Università di Pisa

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Corso di Laurea in Matematica

Minimal Lagrangian diffeomorphisms via Anti-de Sitter Geometry

Tesi di Laurea Magistrale
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Introduction

A general problem in Teichmüller theory consists in finding canonical maps between hyperbolic surfaces. In particular, it is known that in the isotopy class of the identity of a closed surface there are the minimal Lagrangian maps. More precisely, Labourie and Schoen observed independently that, given a closed hyperbolic surface S and two hyperbolic metrics h and h' on it, there exists a unique diffeomorphism isotopic to the identity such that its graph is a minimal Lagrangian submanifold of $S \times S$ with respect to the product metric $\omega_h - \omega_{h'}$. Many proofs have been provided, and in this thesis we will focus on the proof in the context of Anti-de Sitter three dimensional geometry. In fact, minimal Lagrangian maps are closely related to maximal spacelike surfaces in maximal globally hyperbolic three-dimensional Anti-de Sitter spacetimes.

In the first part of thesis we will define the three models of Anti-de Sitter geometry, which is a Lorentzian geometry of constant sectional curvature -1, and we will study their geometric properties. To define them, we will denote by $\mathbb{R}^{n,2}$ the real vector space $\mathbb{R}^{n+2}$ equipped with the quadratic form

$$q_{n,2}(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2 - x_{n+2}^2.$$

The Anti-de Sitter space is defined as

$$\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} := \{x \in \mathbb{R}^{n,2} \mid q_{n,2}(x) = -1\}/\{\pm \mathbb{1}\} .$$

In dimension three there is a special model which endows $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ with a Lie group structure. The vector space $\mathcal{M}(2,\mathbb{R})$ of 2×2 matrices with real entries and the quadratic form $q = -det$ is identified to $(\mathbb{R}^{2,2}, q_{2,2})$. Via this identification $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cong \mathrm{PSL}(2,\mathbb{R})$, $\mathrm{Isom}_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}) \cong \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ and $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, which is the projectivizazion of the matrices of rank 1, is diffeomorphic to $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$. In this model the space of timelike geodesics is identified with $\mathbb{H}^2 \times \mathbb{H}^2$, which will prove useful in the definition of the Gauss map.

In the second part, following the work of Goeffrey Mess in [7], we focus on the classification of maximal globally hyperbolic (MGH) AdS spacetimes of genus $n \geqslant 2$. These are three dimensional AdS manifolds that admit a closed surface Σ of genus n that intersects every inextensible timelike curve exacly once, maximal by isometric embedding. To do so, first we will prove that a spacetime is maximal globally hyperbolic if and only if is obtained as the quotient of the invisible domain of a proper achronal meridian in $\mathbb{AdS}^{2,1}$. Then we will prove that given a pair of representations of hyperbolic metrics $\rho = (\rho_l, \rho_r)$ on a surface Σ there is a unique achronal meridian equivariant under the action of ρ, and that there is a unique MGH AdS spacetime M_ρ with holonomy ρ obtained as invisible domain. More precicely, the classification result, due to Mess, is the following:

Theorem. *Given a closed surface Σ of genus $n \geqslant 2$, the holonomy map*

$$\rho : \mathcal{MGH}(\Sigma) \to \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$$

is a homeomorphism,

where $\mathcal{MGH}(\Sigma)$ denotes the deformation space of MGH AdS metrics on $\Sigma \times \mathbb{R}$ and $\mathcal{T}(\Sigma)$ is the Teichmüller space of Σ.

In the third part we will focus on the geometric properties of spacelike surfaces in AdS 3-manifolds. In analogy with Riemmanian geometry, we will define on a spacelike surface the first fundamental form I, the second fundamental form $I\!I$ and the shape operator B and we will show that they satisfy the Gauss-Codazzi equations

$$K_I = -1 - \det B \ , \qquad d^{\nabla^I} I\!I = 0 \ .$$

We will then define the Gauss map $G_\sigma : S \to \mathbb{H}^2 \times \mathbb{H}^2$ associated to a spacelike surface as $G_\sigma(x) = (p, q)$ where the pair (p, q) identifies the timelike geodesic orthogonal to $\mathrm{Im}(d_x\sigma)$ at $\sigma(x)$. The components of the Gauss map, denoted by $\Pi_l, \Pi_r : S \to \mathbb{H}^2$, are called *left* and *right projections.*

Proposition. *The pull-back metrics on (S, I) via the left and right projections are be given by*

$$\Pi_l^* g_{\mathbb{H}^2} = I((id - \mathcal{J}B)\cdot, (id - \mathcal{J}B)\cdot),$$

and

$$\Pi_r^*(g_{\mathbb{H}^2}) = I((id + \mathcal{J}B)\cdot, (id + \mathcal{J}B)\cdot),$$

where $\mathcal{J}$ is the almost-complex structure associated to S.

This result will prove to be fundamental in the proof of the main theorem.

In the last part we will introduce the last tools necessary for the proof of the main theorem. In particular we will introduce harmonic and minimal maps and study their close relation, and we will prove the existence and uniqueness of maximal surfaces in MGH spacetimes, which are surfaces with null mean curvature. Then given a pair of representations of hyperbolic metrics $\rho = (\rho_l, \rho_r)$ on a surface Σ we will take the unique MGH spacetime M with holonomy ρ, and we will use the Gauss map associated to the unique maximal surface in M to construct minimal Lagrangian diffeomorphism and prove the following:

Theorem. *Given a closed surface Σ and two hyperbolic metrics h, h' on it, there exists a minimal Lagrangian diffeomorphism $f_0 : (\Sigma, h) \to (\Sigma, h')$ isotopic to the identity.*

CHAPTER 1

Preliminaries on Lorentzian geometry

The scope of this introductory chapter is to briefly recall basic facts about Lorentzian manifolds as later on we will work on Lorentzian manifolds with constant sectional curvature $K = -1$. We will show that, as in Riemannian geometry, Lorentzian manifolds of constant sectional curvature are locally isometric. We will focus on Lorentzian manifolds with maximal isometry group and we will show that generally there are manifolds with maximal isometry group that are not simply connected.

1.1 Basic definitions

A *Lorentzian metric* on a manifold of dimension $n + 1$ is a non degenerate symmetric 2-tensor g of signature $(n, 1)$. A *Lorentzian manifold* is a connected manifold M equipped with a Lorentzian metric g.
We say that a vector $0 \neq v \in TM$ is *spacelike, lightlike, timelike* if $g(v, v)$ is respectively positive, zero or negative. More generally we say that a linear subspace $V \in T_xM$ is *spacelike, lightlike, timelike* if the restriction of g_x to V is positive definite, degenerate or indefinite, respectively.
The set of lightlike vectors, together with the null vector, disconnects the tangent space T_xM into three regions: two convex open cones formed by timelike vectors and the region of spacelike vectors. As a consequence the set of timelike vectors in TM is either connected or has two connected components. In the latter case M is said to be *time-orientable*, and a *time orientation* is a

choice of one of these components. Vectors in the chosen component are said to be *future-directed*, and vectors in the other component are said to be *past-directed*. In analogy with tangent vectors, a differetiable curve is *spacelike, lightlike, timelike* if the tangent vector is respectively spacelike, lightlike or timelike at every point. Moreover, we say that a curve is *causal* if the tangent vector is timelike or lightlike at every point. Given a point x in a time-oriented Lorentzian manifold, we also define the *future* of x as the set $I^+(x)$ of points connected to x by a future-directed causal curve. In an analogous way we define the *past* $I^-(x)$ of x.
As in the Riemannian setting, one can define on a Lorentzian manifold (M, g) the *Levi-Civita connection*. Following the analogy with Riemannian geometry, the Levi-Civita connection determines the Riemann curvature tensor defined by:

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w.$$

We then say that a Lorentzian manifold has constant sectional curvature K if

$$g(R(u, v)v, u) = K(g(u, u)g(v, v) - g(u, v)^2) \tag{1.1}$$

for every pair of vectors $u, v \in T_x M$ and every $x \in M$. We recall that in the Lorentzian setting the sectional curvature can be defined only for planes in $T_x M$ where g is non-degenerate.

1.2 Maximal isometry group and geodesic completeness

Lorentzian manifolds of constant curvature K are locally isometric, more precisely

Lemma 1.1. Let M and N be Lorentzian manifolds of constant curvature K. Every linear isometry $L : T_x M \to T_y N$ extends to an isometry $f : U \to V$ where U and V are neighborhoods of x and y respectively. Any two extensions $f : U \to V$ and $f' : U' \to V'$ coincide on $U \cap U'$. Moreover, if M is simply connected and N is geodesically complete L extends to a local isometry $f : M \to N$.

As in the Riemannian case the proof is a consequence of the classical Cartan-Ambrose-Hicks Theorem (for a reference see [8]).
As a consequence of the lemma

Corollary 1.2. *Let M and N be simply connected, geodesically complete Lorentzian manifolds of constant curvature K. Then any linear isometry $L : T_xM \to T_yN$ extends to a global isometry $f : M \to N$. In particular there is a unique simply connected, geodesically complete Lorentzian manifold of constant curvature K up to isometries.*

In Chapter 2 and 3 we will construct explicit models for $K = -1$.
Another consequence is that the group of isometries $\mathrm{Isom}(M)$ can be realized as a subset of $\mathrm{ISO}(T_{x_o}, TM)$, where $x_0 \in M$ and $\mathrm{ISO}(T_{x_o}, TM)$ is the fiber bundle over M whose fiber over x is the space of linear isometries of $T_{x_0}M$ into T_xM. It follows that the maximal dimension of $\mathrm{Isom}(M)$ is $\dim(O(n+1)) + n + 1 = (n+1)(n+2)/2$.

Definition 1.3. A Lorentzian manifold M has *maximal isometry group* if the action of Isom(M) is transitive and for every $x \in M$ every linear isometry $L : T_xM \to T_xM$ extends to an isometry of M.
Equivalently M has maximal isometry group if $\mathrm{Isom}(M) \subset \mathrm{ISO}(T_xM, TM)$ is a bijection.

It follows that if M has maximal isometry group the dimension of the isometry group is maximal.
From the corollary it follows that every simply connected Lorentzian manifold M has maximal isometry group if it is geodesically complete and has constant curvature. The converse holds:

Lemma 1.4. If M is a Lorentzian manifold with maximal isometry group then M has constant sectional curvature and is geodesically complete.

Proof. Fix a point $x \in M$, the identity component of $\mathrm{O}(T_xM) \simeq \mathrm{O}(n,1)$ acts transitively on spacelike planes, therefore there is a constant K such that Equation 1.1 holds for every pair (u, v) of vectors tangent at x which generate a spacelike plane. Now, for every point $x \in M$ both sides of Equation 1.1 are polynomial in $u, v \in T_xM$. Since the set of pairs (u, v) which generate spacelike planes is open in $T_xM \times T_xM$, Equation 1.1 must hold for every pair of vectors $u, v \in T_xM$. Since $\mathrm{Isom}(M)$ acts transitively on M, it follows that the sectional curvature of M is constant.
Let us now show that the manifold is geodesically complete. Suppose γ is a parametrized geodesic with $\gamma(0) = x$ and $\gamma'(0) = v \in T_xM$, which is defined for a finite time $T > 0$. Let $T_0 = T - \epsilon > 0$. By assumption one can find an isometry $f : M \to M$ such that $f(x) = \gamma(T_0)$ and $df_x(v) = \gamma'(T_0)$. Then

$t \to f \circ \gamma(t - T_0)$ is a parametrized geodesic which provides a continuation of γ beyond T. □

For simply connected Lorentzian manifold there is the following classification result:

Proposition 1.5. *Let M_K be a simply connected Lorentzian manifold of constant curvature K with maximal isometry group. If M is a Lorentzian manifold of constant curvature K then:*

- *M is geodesically complete if and only if there is a local isometry $p : M_K \to M$ which is a universal covering.*
- *M has maximal isometry group if and only if $Aut(p : M_K \to M)$ is normal in $Isom(M_K)$*

Proof. Suppose M is geodesically complete, then by lifting the metric to the universal cover $\widetilde{M}$ one gets a simply connected geodesically complete Lorentzian manifold of constant sectional curvature K which is isometric to M_K. The covering map $p : M_K \to M$ is then a local isometry by construction. The converse is straightforward.
Now let Γ be $\mathrm{Aut}(p : M_K \to M)$, which is a discrete subgroup of $\mathrm{Isom}(M_K)$. Thus M is obtained as the quotient $M = M_K/\Gamma$, where Γ acts freely and properly discontinuously on M_K. The isometry group of M is isomorphic to $N(\Gamma)/\Gamma$, where by $N(\Gamma)$ we denote the normalizer of Γ in $\mathrm{Isom}(M_K)$. The isomorphism is based on the observation that any isometry of M_K which normalizes Γ descends to an isometry of M, and conversely the lifting of any isometry of M must be in $N(\Gamma)$.
Hence the condition that M has maximal isometry group is equivalent to the condition that every element f of $\mathrm{Isom}(M_K)$ descends to the quotient to an isometry of M. This is in turn equivalent to the condition that $f\Gamma f^{-1} = \Gamma$ for every $f \in \mathrm{Isom}(M_K)$, which is the same as saying that Γ is normal in $\mathrm{Isom}(M_K)$. □

It follows that any isometry between connected open subsets of a Lorentzian manifold M with maximal isometry group extends to a global isometry. In particular, if M_K is a Lorentzian manifold of constant sectional curvature K with maximal isometry group, than any Lorentzian manifold M of constant sectional curvature K admits a natural $(\mathrm{Isom}(M_K), M_K)$-structure whose charts are isometries between open subsets of M and open subsets of M_K. We will refer to Lorentzian manifolds of constant sectional curvature K with maximal isometry group as models of constant sectional curvature K. In the following chapters we will study several models of constant sectional curvature -1 called models of Anti-de Sitter geometry.

CHAPTER 2

Anti-de Sitter Space

In this chapter we construct models of Anti-de Sitter geometry, pointing out analogies with the models of hyperbolic space.

2.1 The quadric model

Let us start with a model which is the analogue of the hyperboloid model of hyperbolic space. We denote by $\mathbb{R}^{n,2}$ the real vector space $\mathbb{R}^{n+2}$ equipped with the quadratic form

$$q_{n,2}(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2 - x_{n+2}^2,$$

by $\langle v, w \rangle_{n,2}$ the associated symmetric form and by $O(n,2)$ the group of linear transformation of $\mathbb{R}^{n+2}$ which preserve $q_{n,2}$.
We define

$$\mathbb{H}^{n,1} := \{x \in \mathbb{R}^{n,2} \mid q_{n,2}(x) = -1\},$$

that is a smooth connected submanifold of $\mathbb{R}^{n,2}$ of dimension $n+1$. The tangent space $T_x\mathbb{H}^{n,1}$ coincides with the orthogonal space

$$x^{\perp} = \{y \in \mathbb{R}^{n,2} \mid \langle x, y \rangle_{n,2} = 0\}.$$

The restriction of the symmetric form $\langle \cdot, \cdot \rangle_{n,2}$ to $T\mathbb{H}^{n,1}$ induces a Lorentzian metric on $\mathbb{H}^{n,1}$.
We remark that the hyperbolic space $\mathbb{H}^n$ is isometrically embedded in $\mathbb{H}^{n,1}$ as the submanifold defined by $x_{n+2} = 0,\ x_{n+1} > 0$.

The group $O(n,2)$ acts by isometries on $\mathbb{H}^{n,1}$. In particular if $x \in \mathbb{H}^{n,1}$ and $v_1, \cdots, v_{n+1}$ is an orthonormal basis of $T_x\mathbb{H}^{n,1}$ then the linear transformation sending the standard basis to the basis $v_1, \cdots, v_{n+1}, x$ is in $O(n,2)$. This shows that the action of $O(n,2)$ on $\mathbb{H}^{n,1}$ is transitive and the stabilizer of a point x acts transitively on the space of orthonormal basis of $T_x\mathbb{H}^{n,1}$. These facts imply that $\mathbb{H}^{n,1}$ has maximal isometry group and that the isometry group is $O(n,2)$ and, by Lemma 1.4, $\mathbb{H}^{n,1}$ has constant curvature equal to -1.

2.2 The Klein model

Let us define

$$\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} := \mathbb{H}^{n,1}/\{\pm\mathbb{1}\}.$$

Since $\{\pm\mathbb{1}\}$ is the center of $O(n,2)$, $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ has maximal isometry group and is therefore a model of constant sectional curvature -1. It can also be shown that the center of the isometry group of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is trivial, or equivalently that $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is the quotient of its universal covering by the center of its isometry group. It follows that $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is the *minimal* model of AdS geometry, in the sense that any other model is a covering of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$.
By definition $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is identified with the subset of $\mathbb{R}\mathrm{P}^{n+1}$ given by the timelike directions of $\mathbb{R}^{n,2}$:

$$\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} = \{[x] \in \mathbb{R}\mathrm{P}^{n+1} \mid q_{n,2} < 0\}.$$

Like in hyperbolic geometry, the boundary of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,2}$ is the projectivization of the set of lightlike vectors in $\mathbb{R}^{n,2}$:

$$\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} = \{[x] \in \mathbb{R}\mathrm{P}^{n+1} \mid q_{n,2} = 0\}.$$

Isometries of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ induce projective transformation which preserve $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$.

2.3 The Poincaré model for the universal cover

Notice that $\mathbb{H}^{n,1}$ is not simply connected, being homeomorphic to $\mathbb{R}^n \times S^1$. Therefore $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is not simply connected either, being its double quotient. Hence we aim to construct a simply connected model of Anti-de Sitter geometry, that is the universal cover of $\mathbb{H}^{n,1}$ and $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$. Let $\mathbb{H}^n$ be the hyperboloid model of hyperbolic space. Then

$$\pi(y,t) = (y_1, \cdots, y_n, y_{n+1}\cos(t), y_{n+2}\sin(t))$$

is a map $\pi : \mathbb{H}^n \times \mathbb{R} \to \mathbb{H}^{n,1}$ that is a covering with deck transformation of the form $(y,t) \to (y, t+2k\pi)$ for $k \in \mathbb{Z}$. Composing π with the double quotient, the universal cover for the projective model $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is clearly $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1} := \mathbb{H}^n \times \mathbb{R}$. Pulling back the metric of $\mathbb{H}^{n,1}$ over $\mathbb{H}^n \times \mathbb{R}$ we get a simply connected Lorentzian manifold of constant curvature -1. The metric of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$ is a warped product of the form

$$\pi^* g_{\mathbb{H}^{n,1}} = g_{\mathbb{H}^n} - y_{n+1}^2 dt^2.$$

In this setting we have a central extension, that is a non split short exact sequence

$$0 \to \mathbb{Z} \to \mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}) \to O(n,2) \to 1.$$

This tells us that $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$ has maximal isometry group, hence we have found a simply connected model for AdS geometry. Using the Poincaré model of the hyperbolic space (namely $\mathbb{D}^n$), $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$ is isometric to $\mathbb{D}^n \times \mathbb{R}$ equipped with the metric

$$\frac{4}{(1-r^2)^2}(dx_1^2 + \cdots + dx_n^2) - \Big(\frac{1+r^2}{1-r^2}\Big)^2 dt^2.$$

The Poincaré model of the AdS geometry is then the cylinder $\mathbb{D}^n \times \mathbb{R}$ equipped with this metric. It follows that each slice $\{t = c\}$ is a totally geodesic copy of $\mathbb{H}^n$. Moreover, the vector field $\partial/\partial t$ is a timelike non-vanishing vector field on $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$, which shows that $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$ is time-orientable. $\mathbb{H}^n$ and $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ are time orientable too, since time orientation are preserved by the action of the deck transformations of the covering $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1} \to \mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$.

2.4 Geodesics

In the quadric model. Given $x \in \mathbb{H}^{n,1}$ and $v \in T_x\mathbb{H}^{n,1}$ we want to determine the geodesic through x with speed v. If v is lightlike, then

$$\gamma(t) = x + tv$$

is a geodesic of $\mathbb{R}^{n,2}$ contained in $\mathbb{H}^{n,1}$, hence γ is a geodesic for $\mathbb{H}^{n,1}$.
If v is either timelike or spacelike, the geodesic γ is contained in the intersection $Span(x,v) \cap \mathbb{H}^{n,1}$. In fact the linear transformation T that fixes pointwise $Span(x,v)$ and whose restriction to $Span(x,v)^\perp$ is $-\mathbb{1}_{Span(x,v)^\perp}$ is in $O(n,2)$. Hence $T \circ \gamma = \gamma$ and γ is necessarily contained in $Span(x,v) \cap \mathbb{H}^{n,1}$. The geodesic is given by the expression

$$\gamma(t) = \cosh(t)x + \sinh(t)v$$

if $q_{n,2}(v) = 1$ and

$$\gamma(t) = \cos(t)x + \sin(t)v$$

if $q_{n,2}(v) = -1$.

In the Klein model. In analogy with the hyperbolic case, geodesics in the Klein model are intersection of projective lines with the domain $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} \subset \mathbb{R}\mathrm{P}^{n+1}$. From the study of geodesics in the quadric model, it follows that

- Timelike geodesics correspond to projective lines entirely contained in $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$; they are closed non-trivial loops and have length π,
- Spacelike geodesics correspond to lines meeting $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ transversally in two points. They have infinite length,
- Lightlike geodesics correspond to lines tangent to $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$.

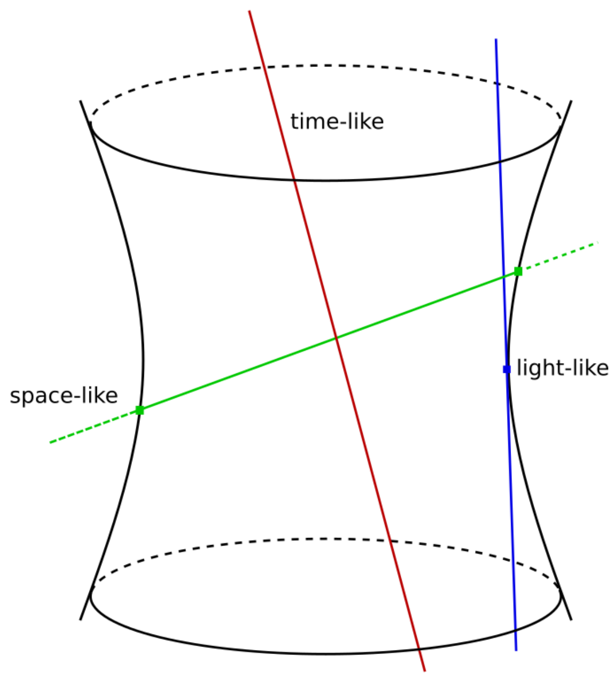

Figure 2.1: Geodesics in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$

In particular, the light cone through a point $[x] \in \mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ coincides with the cone of lines through $[x]$ tangent to $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$.

In an affine chart, timelike geodesics correspond to affine lines which are entirely contained in the Anti de Sitter space, and which are not asymptotic to its boundary; lightlike geodesics are tangent to the one-sheeted hyperboloid, or are asymptotic to it.

Totally geodesic subspaces. Let us briefly discuss more in general totally geodesics subspaces. By an argument analogous to the case of geodesics, totally geodesic subspaces of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ of dimension k are obtained as the intersection of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ with the projectivisation $P(W)$ of a linear subspace W of $\mathbb{R}^{n,2}$ of dimension $k+1$. The negative index of W can be either 2 or 1, for otherwise the intersection $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} \cap P(W)$ would be empty. We have several cases

- If W has signature $(k-1,2)$, then $P(W) \cap \mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is isometric to $\mathbb{A}\mathrm{d}\mathbb{S}^{k-1,1}$.
- If W has signature $(k-2,1)$, then it is a copy of Minkowski space $\mathbb{R}^{k-2,1}$, hence $P(W) \cap \mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is a copy of the Klein model of hyperbolic space.
- If W is degenerate, then $P(W) \cap \mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is a lightlike subspace foliated by lightlike geodesics tangent to the same point of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$.

In the universal cover. In the universal cover $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$ geodesics are the lifts of the geodesics of $\mathbb{H}^{n,1}$ or $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$. Every lightlike or spacelike geodesic in $\mathbb{H}^{n,1}$ and $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is topologically a line, therefore it has a countable number of lifts to $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$. On the other hand in $\mathbb{H}^{n,1}$ and $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ timelike geodesics are topologically circles and are in bijection with timelike geodesics of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$.
Using the Poincaré model we can give an explicit description of lightlike geodesics. In fact, in Lorentzian geometry not only the nature of a vector is conformally invariant but also unparametrized lightlike geodesics are a conformal property ([10], Proposition 2.131):

Theorem 2.1. *If two Lorentzian metrics g, g' on a manifold M are conformal, then they have the same unparametrized lightlike geodesics.*

Because of Theorem 2.1 we can replace the Poincaré metric by the conformally equivalent metric, and often easier to manage in calculation, given by:

$$\frac{4}{(1+r^2)^2}(dx_1^2 + \cdots + dx_n^2) - dt^2 \tag{2.1}$$

Now we observe that the first term in Equation 2.1 is exactly the form of the spherical metric on a hemisphere, pulled-back to the unit disk by the stereographic projection. We call such a metric hemispherical and we denote it by

$g_{\mathbb{S}^n}$. Notice that the boundary of $\mathbb{D}^n$ is an equator for the hemispherical metric, and in fact it is the only equator completely contained in $(\mathbb{D}^n \cup \partial\mathbb{D}^n, g_{\mathbb{S}^n})$, a justification to the fact that we refer to it as *the* equator.
As a consequence, unparametrized lightlike geodesics of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$ going through a point (p_0, t_0) are characterized by the condition that they are mapped to spherical geodesic under the vertical projection $(p, t) \to p$ and moreover

$$t - t_0 = d_{\mathbb{S}^n}(p, p_0)$$

on the geodesic. In particular, these lightlike geodesics meet the boundary of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$ at the point that satisfies the above conditions such that p is on the equator of the hemisphere: as an example, if p_0 is the center of the hemisphere, then the points at infinity of the lightcone over (p_0, t_0) are the horizontal slices $t = t_0 + \pi/2$. This sphere is also the boundary of a hyperplane dual to (p_0, t_0), in a sense that we will explain in the following section.
By an analogous reasoning we can give an explicit description of a lightlike hyperplane in the Poincaré model: the lightlike plane having (p_0, t_0) as a past endpoint, (where now p_0 is on the equator) is precisely $\{(p, t) \mid t - t_0 = d_{\mathbb{S}^n}(p, p_0)\}$ and its future endpoint is $(-p_0, t + \pi.)$

2.5 Polarity in Anti-de Sitter space

The quadratic form $q_{n,2}$ induces a polarity on the projective space $\mathbb{R}\mathrm{P}^{n,1}$, namely the correspondance which associates to the projective subspace $P(W)$ the subspace $P(W^\perp)$. In particular this induces a duality between spacelike totally geodesic subspaces of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$: the dual of a spacelike k-dimensional subspace is a $n-k+1$-dimensional subspace. We indicate with $P_{[x]} = P(x^\perp)$ the hyperplane dual to a point $[x] \in \mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$. Projectively $P_{[x]}$ is characterised as the hyperplane spanned by the intersection of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n-1,1}$ with the lightcone from $[x]$. More geometrically, it can be checked that $P_{[x]}$ is the set of antipodal points to $[x]$ along timelike geodesics through $[x]$. Also, every timelike geodesic through $[x]$ meets $P_{[x]}$ orthogonally at time $\pi/2$. Conversely, given a totally geodesic spacelike hyperplane H, all the timelike geodesics that meet H orthogonally intersect in a single point, which is the dual point of H.
To some extent, this duality between points and planes lifts to the coverings of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$.

In the quadric model. In $\mathbb{H}^{n,1}$ there are two dual planes associated to any point x: the sets

$$P_x^{\pm} = \{\exp_x(\pm(\pi/2)v) \mid q_{n,2}(v) = -1,\, v \text{ future-directed}\}\ .$$

Clearly P_x^+ and P_x^- are antipodal and $P_{-x}^{\pm} = P_x^{\mp}$. The planes $P_x^{\pm}$ disconnect $\mathbb{H}^{n,1}$ in two regions U_x and U_{-x}, where U_x is the region containing x. They can be characterised by

$$U_x = \{y \in \mathbb{H}^{n,1} \,|\, \langle x, y\rangle_{n,1} < 0\}.$$

Spacelike and lightlike geodesics through x do not exit U_x, while all the timelike geodesics through x meet orthogonally $P_x^{\pm}$ and all pass through the point $-x$. More precisely, a point $y \neq x$ is connected to x:

- by a spacelike geodesic if and only if $\langle x, y\rangle_{n,1} < -1$,
- by a lightike geodesic if and only if $\langle x, y\rangle_{(n,1)} = -1$,
- by a timelike geodesic if and only if $|\langle x, y\rangle_{(n,1)}| < 1$.

An immediate consequence is that if y is connected to x by a spacelike geodesic, there is no geodesic joining y to $-x$. Hence the exponential map of $\mathbb{H}^{n,1}$ is not surjective. But as any point $y \in \mathbb{H}^{n,1}$ can be connected through a geodesic either to x or to $-x$, the exponential over $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is surjective.

In the universal cover. Recall that the group of deck transformations for the covering $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1} \to \mathbb{H}^{n,1}$ is $\mathbb{Z}$, where a generator acts by translations of 2π in the $\mathbb{R}$ factor. Hence the preimage of a spacelike plane $P \subset \mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$ is the disjoint union of spacelike planes $(P^k)_{k\in\mathbb{Z}}$, enumerated so that the generator η of $\mathbb{Z}$ acts by sending P^k to P^{k+1}. Moreover, each connected component of $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1} \backslash \bigcup_{k\in\mathbb{Z}} P^k$ is a fundamental domain for the action of deck transformations of the covering $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1} \to \mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$.
Take now $x \in \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1}$ and apply the construction above with $P = P_{\pi'(x)}$ the plane dual to $\pi'(x)$ in $\mathbb{A}\mathrm{d}\mathbb{S}^{n,1}$. We refer to the connected component of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{n,1} \backslash \bigcup_{k\in\mathbb{Z}} P^k_{\pi'(x)}$ containing x as the *Dirichlet domain* R_x of x. By a small abuse of notation, we also denote the planes that bound R_x by P_x^+ and P_x^-.

CHAPTER 3

Anti-de Sitter space in dimension (2+1)

3.1 The PSL$(2,\mathbb{R})$-model

In dimension three there is a special model which endows Anti-de Sitter space with a Lie group structure. Consider the vector space $\mathcal{M}(2,\mathbb{R})$ of 2×2 matrices with real entries and the quadratic form $q = -det$ that has signature $(2,2)$. This gives an identification between $(\mathcal{M}(2,\mathbb{R}), -det)$ and $(\mathbb{R}^{2,2}, q_{2,2})$, unique up to composition by elements in $O(2,2)$, and under this isomorphism $\mathbb{H}^{2,1}$ is identified with the Lie group $SL(2,\mathbb{R})$.
$SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ acts linearly on $\mathcal{M}(2,\mathbb{R})$ by left and right multiplication

$$(A,B)\cdot X := AXB^{-1}$$

preserving the quadratic form $q = -det$. Therefore it induces a representation

$$SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \to O(\mathcal{M}(2,\mathbb{R}), q).$$

The kernel of this representation is given by $K = \{(\mathbb{1},\mathbb{1}), (-\mathbb{1},-\mathbb{1})\}$, and by dimensional argument, the image of this map is the connected component of the identity:

$$\mathrm{Isom}_0(\mathbb{H}^{2,1}) \cong SO_0(\mathcal{M}(2,\mathbb{R}), q) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/K.$$

Using this model there is a natural identification of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ with the Lie group $\mathrm{PSL}(2,\mathbb{R})$ and

$$\mathrm{Isom}_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}) \cong \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$$

acting by left and right multiplication on $\mathrm{PSL}(2,\mathbb{R})$. The stabilizer of the identity in $\mathrm{Isom}_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1})$ is the diagonal subgroup $\Delta < \mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$.

3.2 The boundary of PSL$(2,\mathbb{R})$

From the identification between $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ and $\mathrm{PSL}(2,\mathbb{R})$ we obtain an identification of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ with the boundary of $\mathrm{PSL}(2,\mathbb{R})$ into $P(\mathcal{M}(2,\mathbb{R}))$, which is the projectivization of rank 1 matrices:

$$\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} = \{[X] \in P(\mathcal{M}(2,\mathbb{R})) \mid rank(X) = 1\}.$$

We have a homeomorphism

$$\begin{aligned} \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} &\to \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1 \\ [X] &\to (\mathrm{Im}X, \mathrm{Ker}X) \end{aligned}$$

equivariant under the action of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, acting on $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ as the natural extension of the action on $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, and by left multiplication on $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$.
The equivariance is easily checked observing that $\mathrm{Im}(AXB^{-1}) = A \cdot \mathrm{Im}(X)$ and $\mathrm{Ker}(AXB^{-1}) = B \cdot \mathrm{Ker}(X)$.

Lemma 3.1. The inversion map $\iota[X] = [X]^{-1}$ is a time-reversing isometry of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ which induces the homeomorphism $(x, y) \to (y, x)$ on $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cong \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$.

Proof. The map ι is equivariant with respect to the isomorphism of $\mathrm{PSL}(2,\mathbb{R})\times \mathrm{PSL}(2,\mathbb{R})$ that switches the two factors. To show that it is an time-reversing isometry it thus suffices to check the differential at the identity, which is clearly $d_{\mathbb{1}}\iota = -\mathbb{1}$.
To check the second claim we observe that for an invertible 2×2 matrix we have $(detX)X^{-1} = (trX)\mathbb{1} - X$ by Cayley-Hamilton theorem, so that projectively $[X^{-1}] = [(trX)\mathbb{1} - X]$ along the boundary. This shows that the inversion map on $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ extends to the transormation $[X] \to [(trX)\mathbb{1} - X]$ along the boundary. Let X be a rank 1 matrix, it is traceless if and only if $X^2 = 0$, that is, if and only if $\mathrm{Ker}X = \mathrm{Im}X$, and in this case the statement is easily proven. If $trX \neq 0$ then X is diagonalizable with eigenvalues 0 and trX. Moreover $\mathrm{Ker}X$ and $\mathrm{Im}X$ are the corresponding eigenspaces and it is easily seen that $\mathrm{Ker}((trX)\mathbb{1} - X) = \mathrm{Im}X$ and $\mathrm{Im}((trX)\mathbb{1} - X) = \mathrm{Ker}X$. $\square$

Using the half-plane model for the hyperbolic space $\mathbb{H}^2$, $\mathbb{RP}^1$ is identified with the boundary at infinity $\partial\mathbb{H}^2$ and $\mathrm{PSL}(2,\mathbb{R})$ corresponds to $\mathrm{Isom}_0(\mathbb{H}^2)$, which acts on $\mathbb{RP}^1$ in the canonical way. Therefore, one can identify $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ with $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$ and we can interpret the convergence to $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ in the following way:

Lemma 3.2. A sequence $[X_n] \in \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ converges to $(x,y) \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cong \mathbb{RP}^1 \times \mathbb{RP}^1$ if and only if for every $p \in \mathbb{H}^2$, $X_n(p) \to x$ and $X_n^{-1}(p) \to y$.

Proof. It suffices to check that the condition holds for some $p \in \mathbb{H}^2$ since the action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{H}^2$ is isometric and transitive. Therefore we can take $p = i \in \mathbb{H}^2$. Assume $[X_n]$ converges projectively to $[X] \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. This means that there is a sequence of real numbers λ_n such that

$$\lambda_n X_n = \lambda_n \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} = X.$$

Without loss of generality, we can assume $\lambda_n a_n \to a \neq 0$. Then, using that $det(X) = ad - bc = 0$, we have

$$X(i) = \lim_{n\to\infty} \lambda_n \frac{a_n i + b_n}{c_n i + d_n} = \frac{ai+b}{ci+d} = \frac{ai+b}{ci+\frac{bc}{a}} = \frac{a(ai+b)}{c(ai+b)} = \frac{a}{c}.$$

Using the identification between $\mathbb{RP}^1$ and $\mathbb{R} \cup \{\infty\} = \partial_\infty\mathbb{H}^2$ we have that $\frac{a}{c} = \mathrm{Im}(X)$. The convergence of $X_n^{-1}(i) \to \mathrm{Ker}(X)$ follows from Lemma 3.1. $\square$

In this dimension, $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is double ruled quadric. Given any $(x_0, y_0) \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ we refer to

$$\lambda_{y_0} := \{(x, y_0) \mid x \in \mathbb{RP}^1\}$$

as the *left ruling* through (x_0, y_0), and similarly we refer to

$$\mu_{x_0} := \{(x_0, y) \mid y \in \mathbb{RP}^1\}$$

as the *right ruling* through (x_0, y_0). The left and right rulings describe projective lines in $\mathbb{RP}^3$ which is cointained in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, hence lightlike for the conformal Lorentzian structure of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.

Proposition 3.3. *Let $\pi_l, \pi_r : \mathbb{RP}^1 \times \mathbb{RP}^1 \to \mathbb{RP}^1$ be the canonical projections and $d\theta$ the angular form on $\mathbb{R}$. Then the symmetric product $\pi_l^*(d\theta)\pi_r^*(d\theta)$ is in the conformal class of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.*

Therefore a C^1 curve in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is spacelike when it is locally the graph of an orientation preserving function, and timelike when it is locally the graph of an orientation reversing function.

3.3 Levi-Civita connection and cross-product

Any Lie group G is equipped with two natural connections, the *left-invariant connection* D^l and the *right-invariant connection* D^r. For example, D^l is uniquely determined by the condition that left-invariant vector fields are parallel, and is left-invariant in the sense that

$$(L_g)_*(D^l_V W) = D^l_{(L_g)_*(V)}(L_g)_*(W),$$

where $L_g : G \to G$ is the left multiplication by g. The connection D^l at a point $g \in G$ can be expressed as

$$D^l_V W = \left.\frac{d}{dt}\right|_{t=0} (L_{g\gamma(t)^{-1}})_*(W_{\gamma(t)})$$

where $\gamma(t)$ is a path with $\gamma(0) = g$ and $\gamma'(0) = V$. In an analogous way, one defines the right-invariant connection D^r. Both connections D^l and D^r are flat and compatible with any metric which is left-invariant or right-invariant respectively. But these connections are not torsion-free: by computation one obtains that

$$\tau^l(V, W) = -[V, W]$$

and

$$\tau^r(V, W) = [V, W].$$

A direct computation also shows that

$$D^r_V W - D^l_V W = [V, W].$$

Now, given a bi-invariant pseudo-Riemannian metric on G, its Levi-Civita connection ∇ can be expressed via D^l and D^r as

$$\nabla_V W = \frac{1}{2}(D^l_V W + D^r_V W).$$

In our case, $\mathrm{PSL}(2,\mathbb{R})$ is equipped with a natural Lorentzian cross-product which can be used to rewrite the expression for the Levi-Civita connection. The cross-product on $\mathrm{PSL}(2,\mathbb{R})$ is a $T\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$-valued 2-form $(V, W) \to V \boxtimes W$, which is defined by the equality

$$\langle V \boxtimes W, U\rangle = \Omega(V, W, U) \qquad \forall U \in T\mathbb{A}\mathrm{d}\mathbb{S}^{2,1},$$

where $\langle \cdot, \cdot \rangle$ is the Anti-de Sitter metric and Ω is the associated volume form. In this setting

$$[V, W] = -2V \boxtimes W.$$

This permits to rewrite the expression for the Levi-Civita connection of left-invariant vector fields as

$$\nabla_V W = -V \boxtimes W = D^l_V W - V \boxtimes W = D^r_V W + V \boxtimes W.$$

3.4 Geodesics in $\mathbf{PSL}(2,\mathbb{R})$

Let us start by understanding the geodesics through the identity. Using the Lie group structure of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ it suffices to understand the one-parameter subgroups of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. We get the following:

- Timelike geodesics are, up to conjugacy, of the form

$$\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

  under the identification of $\mathrm{PSL}(2,\mathbb{R})$ with $\mathrm{Isom}(\mathbb{H}^2)$ they are elliptic one-parameter groups fixing a point in $\mathbb{H}^2$.

- Spacelike geodesics are, up to conjugacy,

$$\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} ;$$

  these are hyperbolic one-parameter groups fixing two points in the boundary of $\mathbb{H}^2$

- Lightlike geodesics are the parabolic one-parameter groups conjugate to

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} .$$

Using this description of timelike geodesics through $\mathbb{1}$, we can also interpret the duality in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. Recalling that the dual plane of a point A is the set of antipodal points along timelike geodesics through A, one sees that the dual plane of $\mathbb{1}$ consists of elliptic order-two isometries of $\mathbb{H}^2$. Equivalently, this is the set of (projective classes) of traceless matrices, that is (by the Cayley-Hamilton theorem)

$$P_{\mathbb{1}} = \{[J] \in \mathrm{PSL}(2,\mathbb{R}) \mid J^2 = -\mathbb{1}\}.$$

The boundary at infinity of $P_{\mathbb{1}}$ is made of traceless matrices of rank 1.
The stabilizer of $\mathbb{1}$ is the diagonal subgroup of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, and it also acts on the dual plane $P_{\mathbb{1}}$ by conjugation. Therefore, the following statement is straightforward:

Lemma 3.4. The map from $\mathbb{H}^2$ to $P_{\mathbb{1}}$, sending $p \in \mathbb{H}^2$ to the elliptic order-two element in $\mathrm{PSL}(2,\mathbb{R})$ fixing p, is a $\mathrm{PSL}(2,\mathbb{R})$-equivariant isometry.

Timelike geodesics. To get a complete description of timelike geodesics it suffices to let the isometry group of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ act on geodesics through the identity.

Proposition 3.5. *There is a homeomorphism between the space of timelike geodesics of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ and $\mathbb{H}^2 \times \mathbb{H}^2$. The homeomorphism is equivariant for the action of $Isom_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}) \cong PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$.*

Proof. We define the homeomorphism as follows. Given a pair $(p,q) \in \mathbb{H}^2 \times \mathbb{H}^2$, we associate to it the set

$$L_{p,q} = \{X \in \mathrm{PSL}(2,\mathbb{R}) \; | X \cdot q = p\}.$$

By a previous discussion, geodesics through the identity are of the form $L_{p,p}$ for some $p \in \mathbb{H}^2$. The action of $(A,B) \in \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ sends $L_{p,q}$ to $L_{A \cdot p, B \cdot q}$ which implies that the map is equivariant under the action of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$, that $L_{p,q}$ is an unparametrized timelike geodesic and that all unparametrized timelike geodesics are of this form. This shows in particular that the map defined is surjective. Suppose now that $L_{p,q} = L_{p',q'}$ for some $(p,q) \neq (p',q')$. In particular, there exists an isometry of $\mathbb{H}^2$ sending p to q and p' to q', which is unique. This gives a contradiction and therefore the map is injective. □

Spacelike geodesics. Let l be a geodesic of $\mathbb{H}^2$. The one-parameter group of hyperbolic transformations fixing l as an oriented geodesic is a spacelike geodesic through the identity. By an argument similar to the previous proposition, one proves that every spacelike geodesic is of the form

$$L_{l,j} = \{X \in \mathrm{PSL}(2,\mathbb{R}) \mid X \cdot j = l \;\text{ as oriented geodesics}\}$$

where l and j are oriented geodesics of $\mathbb{H}^2$. Every spacelike geodesic can be expressed in this form in two ways, as one can change the orientation of both l and j. Every such choice corresponds to a choice of orientation for the spacelike geodesic. In other words, we can state the following:

Proposition 3.6. *There is a homeomorphism between the space of oriented spacelike geodesics of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ and the product of two copies of $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2 \setminus \Delta$, the space of oriented geodesics of $\mathbb{H}^2$. The homeomorphism is equivariant for the action of $Isom_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}) \cong PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$.*

CHAPTER 4

Globally hyperbolic spacetimes

From now on we will only work with Lorentzian manifolds of dimension (2+1). The aim of this section is to classify maximal globally hyperbolic spacetimes with compact Cauchy surface of genus $n \geqslant 2$. To do so, we have to study causal properties of Anti-de Sitter geometry and isometric actions, which constitute the fundamental setup for the proofs of Mess' classification results.
Here we will first study achronal sets in the conformal compactification of Anti-de Sitter space, a notion that makes sense in the universal cover $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, and then adapt the notion for subsets of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. We will also introduce the fundamental notions of invisible domain and of domain of dependence, and describe their properties.

4.1 Achronal and acausal set

Definition 4.1. A subset $X \subset \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ is *achronal* (resp. *acausal*) if no pair of points in X are connected by timelike (resp. causal) lines in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$

Consider the Poincaré model $\mathbb{D} \times \mathbb{R}$ of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ equipped with the hemispherical metric $g_{\mathbb{S}^2} - dt^2$ introduced in Section 2.4. The following lemma gives a charecterization of achronal/acausal set:

Lemma 4.2. A subset X of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ is achronal (resp. acausal) if and only if it is the graph of a function $f : D \to \mathbb{R}$ that is 1-Lipschitz (resp.

strictly 1-Lipschitz) with respect to the distance induced by the hemispherical metric $g_{\mathbb{S}^2}$. Here D denotes the projection of X to the $\mathbb{D}$ factor.

Proof. Assume that X is an achronal set. The restriction of the projection $\pi_{\mathbb{D}} : \mathbb{D} \times \mathbb{R} \to \mathbb{D}$ to X is injective, since vertical lines in the Poincaré model are timelike. So X can be regarded as the graph of a function $f : D \to \mathbb{R}$. Imposing that $(x, f(x))$ and $(y, f(y))$ are not related by a timelike curve we can deduce that

$$|f(x) - f(y)| \leqslant d_{\mathcal{S}^2}(x, y), \tag{4.1}$$

where $d_{\mathcal{S}^2}$ is the hemispherical distance. The same argument shows that the graph of a 1-Lipschitz function defined on some subset of $\mathbb{D}$ is achronal.
Moreover, two points (x, t) and (y, s) are on the same lightlike geodesic if and only if $|t - s| = d_{\mathcal{S}^2}(x, y)$. Therefore X is acausal if and only if the inequality in (4.1) is strict. $\square$

A 1-Lipschitz function on a region $D \subset \mathbb{D}$ extends uniquely to the boundary of D. As a consequence of the previous lemma we have:

Lemma 4.3. An achronal subset X in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is properly embedded if and only if it is a global graph over $\mathbb{D}$. In this case it extends uniquely to the graph of a 1-Lipschitz function over $\mathbb{D} \cup \partial\mathbb{D}$.

In the following, we refer to an achronal subset X in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ which is the graph of a 1-Lipschitz function defined on a domain of $\mathbb{D}$ as an *achronal surface*. Recall the following definition

Definition 4.4. Given a surface S and a Lorentzian manifold (M, g), a C^1 immersion $\sigma : S \to M$ is *spacelike* if the pull-back metric $\sigma^* g$ is a Riemannian metric. In this setting $\sigma(S)$ is said to be a *spacelike surface*.

A spacelike surface is locally acausal, but there are examples of spacelike surfaces which are not even achronal. A spacelike surface is acausal when it is properly embedded in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$.

Lemma 4.5. Let S be a properly embedded achronal surface of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$. If a lightlike geodesic segment γ joins two points of S then γ is contained in S.

Proof. Let $f^S : \overline{\mathbb{D}} \to \mathbb{R}$ be the function 1-Lipschitz defining S. If γ joins $(x, f^S(x))$ and $(y, f^S(y))$ then (up to switching the role of x and y) $f^S(y) = f^S(x) + d_{\mathcal{S}^2}(x, y)$. Moreover, γ consists of points of the form $(z, f^S(x) +$

$d_{\mathcal{S}^2}(x,z))$, for z lying on the $g_{\mathcal{S}^2}$-geodesic segment joining x to y. By achronality of S we have

$$f^S(z) - f^S(x) \leqslant d_{\mathcal{S}^2}(x,z)$$

and

$$f^S(x) - f^S(z) \leqslant d_{\mathcal{S}^2}(z,y) = d_{\mathcal{S}^2}(x,y) - d_{\mathcal{S}^2}(x,z),$$

which implies that $f^S(z) \geqslant f^S(x) + d_{\mathcal{S}^2}(x,z)$. Therefore we can conclude $f^S(z) = f^S(x) + d_{\mathcal{S}^2}(x,z)$, proving that γ is contained in S. □

Given a function $f : \overline{\mathbb{D}} \to \mathbb{R}$, we define its oscillation as

$$osc(f) := \max_{y\in\overline{\mathbb{D}}} f(y) - \min_{y\in\overline{\mathbb{D}}} f(y).$$

Lemma 4.6. Let S be a properly embedded achronal surface, defined as the graph of $f^S : \overline{\mathbb{D}} \to \mathbb{R}$, then $osc(f^S) \leqslant \pi$. Moreover $osc(f^S) = \pi$ if and only if S is a lightlike plane.

Proof. As f^S is 1-Lipschitz and the diameter of $\mathbb{D}$ for $g_{\mathcal{S}^2}$ is π, then $osc(f^S)$ is bounded by π. Moreover, if $osc(f^S) = \pi$ then there are two antipodal points $y, y' \in \partial\mathbb{D}$ such that $f^S(y') = f^S(y) + \pi$. Recall that the lightlike plane with past and future points $(y, f^S(y))$ and $(y', f^S(y) + \pi)$ is

$$P = \{(x,t) \mid t = f^S(y) + d_{\mathcal{S}^2}(x,y)\}$$

and it is foliated by lightlike geodesics joining $(y, f^S(y))$ to $(y', f^S(y)+\pi)$. By Lemma 4.5, P is included in S. Since P and S are both global graphs over $\overline{\mathbb{D}}$, $S = P$. □

4.2 Invisible domain

Definition 4.7. Let X be an achronal set in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, the *invisible domain* of X is the subset $\Omega(X) \subset \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ of points which are connected to X by no causal path.

By a result of McShane [6] any 1-Lipschitz function on a subset of a metric space admits a 1-Lipschitz extension everywhere. Hence any achronal set is a subset of a properly embedded achronal surface. Given an achronal set X defined as $f^X : D \to \mathbb{R}$ there are two particular extensions called *extremal extensions*:

$$f^X_-(y) = \sup\nolimits_{x\in D}\{f^X(x) - d_{\mathcal{S}^2}(x,y)\}, \quad f^X_+(y) = \inf\nolimits_{x\in D}\{f^X(x) + d_{\mathcal{S}^2}(x,y)\}$$

Lemma 4.8. Let X be a closed achronal subset of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ and let $S_\pm(X)$ be the graphs of the extremal extensions $f_\pm^X$:

- The properly embedded surfaces $S_-(X)$ and $S_+(X)$ are achronal with $S_\pm(X) \subset \overline{\mathrm{I}^\pm(S_\mp(X))}$, and $\Omega(X) = \mathrm{I}^+(S_-(X)) \cap \mathrm{I}^-(S_+(X))$.
- Every achronal subset containing X is contained in $S_-(X) \cup \Omega(X) \cup S_+(X)$.
- Every point in $S_\pm(X)$ is connected to X by at least one lightlike geodesic segment which is entirely contained in $S_\pm(X)$. Moreover $S_+(X) \cap S_-(X)$ is the union of X and all lightlike geodesic segments joining points of X.

Remark 4.9. *$S_-(X) \cup \Omega(X) \cup S_+(X)$ is the set of points connected to any point of X by spacelike or lightlike geodesic. In particular $\Omega(X)$ consists of points connected to any point of X by a spacelike geodesic.*

Remark 4.10. *The intersection of any properly embedded achronal surface containing X with $S_\pm(X)$ is a union of lightlike geodesic segments with an endpoint in X. In particular any properly embedded acausal surface containing X is contained in the region $\Omega(X)$.*

4.3 Achronal meridians

Definition 4.11. An *achronal meridian* in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ is a subset Λ of $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ that is the graph of a 1-Lipschitz function $f : \partial\mathbb{D} \to \mathbb{R}$.

The importance of these definitions will be evident in the following sections. The invisible domain of a achronal meridian will be a fundamental tool in the study of globally hyperbolic spacetimes.

Lemma 4.12. Let Λ be an achronal meridian in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ different from the boundary of a lightlike plane. Then $S_+(\Lambda) \cap S_-(\Lambda) = \Lambda$. Moreover there is an achronal properly embedded surface in $\Omega(\Lambda)$ whose boundary in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ is Λ.

Proof. By Lemma 4.6 the maximal oscillation of $f : \partial\mathbb{D} \to \mathbb{R}$ that defines Λ is smaller than π. If a lightlike geodesic connects $(x, f(x))$ to $(y, f(y))$ then x and y are not antipodal. Then they are connected by a unique length-minimizing geodesic in $\overline{\mathbb{D}}$ for the hemispherical metric, which lies in $\partial\mathbb{D}$. So the lightlike line connecting $(x, f(x))$ to $(y, f(y))$ is contained in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$. By Lemma 4.8 we conclude that $S_+(\Lambda)$ and $S_-(\Lambda)$ do not meet in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ and

$S_+(\Lambda) \cap S_-(\Lambda) = \Lambda$.
To conclude, the function $(f_- + f_+)/2$ is 1-Lipschitz and defines an achronal properly embedded surface contained in $\Omega(\Lambda)$ whose boundary is Λ. $\square$

To conclude this section let us state two results on the invisible domain of an achronal meridian in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$:

Proposition 4.13. *Let Λ an achronal meridian in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ different from the boundary of a lightlike plane. Then*

- *$x \in \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ lies in $\Omega(\Lambda)$ if and only if Λ is contained in the interior of the Dirichlet region R_x.*
- *The length of the intersection of $\Omega(\Lambda)$ with any timelike geodesic of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ is at most π.*

Proof. By remark 4.9 a point x lies in $\Omega(\Lambda)$ if and only if it is connected to any point of Λ by a spacelike geodesic, and the region of points connected to x by a spacelike geodesic is exactly R_x.
For the second statement, suppose that a timelike geodesic γ meets $\Omega(\Lambda)$ at a point x. By the first statement we have $\Omega(\Lambda) \subset R_x$, so the length of $\gamma \cap \Omega(\Lambda)$ is smaller than the length of $\gamma \cap R_x$, which is at most π. $\square$

Following from this proposition we have:

Proposition 4.14. *Let Λ an achronal meridian in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ different from the boundary of a lightlike plane. The invisible domain $\Omega(\Lambda)$ is contained in a Dirichlet region. Moreover the closure of $\Omega(\Lambda)$ is contained in a Dirichlet region unless Λ is the boundary of a spacelike plane.*

4.4 Domain of dependance

Definition 4.15. Given an achronal subset X in a Lorentzian manifold (M, g), the *domain of dependance* of X is the set

$$\mathcal{D}(X) = \{p \in M \mid \text{every inextensible causal curve through } p \text{ meet } X \text{ exacly once}\}.$$

If $\mathcal{D}(X) = M$ we say that M is *globally hyperbolic* with *Cauchy surface X*.

Globally hyperbolic spacetimes have strong geometric property we summarize in the following theorem:

Theorem 4.16. *Let M be a globally hyperbolic spacetime, then*

- *M is diffeomorphic to $\Sigma \times \mathbb{R}$ where Σ is a Cauchy surface.*
- *Any two Cauchy surfaces are diffeomorphic.*
- *There exists a submersion $\tau : M \to \mathbb{R}$ whose fibers are Cauchy surfaces.*

Remark 4.17. *$\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ is not globally hyperbolic. In fact if X is achronal it is contained in the graph of a 1-Lipschitz function $f : \overline{\mathbb{D}} \to \mathbb{R}$. If $t_0 > \sup f$ and $\xi \in \partial\mathbb{D}$ then any lightlike ray with past end-point (ξ, t_0) does not intersect X.*

Lemma 4.18. Given an achronal meridian Λ in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, any Cauchy surface in $\Omega(\Lambda)$ is properly embedded with boundary at infinity Λ.

Proof. Let S be a Cauchy surface in $\Omega(\Lambda)$, then for every $x \in \mathbb{D}$ the vertical line through x in the Poincaré model meets $\Omega(\Lambda)$ and its intersection with $\Omega(\Lambda)$ meet S by definition of Cauchy surface. Then S is a graph over $\mathbb{D}$, proving S is properly embedded, with $\partial S = \Lambda$. $\square$

Proposition 4.19. *Let Λ be an achronal meridian in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ different from the boundary of a lightlike plane. If S is a properly embedded surface in $\Omega(\Lambda)$ then $\mathcal{D}(S) = \Omega(\Lambda)$. In particular $\Omega(\Lambda)$ is a globally hyperbolic spacetime.*

Proof. Take any inextensible causal path through $x \in \Omega(\Lambda)$. Its future end-point is in $S_+(\Lambda)$ since, by definition of $\Omega(\Lambda)$, x cannot be connected to Λ by a causal path. The same argument shows that the past end-point is in $S_-(\Lambda)$. Since the inextensible causal path meets both $S_+(\Lambda)$ and $S_-(\Lambda)$, then it must meet S, hence $x \in \mathcal{D}(S)$.
Conversely, if x is not in $\Omega(\Lambda)$ then one can find an inextensible causal path joining x to Λ. Therefore x is not in $\mathcal{D}(S)$. $\square$

As a consequence of this Proposition we have that the domain of dependance of a properly embedded surface in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ only depends on the boundary at infinity:

Corollary 4.20. *If S and S' are properly embedded spacelike surface in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ then $\mathcal{D}(S) = \mathcal{D}(S')$ if and only if $\partial S = \partial S'$.*

4.5 Properly achronal set in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$

As $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ contains closed timelike lines, it does not contain any achronal subset. Therefore we have to adapt the definition to work in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$:

Definition 4.21. A subset X of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is a *proper achronal subset* if there exists a spacelike plane P such that X is contained in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \setminus \overline{P}$ and is an achronal subset of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \setminus \overline{P}$.

The definition makes sense since $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \setminus \overline{P}$ does not contain closed causal curves. Moreover $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \setminus \overline{P}$ is simply connected, so it admits an isometric embedding into $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ whose image is a Dirichlet region. Therefore, if X is a proper achronal set in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ it admits a section to $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ whose image $\widetilde{X}$ is achronal. Conversely if $\widetilde{X}$ is an achronal subset of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ different from the boundary of a lightlike plane then, by Lemma 4.6, it is contained in a Dirichlet region whose projection to $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ sends $\widetilde{X}$ to a proper achronal subset.

We now focus on proper achronal meridians of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, which are proper achronal embedded circles of the boundary of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. The following example of proper achronal meridian will be extensively used later to classify maximal globally hyperbolic spacetimes.

Lemma 4.22. Let $\varphi : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ be an orientation preserving homeomorphism. Then the graph of φ, $\Lambda_\varphi \subset \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1 \simeq \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is a proper achronal subset and any lift $\widetilde{\Lambda}_\varphi$ is an achronal meridian in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$.

Proof. Λ_φ is locally achronal. In fact let U and V be intervals around x and $\varphi(x)$ and θ_1, θ_2 positive coordinates on U and V respectively. By Proposition 3.3, timelike curves $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ in $U \times V$ are characterized by the property that $\theta_1'(\gamma_1(t))\theta_2'(\gamma_2(t)) < 0$. Therefore, by the assumption that φ is orientation preserving, points in $\Lambda_\varphi \cap U \times V$ are not related by a timelike curve contained in $U \times V$.
Now let us prove that there exists a spacelike plane P such that $\Lambda_\varphi \cap \overline{P} = \emptyset$, and therefore that Λ_φ is properly achronal. Consider the identification $\mathbb{R}\mathrm{P}^1 \simeq \mathbb{R} \cup \{\infty\}$ and take $\varphi_0 \in \mathrm{PSL}(2,\mathbb{R})$ such that $\varphi_0^{-1}\varphi(0) = 1$, $\varphi_0^{-1}\varphi(1) = \infty$ and $\varphi_0^{-1}\varphi(\infty) = 0$. $\varphi_0^{-1}\varphi$ sends the intervals $(\infty, 0)$, $(0, 1)$ and $(1, \infty)$ respectively to $(0, 1)$, $(1, \infty)$, $(\infty, 0)$, thus $\varphi_0^{-1}\varphi$ has no fixed points. This means that the graph of φ does not meet the graph of φ_0, which is the asymptotic boundary of a spacelike plane $P_{\varphi_0^{-1}}$.
Let now $\widetilde{\Lambda}_\varphi$ be a lift of Λ_φ to $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$. As Λ_φ is contained in a simply connected region of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, $\widetilde{\Lambda}_\varphi$ is a closed locally achronal curve in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$. The projection $\widetilde{\Lambda}_\varphi \to \partial\mathbb{D}$ is locally injective and $\widetilde{\Lambda}_\varphi$ is compact, hence the map is a covering. On the other hand, since Λ_φ is homotopic to the boundary of a

plane in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, $\widetilde{\Lambda}_\varphi$ is homotopic to $\partial\mathbb{D}$ in $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$. Therefore the projection $\widetilde{\Lambda}_\varphi \to \partial\mathbb{D}$ is bijective and it follows that $\widetilde{\Lambda}_\varphi$ is achronal. $\square$

Remark 4.23. *All the results we have proven for achronal sets in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ can be rephrased for proper achronal sets in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.*

Proposition 4.24. *Let Λ be a proper achronal meridian in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ and denote by $\widetilde{\Lambda}$ any lift to $\partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$. Then the universal cover of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ maps $\Omega(\widetilde{\Lambda})$ injectively to the domain*

$$\Omega(\Lambda) := \{x \in \mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \mid P_x \cap \Lambda = \varnothing\}.$$

Proof. By Proposition 4.14, $\Omega(\Lambda)$ is contained in a Dirichlet region $R_{\widetilde{x}}$, hence the restriction of $p : \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ to $\Omega(\widetilde{\Lambda})$ is injective and its image is contained in $p(R_{\widetilde{x}})$ which is the complement in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \cup \partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ of the spacelike plane P_x dual to $x = p(\widetilde{x})$. By the first point of Proposition 4.13 one can pick any $\widetilde{x} \in \Omega(\widetilde{\Lambda})$, which shows that $p(\Omega(\widetilde{\Lambda}))$ is contained in $\Omega(\Lambda) := \{x \in \mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \mid P_x \cap \Lambda = \varnothing\}$.
For the converse inclusion, let $x \in \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ be a point whose dual plane P_x does not meet Λ. The set $p^{-1}(P_x)$ disconnect $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \cup \partial\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ in a disjoint union of Dirichlet regions centered at the preimages of x. $\widetilde{\Lambda}$ is contained in exacly one such region, say $R_{\widetilde{x}}$, and $\widetilde{x} \in \Omega(\widetilde{\Lambda})$ by the first point of Proposition 4.13. This implies thet $x = p(\widetilde{x})$ lies in $p(\Omega(\widetilde{\Lambda}))$. $\square$

When Λ is the graph of an orientation preserving homeomorphism $\varphi : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$, there is yet another characterization of $\Omega(\Lambda)$ using the identification $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \simeq \mathrm{PSL}(2,\mathbb{R})$.

Corollary 4.25. *Let $\varphi : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ be an orientation preserving homeomorphism. Then $x \in \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ lies in $\Omega(\Lambda_\varphi)$ if and only if $x \circ \varphi$ has no fixed point as a homeomorphism of $\mathbb{R}\mathrm{P}^1$.*

Proof. The dual plane of x, as an element of $\mathrm{PSL}(2,\mathbb{R})$, meets $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ along the graph of x^{-1}, say $\lambda_{x^{-1}}$. Then $x \in \Omega(\Lambda_\varphi)$ if and only if $\lambda_{x^{-1}} \cap \Lambda_\varphi = \varnothing$, which is an equivalent to requiring that $x \circ \varphi$ has no fixed point on $\mathbb{R}\mathrm{P}^1$. $\square$

Proposition 4.26. *Let $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ be a proper spacelike immersion. Then*

- *σ is a proper embedding*
- *σ lifts to a proper embedding $\widetilde{\sigma} : S \to \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$*

- *The boundary at infinity of $\sigma(S)$ is a proper achronal meridian Λ in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$*
- $\mathcal{D}(\sigma(S)) = \Omega(\Lambda)$

Proof. Denote by $\widehat{S}$ the covering of S admitting a lift $\widehat{\sigma} : \widehat{S} \to \mathbb{H}^{2,1}$. Either $\widehat{S} = S$ or it is a $2:1$ covering. In both cases $\widehat{\sigma}$ is a proper immersion.
Consider now the identification $\pi : \mathbb{H}^2 \times \mathbb{RP}^1 \to \mathbb{H}^{2,1}$. The induced projection $pr : \mathbb{H}^{2,1} \to \mathbb{H}^2$ is a proper fibration with timelike fibers. In particular $\widehat{\sigma}$ is trasverse to the fibers of pr, and it follows that $pr \circ \widehat{\sigma} : \widehat{S} \to \mathbb{H}^2$ is a proper local diffeomorphisms, hence a covering map. Since $\mathbb{H}^2$ is simply connected, we can deduce that $pr \circ \widehat{\sigma} : \widehat{S} \to \mathbb{H}^2$ is a homeomorphism, $\widehat{\sigma}$ is an embedding, and $\widehat{S}$ is homeomorphic to the plane.
In particular we can lift $\widehat{\sigma}$ to the universal cover $\widetilde{\sigma} : \widehat{S} \to \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$, which is still a proper spacelike embedding. By Lemma 4.2 and Lemma 4.3 the image of $\widetilde{\sigma}$ is an achronal surface whose boundary is an achronal meridian, and by Lemma 4.6 it is contained in a Dirichlet region. It follows that $\widetilde{\sigma}(\widehat{S})$ is contained in a Dirichlet domain of the covering map $\mathbb{H}^{2,1} \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, on which we know that the covering map is injective. In particular σ is also injective, hence $\widehat{S} = S$ and this concludes the proof. $\square$

We have the following analogue version of Corollary in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$:

Corollary 4.27. *If S and S' are properly embedded spacelike surface in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, then $\mathcal{D}(S) = \mathcal{D}(S')$ if and only if $\partial S = \partial S'$.*

4.6 Globally hyperbolic spacetimes

The aim of this section and the following section is to study maximal globally hyperbolic (MGH) Anti-de Sitter spacetimes containing a compact Cauchy surface of genus n (in short we say that the globally hyperbolic spacetime has genus n). We will be interested mainly in the case $n \geqslant 2$ because later on we will study MGH spacetimes whose Cauchy surface is a closed hyperbolic surface.

Remark 4.28. *It can be shown that given $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ spacelike immersion where $\sigma^*(g_{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}})$ is a complete Riemannian metric, then σ is a proper embedding. In particular there are no globally hyperbolic Anti-de Sitter spacetime of genus zero ([2]).*

Proposition 4.29. *Let M be a globally hyperbolic Anti-de Sitter spacetime of genus $n \geqslant 1$. Then*

- *The developing map $dev : \widetilde{M} \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is injective.*
- *If Σ is a Cauchy surface of M, then the image of dev is contained in $\Omega(\Lambda)$ where Λ is the boundary of $dev(\tilde{\Sigma})$.*
- *If $\rho : \pi_1(M) \to Isom(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1})$ is the holonomy representation, $\rho(\pi_1(M))$ acts freely and properly discontinuously on $\Omega(\Lambda)$ and $\Omega(\Lambda)/\rho(\pi_1(M))$ is a globally hyperbolic spacetime containing M.*

Proof. Let $\widetilde{dev} : \widetilde{M} \to \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ be a lift of dev to the universal cover. By Theorem 4.16, the spacetime M admits a foliation by smooth spacelike surfaces $(\Sigma_t)_{t\in\mathbb{R}}$ of genus $n \geqslant 1$, such that Σ_t is contained in the future cone of $\Sigma_{t'}$ for $t > t'$. Let $\tilde{\Sigma}_t$ be the foliation on $\widetilde{M}$ obtained lifting the one on M. Since Σ_t is closed, the metric induced on Σ_t is complete, and so is the one on $\tilde{\Sigma}_t$. By Remark 4.28 we have that the restriction of $\widetilde{dev}$ to $\tilde{\Sigma}_t$ is a proper embedding, since $\widetilde{dev}$ is a local isometry.
Assume now by contradiction that $\tilde{\Sigma}_t \cap \tilde{\Sigma}_{t'} \neq \varnothing$ for $t \geqslant t'$. This means that there is $x \in \tilde{\Sigma}_t$ such that $\widetilde{dev}(x) \in \widetilde{dev}(\tilde{\Sigma}_{t'})$. By assumption x is connected to $\tilde{\Sigma}_{t'}$ by a timelike arc η in $\widetilde{M}$ and $\widetilde{dev}$ is therefore a timelike arc in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1}$ with end-points in $\widetilde{dev}(\tilde{\Sigma}_{t'})$, which contradicts the achronality of $\widetilde{dev}(\tilde{\Sigma}_{t'})$. This prove that $\widetilde{dev}$ is injective. Moreover we conclude that $\widetilde{dev}(\tilde{\Sigma}_t)$ is a Cauchy surface of $\widetilde{dev}(\widetilde{M})$. Therefore, by Proposition 4.26, $\widetilde{dev}(\widetilde{M}) \subset \mathcal{D}(\widetilde{dev}(\tilde{\Sigma}_t)) = \Omega(\tilde{\Lambda})$, where $\tilde{\Lambda}$ is the boundary at infinity of $\widetilde{dev}(\tilde{\Sigma}_t)$, which proves the second point.
The map $\widetilde{dev}$ is $\tilde{\rho}$-equivariant, for a representation $\tilde{\rho} : \pi_1(M) \to \mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1})$ which is a lift of the holonomy of M. As $\widetilde{dev}$ is $\tilde{\rho}$-invariant, then so are $\tilde{\Lambda}$ and $\Omega(\tilde{\Lambda})$. Now we shall prove that the action of $\pi_1(M)$ on $\Omega(\tilde{\Lambda})$ given by $\tilde{\rho}$ is proper. This also proves that the action is free since $\pi_1(M)$ has no torsion.
If K is a relatively compact set in $\Omega(\tilde{\Lambda})$ then

$$X_K := (I^+(K) \cup I^-(K)) \cap \widetilde{dev}(\tilde{\Sigma}_t)$$

is relatively compact as well. Since the action of $\pi_1(M)$ on $\tilde{\Sigma}_t$ is proper, so is the one on $\widetilde{dev}(\tilde{\Sigma}_t)$ and, moreover, $X_{\gamma K} = \gamma(X_K)$. Therefore, we have that $X_{\gamma K} \cap X_K \neq \varnothing$ for finitely many $\gamma \in \pi_1(M)$. On the other hand if $K \cap \gamma K \neq \varnothing$ then $X_{\gamma K} \cap X_K \neq \varnothing$, so the action on $\Omega(\tilde{\Lambda})$ is proper.
By the path lifting property, $\widetilde{dev}(\tilde{\Sigma}_t)/\pi_1(M)$ is a Cauchy surface of $\Omega(\tilde{\Lambda})/\pi_1(M)$,

which is therefore globally hyperbolic. The proof is concluded since by Proposition 4.24 the restriction of the covering map $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^{2,1} \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ to $\Omega(\widetilde{\Lambda}) \cup \widetilde{\Lambda}$ is injective. □

Definition 4.30. A globally hyperbolic Anti-de Sitter spacetime (M, g) is said to be *maximal* if any isometric embedding of (M, g) into a globally hyperbolic Anti-de Sitter spacetime (M', g') which sends a Cauchy surface of (M, g) to a Cauchy surface of (M', g') is surjective.

As a direct consequence of Proposition 4.29 we have:

Corollary 4.31. *A globally hyperbolic Anti-de Sitter spacetime M is maximal if and only if $\widetilde{M}$ is isometric to the invisible domain of a proper achronal meridian in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.*

4.7 Genus $n \geqslant 2$ classification

The purpose of this section is to classify all maximal globally hyperbolic (MGH) spacetimes of genus $n \geqslant 2$. Therefore, let Σ_n be an oriented surface of genus $n \geqslant 2$.

Definition 4.32. A representation $\rho : \pi_1(\Sigma_n) \to \mathrm{PSL}(2, \mathbb{R})$ is *positive Fuchsian* if there is a ρ-equivariant orientation-preserving homeomorphism $\delta : \widetilde{\Sigma_n} \to \mathbb{H}^2$.

The following classical result in Teichmüller theory is essential for the construction of MGH spacetimes of genus $n \geqslant 2$.

Lemma 4.33. Given two positive Fuchsian representations $\rho_l, \rho_r : \pi_1(\Sigma_n) \to \mathrm{PSL}(2, \mathbb{R})$, any (ρ_l, ρ_r)-equivariant orientation-preserving homeomorphism of $\mathbb{H}^2$ extends continuously to an orientation-preserving homeomorphism of $\mathbb{H}^2 \cup \mathbb{R}\mathrm{P}^1$. Moreover, its extension $\varphi : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ is the unique (ρ_l, ρ_r)-equivariant orientation-preserving homeomorphism of $\mathbb{R}\mathrm{P}^1$.

By (ρ_l, ρ_r)-equivariance of φ we mean that for every $\gamma \in \pi_1(\Sigma_n)$:

$$\varphi \circ \rho_l(\gamma) = \rho_r(\gamma) \circ \varphi. \tag{4.2}$$

Given two positive Fuchsian representations $\rho_l, \rho_r : \pi_1(\Sigma_n) \to \mathrm{PSL}(2, \mathbb{R})$ we will consider the associated representation in Anti-de Sitter geometry given by

$$\rho = (\rho_l, \rho_r) : \pi_1(\Sigma_n) \to \mathrm{Isom}_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}) \cong \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R}).$$

In this setting we define $\Lambda(\rho)$ as the graph of $\varphi : \mathbb{RP}^1 \to \mathbb{RP}^1$ defined by ρ, and $\Omega_\rho := \Omega(\Lambda(\rho))$ its invisible domain in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.

Proposition 4.34. *The domain Ω_ρ is invariant under the isometric action of $\pi_1(\Sigma_n)$ on $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ induced by ρ. Moreover $\pi_1(\Sigma_n)$ acts freely and properly discontinuously on Ω_ρ and the quotient is a MGH spacetime of genus n and holonomy ρ.*

Proof. By construction of φ, for any $(x, \varphi(x)) \in \Lambda(\rho)$ and $\gamma \in \pi_1(\Sigma_n)$ we have that

$$\rho(\gamma) \cdot (x, \varphi(x)) = (\rho_l(\gamma) \cdot x, \rho_r(\gamma) \cdot \varphi(x)) = (\rho_l(\gamma) \cdot x, \varphi(\rho_l(\gamma) \cdot x)) \in \Lambda(\rho),$$

proving the invariance of $\Lambda(\rho)$ by the action of $\pi_1(\Sigma_n)$. By Corollary 4.25, Ω_ρ is the set of $x \in \mathrm{PSL}(2, \mathbb{R})$ such that $x \circ \varphi$ have no fixed point on $\mathbb{RP}^1$. The invariance of Ω_ρ follows by the fact that

$$(\rho_l(\gamma) \circ x \circ \rho_r^{-1}) \circ \varphi = (\rho_l(\gamma) \circ x \circ \varphi \circ \rho_l^{-1})$$

acts freely on $\mathbb{RP}^1$ if $x \circ \varphi$ does.
We now show that the action of ρ is properly discontinuous on Ω_ρ. This will also show that the action is free, since $\pi_1(\Sigma_n)$ has no torsion. For this purpose let K be a compact set in Ω_ρ, take a sequence $x_n \in K$ and a sequence $\gamma_n \in \pi_1(\Sigma_n)$ not definitively constant. We claim that up to a subsequence $(\rho(\gamma_n) \cdot x_n)$ converges to some $(\xi_+, \varphi(\xi_+)) \in \Lambda(\rho)$.
Recall that since Fuchsian representations act cocompactly on $\mathbb{H}^2$, the sequence $\rho_l(\gamma_n)$ has no converging subsequence in $\mathrm{PSL}(2, \mathbb{R})$. Up to taking a subsequence, there exist $\xi_-, \xi_+ \in \mathbb{RP}^1$ such that $\rho_l(\gamma_n)^{\pm 1}(\xi) \to \xi_\pm$ for all $\xi \neq \xi_\mp$ and the convergence is uniform on compact sets of $(\mathbb{H}^2 \cup \mathbb{RP}^1) \backslash \{\xi_\pm\}$. By equivariance (4.2), the same holds for $\rho_r(\gamma_n)$ where $\xi_\pm$ is replaced by $\varphi(\xi_\pm)$. To apply the criterion on convergence of Lemma 3.2 pick $p \in \mathbb{H}^2$. By the dinamical property above, for any $\delta > 0$ one can find n_0 such that $\rho_r(\gamma_n)^{-1}(p)$ is in the δ-neighborhood U_δ of $\varphi(\xi_-)$ (for the Euclidean metric on the closed disc). Since x_n lies in the compact K, we can assume that it converges to $x_\infty \in \Omega_\rho$, hence $x_\infty \circ \varphi$ has no fixed point, and in particular $x_\infty \circ \varphi(\xi_-) \neq \xi_-$. Up to taking δ sufficiently small and n_0 large, $x_n(U_\delta)$ lies in a neighborhood V_ϵ of $x_\infty \circ \varphi(\xi_-)$ such that the closure of V_ϵ does not contain ξ_-. By construction $x_n \circ \rho_r(\gamma_n)^{-1}(p) \in V_\epsilon$ and by the uniform convergence on compact sets on the complement of ξ_-, $(\rho(\gamma_n) \cdot x_n)(p) = \rho_l(\gamma_n) \circ x_n \circ \rho_r(\gamma_n)^{-1}(p)$ converges to ξ_+. The same argument shows that $(\rho(\gamma_n) \circ x_n)^{-1}(p) = \rho_r(\gamma_n) \circ x_n \circ \rho_l(\gamma_n)^{-1}(p)$ converges to $\varphi(\xi_+)$. By Lemma 3.2 we conclude that $(\rho(\gamma_n) \cdot x_n)$ converges to $(\xi_+, \varphi(\xi_+)) \in \Lambda(\rho)$.

Now we prove that the quotient of Ω_ρ by the action of $\rho(\pi_1(\Sigma_n))$ is a MGH spacetime. The past and future boundary components $\partial_\pm C(\Lambda(\rho))$ are contained in Ω_ρ since $\Lambda(\rho)$ is the graph of an orientation-preserving homeomorphism. Hence they are ρ-invariant properly embedded Cauchy surfaces in Ω_ρ and project to Cauchy surfaces of the quotient by the action of $\rho(\pi_1(\Sigma_n))$, which are homeomorphic to Σ_n. By Proposition 4.29 and Corollary 4.31, $\Omega_\rho/\rho(\pi_1(\Sigma_n))$ is a MGH spacetime. □

Now we prove that the MHG constructed in Proposition 4.34 are all the MGH spacetime of genus n.

Lemma 4.35. Let $\rho = (\rho_l, \rho_r)$ be a pair of positive Fuchsian representations, and $\varphi : \mathbb{RP}^1 \to \mathbb{RP}^1$ be the unique (ρ_l, ρ_r)-equivariant orientation-preserving homeomorphism of $\mathbb{RP}^1$. Then $\Lambda(\rho)$ is the unique proper achronal meridian in $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ invariant under the action of $\rho(\pi_1(\Sigma_n))$.

Proof. Let Λ be a proper achronal meridian invariant under the action of $\rho(\pi_1(\Sigma_n))$. We claim $\Lambda \cap \Lambda(\rho)$ is not empty.
Let γ be a non-trivial element in $\pi_1(\Sigma_n)$. $\rho_l(\gamma)$ and $\rho_r(\gamma)$ are necessarely hyperbolic elements in $\mathrm{PSL}(2,\mathbb{R})$, being Σ_n compact, and we denote by $\xi_l^+(\gamma)$ and $\xi_r^+(\gamma)$ their attractive fixed points respectively. We have that $\xi_r^+(\gamma) = \varphi(\xi_l^+(\gamma))$, hence

$$(\xi_l^+(\gamma), \xi_r^+(\gamma)) \in \Lambda(\rho).$$

Now, the curve Λ must meet the leaf of the left ruling of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$

$$\lambda_{\xi_r^+(\gamma)} = \{(x, \xi_r^+(\gamma)) \mid x \in \mathbb{RP}^1\}$$

in a point $(x_0, \xi_r^+(\gamma))$. Then $(\rho_l(\gamma)^k \cdot x_0, \xi_r^+(\gamma)) \in \Lambda$ for $k > 0$ and, if $x_0 \neq \xi_l^-(\gamma)$, passing to the limit on k, we have that $(\xi_l^+(\gamma), \xi_r^+(\gamma))$ lies in Λ.
To conclude assume by contradiction that for every $\gamma \in \pi_1(\Sigma_n)$ the point $(\xi_l^-(\gamma), \xi_r^+(\gamma))$ lies in Λ. Take $\alpha, \beta \in \pi_1(\Sigma_n)$ so that the axis of $\rho_l(\alpha)$ and $\rho_l(\beta)$ do not intersect. We may assume that the cyclic order of the end points of those axis is

$$\xi_l^+(\alpha) < \xi_l^+(\beta) < \xi_l^-(\beta) < \xi_l^-(\alpha). \tag{4.3}$$

Since $\xi_r^\pm(\alpha) = \varphi(\xi_l^\pm(\alpha))$ and $\xi_r^\pm(\beta) = \varphi(\xi_l^\pm(\beta))$, Equation 4.3 holds if we replace ρ_l with ρ_r. On the other hand we assumed that Λ contains $(\xi_l^+(\alpha), \xi_r^-(\alpha))$, $(\xi_l^+(\beta), \xi_r^-(\beta))$, $(\xi_l^-(\beta), \xi_r^+(\beta))$, $(\xi_l^-(\alpha), \xi_r^+(\alpha))$. By achronality of Λ, the cyclic order of the first and second components must be the same, hence

$$\xi_l^-(\alpha) \leqslant \xi_l^-(\beta) \leqslant \xi_l^+(\beta) \leqslant \xi_l^+(\alpha),$$

which contradicts Equation 4.3. □

We proved that given a pair $\rho = (\rho_l, \rho_r)$ of positive Fuchsian representations of $\pi_1(\Sigma_n)$ the MGH spacetime $M_\rho := \Omega_\rho/\rho(\pi_1(\Sigma_n))$ is the unique MGH spacetime with holonomy ρ.
The last step for the classification result is that given a MGH spacetime, the left and right components of the holonomy are necessarely positive Fuchsian.

Remark 4.36. *By a result of Goldman [5], a representation ρ is positive Fuchsian if and only if the associated flat $\mathbb{R}\mathrm{P}^1$ bundle E_ρ, constructed as the quotient of $\widetilde{\Sigma}_n \times \mathbb{R}\mathrm{P}^1$ by the diagonal action of $\pi_1(\Sigma_n)$ given by deck transformation on the first factor and by ρ on the second factor, has Euler class $2-2n$. This is also equivalent to the existence of an orientation-preserving fiber bundle isomorphism between E_ρ and the unit tangent bundle of Σ_n.*

Proposition 4.37. *Let M be an oriented, time-oriented, globally hyperbolic spacetime of genus $n \geqslant 2$ and let us endow a Cauchy surface Σ with the orientation induced by the future normal vector. Then the left and right components of the holonomy $\rho = (\rho_l, \rho_r) : \pi_1(\Sigma) \to PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ are positive Fuchsian representations.*

Proof. By Remark 4.36 we have to prove that the $\mathbb{R}\mathrm{P}^1$-flat bundles with holonomy ρ_l and ρ_r are isomorphic to the unit tangent bundle. We focus on ρ_l, the proof for ρ_r is analogous.
Define $\Phi_l : T^1\widetilde{\Sigma} \to \widetilde{\Sigma} \times \mathbb{R}\mathrm{P}^1$ in the following way: for an element $(x,v) \in T^1\widetilde{\Sigma}$ let

$$\xi(x,v) = (\xi^l(x,v), \xi^r(x,v)) \in \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$$

be the end-point of the spacelike geodesic ray $exp_x(tv)$ in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, for positive t. We define $\Phi_l(x,v) = (x, \xi^l(x,v))$. This map is clearly continuous, proper, equivariant and fiber preserving.
To prove that it is bijective it is sufficient to notice that for any $x \in \widetilde{\Sigma}$ the map $\xi_x : T^1_x\widetilde{\Sigma} \to \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1$ is an embedding with image the boundary of the totally geodesic plane tangent to $\widetilde{\Sigma}$ at x. This boundary is the graph of an orientation-preserving map of $\mathbb{R}\mathrm{P}^1$, so the projection $v \to \xi^l(x,v)$ is bijective. Moreover, by the choice of the orientation on Σ, the orientation on $T^1_x\widetilde{\Sigma}$ corresponds to the orientation induced on $\xi_x(T^1_x\widetilde{\Sigma})$ as graph of an orientation-preserving homeomorphism. $\square$

We conclude this section by stating the classification result. Let the *deformation space* of MGH spacetimes of genus n be:

$$\mathcal{MGH}(\Sigma_n) = \{g \text{ MGH AdS metric on } \Sigma_n \times \mathbb{R}\}/\mathrm{Diff}_0(\Sigma_n \times \mathbb{R}) \ ,$$

where the group of diffeomorphisms isotopic to the identity acts by pull-back. The holonomy map takes value in the space of representations of $\pi_1(\Sigma_n)$ into $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ up to conjugation and is well-defined on the quotient $\mathcal{MGH}(\Sigma_n)$. We proved that the left and right components of the holonomy of elements of $\mathcal{MGH}(\Sigma_n)$ are positive Fuchsian representations, and the space of these representations up to conjugacy is identified with the Teichmüller space of Σ_n

$$\mathcal{T}(\Sigma_n) \cong \{\rho : \pi_1(\Sigma_n) \to \mathrm{PSL}(2,\mathbb{R}) \text{ positive Fuchsian representations}\}/\mathrm{PSL}(2,\mathbb{R})\,.$$

Therefore the holonomy map can be considered as a map from $\mathcal{MGH}(\Sigma_n)$ with values in $\mathcal{T}(\Sigma_n) \times \mathcal{T}(\Sigma_n)$. Hence, we can summarize this section with the following theorem of Mess.

Theorem 4.38. *The holonomy map*

$$\rho : \mathcal{MGH}(\Sigma_n) \to \mathcal{T}(\Sigma_n) \times \mathcal{T}(\Sigma_n)$$

is a homeomorphism.

CHAPTER 5

Gauss map and spacelike immersions

5.1 Spacelike surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$

In this section we will briefly talk about geometric properties of immersed spacelike surfaces that is analogous to the theory for the Euclidean space.
Let us denote with ∇ the Levi-Civita connection of the Lorentzian metric of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. Given a spacelike immersion $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ the pull-back metric $I = \sigma^*(g_{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}})$ is called *first fundamental form* of σ. The tangent bundle TS is naturally identified to a subbundle of the pull-back $\sigma^*(TM)$, therefore we can define its normal bundle N_σ. Using the decomposition

$$\sigma^* TM = TS \oplus N_\sigma \ ,$$

the pull-back of the Levi-Civita connection ∇, restricted to sections tangent to S splits as the sum of the Levi-Civita connection of the first fundamental form I and a symmetric 2-form with value in N_σ. Given the time-orientability, we take the future-directed unit normal vector field ν of S. The Levi-Civita connection ∇^I of the first fundamental form I of S is defined by the relation

$$\nabla_V W = \nabla^I_V W + I\!I(V, W)\nu \ , \tag{5.1}$$

where the $(2,0)$-tensor $I\!I$ is called *second fundamental form* of σ and is defined as $I\!I(V, W) = \langle \nu, \nabla_V W \rangle$. The *shape operator* of σ is the $(1,1)$-tensor B defined as

$$B(V) = \nabla_V \nu \ .$$

The shape operator is related to the second fundamental form by

$$I\!I(V, W) = I(B(V), W) \ .$$

In particular, B is diagonalisable and its eigenvalues are called principal curvatures.

Theorem 5.1. *The first and second fundamental form of an immersion σ satisfy the Gauss equation*

$$K_I = -1 - detB \ ,$$

and the Codazzi equation

$$d^{\nabla^I} I\!I = 0 \ ,$$

where d^{∇^I} is the operator defined by

$$(d^{\nabla^I} I\!I)(X, Y, Z) = (\nabla^I_X I\!I)(Y, Z) - (\nabla^I_Y I\!I)(X, Z).$$

Proof. For easier computations, suppose $S \subset \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. For an $X \in T\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, denote by $X^\top$ and $X^\perp$ the tangent and normal part of X with respect to TS. Using Equation 5.1 we have

$$(\nabla_X \nabla_Y Z)^\top = \nabla^I_X \nabla^I_Y Z + I\!I(Y, Z)\nabla_X \nu \ ,$$
$$(\nabla_X \nabla_Y Z)^\perp = I\!I(X, \nabla^I_Y Z)\nu + \partial_X(I\!I(Y, Z))\nu \ ,$$

and

$$(\nabla_{[X,Y]} Z)^\top = \nabla^I_{[X,Y]} Z \ ,$$
$$(\nabla_{[X,Y]} Z)^\perp = I\!I([X, Y], Z)\nu = I\!I(\nabla^I_X Y, Z)\nu - I\!I(\nabla^I_Y X, Z)\nu \ .$$

Computing the tangential part of the Riemann tensor we have

$$\begin{aligned}\langle R(X, Y)Z, W\rangle &= \langle R^I(X, Y)Z, W\rangle + \langle I\!I(Y, Z)\nabla_X \nu, W\rangle - \langle I\!I(X, Z)\nabla_Y \nu, W\rangle \\ &= \langle R^I(X, Y)Z, W\rangle - \langle I\!I(Y, Z)\nu, \nabla^I_X W\rangle + \langle I\!I(X, Z)\nu, \nabla^I_Y W\rangle \\ &= \langle R^I(X, Y)Z, W\rangle - I\!I(Y, Z)I\!I(X, W) + I\!I(X, Z)I\!I(Y, W) \ .\end{aligned}$$

Therefore the sectional curvature satisfies

$$-1 = K_{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}} = K_I + \det(I^{-1}I\!I) = K_I + \det B \ .$$

Looking at the normal part of $R(X, Y)Z$ we have

$$\begin{aligned}0 = \langle R(X, Y)Z, \nu\rangle &= \partial_Y(I\!I(X, Z)) - I\!I(\nabla^I_Y X, Z) - I\!I(X, \nabla^I_Y Z) - \partial_X(I\!I(Y, Z)) \\ &\quad + I\!I(\nabla^I_X Y, Z) + I\!I(Y, \nabla^I_X Z) = (\nabla^I_Y I\!I)(X, Z) - (\nabla^I_X I\!I)(Y, Z).\end{aligned}$$

where we used that the normal part of the Riemann tensor is null when the sectional curvature is constant (Equation 1.1). $\square$

As in the Euclidean space, the embedding data I and $I\!I$ of a simply connected surface determines the immersion uniquely up to isometries of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$

Theorem 5.2. *Let S be a simply connected surface, let I be a Riemannian metric on S and $I\!I$ be a symmetric $(2,0)$-tensor on S. If I and $I\!I$ satisfy the Gauss-Codazzi equations, then there exists a spacelike immersion $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ having I and $I\!I$ as first and second fundamental form. Moreover, if σ and σ' are two such immersions, then there exists a time-preserving isometry φ of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ such that $\sigma' = \varphi \circ \sigma$.*

5.2 Germs of spacelike immersions

Let us now consider che case in which S is an oriented surface, not necessarily simply connected. Let $\sigma : S \to (M, g)$ a spacelike immersion where (M, g) is an oriented AdS spacetime. As in the previous section, we can associate to σ the pair $(I, I\!I)$ of first and second fundamental form, where $I\!I$ is computed with respect to the unit future pointing normal vector ν of σ. We will always assume that the orientation of S and ν are compatible with the one on M.
We want to prove that the pair $(I, I\!I)$ depends only on the *germ* of σ, which is defined as follow:

Definition 5.3. A *germ* of a spacelike immersion of S into an AdS spacetime is an equivalent class of spacelike immersions $\sigma : S \to (M, g)$ by the following relation: $\sigma : S \to (M, g)$ and $\sigma' : S \to (M', g')$ are equivalent if there exist open subsets $U \supset \sigma(S)$ in M and $U' \supset \sigma'(S)$ in M' and a time-preserving isometry $f : (U, g) \to (U', g')$ such that $\sigma' = f \circ \sigma$.

Given a pair $(I, I\!I)$ on S and $\pi : \widetilde{S} \to S$ a universal cover, one can take the pair $(\pi^* I, \pi^* I\!I)$ on $\widetilde{S}$ that clearly satisfies the Gauss-Codazzi equations. By the existence part of Theorem 5.2 there exists a spacelike immersion $\widetilde{\sigma} : \widetilde{S} \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ having immersion data $(\pi^* I, \pi^* I\!I)$. As a coscequence of the uniqueness part of Theorem 5.2 we have that associeted to $\widetilde{\sigma}$ there is a map $\rho : \pi_1(S) \to \mathrm{Isom}_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1})$ such that for every $\gamma \in \pi_1(S)$, $\widetilde{\sigma} \circ \gamma = \rho(\gamma) \circ \widetilde{\sigma}$. Moreover changing $\widetilde{\sigma}$ by post-composition with an isometry f of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ has the effect of conjugating ρ by f.

Given $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ one can extend the immersion to an immersion of an open neighbourhood of $S \times \{0\}$ in $S \times \mathbb{R}$ into $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, by mapping (x, t) to the point $\gamma(t)$ on the timelike geodesic γ such that $\gamma(0) = \sigma(p)$ and $\gamma'(0)$ is the future unit normal vector ν of σ at x.

Lemma 5.4. Given a spacelike immersion $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, the pull-back of the ambient metric by means of the map $\sigma_t : x \to exp_{\sigma(x)}(t\nu(x))$ is given by

$$I_t = I((\cos(t)id + \sin(t)B)\cdot, (\cos(t)id + \sin(t)B)\cdot).$$

The second fundamental form and the shape operator are given by

$$I\!I_t = I((-\sin(t)id + \cos(t)B)\cdot, (\cos(t)id - \sin(t)B)\cdot).$$

$$B_t = (\cos(t)id + \sin(t)B)^{-1}(-\sin(t)id + \cos(t)B).$$

Proof. By Section 2.4 we have that $\sigma_t(x) = \cos(t)\sigma(x) + \sin(t)\nu(x)$. Then

$$\begin{aligned} I_t(v, w) &= \langle d(\sigma_t)_x(v), d(\sigma_t)_x(w)\rangle \\ &= \langle \cos(t)d\sigma_x(v) + \sin(t)d\nu_x(v), \cos(t)d\sigma_x(w) + \sin(t)d\nu_x(w)\rangle \\ &= I(\cos(t)v + \sin(t)B(v), \cos(t)w + \sin(t)B(w)). \end{aligned}$$

The formula for $I\!I_t$ follows from the fact that $I\!I_t = \frac{1}{2}\frac{dI_t}{dt}$, and the formula for B_t follows from $B_t = I_t^{-1}I\!I_t$. $\square$

Corollary 5.5. *Given a spacelike immersion $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, the pull-back metric by means of the map $(p, t) \to exp_{\sigma(x)}(t\nu(x))$ is given by*

$$-dt^2 + cos^2(t)I + 2cos(t)sin(t)I\!I + sin^2(t)I\!I\!I, \tag{5.2}$$

where I, $I\!I$, $I\!I\!I$ are the first, second and third fundamental form of σ and third fundamental form is defined as $I\!I\!I(\cdot,\cdot) = I(B(\cdot), B(\cdot))$.

Therefore, given a pair $(I, I\!I)$, Equation 5.2 provides a Lorentzian metric of constant curvature -1 on an open neighbourhood of $S \times \{0\}$ in $S \times \mathbb{R}$ into $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, and thus a germ of immersion of S into an AdS spacetime with immersion data $(I, I\!I)$. We conclude the above discussion with the following:

Proposition 5.6. *Given a surface S, there are natural identifications between the following spaces:*

- *The space of pairs $(I, I\!I)$ on S which are solutions of the Gauss-Codazzi equations,*
- *The space of germs of spacelike immersions of S into Anti-de Sitter manifolds,*
- *The space of spacelike immersions of $\widetilde{S}$ into $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, equivariant with respect to a representation $\rho : \pi_1(S) \to Isom_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1})$, up to the action of $Isom_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1})$ by post-composition.*

When S is a closed surface, by Proposition 4.26, the border at infinity of $\widetilde{\sigma}(\widetilde{S})$ is a proper achronal meridian Λ. So, $\widetilde{\sigma}(\widetilde{S})$ is contained in the globally hyperbolic spacetime $\Omega(\Lambda) = \widetilde{\sigma}(\widetilde{\mathcal{S}})$, and therefore is contained in a maximal one. Remarkably, the embedding data $(I, I\!I)$ permits to recover an explicit pair of elements in $\mathcal{T}(S) \times \mathcal{T}(S)$ which parametrizes MGH Anti-de Sitter manifolds with compact Cauchy surfaces and in the next section we will show an explicit formula for such pair.

5.3 Gauss map

We can now define the Gauss map for spacelike surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. Recall that the space of timelike geodesics in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ is identified with $\mathbb{H}^2 \times \mathbb{H}^2$, where the identification maps a geodesic of the form

$$L_{p,q} = \{X \in \mathrm{PSL}(2,\mathbb{R}) \mid X \cdot q = p\}$$

to the pair $(p,q) \in \mathbb{H}^2 \times \mathbb{H}^2$.

Definition 5.7. Let $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ be a spacelike immersion. The *Gauss map* $G_\sigma : S \to \mathbb{H}^2 \times \mathbb{H}^2$ is defined as $G_\sigma(x) = (p,q)$ such that $L_{p,q}$ is the timelike geodesic orthogonal to $\mathrm{Im}(d_x\sigma)$ at $\sigma(x)$.
The components of the Gauss map, denoted by $\Pi_l, \Pi_r : S \to \mathbb{H}^2$, are called *left* and *right projections*.

The Gauss map G_σ is natural with respect to the action of the isometry group, meaning that $G_{f\circ\sigma} = f \cdot G_\sigma$ for every $f \in \mathrm{Isom}_0(\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}) = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. The Gauss map is also invariant by reparametrization, in the sense that $G_{\sigma\circ\phi} = G_\sigma \circ \phi$ for a diffeomorphism $\phi : S' \to S$. Hence it makes sense to talk about the Gauss map of a spacelike surface in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.

Example 5.8: In Lemma 3.4 we gave an isometric embedding of $\mathbb{H}^2$ in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ with image the plane $P_{\mathbb{1}}$ dual to the identity. This isometric embedding was defined by sending $p \in \mathbb{H}^2$ to the unique order-two element in $\mathrm{PSL}(2,\mathbb{R})$ fixing p, which by definition lies on the geodesic $L_{p,p}$. Moreover the geodesic $L_{p,p}$ is orthogonal to $P_{\mathbb{1}}$. Hence the Gauss map associated to this isometric embedding of $\mathbb{H}^2$ is just $p \mapsto (p,p)$.

The geodesic leaving from $\mathbb{1}$ with velocity $\nu(p)$ meets $P_{\mathbb{1}}$ orthogonally at $\exp((\pi/2)\nu(p))$, therefore, using Example 5.8, one can recover the following:

Lemma 5.9. Given a spacelike immersion $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$, with future unit normal vector field ν, if $\sigma(p) = \mathrm{Id}$, then

$$G_\sigma(p) = G_{P_{\mathbb{1}}}(\exp(\frac{\pi}{2}\nu(p))). \tag{5.3}$$

Now we denote with $T^{1,+}_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ the hyperboloid of future unit timelike vectors in $T_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ and consider the following map:

$$\mathrm{Fix} : T^{1,+}_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^{2.1} \to \mathbb{H}^2$$

defined so that $\mathrm{Fix}(\nu)$ is the fixed point of the one-parameter elliptic group $\{\exp(t\nu) \mid t \in \mathbb{R}\}$. This map is equivariant for the action of $\mathrm{PSL}(2,\mathbb{R})$, which acts on the hyperboloid $T^{1,+}_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^{2.1}$ by the adjoint representation and on $\mathbb{H}^2$ by the obvious action. Since both $T^{1,+}_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^{2.1}$ and $\mathbb{H}^2$ have constant curvature -1, it follows from equivariance that Fix is an isometry.
In terms of the map Fix, Equation 5.3 reads as:

$$G_\sigma(p) = (\mathrm{Fix}(\nu(p)), \mathrm{Fix}(\nu(p))), \tag{5.4}$$

provided that $\sigma(p) = \mathrm{Id}$.
Via Lemma 5.9 and the naturality, we can recover a different description of the Gauss map. Recalling the structure of Lie group of $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1} \simeq \mathrm{PSL}(2,\mathbb{R})$ we will denote by R_γ and L_γ the right and left multiplication by $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ respectively.

Lemma 5.10. Given a spacelike immersion $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2.1}$ with future unit normal vector field ν,

$$G_\sigma(p) = (\mathrm{Fix}((R_{\sigma(p)^{-1}})_*(\nu(p)), \mathrm{Fix}((L_{\sigma(p)^{-1}})_*(\nu(p)))).$$

Proof. If $\sigma(p) = \mathrm{Id}$, then the equality holds true by Equation 5.4. For the general case, the immersion $\sigma' = (\mathrm{Id}, \sigma(p)) \circ \sigma$ is such that $\sigma'(p) = \mathrm{Id}$, and the future normal vector at $\sigma'(p)$ equals $\nu'(p) = (R_{\sigma(p)^{-1}}))_*(\nu(p))$. Therefore:

$$G_{\sigma'}(p) = (\mathrm{Fix}((R_{\sigma(p)^{-1}})_*(\nu(p)), \mathrm{Fix}((R_{\sigma(p)^{-1}})_*(\nu(p))))).$$

Now by the naturality of the Gauss map it follows,

$$\begin{aligned} G_\sigma(p) &= (\mathrm{Id}, \sigma(p)^{-1}) \cdot G_{\sigma'}(p) \\ &= (\mathrm{Fix}((R_{\sigma(p)^{-1}})_*(\nu(p)), \sigma(p)^{-1} \circ \mathrm{Fix}((R_{\sigma(p)^{-1}})_*(\nu(p))))) \\ &= (\mathrm{Fix}((R_{\sigma(p)^{-1}})_*(\nu(p)), \mathrm{Fix}((L_{\sigma(p)^{-1}})_*(\nu(p))))). \end{aligned}$$

where we used that Fix is equivariant with respect to the adjoint action on the hyperboloid $T^{1,+}_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^{2.1}$. ∎

We want now to prove formulae which express the pull-back of the hyperbolic metrics by the left and right projections. When applying these formulae to the embedding data $(I, I\!I)$ of a closed surface S, we obtain a pair of hyperbolic metrics. By Proposition 4.37 the isotopy classes of these two metric identify the parameters in $\mathcal{T}(S) \times \mathcal{T}(S)$ of a MGH spacetime M

Remark 5.11. *Under the identification given by Fix, the left and right projections can be interpreted in the following way. Given $p \in S$, $\Pi_l(p)$ is the parallel transport in Id of the future unit vector $\nu(p)$ at $\sigma(p)$ with respect to the right-invariant connection D^r. The right projection is then obtained using the left-invariant connection.*

Proposition 5.12. *Let $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ be a spacelike immersion, let $\Pi_l, \Pi_r : S \to \mathbb{H}^2$ be the left and right projections and let $g_{\mathbb{H}^2}$ be the hyperbolic metric. Then*

$$\Pi_l^* g_{\mathbb{H}^2} = I((id - \mathcal{J}B)\cdot, (id - \mathcal{J}B)\cdot),$$

and

$$\Pi_r^*(g_{\mathbb{H}^2}) = I((id + \mathcal{J}B)\cdot, (id + \mathcal{J}B)\cdot),$$

where I is the first fundamental form of σ, $\mathcal{J}$ its associated almost-complex structure, and B the shape operator.

Proof. Let us check the formula for Π_r, the proof is analogous for Π_l. By Lemma 5.10

$$\Pi_r(x) = \mathrm{Fix}((L_{\sigma(x)^{-1}})_*(\nu(x))).$$

Since Fix is an isometry $\Pi_r^* g_{\mathbb{H}^2}$ equals the pull-back of the Anti-de Sitter metric through the map defined by $\widehat{\Pi}_r(x) = (L_{\sigma(x)^{-1}})_*(\nu(x))$.
Let X_1, X_2, X_3 be an orthonarmal basis of left-invariant vector fields on $T\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. Then the unit normal vector can be expressed as $\nu(x) = \sum_i \nu_i(x) X_i(\sigma(x))$. By Remark 5.11

$$\widehat{\Pi}_r(x) = \sum_i \nu_i(x) X_i(\mathbb{1}).$$

Differentiating we have

$$d\widehat{\Pi}_r(v) = \sum_i d\nu_i(v) X_i(\mathbb{1}).$$

On the other hand, since left-invariant vector fields are parallel for the left-invariant connection D^l, we have

$$D_v^l \nu = \sum_i d\nu_i(v) X_i(\sigma(x)).$$

The last two identities show that

$$\Pi_r^* g_{\mathbb{H}^2}(v,w) = g_{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}}(d\widehat{\Pi}_r(v), d\widehat{\Pi}_r(w)) = g_{\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}}(D^l_v\nu, D^l_w\nu).$$

Recalling that $\nabla_V W = D^l_V W - V \boxtimes W$ and $[V,W] = -2V \boxtimes W$, we have that

$$D^l_v\nu = \nabla_v \nu + v \boxtimes \nu = B(v) - \nu \boxtimes v = (B - \mathcal{J})v.$$

The last equality holds since ν, v, $-Jv$ form a positive basis of $T\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ and the norm of $\nu \boxtimes v$ it the product of those of ν and v, which together imply that $\nu \boxtimes v = -Jv$. We can therefore conclude that

$$\Pi_r^* g_{\mathbb{H}^2}(v,w) = I((B-\mathcal{J})v, (B-\mathcal{J})w) = I((id + \mathcal{J}B)v, (id + \mathcal{J}B)w).$$

□

We collect here some consequences and remarks around Proposition 5.12.

- Proposition 5.12 shows that,up to post-composing with an isometry sending the image of $d_x\sigma$ to a fixed copy of $\mathbb{H}^2$, the differential of the left and right projections essentially has the expression

  $$d_x\sigma \circ (B \pm \mathcal{J}) \ .$$

  Since B is I-symmetric $\mathcal{J} \circ B$ is traceless, and therefore

  $$\det(B \pm \mathcal{J}) = 1 + \det B = -K_I \ . \tag{5.5}$$

  This shows that Π_l is a local diffeomorphism at a point x if and only if Π_r is, which is the case if and only if the intrinsic curvature of I at x is different from 0.

- Since the trace of $id \pm \mathcal{J}B$ equals 2, the differentials of Π_l and Π_r have either rank 2 or rank 1. (In fact, by Equation (5.5), when the differential of Π_l has rank 1, the same holds for the differential of Π_r.) Hence the differential of the Gauss map $G : S \to \mathbb{H}^2 \times \mathbb{H}^2$ is always non-singular.

- If an immersed surface has the property that the curvature of the first fundamental form never vanishes, and if Π_l and Π_r are globally injective, then the image of G is the graph of a diffeomorphism F_σ between two subsets of $\mathbb{H}^2$, called the *associated map*. From Equation (5.5), the Jacobians of Π_l and Π_r are equal, hence the associated map is area-preserving. When Π_l and Π_r are only locally injective, but not globally, we still obtain an area-preserving local diffeomorphism F_σ which is now defined between two hyperbolic surfaces.

- More generally, as a consequence of the previous points, the image of G is always a *Lagrangian submanifold* in $\mathbb{H}^2 \times \mathbb{H}^2$ with respect to the symplectic form
$$\Omega = \pi_l^* \omega_{\mathbb{H}^2} - \pi_r^* \omega_{\mathbb{H}^2} \ , \tag{5.6}$$
where $\omega_{\mathbb{H}^2}$ is the hyperbolic area form. This result has been proved in several works with different methods: see [3], [9]. Moreover, the Lagrangian condition is *locally* the only obstruction to inverting this construction, that is, to realizing an immersed surfaces in $\mathbb{H}^2 \times \mathbb{H}^2$ locally as the image of the Gauss map of a spacelike immersion in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$.

- Given a spacelike immersion σ, the normal evolution of σ is defined as
$$\sigma_t(x) = \exp_{\sigma(x)}(t\nu(x)) \ ,$$
where ν is the future unit normal vector field. When it is an immersion, the computation of the metric in Lemma 5.4 shows that the image of σ_t at x is orthogonal to the geodesic $\gamma(t) = \exp_{\sigma(x)}(t\nu(x))$. In other words, the Gauss map of σ_t is equal to the Gauss map of σ.

We conclude this section with the following:

Proposition 5.13. *Let S be a surface with a Riemannian metric I. Let A be a $(1,1)$-tensor which is I-symmetric and I-Codazzi, which means that $d^{\nabla^I} A = 0$. Then the curvature of $g = I(A\cdot, A\cdot)$ is given by*
$$K_g = \frac{K_I}{det(A)}.$$

Proof. First we prove that the Levi-Civita connection of the metric g is given by
$$\nabla^g_X Y = A^{-1} \nabla^I_X (AY) \ .$$
It is torsion-free, for all X, Y
$$\begin{aligned}\nabla^g_X Y - \nabla^g_Y X &= A^{-1}\nabla^I_X(AY) - A^{-1}\nabla^I_Y(AX) \\ &= A^{-1}(\nabla^I_X A)Y + \nabla^I_X Y - A^{-1}(\nabla^I_Y A)X - \nabla^I_Y X \\ &= A^{-1} d^{\nabla^I} A(X,Y) + \nabla^I_X Y - \nabla^I_Y X = 0 \ ,\end{aligned}$$
since ∇^I is torsion-free and $d^{\nabla^I} A = 0$. ∇^g is also compatible with g since for all X, Y, Z
$$\begin{aligned}\partial_Z g(X,Y) &= \partial_Z I(AX, AY) \\ &= I(\nabla^I_Z(AX), AY) + I(AX, \nabla^I_Z(AY)) \\ &= g(A^{-1}\nabla^I_Z(AX), Y) + g(X, A^{-1}\nabla^I_Z(AY)) \\ &= g(\nabla^g_Z X, Y) + g(X, \nabla^g_Z Y) \ .\end{aligned}$$

Let now (e_1, e_2) be an orthonormal frame on S for the metric I. It is known that for an orthonormal frame on a surface

$$\nabla^I_X e_1 = \beta(X) e_2 \ , \qquad \nabla^I_X e_2 = -\beta(X) e_1 \ ,$$

where β is the connection 1-form for the metric I, and the curvature K_I can be defined as $d\beta = -K_I \omega_I$, where ω_I is the volume form on S. Let now $(e'_1, e'_2) = (A^{-1}e_1, A^{-1}e_2)$ be the orthonormal frame for g, then

$$\nabla^g_X e'_1 = A^{-1} \nabla^I_X A e'_1 = \beta(X) A^{-1} e_2 = \beta(X) e'_2 \ ,$$

which shows that β is also the connection form for g. It follows that

$$d\beta = -K_g \omega_g = -K_I \omega_I \ ,$$

and therefore

$$K_g = K_I \frac{\omega_I}{\omega_g} = \frac{K_I}{det A} \ .$$

□

Applying the proposition to the tensors $id \pm \mathcal{J} \circ B$, Equation (5.5) shows that the pull-back metrics defined in Proposition 5.12, when non-degenerate, are always hyperbolic.

CHAPTER 6

Minimal Lagrangian diffeomorphism

6.1 Harmonic and minimal maps

In this section we will briefly introduce harmonic and minimal maps, which will prove useful in the proof of the main theorem.
For this purpose let (M, g) and (N, h) be Riemannian manifolds. In local coordinates the metric tensors g and h are given by $(g_{\alpha\beta})_{\alpha,\beta=1,\cdots,m}$ and $(h_{ij})_{i,j=1,\cdots,n}$. Moreover, let $(g^{\alpha\beta}) = (g^{-1})_{\alpha\beta}$ be the inverse metric tensor on M and let Γ^k_{ij} be the Christoffel symbols of the Levi Civita connection of (N, h).

Definition 6.1. The *energy density* of a smooth map $f : M \to N$ is defined as

$$e(f)(x) := \frac{1}{2} g^{\alpha\beta}(x) h_{ij}(f(x)) \frac{\partial f^i(x)}{\partial x^\alpha} \frac{\partial f^j(x)}{\partial x^\beta} \ . \tag{6.1}$$

The *energy* of f is

$$E(f) := \int_M e(f) dM \tag{6.2}$$

where $dM = \sqrt{det(g)} dx^1 \wedge \cdots \wedge dx^m$ is the volume form on M.

Remark 6.2. *Intrinsically $e(f)$ is the trace (up to the factor $\frac{1}{2}$) of f^*h with respect to g, and therefore its value on x does not depend on the choices of local coordinates.*

Definition 6.3. A map $f : M \to N$ is *harmonic* if it is a critical point of the energy functional.

As harmonic maps are critical points of E, a direct computation of

$$\frac{d}{dt}E(f + t\varphi)|_{t=0} = 0$$

shows that harmonic maps are the solutions of

$$\frac{1}{\sqrt{det(g)}}\frac{\partial}{\partial x^\alpha}(\sqrt{det(g)}g^{\alpha\beta}\frac{\partial f^i}{\partial x^\beta}) + g^{\alpha\beta}(x)\Gamma^k_{ij}(f(x))\frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta} = 0 \ . \qquad (6.3)$$

We turn now our attention to the case where M is a compact surface. As a complex manifold, M admits complex coordinate $z = x + iy$ which defines on the complexified tangent space the vector fields

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \ , \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big),$$

and on the complexified cotangent space the 1-forms

$$dz = dx + idy \ , \qquad d\overline{z} = dx - idy \ .$$

Definition 6.4. A Riemannian metric $\langle\cdot,\cdot\rangle$ on a surface M is called *conformal* if in local coordinates it can be written as

$$\rho^2(z)dzd\overline{z}$$

for a positive, real valued function $\rho(z)$. In particular, we have that

$$\Big\langle\frac{\partial}{\partial z},\frac{\partial}{\partial z}\Big\rangle = 0 = \Big\langle\frac{\partial}{\partial \overline{z}},\frac{\partial}{\partial \overline{z}}\Big\rangle \ ,$$
$$\Big\langle\frac{\partial}{\partial z},\frac{\partial}{\partial \overline{z}}\Big\rangle = 2\rho^2(z).$$

Remark 6.5. *Using real coordinates a conformal metric is written as*

$$\rho^2(z)(dxdx + dydy)$$

and

$$\Big\langle\frac{\partial}{\partial x},\frac{\partial}{\partial x}\Big\rangle = \rho^2(z) = \Big\langle\frac{\partial}{\partial y},\frac{\partial}{\partial y}\Big\rangle \ ,$$
$$\Big\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y}\Big\rangle = 0.$$

In complex coordinates Equation 6.3 is written as

$$\frac{\partial^2 f^i}{\partial z \partial \overline{z}} + \Gamma^i_{jk}(f(z)) \frac{\partial f^j}{\partial z} \frac{\partial f^k}{\partial \overline{z}} = 0 \qquad \forall i = 1, \cdots, n \ .$$

In particular whether a map is harmonic depends only on the Riemann surface structure of M, and not on the choice of a conformal metric.
When N is a surface with complex coordinates $w = u + iv$ in a neighbourhood of $f(z)$ and conformal metric $h(w) = \sigma(w) dw d\overline{w}$, and f is a local deffeomorphism, Equation 6.3 becomes

$$\frac{\partial^2 f}{\partial z \partial \overline{z}} + \frac{1}{\sigma(f(z))} \frac{\partial \sigma}{\partial w} \frac{\partial f}{\partial z} \frac{\partial f}{\partial \overline{z}} = 0 \ , \tag{6.4}$$

where $w(z) = f^1(z) + i f^2(z)$.

Theorem 6.6. *Let M be a surface and N a Riemannian manifold with metric $\langle \cdot, \cdot \rangle_N$ or $(g_{ij})_{i,j=1,\cdots,n}$ in local coordinates. If $f : M \to N$ is harmonic, then the Hopf differential of f*

$$\varphi(z) dz^2 = \Big\langle \frac{\partial f}{\partial z}, \frac{\partial f}{\partial z} \Big\rangle_N dz^2$$

is a holomorphic quadratic differential, i.e $\varphi(z)$ is a holomophic function. If f is a diffeomorphism the converse holds.

Proof. In local coordinates

$$\varphi(z) dz^2 = g_{ij}(f(z)) \frac{\partial f^i}{\partial z} \frac{\partial f^j}{\partial z} dz^2.$$

Differentiating in $\overline{z}$ we have

$$\begin{aligned}
\frac{\partial}{\partial \overline{z}} \Big(g_{ij}(f(z)) \frac{\partial f^i}{\partial z} \frac{\partial f^j}{\partial z} \Big) &= 2 g_{ij} \frac{\partial^2 f^i}{\partial z \partial \overline{z}} \frac{\partial f^j}{\partial z} + \frac{\partial g_{ij}}{\partial x^k} \frac{\partial f^k}{\partial \overline{z}} \frac{\partial f^i}{\partial z} \frac{\partial f^j}{\partial z} \\
&= 2 g_{ij} \frac{\partial^2 f^i}{\partial z \partial \overline{z}} \frac{\partial f^j}{\partial z} + \Big(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^l} \Big) \frac{\partial f^k}{\partial \overline{z}} \frac{\partial f^l}{\partial z} \frac{\partial f^j}{\partial z} \\
&= 2 g_{ij} \frac{\partial f^j}{\partial z} \Big(\frac{\partial^2 f^i}{\partial z \partial \overline{z}} + \Gamma^i_{kl} \frac{\partial f^k}{\partial \overline{z}} \frac{\partial f^l}{\partial z} \Big) \ ,
\end{aligned}$$

which is 0 when f is harmonic.
Let now N be a surface with complex coordinates $w = f^1 + i f^2$ in a neighbourhood of $f(z)$ and conformal metric $h(w) = \sigma(w) dw d\overline{w}$. Now $\varphi(z)$ can be

written as $\sigma(f(z))\frac{\partial f}{\partial z}\frac{\partial \overline{f}}{\partial z}$ and differentiating in $\overline{z}$ we have

$$\begin{aligned}\frac{\partial}{\partial \overline{z}}\Big(\sigma(f(z))\frac{\partial f}{\partial z}\frac{\partial \overline{f}}{\partial z}\Big) &= \frac{\partial \sigma}{\partial w}\frac{\partial f}{\partial \overline{z}}\frac{\partial f}{\partial z}\frac{\partial \overline{f}}{\partial z} + \frac{\partial \sigma}{\partial \overline{w}}\frac{\partial \overline{f}}{\partial \overline{z}}\frac{\partial f}{\partial z}\frac{\partial \overline{f}}{\partial z} + \sigma(f(z))\frac{\partial^2 f}{\partial \overline{z}\partial z}\frac{\partial \overline{f}}{\partial z} + \sigma(f(z))\frac{\partial f}{\partial z}\frac{\partial^2 \overline{f}}{\partial \overline{z}\partial z} \\ &= \sigma(f(z))\frac{\partial \overline{f}}{\partial z}\Big(\frac{\partial^2 f}{\partial \overline{z}\partial z} + \frac{1}{\sigma}\frac{\partial \sigma}{\partial w}\frac{\partial f}{\partial \overline{z}}\frac{\partial f}{\partial z}\Big) + \sigma(f(z))\frac{\partial f}{\partial z}\overline{\Big(\frac{\partial^2 f}{\partial \overline{z}\partial z} + \frac{1}{\sigma}\frac{\partial \sigma}{\partial w}\frac{\partial f}{\partial \overline{z}}\frac{\partial f}{\partial z}\Big)} \\ &= \sigma(f(z))\frac{\partial \overline{f}}{\partial z}P + \sigma(f(z))\frac{\partial f}{\partial z}\overline{P}\ ,\end{aligned}$$

where P is the left hand side of Equation 6.4.
Suppose now that f is a diffeomorphism and that φ is holomorphic. We therefore have

$$\frac{\partial}{\partial \overline{z}}\Big(\sigma(f(z))\frac{\partial f}{\partial z}\frac{\partial \overline{f}}{\partial z}\Big) = 0\ .$$

If f is not harmonic, $P \neq 0$. Since $|P| = |\overline{P}|$ and $\frac{\partial}{\partial \overline{z}}\varphi = 0$, we have

$$\Big|\frac{\partial f}{\partial z}\Big| = \Big|\frac{\partial \overline{f}}{\partial z}\Big| = \Big|\frac{\partial f}{\partial \overline{z}}\Big|\ .$$

This implies that the Jacobian of f is singular which contradics the hypothesis, hence $P = 0$ and f is harmonic. $\square$

Before moving on and talk about minimal maps we prove the following result on quadratic differential :

Lemma 6.7. Given a Riemannian metric g on a surface S and a $(1,1)$-tensor A which is g-symmetric, A is traceless if and only if $g(A\cdot,\cdot)$ is the real part of a quadratic differential for the conformal structure of g. Moreover the quadratic differential is holomorphic if and only if A is g-Codazzi.

Proof. Take normal conformal coordinates around a point x. In this coordinates $g = \delta_{ij}$ and $A = (A_{ij})_{i,j=1,2}$ where $A_{12} = A_{21}$ since A is g-symmetric. Then

$$g(A\cdot,\cdot) = A_{11}(dx^1)^2 + 2A_{12}dx^1dx^2 + A_{22}(dx^2)^2\ .$$

Using complex coordinates $z = x^1 + ix^2$ we have

$$(dx^1)^2 = \frac{dz^2 + 2dzd\overline{z} + d\overline{z}^2}{4}\ , \quad (dx^2)^2 = -\frac{dz^2 - 2dzd\overline{z} + d\overline{z}^2}{4}\ , \quad dx^1dx^2 = i\frac{d\overline{z}^2 - dz^2}{4}\ ,$$

and therefore

$$g(A\cdot,\cdot) = \Big(\frac{A_{11} - A_{22}}{4} - i\frac{A_{12}}{2}\Big)dz^2 + \Big(\frac{A_{11} + A_{22}}{4}\Big)dzd\overline{z} + \Big(\frac{A_{11} - A_{22}}{4} + i\frac{A_{12}}{2}\Big)d\overline{z}^2\ .$$

So $g(A\cdot,\cdot)$ is the real part of a quadratic differential if and only if $\frac{A_{11}+A_{22}}{4}=0$ which happens if and only if A is traceless.
Let now A be traceless, $A_{22}=-A_{11}$ and $g(A\cdot,\cdot)$ writes as

$$g(A\cdot,\cdot)=\Big(\frac{A_{11}}{2}-i\frac{A_{12}}{2}\Big)dz^2+\Big(\frac{A_{11}}{2}+i\frac{A_{12}}{2}\Big)d\overline{z}^2=\varphi dz^2+\overline{\varphi}d\overline{z}^2\ .$$

Recall that $d^\nabla A=0$ means that $\nabla_Y(AX)-\nabla_X(AY)-A([X,Y])=0$ for all $X,Y\in\Gamma(TS)$, which in normal coordinates is

$$Y^iX^j\frac{\partial A_{kj}}{\partial x^i}-X^iY^j\frac{\partial A_{kj}}{\partial x^i}=0\quad\text{for } k=1,2\ , \tag{6.5}$$

since the Christofell symbols are null. Evaluating $d^\nabla A$ on the pairs $(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})$ we have that $d^\nabla A=0$ if and only if

$$\frac{\partial A_{11}}{\partial x^2}=\frac{\partial A_{12}}{\partial x^1}\ ,\qquad \frac{\partial A_{11}}{\partial x^1}=-\frac{\partial A_{12}}{\partial x^2}\ ,$$

which happens if and only if φ is holomorphic. $\square$

We now turn our attention to minimal maps. Let M be a Riemannian manifold, N a Riemannian manifold with metric $\langle\cdot,\cdot\rangle$ and $f:M\to N$ a local embedding. Let $\widetilde{\omega}$ be the volume form on $f(M)$, and ω, $e_1,\cdots,e_m$ a volume form and a frame positively oriented orthonormal frame on M. Then

$$\begin{aligned}\mathrm{Vol}(f(M))&=\int_{f(M)}\widetilde{\omega}=\int_M f^*\omega=\int_M|f_*e_1\wedge\cdots\wedge f_*e_m|\omega=\\&=\int_M\langle f_*e_1\wedge\cdots\wedge f_*e_m,f_*e_1\wedge\cdots\wedge f_*e_m\rangle^{\frac{1}{2}}\omega\ ,\end{aligned}$$

where $\langle v_1\wedge\cdots\wedge v_m,w_1\wedge\cdots\wedge w_m\rangle=det(\langle v_i,v_j\rangle)$.
Let now $F:M\times(-\epsilon,\epsilon)\to N$ be such that $f_t(\cdot)=F(\cdot,t)$ is a local embedding for all t, $f_0=f$ and

$$suppF:=\overline{\{x\in M\ :\ F(x,t)\neq f(x)\text{ for some }t\}}$$

is a compact subset of M. For an easier computation, suppose that M is a submanifold of N so that $f=id_M$. The variation of volume is then

$$\frac{d}{dt}\mathrm{Vol}(f_t(M))\Big|_{t=0}=\sum_{i=1}^m\int_M\frac{\langle f_{t*}e_1\wedge\cdots\wedge\frac{\partial}{\partial t}f_{t*}e_i\wedge\cdots\wedge f_{t*}e_m,f_{t*}e_1\wedge\cdots\wedge f_{t*}e_m\rangle}{|f_{t*}e_1\wedge\cdots\wedge f_{t*}e_m|}\omega$$

and putting $X = \frac{\partial}{\partial t} f_{t*}|_{t=0}$ we have

$$\begin{aligned}\frac{d}{dt}\mathrm{Vol}(f_t(M))\Big|_{t=0} &= \sum_{i=1}^{m}\int_M \frac{\langle e_1 \wedge \cdots \wedge \nabla^N_{e_i}X \wedge \cdots \wedge e_m, e_1 \wedge \cdots \wedge e_m\rangle}{|e_1 \wedge \cdots \wedge e_m|}\omega \\ &= \int_M \langle \nabla^N_{e_i}X, e_i\rangle\, \omega = -\int_M \langle X, \nabla^N_{e_i}e_i\rangle\, \omega\ .\end{aligned}$$

Since the volume in invariant under reparametrization, we can take assume that $X^\top = 0$ and therefore

$$\frac{d}{dt}\mathrm{Vol}(f_t(M))\Big|_{t=0} = -\int_M \langle X^\perp, \nabla^N_{e_i}e_i\rangle\, \omega\ .$$

Definition 6.8. A submanifold M of a Riemannian manifold N is called *minimal* if it is a critical point of the volume functional i.e

$$\frac{d}{dt}\mathrm{Vol}(f_t(M))\Big|_{t=0} = 0$$

for all local variations of M.
Moreover, we say a map $f : M \to N$ is *minimal* is $f(M)$ is a minimal submanifold of N.

Similarly to the case of codimension one, for every choice of a vector field ν normal to M one has

$$B_\nu : TM \to TM$$

defined by $B_\nu(X) = (\nabla^N_X \nu)^\top$, and the *mean curvature* of M in the direction ν is

$$H_\nu := \frac{1}{m} tr(B_\nu)\ .$$

Theorem 6.9. *A submanifold M of a Riemannian manifold N is minimal if and only if the mean curvature of M vanishes for all normal directions.*

Proof. We fix an orthonormal basis $\nu_1, \cdots, \nu_k$ of $T_xM^\perp$ and write $X^\perp = \xi^j \nu_j$. Then

$$-\langle X^\perp, \nabla^N_{e_i}e_i\rangle = \xi^j tr(B_{\nu_j}) = m \cdot \xi^j H_{\nu_j}\ ,$$

and M is therefore minimal if and only if $m \cdot \xi^j H_{\nu_j}$ vanishes for all choices of ξ^j, and the claim follows. □

Let now $f : M \to N$ be an isometric immersion. We want to write the condition for the vanishing of the mean curvature, namely

$$(\nabla^N_{e_\alpha} e_\beta)^\perp = 0 \qquad \forall \alpha, \beta\ .$$

Again we can take normal coordinates on M, which, at a point, satisfy the following
$$\nabla^M_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^\beta} = 0 \ ,$$
for all $\alpha, \beta = 1, \cdots, m$, and we can take
$$e_\alpha = f_* \Big(\frac{\partial}{\partial x^\alpha} \Big) = \frac{\partial f^i}{\partial x^\alpha} \frac{\partial}{\partial f^i} \ ,$$
where $(f^1, \cdots, f^n)$ are local coordinates near $f(x)$. Since f is an isometric immersion we have $f_* \nabla^M_X Y = \nabla^{f(M)}_{f_* X} f_* Y$, and
$$\begin{aligned} (\nabla^N_{e_i} e_i)^\perp = \nabla^N_{e_i} e_i = \nabla^N_{\frac{\partial f^i}{\partial x^\alpha} \frac{\partial}{\partial f^i}} \frac{\partial f^j}{\partial x^\alpha} \frac{\partial}{\partial f^j} = \\ = \frac{\partial^2 f^j}{(\partial x^\alpha)^2} \frac{\partial}{\partial f^j} + \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\alpha} \Gamma^j_{ik} \frac{\partial}{\partial f^j} \ , \end{aligned}$$
where Γ^j_{ik} are the Christoffel symbols of N. Therefore we conclude that $f(M)$ has vanishing mean curvature if and only if
$$\frac{\partial^2 f^j}{(\partial x^\alpha)^2} + \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\alpha} \Gamma^j_{ik} = 0 \quad \text{for } j = 1, \cdots, n \ . \tag{6.6}$$
In arbitrary coordinates Equation 6.6 is written as
$$\frac{1}{\sqrt{det(g)}} \frac{\partial}{\partial x^\alpha} (\sqrt{det(g)} g^{\alpha\beta} \frac{\partial f^j}{\partial x^\beta}) + g^{\alpha\beta}(x) \Gamma^j_{ik}(f(x)) \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} = 0 \ \text{ for } j = 1, \cdots, n \ ,$$
where $(g_{\alpha\beta})_{\alpha,\beta=1,\cdots,m}$ is the metric tensor on M. We can therefore conclude that an isometric immersion is minimal if and only if it is harmonic.

Now let M be a surface with metric g and N a Riemannian manifold with metric h. We say that a map $f : M \to N$ is *conformal* if $f^* h = \lambda g$ where λ is some positive function, namely the pull-back of the metric h is *conformal* to g. We already observed that the harmonicity of $f : M \to N$ when M is a surface does not depend on the choice of conformal metric, we can therefore observe that an immersion $f : M \to N$ that is conformal and harmonic, is minimal. It suffices to substitute the metric g on M with the conformal metric $f^* h$, making f an isometric immersion, and now the fact that f is harmonic is now equivalent to it being minimal.

To summarize we state the following

Theorem 6.10. *Let M be a surface with metric g and N a Riemmanian manifold. If $f : M \to N$ is a conformal, harmonic immersion then it is minimal.*

6.2 CMC Surfaces

We now turn back out attention to smoothly embedded spacelike surfaces in globally hyperbolic AdS spacetime. A surface S has *constant mean curvature* (CMC) if the trace of the shape operator B is constant. A *maximal surface* is a surface S whose mean curvature is constantly equal to zero.

Theorem 6.11. *Every MGH Anti-de Sitter manifold with compact Cauchy surface admits a maximal surface.*

The proof of Theorem 6.11 revolves around the existence of a pair of barriers which implies existence of a maximal surface. A pair of barriers is a pair of surfaces Σ^+, Σ^- with everywhere positive/negative mean curvature, respectively, and such that Σ^+ is contained in the future of Σ^-. To do this one can take convex hull C of a proper achronal meridian Λ that defines M. This definition makes sense in affine chart, and in this setting it is well defined since $C \subset \Omega(\Lambda)$ which is contained in a Dirichlet domain. Now ∂C is the union of Λ and two surfaces $\partial_+ C$, $\partial_- C$ which are convex and concave respectively. Here a set is convex (resp. concave) if it is contained in the future (resp. past) of all its support planes. Convex (resp. concave) surfaces have non-positive (resp. non-negative) mean curvature. The quotient of $\partial_+ C$, $\partial_- C$ do not work as barriers since they are not smooth, but close to them one can find a pair of smooth barriers. A detailed proof can be found in [1]. One general fact that follows is that a maximal surface is always contained in the convex hull of its border at infinity.
Now we want to prove that such maximal surface is also unique. We already know that given two surfaces S_1, S_2 in a MGH spacetime M, then the border at infinity of $dev(\widetilde{S_1})$ and $dev(\widetilde{S_2})$ coincide in $dev(\widetilde{M}) \subset \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. Therefore, is suffices to prove the uniqueness in $\mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$. Moreover, we can take the problem in $\mathbb{H}^{2,1}$, once we choose a lifting of the border of $dev(\widetilde{S_1})$ and $dev(\widetilde{S_2})$.
Let therefore S_1 and S_2 be lifting to $\mathbb{H}^{2,1}$ of two maximal surfaces in a MGH spacetime (M, g), such that $\partial_\infty S_1 = \partial_\infty S_2$, and let ρ be the lifting to $\mathrm{Isom}(\mathbb{H}^{2,1})$ of the holonomy of (M, g).

Lemma 6.12. The function

$$\begin{array}{rcl} B : S_1 \times S_2 & \to & \mathbb{R} \\ (x, y) & \to & \langle x, y \rangle_{2,2} \end{array}$$

is negative. Moreover, if $S_1 \neq S_2$ there is a pair (x, y) such that $B(x, y) > -1$ and the function B achieves its maximum.

Proof. By Proposition 4.13 we know that for any $x \in S_1$ the border at infinity $\partial_\infty S_1$ is contained in the region U_x defined in Section 2.5. By definition of U_x we have that $\langle x, \cdot \rangle_{2,2}$ is negative on $\partial_\infty S_1$. Since $\partial_\infty S_1 = \partial_\infty S_2$, S_2 is included in $Conv(\partial_\infty S_2) = Conv(\partial_\infty S_1)$ and therefore $\langle x, \cdot \rangle_{2,2}$ is negative on S_2.
Let now $S_1 \neq S_2$ and let x be a point in S_1 which is not in S_2. Choose now an identification of $\mathbb{H}^{2,1}$ with $\mathbb{D} \times \mathbb{S}^1$ in which $x = (0, v_1)$ for some $v_1 \in S^1$. Recalling that S_2 is obtained as the graph of a function $f : \mathbb{D} \to \mathbb{S}^1$, there is a point in S_2 of the form $y = (0, v_2) = (0, f(0))$. Since $x \notin S_2$ we have $v_2 \neq v_1$. Back in the quadric model we have that $x = (0, 0, v_1)$ and $y = (0, 0, v_2)$, and therefore

$$B(x, y) = \langle x, y \rangle_{2,2} = -\langle v_1, v_2 \rangle_{\mathbb{S}^1} > -1 \ .$$

We now have to prove that B achieves its maximum. Let $(x_n, y_n) \in S_1 \times S_2$ be a maximazing sequence of B. Since ρ acts co-compacly on S_1 and preserves B, we can suppose that x_n is bounded in S_1. Up to extracting a subsequence, we can also assume that x_n converges to $x \in S_1$. Assume by contradiction that y_n is unbounded in S_2. Then, up to extracting a subsequence, there exists a sequence of real number $\epsilon_n \to 0$ such that $\epsilon_n y_n$ converges to a vector $y \in \partial_\infty S_2$. This implies that $-1 < B(x_n, y_n) < 0$ for n sufficiently large, then $B(x_n, y_n)$ is bounded and we have

$$B(x, y) = \lim_{n \to +\infty} \epsilon_n B(x_n, y_n) = 0.$$

The vector y is thus in $x^\perp$, so y is not in U_x which by Proposition 4.13 contains $\partial_\infty S_1 = \partial_\infty S_2$. So $y \notin \partial_\infty S_2$, hence a contradiction. ☐

We can now prove the uniqueness of the maximal surface:

Theorem 6.13. *Given a proper achronal meridian $\Lambda \subset \mathbb{H}^{2,1}$ there is a unique maximal surface S such that $\partial_\infty S = \Lambda$.*

Proof. Let S_1, S_2 be two different maximal surface. By Lemma 6.12 there is a point $(x_0, y_0) \in S_1 \times S_2$ where B achieves its maximum and

$$-1 < \langle x_0, y_0 \rangle_{2,2} < 0 \ .$$

Let $\dot{x}_0 \in T_{x_0} S_1$ and $\dot{y}_0 \in T_{y_0} S_2$, and let $(x(t))_{t \in (-\epsilon, \epsilon)}$ and $(y(t))_{t \in (-\epsilon, \epsilon)}$ be geodesics on S_1 and S_2 respectively, satisfying $x(0) = x_0$, $x'(0) = \dot{x}_0$ and $y(0) = y_0$, $y'(0) = \dot{y}_0$. Since $B(x(t), y_0)$ is maximal at $t = 0$, we have $\langle \dot{x}_0, y_0 \rangle_{2,2}$=0. Similarly. we have $\langle x_0, \dot{y}_0 \rangle_{2,2}$=0. We thus have that $T_{x_0} S_1$ and $T_{y_0} S_2$ are orhogonal to both x_0 and y_0. Since $x_0, y_0 \in \mathbb{H}^{2,1}$ are not parallel we also have that $T_{x_0} S_1 = T_{y_0} S_2$.

Denote now by $I\!I_1$ and $I\!I_2$ the second fundamental form of S_1 and S_2. The orthogonal projection of $x''(0)$ to $T_{x_0}S$ is given by $I\!I(\dot{x}_0, \dot{x}_0)\nu$, hence we have

$$I\!I_1(\dot{x}_0, \dot{x}_0)\nu = x''(0) - q_{2,2}(\dot{x}_0)x_0 \ ,$$

and similarly

$$I\!I_2(\dot{y}_0, \dot{y}_0)\nu = y''(0) - q_{2,2}(\dot{y}_0)y_0 \ .$$

Then we have

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0} B(x(t), y(t)) &= B(x''(0), y_0) + 2B(\dot{x}_0, \dot{y}_0) + B(x_0, y''(0)) \\ &= 2\langle \dot{x}_0, \dot{y}_0\rangle_{2,2} + (q_{2,2}(\dot{x}_0) + q_{2,2}(\dot{y}_0))\langle x_0, y_0\rangle_{2,2} \\ &\quad + \langle I\!I_1(\dot{x}_0, \dot{x}_0)\nu, y_0\rangle_{2,2} + \langle I\!I_2(\dot{y}_0, \dot{y}_0)\nu, x_0\rangle_{2,2}\end{aligned}$$

Our goal now is to find $\dot{x}_0$ and $\dot{y}_0$ such that this second derivative is positive, in contradiction with the fact that (x_0, y_0) is a point of maximum.
Consider the quadratic form $\beta_1 : w \to \langle I\!I_1(w, w)\nu, y_0\rangle_{2,2}$. Since S_1 is maximal, $I\!I_1$ is traceless and so is β_1. Hence β_1 has opposite eigenvalues $\lambda, -\lambda$. Similarly $\beta_2 : w \to \langle I\!I_2(w, w)\nu, x_0\rangle_{2,2}$ has opposite eigenvalues $\mu, -\mu$. Up to switching S_1 and S_2 we can suppose $\lambda \geqslant \mu \geqslant 0$. Now we can choose $\dot{x}_0$ and $\dot{y}_0$ such that

$$q_{2,2}(\dot{x}_0) = 1 \ , \qquad \beta_1(\dot{x}_0) = \lambda \ , \qquad \dot{y}_0 = \dot{x}_0 \ .$$

Since $\dot{y}_0$ is unitary, $\beta_2(\dot{y}_0) \geqslant -\mu \geqslant -\lambda$, and therefore

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0} B(x(t), y(t)) &= 2\langle \dot{x}_0, \dot{y}_0\rangle_{2,2} + 2\langle x_0, y_0\rangle_{2,2} + \beta_1(\dot{x}_0) + \beta_2(\dot{y}_0) \\ &\geqslant 2\langle x_0, y_0\rangle_{2,2} + 2 > 0 \ .\end{aligned}$$

This contradics the maximality of B at (x_0, y_0), hence the maximal surface is unique. □

Given now a spacelike immersion σ, recall by Lemma 5.4 that

$$B_t = (\cos(t)id + \sin(t)B)^{-1}(-\sin(t)id + \cos(t)B) \ .$$

Let $\lambda(x)$, $\mu(x)$ be the eigenvalues of B. By direct computation one finds that

$$\lambda_t(x) = \frac{\lambda(x) + \tan(t)}{1 - \lambda(x)\tan(t)} \qquad \mu_t(x) = \frac{\mu(x) + \tan(t)}{1 - \mu(x)\tan(t)} \ . \tag{6.7}$$

In general one can find a relation between CMC surfaces and constant Gaussian curvature (CGC) surfaces, namely surfaces where $K = -1 - detB$ is constant, see [4]. Here we show a weaker result which will prove useful in the proof of the main result:

Proposition 6.14. *Let $\sigma : S \to \mathbb{A}\mathrm{d}\mathbb{S}^{2,1}$ be a maximal surface. Then the normal evolution σ_t at the time $t = -\pi/4$ is an immersion of constant Gaussian curvature $K = -2$.*

Proof. Since σ determines a maximal surface $\mu(x) = -\lambda(x)$. By Equation 6.7 the eigenvalues of B_t at time $t = -\pi/4$ are

$$\lambda_{-\frac{\pi}{4}}(x) = \frac{\lambda(x) - 1}{1 + \lambda(x)} \qquad \mu_{-\frac{\pi}{4}}(x) = \frac{\mu(x) - 1}{1 + \mu(x)} = \frac{\lambda(x) + 1}{-1 + \lambda(x)} \ .$$

The Gaussian curvature at x is therefore given by

$$K_{-\frac{\pi}{4}}(x) = -1 - \frac{\lambda^2(x) - 1}{\lambda^2(x) - 1} = -1 - 1 = -2 \ .$$

□

6.3 Minimal Lagrangian diffeomorphisms

Definition 6.15. Let h and h' be two hyperbolic metrics on the closed surface S. A smooth map $f : (S, h) \to (S, h')$ is *minimal Lagrangian* if its graph is a minimal Lagrangian submanifold of $S \times S$ with respect to the product metric $h \oplus h'$ and the symplectic form $\pi_l^* \omega_h - \pi_r^* \omega_{h'}$.

Theorem 6.16. *Given a closed surface Σ and two hyperbolic metrics h, h' on it, there exists a minimal Lagrangian diffeomorphism $f_0 : (\Sigma, h) \to (\Sigma, h')$ isotopic to the identity.*

Proof. Consider the MGH spacetime M, with compact Cauchy surface Σ, whose holonomy $\rho = (\rho_l, \rho_r)$ induces h and h' as quotient metric in $\mathbb{H}^2/\rho_l(\pi_1\Sigma)$ and $\mathbb{H}^2/\rho_r(\pi_1\Sigma)$. Let now Σ_0 be a maximal surface in M, we want to prove that the associated map f_0 is a minimal Lagrangian diffeomorphism.
First we prove that f_0 is a diffeomorphism. By Proposition 6.14, Σ_0 is obtained from a surface Σ_1 of constant Gaussian curvature equal to -2 by normal evolution. The right and left projection of the Gauss map are always local diffeomorphism on a surface with curvature everywhere different from 0, and, since Σ_1 is compact, the projections are proper which makes them embeddings, and ebbeddings from a compact surface to a connected surface are diffeomorphisms. Now, recalling from Section 5.3 that spacelike immersions which differ from one another by the normal evolution have the same image, we have that the projections are diffeomorphism for Σ_0, and therefore f_0 is a diffeomorphism. Moreover the Lagrangian condition is equivalent to the fact

that f_0 is area-preserving, and we proved in Section 5.3 that this is always verified by the image of the Gauss map.
We now want to prove that the Gauss map is harmonic and conformal, which, by Theorem 6.10 implies that it is minimal.
The pull-back metric via the Gauss map is the sum of $\Pi_l^* g_{\mathbb{H}^2}$ and $\Pi_r^* g_{\mathbb{H}^2}$, which, by Proposition 5.12, is given by

$$\Pi_l^* g_{\mathbb{H}^2} + \Pi_r^* g_{\mathbb{H}^2} = 2I + 2I(\mathcal{J}B\cdot, \mathcal{J}B\cdot) = 2I + 2I(B\cdot, B\cdot) = 2(I + \mathit{III}) \ .$$

Being B traceless, by the Cayley-Hamilton theorem $B^2 + (\det B)id = 0$. Now, using that B is I-symmetric, we have that $\mathit{III} = -(\det B)I$, showing conformality.
To prove the harmonicity of the Gauss map it suffices to prove that the left and right projection are harmonic. We check the harmonicity of Π_l, since the proof is the same for Π_r. To see this we can write the target metric

$$\Pi_l^* g_{\mathbb{H}^2} = I((\mathrm{id} - \mathcal{J}B)\cdot, (\mathrm{id} - \mathcal{J}B)\cdot) = I(\cdot, \cdot) + \mathit{III}(\cdot, \cdot) - 2I(\mathcal{J}B\cdot, \cdot),$$

where we used that B is traceless and therefore $\mathcal{J}B$ is I-symmetric. Moreover $\mathcal{J}B$ is I-Codazzi and by Lemma 6.7 $-2I(\mathcal{J}B\cdot, \cdot)$ is the real part of the Hopf differential of Π_l which is holomorphic since $\mathcal{J}B$ is I-Codazzi. Hence, by Theorem 6.6 we have that Π_l is harmonic. □

For the uniqueness of the minimal Lagrangian diffeomorphism we need the following:

Lemma 6.17. Given a minimal Lagrangian diffeomorphism $\Phi : (S, h) \to (S, h')$ there is a $(1,1)$-tensor b that is h-symmetric, $\det(b) = 1$, h-Codazzi and satisfies

$$\Phi^* h' = h(b\cdot, b\cdot).$$

Proof. One can always find a b that is h-symmetric and such that $\Phi^* h' = h(b\cdot, b\cdot)$. Moreover $\det(b) = 1$ since Φ is area preserving. Hence, we just need to prove that b is h-Codazzi.
Since graph(Φ) is minimal in $(S, h) \times (S, h')$, the inclusion map is harmonic. A map is harmonic if and only if its factors are, therefore, since graph$(\Phi) \cong S$, we have that $id : (S, g) \to (S, h)$ and $\Phi : (S, g) \to (S, h')$ are harmonic, where g is the metric induced on graph(Φ). Since the product metric is the sum of the metric tensor on each factor, we have that

$$g = h + \Phi^* h' = h + h(b\cdot, b\cdot) \ .$$

Define now $G = h((id + b)\cdot, (id + b)\cdot)$ and $B = (id + b)^{-1}(b - id)$. B is G-symmetric, indeed for v, w tangent vectors we have

$$\begin{aligned} G(Bv, w) &= h((id+b)(id+b)^{-1}(b-id)v, (id+b)w) \\ &= h((b-id)v, (id+b)w) = h((id+b)v, (b-id)w) \\ &= G(v, Bw) \ . \end{aligned}$$

Moreover, since $\det(b) = 1$, a direct computation in a basis where b is diagonal shows that B is traceless. Now, since the minimal polynomial of b is $b^2 - tr(b)b + id = 0$, we have that $(id + b)^2 = id + 2b + b^2 = (2 + tr(b))b$, hence G is conformal to g because

$$G = h((id+b)^2\cdot, \cdot) = (2 + tr(b))h(b\cdot, \cdot) \ ,$$

and

$$g = h + h(b^2\cdot, \cdot) = tr(b)h(b\cdot, \cdot) \ .$$

Therefore $id : (S, G) \to (S, h)$ is harmonic too. Moreover, using that the minimal polynomial of B is $B^2 + \det(B)id = 0$ since B is traceless, and that $id - B = 2(id + b)^{-1}$, we can write

$$\frac{h}{4} = G((id - B)\cdot, (id - B)\cdot) = (1 - \det(B))G - 2G(B\cdot, \cdot) \ .$$

Hence, by Lemma 6.6 we have that $G(B\cdot, \cdot)$ is the real part of a harmonic quadratic differential and, by Lemma 6.7, B is G-Codazzi. By Proposition 5.13 the Levi-Civita connection of h is given by

$$\nabla^h_Y X = (id - B)^{-1}(\nabla^G_Y (id - B)X) \ ,$$

hence

$$\begin{aligned} (d^{\nabla^h} b)(X, Y) &= (id - B)^{-1}(\nabla^G_Y((id - B)bX) - \nabla^G_X((id - B)bY)) - b\,[X, Y] \\ &= (id - B)^{-1}(\nabla^G_Y((id + B)X) - \nabla^h_X((id + B)Y) - (id + B)\,[X, Y]) \\ &= (id - B)^{-1}(d^{\nabla^G}(id + B))(X, Y) = 0 \ . \end{aligned}$$

where we used that $b = (id - B)^{-1}(id + B)$, so b is h-Codazzi. □

We are now ready to prove the uniqueness

Theorem 6.18. *Given a surface Σ and two hyperbolic metrics h, h' on it, the minimal Lagrangian diffeomorphism $\Phi : (\Sigma, h) \to (\Sigma, h')$ is unique.*

Proof. Let b be the unique $(1,1)$-tensor associated to Φ as in Lemma 6.17. We now want to construct a pair $(I, I\!I)$ that define a maximal immersion such that the associated map f_0 respect $f_0^* h' = h(b\cdot, b\cdot)$.
Let I be the metric on Σ defined by

$$4I = h((id + b)\cdot, (id + b)\cdot) ,$$

let $\mathcal{J}$ be the complex structure defined on Σ by I and B the $(1,1)$-tensor defined by

$$\mathcal{J}B = (id + b)^{-1}(b - id) .$$

Here I and B are well-defined at all points since b has positive eigenvalues. Since $\det(b) = 1$, a direct computation in a basis where b is diagonal shows that $\mathrm{tr}(\mathcal{J}B) = 0$, hence B is I-symmetric. Hence we can set $I\!I = I(B\cdot, \cdot)$. We can also write b as

$$b = (id - \mathcal{J}B)^{-1}(id + \mathcal{J}B) .$$

We now have to prove that I and B satisfy the Gauss-Codazzi equations, and that B is traceless. $\mathcal{J}B$ is I-symmetric, indeed for v, w tangent vectors we have

$$\begin{aligned} 4I(\mathcal{J}Bv, w) &= h((id + b)(id + b)^{-1}(b - id)v, (id + b)w) \\ &= h((b - id)v, (id + b)w) = h((id + b)v, (b - id)w) \\ &= 4I(v, \mathcal{J}Bw) , \end{aligned}$$

and it follows that B is traceless.
By Proposition 5.13 the Levi-Civita connection ∇^I of I is

$$\nabla^I_Y X = (id + b)^{-1}\nabla^h_Y((id + b)X) .$$

Since b is h-Codazzi, $(b - id)$ is h-Codazzi, and therefore

$$\begin{aligned} (d^{\nabla^I}\mathcal{J}B)(X, Y) &= (id + b)^{-1}(\nabla^h_Y((id + b)\mathcal{J}BX) - \nabla^h_X((id + b)\mathcal{J}BY)) - \mathcal{J}B\,[X, Y] \\ &= (id + b)^{-1}(\nabla^h_Y((b - id)X) - \nabla^h_X((b - id)Y) - (b - id)\,[X, Y]) \\ &= (id + b)^{-1}(d^{\nabla^h}(b - id))(X, Y) = 0 . \end{aligned}$$

Now, since $\mathcal{J}\nabla^I_Y(BX) = (\nabla^I_Y\mathcal{J})(BX)$, we have that $d^{\nabla^I}(\mathcal{J}B) = \mathcal{J}d^{\nabla^I}B$, and therefore B is h-Codazzi.
A simple computation shows that

$$(id + \mathcal{J}B) = 2(id + b)^{-1}b , \qquad (id - \mathcal{J}B) = 2(id + b)^{-1} , \tag{6.8}$$

hence, by Proposition 5.13 the curvature of I is given by

$$K_I = \frac{-1}{\det((id+b)/2)} = -\det(id - \mathcal{J}B) = -1 - \det(B) \ .$$

We have shown that the pair $(I, I\!I)$ defines a maximal immersion. Let now f_0 be the minimal Lagrangian diffeomorphism associated to the pair $(I, I\!I)$, the $(1,1)$ tensor associated to f_0 in Lemma 6.17 is clearly $b_{f_0} = (id + \mathcal{J}B)(id - \mathcal{J}B)^{-1} = b$, hence $f_0 = \Phi$. The uniqueness of the maximal surface ensures that any other minimal Lagrangian diffeomorphism Φ' is equal, via the same construction, to $f_0 = \Phi$, hence the uniqueness. $\square$

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