Z = m)$

Тогава $X(\Omega) = \{0, 1, 2, 3\}$

$Y(\Omega) = \{0, 1, 2, 3\}$

$w = (a, b, c) \in \Omega$

$\Omega = V(6,1) \times V(6,1) \times V(6,1) \Rightarrow |\Omega| = |V(6,1)| \cdot |V(6,1)| \cdot |V(6,1)| = 6^3 = 216$

$\underbrace{\begin{matrix} \cancel{1} & \cancel{1} & \cancel{1} \\ 3 & 3 & 3 \end{matrix}}_{8н.}$ $p_{0,0} = \frac{8}{6^3} = \frac{8}{216}$

$p_{0,3} = \frac{1}{216}$ ← $\begin{matrix} 5 & 5 & 5 \end{matrix}$

$3 \times \underbrace{\begin{pmatrix} 5 & \begin{matrix}1\\3\end{matrix} & \begin{matrix}1\\3\end{matrix} \end{pmatrix}}_{4н.}$ $p_{0,1} = \frac{3 \cdot 4}{6^3} = \frac{12}{216}$

$p_{1,0} = \frac{36}{216}$ $3 \times \underbrace{\begin{pmatrix} \begin{matrix}2\\4\\6\end{matrix} & \begin{matrix}1\\3\end{matrix} & \begin{matrix}1\\3\end{matrix} \end{pmatrix}}_{3.2.2}$

$p_{1,1} = \frac{3 \cdot 2 \cdot (3 \cdot 2)}{216} = \frac{36}{216}$ ← $3 \times \begin{pmatrix} 5 & \begin{matrix}2\\4\\6\end{matrix} & \begin{matrix}1\\3\end{matrix} \end{pmatrix}$

$p_{1,2} = 9/216$ ← $3 \times \begin{pmatrix} 5 & 5 & \begin{matrix}2\\4\\6\end{matrix} \end{pmatrix}$

$3 \times \underbrace{\begin{pmatrix} 5 & 5 & \begin{matrix}1\\3\end{matrix} \end{pmatrix}}_{2н.}$ $p_{0,2} = \frac{3 \cdot 2}{6^3} = \frac{6}{216}$

$p_{1,3} = \{$и 3-те зара да са 5-ици, че да няма 1 четно$\} = 0$

$$P_{3,0} = \frac{3.3.3}{216} = \frac{27}{216} \quad \left\{ \begin{array}{ccc} 2 & 2 & 2 \\ 4 & 4 & 4 \\ 6 & 6 & 6 \end{array} \right.$$

$$3\times\underbrace{\begin{pmatrix} 2 & 2 & 1 \\ 4 & 4 & 3 \\ 6 & 6 & \end{pmatrix}}_{3.3.2} \quad \boxed{P_{2,0} = \frac{3.18}{216} = \frac{54}{216}}$$

$P_{3,1} = \{3$ четни и 1 5-ица$\} = 0$

$$3\times\begin{pmatrix} 2 & 2 & \\ 4 & 4 & 5 \\ 6 & 6 & \end{pmatrix} \quad \begin{matrix} 9\text{н.} & 1\text{н.} \end{matrix} \quad P_{2,1} = \frac{3.9}{216} = \frac{27}{216}$$

$P_{3,2} = \{3$ четни и 2 5-ици$\} = 0$

$P_{2,2} = \{$да има 2 5-ици от 3-те и едновременно 2 четни$\} = 0$

$P_{2,3} = \{$всички да са 5-ици и едновременно да има 2-четни$\} = 0$

$P_{3,3} = \{3$ четни и 3 5-ици$\} = 0$

Съставяме следната таблица:

x \ y	0	1	2	3	Σ
0	$\frac{8}{216}$	$\frac{12}{216}$	$\frac{6}{216}$	$\frac{1}{216}$	$\frac{27}{216}$
1	$\frac{36}{216}$	$\frac{36}{216}$	$\frac{9}{216}$	0	$\frac{81}{216}$
2	$\frac{54}{216}$	$\frac{27}{216}$	0	0	$\frac{81}{216}$
3	$\frac{27}{216}$	0	0	0	$\frac{27}{216}$
Σ	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$	1

получаваме съвместното разпределение на X и Y

правилно сме пресметнали

б) Сега нека пресметнем разпр. на $Z = \max\{X, Y\}$:

за $Z = 0 \Rightarrow S_0 = P_{0,0} = 8/216$

за $Z = 1 \Rightarrow S_1 = P_{0,1} + P_{1,0} + P_{1,1} = \frac{12+36+36}{216} = \frac{84}{216}$

за $Z = 2 \Rightarrow S_2 = P_{0,2} + P_{2,0} + P_{1,2} + P_{2,1} + P_{2,2} =$
$= \frac{6+54+9+27+0}{216} = \frac{96}{216}$

за $Z = 3 \Rightarrow S_3 = P_{0,3} + P_{1,3} + P_{2,3} + P_{3,3} + P_{3,0} + P_{3,1} + P_{3,2} =$
$= \frac{1+0+0+0+27+0+0}{216} = \frac{28}{216}$

$$E(Z \mid Y=1) = ?$$

$$E(Z \mid Y=1) = \sum_{m=0}^{3} m\,\mathbb{P}(Z=m \mid Y=1) =$$

$$= 0 + \frac{1 \cdot \mathbb{P}(Z=1 \mid Y=1)}{\mathbb{P}(Y=1)} + \frac{2 \cdot \mathbb{P}(Z=2 \mid Y=1)}{\mathbb{P}(Y=1)} + \frac{3 \cdot \mathbb{P}(Z=3 \mid Y=1)}{\mathbb{P}(Y=1)} =$$

$$= 0 + \frac{p_{0,1} + p_{1,1}}{r_1} + \frac{2 \cdot p_{2,1}}{r_1} + \frac{3 \cdot p_{3,1}}{r_1} =$$

$$= 0 + \frac{\frac{12}{\cancel{216}} + \frac{36}{\cancel{216}}}{\frac{75}{\cancel{216}}} + \frac{2 \cdot \frac{27}{\cancel{216}}}{\frac{75}{\cancel{216}}} + \frac{3 \cdot 0}{\frac{75}{216}} = \frac{48}{75} + \frac{54}{75} = \boxed{\frac{102}{75} = 1{,}36}$$

Зад. 2 $Z=(X,Y)$ с $f(x,y)=\begin{cases} c(x+y)^2 & \text{при } 0<x<y<1 \\ 0 & \text{иначе} \end{cases}$

а) $c=?$

б) $f_{X+Y}=?$ $E(Y \mid X=\frac{1}{2})=?$

Р-е: а) Нека $D=\{(x,y)\in\mathbb{R}^2 \mid 0<x<y<1\}$

От свойството $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy=1$ намираме $\underline{\underline{c}}$:

$$1=\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy=\iint_D c(x+y)^2\,dx\,dy=$$

$$=c\int_0^1\left(\int_x^1 (x+y)^2\,dy\right)dx=c\int_0^1\left(\frac{(x+y)^3}{3}\right)\Bigg|_{y=x}^{1}dx=$$

$$=c\int_0^1\left(\frac{1-7x^3}{3}+x^2+x\right)dx=\frac{7}{12}\cdot c \Rightarrow c=\frac{12}{7}$$

б) Нека положим $\underset{\in(0,2)}{Z_1=X+Y}$ и $\underset{\in(0,1)}{Z_2=X}$

$\Rightarrow X=Z_2$

$Y=Z_1-X=Z_1-Z_2$

$$J=\begin{vmatrix} \frac{\partial X(Z_1,Z_2)}{\partial Z_1} & \frac{\partial X(Z_1,Z_2)}{\partial Z_2} \\ \frac{\partial Y(Z_1,Z_2)}{\partial Z_1} & \frac{\partial Y(Z_1,Z_2)}{\partial Z_2} \end{vmatrix}=\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}=1$$

От теоремата:

$$f_{U,V}(u,v)=\begin{cases} f_{X,Y}(x(u,v),y(u,v))\,|J_{\varphi^{-1}}(u,v)| & \text{за } (u,v)\in\varphi(D) \\ 0 & \text{иначе} \end{cases}$$

$$\Rightarrow f_{Z_1,Z_2}(Z_1,Z_2)=f_{X,Y}(Z_2,Z_1-Z_2)\cdot|J|=$$

$$=\frac{12}{7}\cdot(\cancel{Z_2}+Z_1-\cancel{Z_2})^2\cdot 1=\frac{12}{7}Z_1^2$$

$$f_{Z_1}(Z_1)=\int_{\mathbb{R}}\frac{12}{7}Z_1^2\,dZ_2$$

$$Z_2<Z_1-Z_2<1 \Rightarrow 2Z_2<Z_1<1+Z_2 \Rightarrow \left|\begin{array}{l} Z_2<\frac{Z_1}{2} \\ Z_2>Z_1-1 \end{array}\right.$$

$$\Rightarrow \begin{cases} z_2 < z_1/2 \\ z_2 > z_1 - 1 \\ z_2 \in (0,1) \\ z_1 \in (0,2) \end{cases}$$

I сл. за $z_1 \in [1;2)$: $z_1 - 1 < z_2 < z_1/2$

II сл. за $z_1 \in (0,1)$: $\underbrace{0 < z_2 < z_1/2}_{\text{границите}}$

I сл. $z_1 \in [1;2)$

$$f_{z_1}(z_1) = \int_{z_1-1}^{z_1/2} 12/7\, z_1^2\, dz_2 = \frac{12}{7} z_1^2 z_2 \Big|_{z_2 = z_1 - 1}^{z_2 = \frac{z_1}{2}} =$$

$$= \frac{12}{7} z_1^2 \left(\frac{z_1}{2} - z_1 + 1\right) = \frac{12}{7} z_1^2 \left(1 - \frac{z_1}{2}\right)$$

II сл. $z_1 \in (0,1)$

$$f_{z_1}(z_1) = \int_0^{z_1/2} 12/7\, z_1^2\, dz_2 = 12/7\, z_1^2 z_2 \Big|_{z_2=0}^{z_2 = z_1/2} =$$

$$= \frac{12}{7} z_1^2 \frac{z_1}{2} = \frac{12}{7} \frac{z^3}{2} = \frac{6z_1^3}{7}$$

Получаваме:

$$f_{z_1}(t) = f_{x+y}(t) = \begin{cases} 0 & \text{за } t \le 0 \\ 6t^3/7 & \text{за } t \in (0;1) \\ 12/7 \cdot t^2 \left(1 - \frac{t}{2}\right) & \text{за } t \in [1;2) \\ 0 & \text{за } t \ge 2 \end{cases}$$

Сега нека пресметнем $E\left(y \mid x = \frac{1}{2}\right)$:

$$f_x(x) = \int_{\mathbb{R}} f_{x,y}(x,v)\,dv = \int_x^1 \frac{12}{7}(x+v)^2\,dv = \frac{12}{7}\int_x^1 (x^2 + 2xv + v^2)\,dv =$$

$$= \frac{12}{7}\left(x^2 v + x v^2 + v^3/3\right)\Big|_x^1 = \frac{12}{7}\left(x^2 + x + \frac{1}{3} - x^3 - x^3 - \frac{x^3}{3}\right) =$$

$$= \frac{12}{7}\left(x^2 + x + \frac{1}{3} - \frac{7}{3}x^3\right) = \frac{4}{7} + \frac{12}{7}x + \frac{12}{7}x^2 - 4x^3$$

$$\Rightarrow f_x\left(\frac{1}{2}\right) = \frac{4}{7} + \frac{6}{7} + \frac{3}{7} - \frac{1}{2} = 1 + \frac{6}{7} - \frac{1}{2} = \frac{1}{2} + \frac{6}{7} = \frac{19}{14}$$

$$\Rightarrow E\left(y \mid x=\frac{1}{2}\right)=\frac{1}{f_X\left(\frac{1}{2}\right)} \int_{\mathbb{R}} y \frac{12}{7}\left(\frac{1}{2}+y\right)^2 dy =$$

$$=\frac{14}{19} \int_{1/2}^{1} y \frac{12}{7}\left(\frac{1}{2}+y\right)^2 dy = \frac{24}{19} \int_{1/2}^{1} \frac{1}{4}y + y^2 + y^3 \, dy =$$

$$=\frac{24}{19}\left(\frac{y^2}{8}+\frac{y^3}{3}+\frac{y^4}{4}\right)\Bigg|_{1/2}^{1} = \frac{24}{19}\left(\frac{1}{8}+\frac{1}{3}+\frac{1}{4}-\frac{1}{32}-\frac{1}{24}-\frac{1}{64}\right)=$$

$$=\frac{24}{19}\left(\frac{3+8+6-1}{24}-\frac{3}{64}\right)=\frac{24}{19}\left(\frac{2}{3}-\frac{3}{64}\right)=\frac{24}{19}\cdot\frac{119}{192} \approx$$

$$\approx 0,7828$$

Зад. 3 $\mathcal{N}(170, 4^2)$ – момичета

$\mathcal{N}(174, 4^2)$ – момчета

а) $\mathbb{P}$(от 3-ма студенти поне 1 с ръст $\in [160; 172]$) = ?

б) $\mathbb{P}$(студент с ръст > 170 | ръстът му > 165) = ?

Р-е: Нека $X_1 \in \mathcal{N}(\overbrace{170}^{\mu_1}, \overbrace{4^2}^{\sigma_1^2})$ – момичета, $X_2 \in \mathcal{N}(\overbrace{174}^{\mu_2}, \overbrace{4^2}^{\sigma_2^2})$ – момчета

X – сл. избран $\Rightarrow X_1 = 4X + 170$ $\quad X_2 = 4X + 174$

а) Нека $A = \{$от 5-ма студенти поне 1 да е с ръст $\in [160; 172]\}$

Тогава $\overline{A} = \{$от 5-ма, нито 1 да не е с ръст $\in [160; 172]\}$

Нека $B_i = \{i$-тият студент < 160 или $> 172\}$, $i = \overline{1, 5}$

$\overline{A} = B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5$

$\Rightarrow \mathbb{P}(A) = 1 - \mathbb{P}(\overline{A}) = 1 - \mathbb{P}(B_1 \cap B_2 \cap B_3 \cap B_4 \cap B_5) \overset{\substack{B_1 \perp\!\!\!\perp B_2 \perp\!\!\!\perp B_3 \perp\!\!\!\perp B_4 \perp\!\!\!\perp B_5 \\ \mathbb{P}(B_1) = \mathbb{P}(B_2) = \mathbb{P}(B_3) = \mathbb{P}(B_4) = \mathbb{P}(B_5)}}{=\!=\!=\!=} 1 - \prod_{i=1}^{5} \mathbb{P}(B_i) =$

$= 1 - (\mathbb{P}(B_1))^5$

Като приемем, че $\mathbb{P}$(да изберем момче) $= \mathbb{P}$(момиче) $= \frac{1}{2}$

$\mathbb{P}(B_1) = \mathbb{P}(X < 160 \cup X > 172) = \mathbb{P}(X < 160) + \mathbb{P}(X > 172)$

Нека ги разпишем поотделно:

- $\mathbb{P}(X < 160) = \mathbb{P}(X < 160 \mid \text{момиче}) \underbrace{\mathbb{P}(\text{момиче})}_{1/2} + \mathbb{P}(X < 160 \mid \text{момче}) \underbrace{\mathbb{P}(\text{момче})}_{1/2} =$

$= \frac{1}{2}\mathbb{P}(4X + 170 < 160) + \frac{1}{2}\mathbb{P}(4X + 174 < 160) =$

$= \frac{1}{2}\mathbb{P}\left(X < \frac{160 - 170}{4}\right) + \frac{1}{2}\mathbb{P}\left(X < \frac{160 - 174}{4}\right) =$

$= \frac{1}{2}\mathbb{P}(X < -2{,}5) + \frac{1}{2}\mathbb{P}(X < -3{,}5) =$

$= \frac{1}{2}(\Phi(-2{,}5) + \Phi(-3{,}5)) = \frac{1}{2}(0{,}0062 + 0{,}0001) = \frac{1}{2} \cdot 0{,}0063 =$

$= 0{,}00315$

- $\mathbb{P}(X > 172) = 1 - \mathbb{P}(X \leq 172)$

Аналогично:

$\mathbb{P}(X \leq 172) = \mathbb{P}(X \leq 172 \mid \text{момиче})\mathbb{P}(\text{момиче}) + \mathbb{P}(X \leq 172 \mid \text{момче})\mathbb{P}(\text{момче}) =$

$= \frac{1}{2}\mathbb{P}(4X + 170 \leq 172) + \frac{1}{2}\mathbb{P}(4X + 174 \leq 172) =$

$= \frac{1}{2}\mathbb{P}\left(X \leq \frac{172 - 170}{4}\right) + \frac{1}{2}\mathbb{P}\left(X < \frac{172 - 174}{4}\right) = \frac{1}{2}\left(\Phi\left(\frac{1}{2}\right)\right) + \frac{1}{2}\left(\Phi\left(-\frac{1}{2}\right)\right) =$

$= \frac{1}{2} \cdot 1 = \frac{1}{2} \Rightarrow \mathbb{P}(X > 172) = 1 - \mathbb{P}(X \leq 172) = 1 - 0{,}5 = 0{,}5$

Накрая пресмятаме:

$\mathbb{P}(B_1) = \mathbb{P}(X < 160) + \mathbb{P}(X > 172) = 0{,}00315 + 0{,}5 = 0{,}50315$

$\Rightarrow \mathbb{P}(A) = 1 - (\mathbb{P}(B_1))^5 = 1 - \underbrace{(0{,}50315)^5}_{\approx 0{,}0322468565111 38 \to 0{,}03225} = 1 - 0{,}03225 = 0{,}96775$

б) Нека $M = \{\text{студент с ръст} > 170\}$

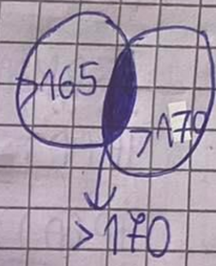
$N = \{\text{студент с ръст} > 165\}$

$X \in \mathcal{N}(0,1) \Rightarrow X_1 = 4X + 170 \quad X_2 = 4X + 174$

Търсим $\mathbb{P}(M|N)$:

$$\mathbb{P}(M|N) = \frac{1}{2}\mathbb{P}(\text{момче} > 170 \mid \text{момчето} > 165) + \frac{1}{2}\mathbb{P}(\text{момиче} > 170 \mid \text{момичето} > 165) =$$

$$= \frac{1}{2}\cdot\frac{\mathbb{P}(X_2 > 170)}{\mathbb{P}(X_2 > 165)} + \frac{1}{2}\cdot\frac{\mathbb{P}(X_1 > 170)}{\mathbb{P}(X_1 > 165)} =$$



$$= \frac{1}{2}\cdot\frac{\overset{4X > -4}{\mathbb{P}(4X + 174 > 170)}}{\underset{4X > -9}{\mathbb{P}(4X + 174 > 165)}} + \frac{1}{2}\cdot\frac{\overset{4X > 0}{\mathbb{P}(4X + \cancel{170} > \cancel{170})}}{\underset{4X > -5}{\mathbb{P}(4X + 170 > 165)}} =$$

$$= \frac{1}{2}\cdot\frac{\mathbb{P}(X > -1)}{\mathbb{P}(X > -9/4)} + \frac{1}{2}\cdot\frac{\mathbb{P}(X > 0)}{\mathbb{P}(X > -5/4)} =$$

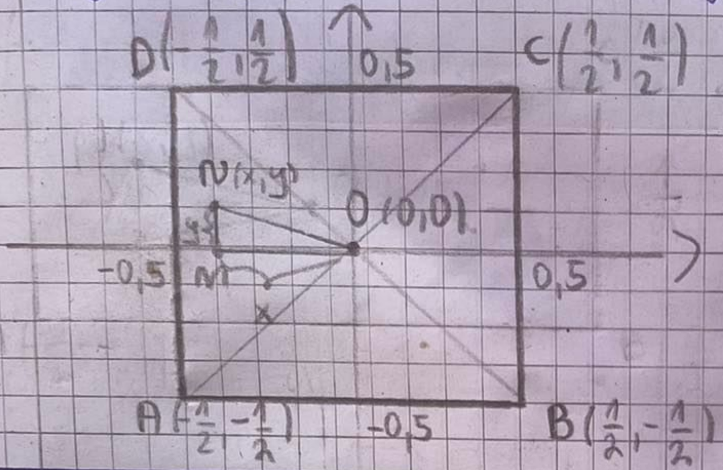
$$= \frac{1}{2}\cdot\frac{1 - \mathbb{P}(X \leq -1)}{1 - \mathbb{P}(X \leq -\frac{9}{4})} + \frac{1}{2}\cdot\frac{1 - \mathbb{P}(X \leq 0)}{1 - \mathbb{P}(X \leq -5/4)} =$$

$$= \frac{1}{2}\cdot\frac{1 - \Phi(-1)}{1 - \Phi(-\frac{9}{4})} + \frac{1}{2}\cdot\frac{1 - \Phi(0)}{1 - \Phi(-5/4)} =$$

$$= \frac{1}{2}\cdot\frac{\Phi(1)}{\Phi(\frac{9}{4})} + \frac{1}{2}\cdot\frac{\Phi(0)}{\Phi(5/4)} = \underbrace{\frac{1}{2}\cdot\frac{0{,}8413}{0{,}9878}}_{1{,}9756} + \underbrace{\frac{1}{2}\cdot\frac{0{,}5}{0{,}8944}}_{1{,}7888} =$$

$$= 0{,}42584531 + 0{,}27951699 = 0{,}7053623 \approx 0{,}71$$

Зад. 4 Във вътрешността на квадрат с лице 1 по случаен начин попада точка. Да се намери средната стойност и дисперсията на разстоянието от точката до центъра на квадрата.



$S_{ABCD} = a^2 = 1 \Rightarrow a = 1$

т. O – център

т. N – случайна

$E(NO) = ?$

$D(NO) = ?$

Р-е: Нека б.о.о. приемем, че се намираме в Oxy, като т. $O(0,0)$ е център на квадрата. Нека т. $N(x,y)$ попада случайно вътре в $\square ABCD$.
$\Rightarrow X, Y$ – случайни величини
(X,Y) е равномерно по усл. и $(X,Y) \in [-\frac{1}{2};\frac{1}{2}] \times [-\frac{1}{2};\frac{1}{2}]$

В $\triangle ONM$: $ON^2 = NM^2 + MO^2 \Rightarrow ON = \sqrt{y^2+x^2} = \underbrace{\sqrt{x^2+y^2}}_{g}$

Сега нека изразим плътността:

$$f_{X,Y}(x,y) = \begin{cases} c & \text{при } (X,Y) \in \text{ вътре в } \square ABCD \\ 0 & \text{иначе} \end{cases}$$

$$1 = \iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dxdy = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} c\,dxdy = c\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} 1\,dxdy =$$

$$= c \cdot S_{\square ABCD} = c \cdot a^2 = c \cdot 1^2 \Rightarrow c = 1$$

Дотук имаме, че (X,Y) е непрекъсната сл. вел. с плътност $f_{X,Y}(x,y)$ и $g: \mathbb{R}^2 \to \mathbb{R}$, като $g = \sqrt{x^2+y^2}$, е интегруема по Лебег ф-я.

Тогава от Теоремата следва, че $g \circ (X,Y)$ е едномерна случайна величина със средно $E\, g\circ(X,Y) = \iint_{\mathbb{R}^2} g(x,y) f_{X,Y}(x,y)\,dxdy$.

$$\Rightarrow E(NO) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \sqrt{x^2+y^2} \cdot 1\,dxdy \approx 0{,}38260$$

$$E(NO^2) = \int_{-1/2}^{1/2}\int_{1/2}^{1/2} (x^2+y^2)\cdot 1\,dxdy = 1/6 \approx 0{,}16667$$

$$\Rightarrow DX = E(NO^2) - E^2(NO) \approx \frac{1}{6} - (0{,}38260)^2 \approx 0{,}16667 - 0{,}14638276 \approx 0{,}02028724$$