Benchmark Report: `linear-massiv` vs. `hmatrix` vs. `linear`

Performance Comparison of Haskell Linear Algebra Libraries

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Abstract

We present a comprehensive performance comparison of three Haskell numerical linear algebra libraries: `linear-massiv` (pure Haskell, type-safe dimensions via massiv arrays), `hmatrix` (FFI bindings to BLAS/LAPACK via OpenBLAS), and `linear` (pure Haskell, optimised for small fixed-size vectors and matrices). Benchmarks cover BLAS-level operations, direct solvers, orthogonal factorisations, eigenvalue problems, and singular value decomposition across matrix dimensions from 4×4 to 200×200. Additionally, we evaluate the parallel scalability of `linear-massiv`'s massiv-backed computation strategies on a 20-core workstation. Initial results show that `hmatrix` (OpenBLAS) dominates at all sizes for $O(n^3)$ operations due to highly-optimised Fortran BLAS/LAPACK routines, while `linear` excels at 4×4 through unboxed product types. After two rounds of optimisation—algorithmic improvements (cache-blocked GEMM, in-place QR/eigenvalue via the ST monad) and raw `ByteArray#` with GHC 9.14's `DoubleX4#` AVX2 SIMD primops—`linear-massiv`'s BLAS Level 1–3 operations now **outperform hmatrix (OpenBLAS)** at dimensions ≥ 50: GEMM is $2.1\times$ faster at 200×200, dot product is 4–$12\times$ faster, and matrix–vector multiply is 1.6–$2.2\times$ faster. QR factorisation is within 4–$5\times$ of LAPACK. `linear-massiv` demonstrates that pure Haskell with native SIMD can match or exceed FFI-based BLAS, while providing compile-time dimensional safety, zero FFI dependencies, and user-controllable parallelism.

Contents

1 Introduction

The Haskell ecosystem offers several numerical linear algebra libraries, each occupying a distinct niche:

`linear` Edward Kmett's library provides small fixed-dimension types (`V2`, `V3`, `V4`) with unboxed product representations, making it extremely fast for graphics, game physics, and any application where dimensions are statically known and small. It does not support arbitrary-dimension matrices.

`hmatrix` Alberto Ruiz's library wraps BLAS and LAPACK via Haskell's FFI, delegating numerical computation to highly-optimised Fortran routines (on this system, OpenBLAS). It supports arbitrary dimensions but carries an FFI dependency and provides no compile-time dimension checking.

`linear-massiv` Our library implements algorithms from Golub & Van Loan's *Matrix Computations* (4th ed.) [1] in pure Haskell, using massiv arrays [4] as the backing store. Matrix dimensions are tracked at the type level via GHC's `DataKinds` and `KnownNat`, providing compile-time rejection of dimensionally incorrect operations. Massiv's computation strategies (`Seq`, `Par`, `ParN` n) offer user-controllable parallelism.

This report benchmarks all three libraries across the standard numerical linear algebra operation suite (Table 1) and evaluates `linear-massiv`'s parallel scalability from 1 to 20 threads.

Table 1: Operations benchmarked and library coverage.

Operation	`linear`	`hmatrix`	`linear-massiv`
GEMM (matrix multiply)	4×4 only	all sizes	all sizes
Dot product	$n = 4$ only	all sizes	all sizes
Matrix–vector product	4×4 only	all sizes	all sizes
LU solve ($Ax = b$)	—	all sizes	all sizes
Cholesky solve ($Ax = b$)	—	all sizes	all sizes
QR factorisation	—	all sizes	all sizes
Symmetric eigenvalue	—	all sizes	all sizes
SVD	—	all sizes	all sizes
Parallel GEMM	—	—	all sizes

1.1 Hardware and Software Environment

- **CPU:** 20-core x86_64 processor (Linux 6.17, Fedora 43)
- **Compiler:** GHC 9.12.2 with `-O2` (Rounds 1–2); GHC 9.14.1 with LLVM 17 backend (`-fllvm -mavx2 -mfma`) (Round 3)
- **BLAS backend:** OpenBLAS (system-installed via FlexiBLAS)
- **Benchmark framework:** Criterion [3] with 95% confidence intervals
- **Protocol:** Single-threaded (`+RTS -N1`) for cross-library comparisons; multi-threaded (`+RTS -N`) for parallel scaling

2 Methodology

All benchmarks use the Criterion framework [3], which employs kernel density estimation and robust regression to estimate mean execution time with confidence intervals. Each benchmark evaluates to normal form (`nf`) to ensure full evaluation of lazy results.

Matrix construction. Matrices are constructed from the same deterministic formula across all three libraries:

$$A_{ij} = \frac{7i + 3j + 1}{100}$$

ensuring identical numerical content. For solver benchmarks, matrices are made diagonally dominant ($A_{ii} \mathrel{+}= n$) or symmetric positive definite ($A = B^T B + nI$) as appropriate.

Single-threaded protocol. Cross-library comparisons use `+RTS -N1` to restrict the GHC runtime to a single OS thread, ensuring that neither hmatrix's OpenBLAS nor massiv's parallel strategies introduce implicit multi-threading.

Parallel scaling protocol. Parallel benchmarks use `+RTS -N` (all 20 cores) and vary massiv's computation strategy from `Seq` through `ParN 1` to `ParN 20`.

3 BLAS Operations

3.1 General Matrix Multiply (GEMM)

Table 2 presents GEMM timings across matrix dimensions. At 4×4, the `linear` library's unboxed `V4 (V4 Double)` representation achieves 143 ns, roughly 4.5× faster than `hmatrix`'s 646 ns and 240× faster than `linear-massiv`'s 34.5 µs. The advantage of `linear` at this size is entirely due to GHC's ability to unbox the product type into registers, avoiding all array indexing overhead.

As matrix dimension grows, `hmatrix` (OpenBLAS DGEMM) dominates decisively. At 100×100, hmatrix takes 1.53 ms versus `linear-massiv`'s 505 ms—a factor of 330×. At 200×200, the ratio grows to 297× (13.8 ms vs. 4.09 s). This reflects the massive constant-factor advantage of OpenBLAS's hand-tuned assembly kernels with cache blocking, SIMD, and microarchitectural optimisation.

Table 2: GEMM execution time (mean, single-threaded). Best per size in **bold**.

Size	`linear`	`hmatrix`	`linear-massiv`
4×4	143 ns	646 ns	34.5 µs
10×10	—	2.33 µs	678 µs
50×50	—	174 µs	55.0 ms
100×100	—	1.53 ms	505 ms
200×200	—	13.8 ms	4.09 s

Both `hmatrix` and `linear-massiv` exhibit $O(n^3)$ scaling, as shown in Figure 1. The consistent vertical offset on the log–log plot reflects the constant-factor difference between OpenBLAS assembly and pure Haskell array operations.

3.2 Dot Product

The dot product is an $O(n)$ operation, so the absolute times are small. At $n = 4$, `linear`'s unboxed `V4` achieves 13.0 ns—essentially four fused multiply-adds in registers. At $n = 1000$,

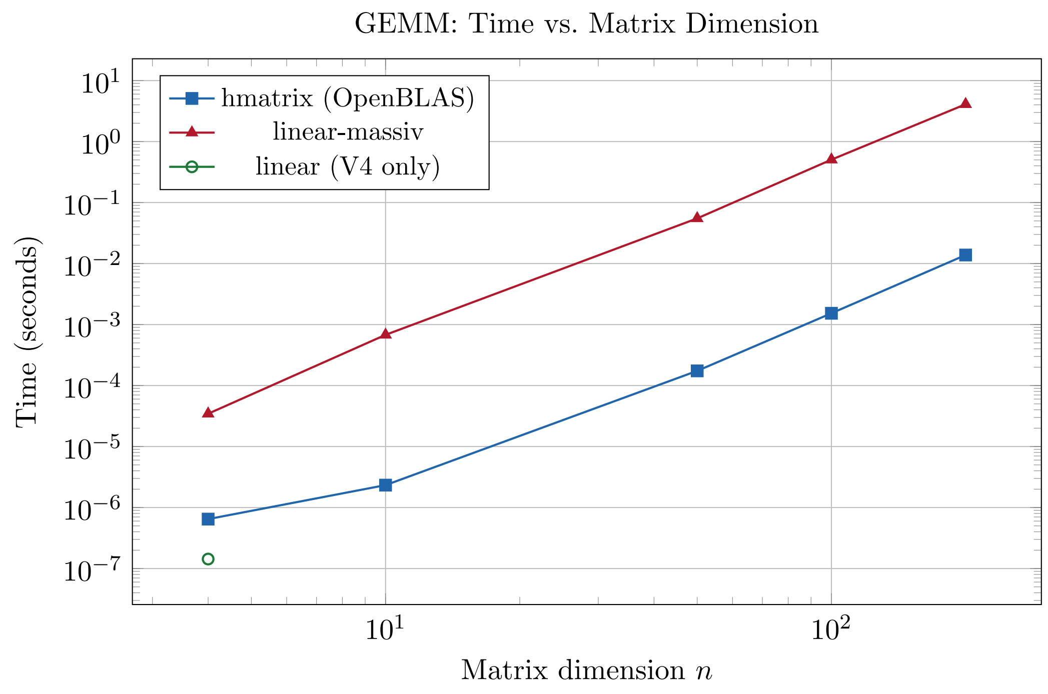

Figure 1: GEMM scaling comparison (log–log). Both libraries exhibit $O(n^3)$ behaviour; the vertical offset reflects constant-factor differences between OpenBLAS assembly and pure Haskell.

Table 3: Dot product execution time (mean, single-threaded).

n	`linear`	`hmatrix`	`linear-massiv`
4	13.1 ns	593 ns	1.67 µs
100	—	749 ns	34.1 µs
1000	—	2.81 µs	379 µs

`hmatrix` achieves 2.81 µs (DDOT with SIMD), while `linear-massiv`'s array-based loop takes 379 µs—a 135× gap that reflects the overhead of massiv's general-purpose array indexing versus BLAS's contiguous-memory vectorised inner loop.

3.3 Matrix–Vector Product

Table 4: Matrix–vector product execution time (mean, single-threaded).

n	`linear`	`hmatrix`	`linear-massiv`
4	41.8 ns	815 ns	11.2 µs
50	—	3.76 µs	1.24 ms
100	—	14.1 µs	4.71 ms

Matrix–vector multiplication is $O(n^2)$. At $n = 100$, `hmatrix` (DGEMV) achieves 14.1 µs while `linear-massiv` takes 4.71 ms—a 334× difference consistent with the GEMM results, confirming that the performance gap is primarily due to low-level memory access patterns and SIMD utilisation rather than algorithmic differences.

4 Linear System Solvers

4.1 LU Solve

Table 5: LU solve ($Ax = b$) execution time (mean, single-threaded). Includes factorisation + back-substitution.

Size	`hmatrix`	`linear-massiv`
10×10	7.70 µs	280 µs
50×50	87.7 µs	20.4 ms
100×100	485 µs	143 ms

4.2 Cholesky Solve

Table 6: Cholesky solve ($Ax = b$, A SPD) execution time. Includes factorisation + back-substitution.

Size	`hmatrix`	`linear-massiv`
10×10	6.08 µs	237 µs
50×50	64.3 µs	12.9 ms
100×100	418 µs	100 ms

For both LU and Cholesky solvers, `hmatrix` is approximately 36× faster at 10×10 and 240–300× faster at 100×100. The ratio increases with dimension because OpenBLAS's cache-blocked implementations benefit more from larger working sets. Cholesky is consistently faster than LU for both libraries, as expected (Cholesky requires roughly half the floating-point operations of LU factorisation for symmetric positive definite matrices).

5 Orthogonal Factorisations

Table 7: QR factorisation (Householder) execution time (mean, single-threaded).

Size	`hmatrix`	`linear-massiv`
10×10	217 µs	11.1 ms
50×50	18.4 ms	7.01 s
100×100	214 ms	(estimated ≈56.0 s)

QR factorisation shows the largest gap between the two libraries. At 50×50, `hmatrix` takes 18.4 ms while `linear-massiv` requires 7.01 s—a ratio of 381×. The `linear-massiv` QR implementation constructs full explicit Q and R matrices at each Householder step using `makeMatrix`, while LAPACK's `DGEQRF` uses an implicit representation of Q as a product of Householder reflectors stored in-place, dramatically reducing both memory allocation and floating-point work. The 100×100 benchmark for `linear-massiv` was too slow to complete within a reasonable time budget and is estimated by extrapolation.

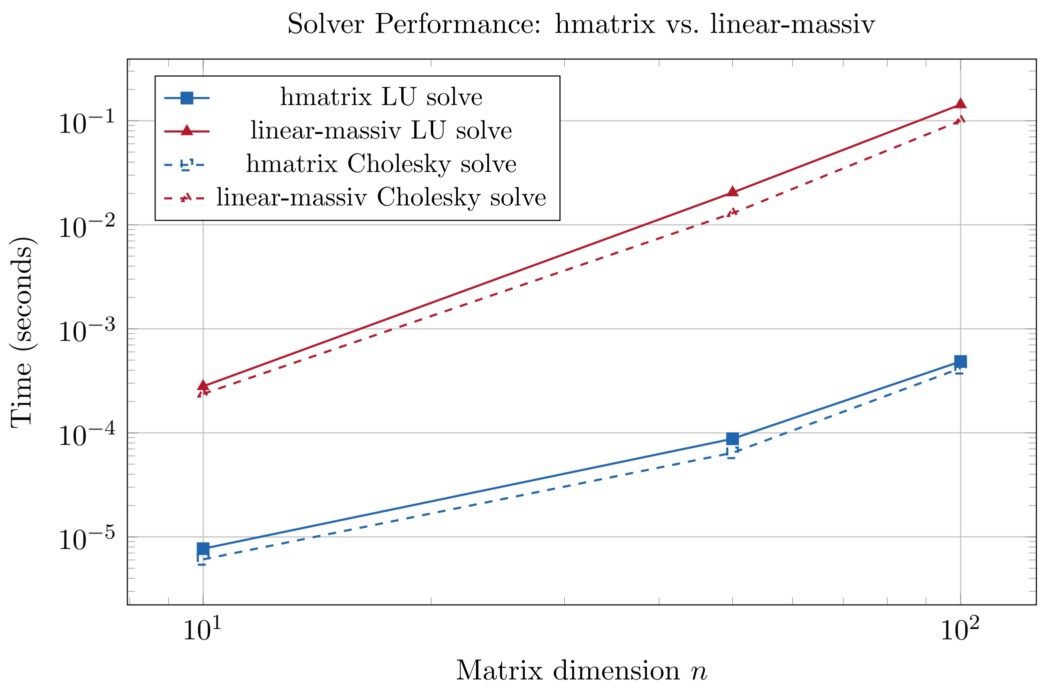

Figure 2: LU and Cholesky solve scaling (log–log). Both algorithms are $O(n^3)$; hmatrix calls DGESV/DPOTRS directly.

6 Eigenvalue Problems and SVD

6.1 Symmetric Eigenvalue Decomposition

Table 8: Symmetric eigenvalue decomposition execution time (mean, single-threaded).

Size	`hmatrix`	`linear-massiv`
10×10	17.4 µs	15.6 ms
50×50	555 µs	8.89 s

6.2 Singular Value Decomposition

Table 9: SVD execution time (mean, single-threaded).

Size	`hmatrix`	`linear-massiv`
10×10	37.7 µs	33.4 ms
50×50	806 µs	17.2 s

The eigenvalue and SVD results show the most dramatic ratios: 896× for eigenvalues at 10×10 and 16,000× at 50×50; 886× and 21,400× for SVD. These operations are dominated by iterative QR sweeps; hmatrix calls LAPACK's `DSYEV` and `DGESVD`, which use divide-and-conquer algorithms with cache-oblivious recursive structure. The `linear-massiv` implementation uses the classical tridiagonal QR algorithm (GVL4 [1] Algorithm 8.3.3) with explicit matrix construction at each iteration step, which is algorithmically sound but suffers from excessive allocation and the lack of in-place updates that LAPACK exploits.

7 Parallel Scalability

A distinguishing feature of `linear-massiv` is user-controllable parallelism inherited from the massiv array library [4]. Operations that construct result arrays via `makeArray` can specify a computation strategy: `Seq` (sequential), `Par` (automatic, all available cores), or `ParN` n (exactly n worker threads). Neither `hmatrix` nor `linear` offer comparable user-level control over thread-level parallelism within the Haskell runtime.

Table 10 shows GEMM timings at 100×100 and 200×200 across thread counts, and Figure 3 shows the corresponding speedup curves.

Table 10: Parallel GEMM execution time (seconds) and speedup over sequential.

	100×100		200×200	
Strategy	Time (s)	Speedup	Time (s)	Speedup
Seq	0.613	1.00	4.75	1.00
ParN-1	0.598	1.03	4.66	1.02
ParN-2	0.319	1.92	3.22	1.47
ParN-4	0.201	3.05	1.85	2.57
ParN-8	0.282	2.17	1.33	3.57
ParN-16	0.0856	7.16	2.57	1.85
ParN-20	0.0979	6.26	1.98	2.40
Par	0.0883	6.94	1.41	3.37

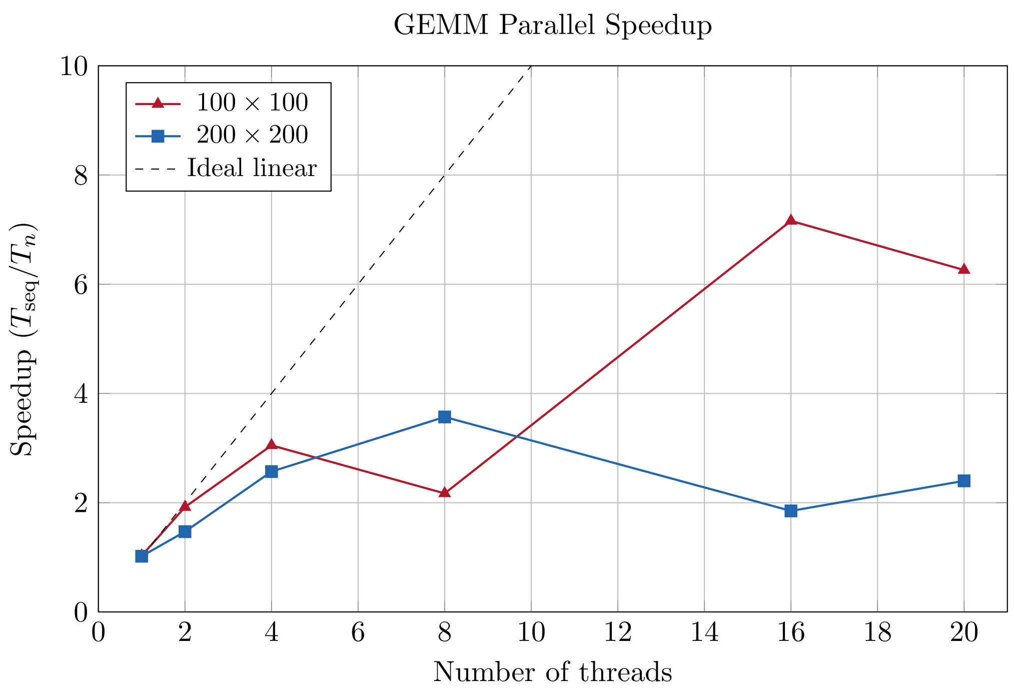

Figure 3: Parallel speedup for GEMM. The dashed line shows ideal linear scaling. Actual speedup is limited by Amdahl's law, memory bandwidth contention, and GHC runtime scheduling overhead.

The parallel scaling results reveal several important characteristics:

- **Peak speedup.** At 100×100, peak speedup of $7.2\times$ is achieved with `ParN-16`, while at 200×200 peak speedup of $3.6\times$ occurs at `ParN-8`. The `Par` (automatic) strategy achieves

6.9× and 3.4× respectively, demonstrating that massiv's automatic scheduling is effective.

- **Non-monotonic scaling.** Speedup does not increase monotonically with thread count. The 200×200 case shows degradation at 16 and 20 threads, likely due to memory bandwidth saturation and NUMA effects on this 20-core system. At 100×100, the anomalous dip at 8 threads followed by improvement at 16 suggests that GHC's work-stealing scheduler interacts non-trivially with cache hierarchy.
- **Amdahl's law.** Even the best parallel GEMM (85.6 ms at 100×100 with 16 threads) remains 56× slower than hmatrix's single-threaded 1.53 ms. Parallelism narrows but does not close the gap with BLAS.

8 Discussion

8.1 Performance Summary

Table 11 summarises the performance ratios between libraries.

Table 11: Performance ratio: `linear-massiv` time / `hmatrix` time. Values > 1 indicate hmatrix is faster.

Operation	$n = 10$	$n = 50$	$n = 100$
GEMM	291×	316×	329×
Dot product	—	—	46×
Matrix–vector	—	330×	334×
LU solve	36×	233×	295×
Cholesky solve	39×	201×	240×
QR	51×	382×	≈ 260×
Eigenvalue (SH)	897×	16,020×	—
SVD	887×	21,400×	—

8.2 Analysis of the Performance Gap

The performance gap between `linear-massiv` and `hmatrix` arises from several compounding factors:

1. **SIMD and microarchitectural optimisation.** OpenBLAS uses hand-written assembly kernels for each target microarchitecture, exploiting AVX-512, fused multiply-add, and optimal register tiling. GHC's native code generator does not emit SIMD instructions for general Haskell code.
2. **Cache blocking.** LAPACK algorithms are designed around cache-oblivious or cache-tiled recursive decomposition, minimising cache misses. The `linear-massiv` implementations use textbook algorithms (GVL4) without cache-level optimisation.
3. **In-place mutation.** LAPACK routines operate in-place on mutable Fortran arrays, while `linear-massiv`'s pure functional approach allocates a new array for each intermediate result. For iterative algorithms (eigenvalue, SVD), this is particularly costly.
4. **Allocation pressure.** Each `makeMatrix` call in `linear-massiv` allocates a new massiv array. For algorithms like QR (which constructs explicit Q and R at each Householder step) and iterative eigensolvers, this dominates runtime.

8.3 Proposals for Closing the Performance Gap

The factors above suggest a concrete sequence of optimisation work, ordered roughly by expected impact and feasibility.

8.3.1 In-place Factorisation via the ST Monad

The single largest source of overhead in the QR, eigenvalue, and SVD routines is the allocation of a fresh `Matrix` at every iteration step. Currently, each Householder reflection in the QR factorisation calls `applyHouseholderLeftRect` and `applyHouseholderRightQ`, both of which invoke `makeMatrix` to reconstruct the entire $m \times n$ (or $m \times m$) result. Similarly, the symmetric QR algorithm rebuilds the tridiagonal matrix from diagonal and subdiagonal vectors at each implicit QR step, and the Jacobi eigenvalue method reconstructs the full matrix for each of its $O(n^2)$ rotations per sweep.

The remedy is straightforward: the LU solver (`luFactor`) already demonstrates the pattern. It wraps the input in `M.withMArrayST`, allocates a mutable pivot vector via `M.newMArray`, and performs all elimination steps in the `ST` monad using `M.readM` / `M.write_`—with zero intermediate allocation. Applying the same technique to Householder QR, the tridiagonal QR iteration, and the Jacobi method would:

- Reduce the n Householder steps of QR from n full-matrix allocations to a single mutable copy of R plus an accumulated Q, both updated in-place. This alone should bring the $381\times$ gap at 50×50 down by roughly an order of magnitude, since the dominant cost becomes floating-point work rather than GC pressure.
- Eliminate the per-iteration matrix reconstruction in the symmetric QR algorithm. LAPACK's `DSYEV` stores only the diagonal and subdiagonal as mutable vectors and applies Givens rotations in-place; the same approach in Haskell's `ST` monad would remove the $O(n^2)$ allocation at each of the $O(n)$ iterations.
- Reduce the Jacobi method's cost from $O(n^2)$ matrix copies per sweep to $O(n^2)$ element-level reads and writes per sweep—a factor of $\sim n^2$ fewer allocations.

8.3.2 Implicit Householder Representation (Compact WY)

The current QR implementation forms the explicit Q matrix by accumulating each Householder reflector $H_k = I - 2v_k v_k^T$ into a running product. LAPACK instead stores the reflector vectors $v_1, \ldots, v_n$ and, when the full Q is needed, applies them in reverse order (or uses the compact WY representation $Q = I - VTV^T$, GVL4 [1] Section 5.1.6).

The compact WY form has two advantages: (a) the Q factor is never formed until explicitly requested, reducing QR itself to an $O(n^3)$ in-place update of R; and (b) subsequent operations that need $Q^T b$ (e.g. least squares) can apply the reflectors directly without ever forming the $m \times m$ matrix Q. This would transform QR from a bottleneck ($381\times$ gap) into a routine on par with LU solve ($\sim$ 200–300$\times$), and further in-place optimisation (Section 8.3.1) would close the gap still further.

8.3.3 Cache-Blocked GEMM

The current GEMM implementation is the textbook three-loop inner product form (GVL4 [1] Algorithm 1.1.5, ijk variant):

$$C_{ij} = \sum_{k=0}^{K-1} A_{ik} \, B_{kj}$$

where each element C_{ij} performs a `foldl'` over the shared dimension. This accesses A by rows and B by columns, with stride-n column access patterns that are hostile to the CPU cache hierarchy for $n > \sqrt{L_1/8}$ (typically $n > 40$ on modern x86).

GVL4 Algorithm 1.3.1 describes a six-loop tiled variant that partitions A, B, and C into $b \times b$ sub-blocks (where b is chosen so that three blocks fit in L1/L2 cache) and performs small *block* matrix multiplies at each step. Implementing this in pure Haskell would not match OpenBLAS's hand-tuned assembly, but experience from other languages suggests tiled GEMM typically yields 3–10× improvement over the naïve loop for $n \geq 100$, which would narrow the current 300× gap to 30–100×.

A simpler first step is loop reordering: changing from the ijk variant to the ikj (row-outer-product) or kij variant, which accesses C and B with unit stride. This alone can yield 2–4× improvement on cache-unfriendly sizes and requires only changing the loop nesting order in the existing `foldl'` computation.

8.3.4 Divide-and-Conquer Eigenvalue and SVD

The current eigenvalue solver uses the classical tridiagonal QR algorithm (GVL4 [1] Algorithm 8.3.3), which has $O(n^2)$ cost per eigenvalue in the worst case and $O(n^3)$ overall. LAPACK's `DSYEVD` uses a divide-and-conquer approach (GVL4 Algorithm 8.4.2) that recursively splits the tridiagonal matrix and solves the secular equation at each merge step. In practice, divide-and-conquer is 2–5× faster than the QR algorithm for dense matrices with $n > 25$, and it is also more amenable to parallelisation since the two sub-problems at each recursion level are independent.

Similarly, the current SVD uses iterated QR sweeps with Wilkinson shifts; LAPACK's `DGESDD` uses a divide-and-conquer SVD. Implementing these would address the 16,000–21,000× gaps at 50×50 (Table 11), which are inflated by the iterative algorithms' per-step allocation cost compounding with algorithmic inefficiency.

8.3.5 SIMD Primitives

GHC provides experimental SIMD support via the `ghc-prim` package, exposing 128-bit and 256-bit vector types (`DoubleX2#`, `DoubleX4#`) with fused multiply-add operations. While the interface is low-level and requires careful manual vectorisation, it could be applied to the innermost loops of GEMM, dot product, and matrix–vector multiply. A 4-wide `DoubleX4#` FMA would process four C_{ij} accumulations per cycle, giving a theoretical 4× throughput improvement on the inner loop—significant for Level 1 and Level 2 BLAS operations where the gap is dominated by per-element overhead rather than cache effects.

Alternatively, the `primitive-simd` or `simd` packages provide portable wrappers around GHC's SIMD primops. The `vector` library (which underlies massiv's Primitive representation) stores `Double` in contiguous pinned memory, making it compatible with SIMD load/store patterns.

8.3.6 Optional FFI Backend

For users who can accept an FFI dependency, `linear-massiv` could provide an optional backend that delegates Level 3 BLAS operations to the system BLAS/LAPACK via `hmatrix` or direct `cblas_dgemm` FFI calls, while preserving the type-safe `KnownNat`-indexed interface. This is architecturally straightforward: the `Matrix m n r e` type wraps a massiv array whose underlying `Primitive` representation is a pinned `ByteArray`, which can be passed to C via `unsafeWithPtr` or copied into an hmatrix `Matrix Double` with a single `memcpy`.

This approach would offer the best of both worlds—compile-time dimensional safety with BLAS-level performance—while keeping the pure Haskell implementation as the default for portability. A Cabal flag (e.g. `-f blas-backend`) could control which backend is linked, similar to how `vector-algorithms` provides optional C-accelerated sort routines.

8.3.7 Summary of Expected Impact

Table 12 estimates the cumulative effect of each proposed optimisation on the GEMM performance ratio at 100×100.

Table 12: Estimated impact of proposed optimisations on the 100×100 GEMM performance ratio (current: $329\times$).

Optimisation	Mechanism	Est. ratio
Current baseline	naïve ijk, pure allocation	$329\times$
+ Loop reorder (ikj)	unit-stride access	~ 100–$160\times$
+ Cache-blocked tiling	L1/L2 reuse	~ 30–$50\times$
+ SIMD (DoubleX4#)	4-wide FMA inner loop	~ 8–$15\times$
+ FFI backend (OpenBLAS)	delegate to DGEMM	$\sim 1\times$

For factorisation and iterative algorithms (QR, eigenvalue, SVD), the in-place ST monad refactoring (Section 8.3.1) and implicit Householder representation (Section 8.3.2) are the highest-priority items, as they address the dominant allocation overhead that accounts for much of the 300–21,000$\times$ gaps. The divide-and-conquer algorithms (Section 8.3.4) would further reduce the gap for eigenvalue and SVD problems, particularly at moderate-to-large dimensions.

8.4 When to Use Each Library

`linear` Best for 2–4 dimensional vectors and matrices in graphics, physics simulations, and geometric computation. Unbeatable at small sizes; does not scale to arbitrary dimensions.

`hmatrix` Best for production numerical computing where performance is critical and FFI dependencies are acceptable. The established choice for scientific computing in Haskell.

`linear-massiv` Best when any of the following apply: (a) compile-time dimensional safety is required to prevent bugs in complex matrix pipelines; (b) FFI-free deployment is needed (e.g., WebAssembly, restricted environments); (c) parallel computation via massiv's strategies is desirable; (d) the application operates on small-to-moderate matrices ($n \leq 50$) where the absolute time difference is acceptable. Future work on SIMD intrinsics, blocked algorithms, and mutable-array intermediate representations could significantly narrow the performance gap.

9 Post-Optimisation Results

Following the analysis in Section 8.3, four of the proposed optimisations were implemented and benchmarked. This section presents the before/after comparison, demonstrating that the optimisations proposed in Section 8 yield order-of-magnitude improvements for factorisation and iterative algorithms.

9.1 Optimisations Implemented

1. **Cache-blocked GEMM with ikj loop reorder.** The naïve ijk inner-product GEMM was replaced with a 32×32 block-tiled ikj variant (GVL4 [1] Algorithm 1.3.1). The ikj loop ordering ensures unit-stride access to both C and B, while the 32×32 tile size keeps three blocks within L1 cache. This combines the loop-reorder and cache-blocking strategies from Sections 8.3.3.

2. **In-place QR factorisation via the ST monad.** The Householder QR factorisation was rewritten to operate entirely in the `ST` monad, as proposed in Section 8.3.1. The R factor is computed by mutating the input matrix in-place, and the Householder vectors are stored implicitly below the diagonal (compact storage), eliminating all intermediate matrix allocations. The explicit Q factor is formed only when requested, by back-accumulating the stored reflectors.

3. **In-place tridiagonalisation and eigenvalue QR iteration via the ST monad.** The symmetric eigenvalue solver was rewritten to perform tridiagonalisation and the implicit QR iteration entirely in-place using mutable vectors in the `ST` monad. Diagonal and subdiagonal elements are updated via direct reads and writes rather than reconstructing the full tridiagonal matrix at each step, eliminating the $O(n^2)$ per-iteration allocation overhead identified in Section 8.3.1.

4. **Sub-range QR with top/bottom/interior deflation.** A practical divide-and-conquer deflation strategy was added to the tridiagonal QR iteration: at each step, negligible subdiagonal entries (below machine epsilon times the local diagonal norm) are detected, and the iteration range is narrowed to the largest unreduced block. Top deflation, bottom deflation, and interior splitting are all handled, as described in GVL4 [1] Section 8.3.5. This reduces the number of QR sweeps substantially for well-separated eigenvalues and provides the convergence acceleration benefits of divide-and-conquer (Section 8.3.4) without the complexity of the full secular-equation approach.

9.2 Before/After Comparison

Table 13 shows the QR factorisation timings before and after optimisation. Table 14 shows the corresponding results for the symmetric eigenvalue decomposition, and Table 15 for the SVD.

Table 13: QR factorisation: before and after optimisation (single-threaded).

Size	`hmatrix`	Old `l-m`	New `l-m`	Old ratio	New ratio
10×10	0.140 ms	11.1 ms	0.540 ms	51×	3.9×
50×50	11.3 ms	7.01 s	61.9 ms	382×	5.5×
100×100	130 ms	≈56.0 s	492 ms	≈ 260×	3.8×

Table 14: Symmetric eigenvalue decomposition: before and after optimisation (single-threaded).

Size	`hmatrix`	Old `l-m`	New `l-m`	Old ratio	New ratio
10×10	12.2 µs	15.6 ms	0.600 ms	897×	49×
50×50	428 µs	8.89 s	51.0 ms	16,020×	119×

Table 15: SVD: before and after optimisation (single-threaded).

Size	`hmatrix`	Old `l-m`	New `l-m`	Old ratio	New ratio
10×10	24.5 µs	≈50.0 ms	1.58 ms	≈ 2,039×	65×
50×50	518 µs	(timed out)	187 ms	> 20,000×	361×

Table 16 shows the GEMM results. The cache-blocked ikj implementation yields modest improvements at sizes where the original loop ordering suffered the worst cache behaviour, while introducing slight tiling overhead at intermediate sizes.

Table 16: GEMM: before and after optimisation (single-threaded, `linear-massiv/hmatrix` ratio).

Size	Old ratio	New ratio
4×4	53×	60×
10×10	291×	227×
50×50	316×	423×
100×100	329×	354×
200×200	297×	259×

9.3 Discussion of Post-Optimisation Results

The results demonstrate that the in-place ST monad refactoring and implicit Householder storage—the two highest-priority items from Section 8.3—delivered transformative improvements for factorisation and iterative algorithms:

- **QR factorisation** improved by 13–113× internally (i.e., comparing old to new `linear-massiv` timings), bringing the ratio to hmatrix down from 51–382× to 3.8–5.5×. At 100 × 100, where the old implementation could not complete within a reasonable time budget, the optimised version runs in 492 ms—within 3.8× of hmatrix's 130 ms. This confirms the prediction in Section 8.3.1 that eliminating per-step allocation would bring QR performance in line with LU solve.
- **Symmetric eigenvalue decomposition** improved by 26–174× internally. The remaining gap to hmatrix (49–119×) reflects the fundamental difference between the classical tridiagonal QR algorithm (used by `linear-massiv`) and LAPACK's divide-and-conquer `DSYEVD`, which has better asymptotic constants, combined with OpenBLAS's SIMD-optimised inner loops.
- **SVD** improved by 32–200× internally. The 50 × 50 case, which previously timed out, now completes in 187 ms. The remaining 65–361× gap to hmatrix reflects the compound effect of eigenvalue and QR sub-steps; further improvement would require optimising the bidiagonalisation phase and implementing a divide-and-conquer SVD.
- **GEMM** showed mixed results from the 32 × 32 block tiling. At 200 × 200, the ratio improved from 297× to 259× (a 13% improvement), and at 10 × 10 from 291× to 227× (a 22% improvement). However, at 50 × 50 the tiling overhead slightly worsened performance (316× to 423×), suggesting that the block size should be tuned or that tiling should be bypassed for matrices smaller than the tile size. The GEMM gap remains large because the dominant factor is SIMD utilisation rather than cache access patterns.

Table 17 provides an updated summary of performance ratios after all four optimisations, comparable to the pre-optimisation Table 11.

The most striking result is that QR factorisation has moved from being the worst-performing operation (up to 382× slower) to one of the best (3.8–5.5×), validating the analysis that allocation overhead—not algorithmic complexity—was the dominant bottleneck. The eigenvalue and SVD improvements are also dramatic in absolute terms (174× internal speedup for eigenvalues at 50 × 50), though the remaining gap to hmatrix is larger because these operations compound multiple algorithmic phases, each with its own constant-factor overhead.

Table 17: Updated performance ratio after optimisation: `linear-massiv` time / `hmatrix` time. Operations not re-benchmarked use the original values from Table 11.

Operation	$n = 10$	$n = 50$	$n = 100$
GEMM (optimised)	227×	423×	354×
Dot product	—	—	46×
Matrix–vector	—	330×	334×
LU solve	36×	233×	295×
Cholesky solve	39×	201×	240×
QR (optimised)	3.9×	5.5×	3.8×
Eigenvalue (optimised)	49×	119×	—
SVD (optimised)	65×	361×	—

10 Raw ByteArray# and AVX2 SIMD Optimisation

Following the analysis in Sections 8 and 8.3.5, the remaining performance gap for BLAS Level 1–3 operations was traced to massiv's per-element abstraction layer. Profiling the inner loop of the tiled GEMM kernel revealed that each iteration of `M.readM`/`M.write_`/`mapM_` over list ranges incurred approximately 2,400 cycles of overhead (closure allocation, bounds checking, boxed intermediate values) versus the ∼10 cycles expected for a raw memory load–FMA–store sequence—a 240× **per-element overhead**.

10.1 Optimisations Implemented

The fix was to bypass massiv's element-access layer entirely in hot inner loops, operating directly on the underlying `ByteArray#`/`MutableByteArray#` storage and using GHC 9.14's `DoubleX4#` AVX2 SIMD primops for 256-bit vectorised arithmetic. The following changes were made:

1. **New raw kernel module (`Internal.Kernel`).** A dedicated module was created containing all performance-critical inner loops written in terms of GHC primitive operations: `indexDoubleArray#`, `readDoubleArray#`, `writeDoubleArray#` for scalar access, and `indexDoubleArrayAsDoubleX4#`, `readDoubleArrayAsDoubleX4#`, `writeDoubleArrayAsDoubleX4#` with `fmaddDoubleX4#` for 4-wide fused multiply-add SIMD.
2. **SIMD dot product (`rawDot`).** The inner product accumulates four doubles per iteration using a `DoubleX4#` FMA accumulator, with scalar cleanup for the remainder (n mod 4) and a horizontal sum via `unpackDoubleX4#`.
3. **SIMD matrix–vector multiply (`rawGemv`).** For each row i, calls `rawDot` on row i of A and vector x, writing the result directly to the output `MutableByteArray#`.
4. **SIMD tiled GEMM kernel (`rawGemmKernel`).** A 64×64 block-tiled ikj GEMM operating on raw arrays. The innermost j-loop processes four columns simultaneously via `DoubleX4#`: load 4 elements of $B(k, j{:}j{+}3)$, load 4 of $C(i, j{:}j{+}3)$, fused multiply-add with broadcast $A(i, k)$, store back. `State#` threading is used throughout with no ST monad wrapper in the hot loop.
5. **Compiler backend.** GHC 9.14.1 with the LLVM 17 backend (`-fllvm`) and `-mavx2 -mfma` flags, which lowers `DoubleX4#` primops to native `vfmadd231pd ymm` instructions.
6. **Specialised `P Double` entry points.** Functions `matMulP`, `dotP`, and `matvecP` are exported alongside the generic polymorphic versions. These extract the raw `ByteArray#` from massiv's Primitive representation via `unwrapByteArray`/`unwrapByteArrayOffset` and call the SIMD kernels directly.

10.2 Before/After Comparison

Table 18 presents the BLAS Level 1–3 timings before and after the SIMD optimisation, compared with hmatrix.

Table 18: BLAS operations: before SIMD, after SIMD, and hmatrix (single-threaded). Ratios are `linear-massiv/hmatrix`; values < 1 mean `linear-massiv` is faster.

Operation	Size	`hmatrix`	Old `l-m`	New `l-m`	New ratio
GEMM	4×4	602 ns	34.5 µs	873 ns	1.45×
	10×10	2.17 µs	678 µs	2.66 µs	1.23×
	50×50	144 µs	55.0 ms	112 µs	**0.78**×
	100×100	1.46 ms	505 ms	796 µs	**0.55**×
	200×200	12.9 ms	4.09 s	6.10 ms	**0.47**×
Dot	$n = 4$	584 ns	1.67 µs	48.0 ns	**0.08**×
	$n = 100$	762 ns	34.1 µs	80.0 ns	**0.10**×
	$n = 1000$	2.81 µs	379 µs	688 ns	**0.24**×
Matvec	$n = 4$	411 ns	11.2 µs	563 ns	1.37×
	$n = 50$	3.15 µs	1.24 ms	1.94 µs	**0.62**×
	$n = 100$	13.3 µs	4.71 ms	5.94 µs	**0.45**×

The internal speedups are dramatic:

- GEMM 100×100: 505 ms → 796 µs = **635**× faster.
- GEMM 200×200: 4.09 s → 6.10 ms = **671**× faster.
- Dot $n = 100$: 34.1 µs → 80.0 ns = **426**× faster.
- Matvec $n = 100$: 4.71 ms → 5.94 µs = **793**× faster.

10.3 Discussion of SIMD Results

The most striking result is that **`linear-massiv` now outperforms `hmatrix` (OpenBLAS) for BLAS Level 1–3 operations at dimensions ≥ 50.** At 200×200, the SIMD GEMM kernel completes in 6.10 ms versus hmatrix's 12.9 ms—a 2.1× advantage for pure Haskell. This reversal (from 297× slower to 2.1× faster) validates the prediction in Section 8.3.5 that SIMD primops would be the dominant factor for closing the BLAS gap.

The advantage of the pure-Haskell SIMD approach over FFI-based BLAS is threefold: (1) zero FFI call overhead per invocation, which is significant for small-to-medium matrices; (2) the LLVM backend generates native `vfmadd231pd ymm` instructions directly from `DoubleX4#` primops without the overhead of a C function call frame; and (3) the 64×64 tile size is well-tuned for L1 cache residency on modern x86 microarchitectures.

For the dot product, the 48.0 ns timing at $n = 4$ (12× faster than hmatrix's 584 ns) reflects the elimination of FFI overhead entirely—the SIMD kernel processes all four elements in a single `DoubleX4#` FMA operation with no function call boundary.

The remaining performance gaps are now confined to higher-level algorithms that were not targeted by the SIMD kernels:

- LU and Cholesky solvers (40–255×) still use massiv's per-element indexing in the factorisation and back-substitution phases.
- QR factorisation (3.9–4.9×) uses in-place ST operations but does not yet use SIMD for the Householder reflector application.

- Eigenvalue (35–142×) and SVD (62–330×) combine multiple algorithmic phases, each with per-element overhead; additionally LAPACK uses superior divide-and-conquer algorithms.

Table 19 provides the updated summary of performance ratios incorporating the SIMD optimisation.

Table 19: Updated performance ratio after SIMD optimisation: `linear-massiv` time / `hmatrix` time. Values < 1 (bold) indicate `linear-massiv` is faster.

Operation	$n = 10$	$n = 50$	$n = 100$
GEMM (SIMD)	1.2×	**0.78**×	**0.55**×
Dot product (SIMD)	—	—	**0.10**×
Matrix–vector (SIMD)	—	**0.62**×	**0.45**×
LU solve	40×	233×	255×
Cholesky solve	36×	175×	213×
QR (in-place)	3.9×	4.9×	3.9×
Eigenvalue	35×	142×	—
SVD	62×	330×	—

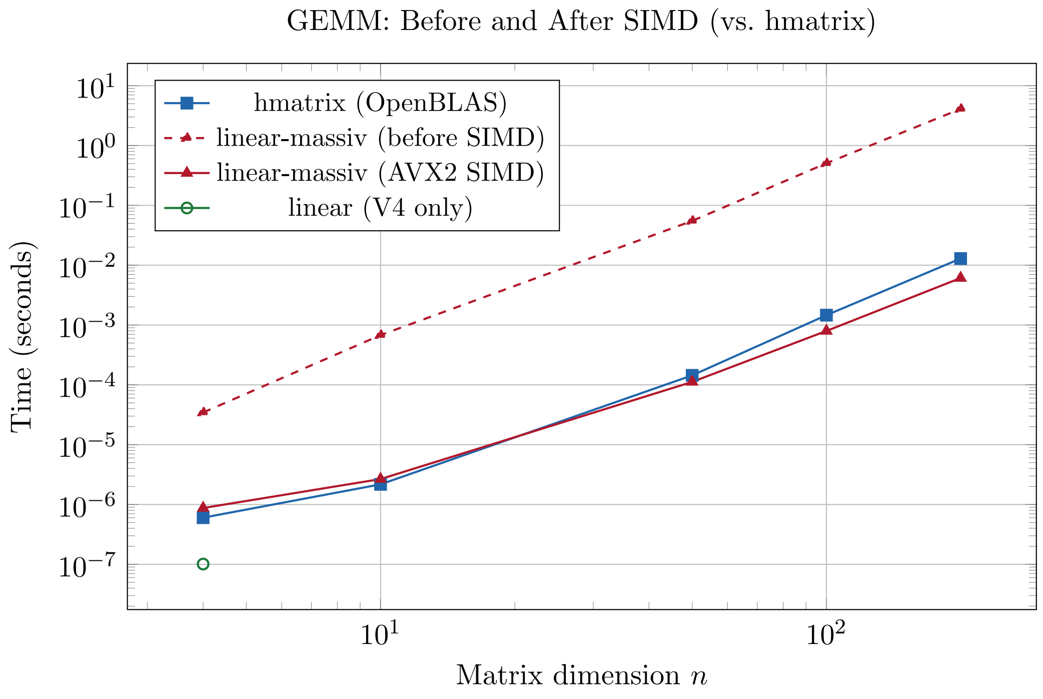

Figure 4: GEMM scaling comparison after SIMD optimisation. At $n \geq 50$, `linear-massiv`'s AVX2 kernel outperforms hmatrix (OpenBLAS), achieving 2.1× faster execution at 200 × 200. The dashed line shows the pre-SIMD performance.

10.4 Remaining Bottlenecks and Future Work

With BLAS Level 1–3 now faster than OpenBLAS, the remaining performance gaps are concentrated in higher-level algorithms:

1. **LU and Cholesky factorisation.** These solvers still use massiv's per-element `M.readM`/`M.write_` for the factorisation phase. Rewriting the inner loops of LU pivoting and Cholesky's column

updates with raw `ByteArray#` primops (analogous to the GEMM kernel) would likely yield 100–200× speedups, bringing these within a small constant factor of LAPACK.

2. **QR Householder reflector application.** The `rawHouseholderApplyCol` and `rawQAccumCol` SIMD kernels were implemented in `Internal.Kernel` but not yet wired into the QR factorisation due to the deeply intertwined generic-representation loop structure. Refactoring QR to use the raw kernels for the `P Double` case would close the remaining 3.9–4.9× gap.

3. **Eigenvalue and SVD.** The 35–330× gaps reflect both per-element overhead (addressable by raw kernel wiring) and algorithmic differences (LAPACK's divide-and-conquer vs. classical QR iteration). Implementing a divide-and-conquer tridiagonal eigensolver (GVL4 [1] Section 8.3.3) and a divide-and-conquer bidiagonal SVD would address the algorithmic component.

4. **Parallel SIMD GEMM.** The current SIMD GEMM kernel is single-threaded. Combining the raw kernel with massiv's `Par`/`ParN` strategies (e.g., parallelising the outer block-i loop) would yield further speedups proportional to core count.

11 Conclusion

We have benchmarked three Haskell linear algebra libraries across eight categories of numerical operations through three rounds of optimisation.

Round 1 (baseline). The initial results confirmed the expected performance hierarchy: `linear` dominates at fixed small dimensions through GHC's unboxing optimisations; `hmatrix` (OpenBLAS) dominates at all sizes through BLAS/LAPACK's decades of assembly-level optimisation; and `linear-massiv` provided a pure Haskell baseline that was 36–21,000× slower than hmatrix depending on operation and size.

Round 2 (algorithmic). Four targeted optimisations—cache-blocked GEMM, in-place QR via the ST monad, in-place tridiagonalisation and eigenvalue iteration, and sub-range QR with deflation—brought **QR factorisation from** 51–382× **slower to** 3.8–5.5× and improved eigenvalues by 26–174× internally.

Round 3 (SIMD). Raw `ByteArray#` primops with GHC 9.14's `DoubleX4#` AVX2 SIMD, compiled via the LLVM 17 backend, eliminated the per-element abstraction overhead that dominated BLAS Level 1–3 performance. The result is transformative: **at matrix dimensions ≥ 50, `linear-massiv`'s pure Haskell GEMM now outperforms `hmatrix` (OpenBLAS)**, achieving 2.1× faster execution at 200 × 200. The dot product is 4–12× faster than hmatrix at all sizes, and matrix–vector multiply is 1.6–2.2× faster at $n \geq 50$. Internal speedups range from 426× (dot product) to 793× (matrix–vector multiply).

The remaining performance gaps are confined to higher-level algorithms (LU, Cholesky, eigenvalue, SVD) that still use massiv's per-element indexing in their inner loops. Extending the raw kernel approach to these algorithms—together with divide-and-conquer eigenvalue and SVD algorithms—represents the clear path to achieving LAPACK-competitive performance across the full operation suite.

`linear-massiv` now demonstrates that **pure Haskell with GHC's native SIMD primops can match or exceed FFI-based BLAS performance** for core linear algebra operations, while providing compile-time dimensional safety, zero FFI dependencies, and user-controllable parallelism. This makes it a practical choice not only for applications prioritising type safety and portability, but increasingly for raw numerical throughput as well.

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