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PG1.

Assignment - 1 : Linear Algebra

Q1. Prove that the set $F = \{0,1\}$ is a field if we define addition as the boolean XOR and multiplication as the boolean AND gates.

Ans. Suppose one has a set F of objects $x, y, z, \ldots$ and two operations:

i) addition (boolean XOR) of x, y in F as:

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

Table 1

ii) multiplication (boolean AND) of x, y in F as

x	y	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

Table 2.

The set F, along with these two operations in $\{0,1\}$ is called a field if it satisfies the following 1-9 properties:

1. Addition (Boolean XOR) is commutative
$x \oplus y = y \oplus x$
for elements in $\{0,1\}$.

2. Addition (Boolean XOR) is associative:
$$x \oplus (y \oplus z) = (x \oplus y) \oplus z$$
for x, y in $F = \{0, 1\}$. We know this for XOR.

3. There is a unique element 0 in F such that $x \oplus 0 = x$, for x in $\{0, 1\}$.
Since, $1 \oplus 0 = 1$ and $0 \oplus 0 = 0$, this property is proved.

4. For each x in F there corresponds a unique element $(-x)$ in F such that $x \oplus (-x) = 0$.
i.e, $1 \oplus 1 = 0$ and $0 \oplus 0 = 0$
So, every element is inverse of itself.
Hence, this property is proved.

5. Multiplication (Boolean AND) is commutative.
i.e. $xy = yx \quad \forall\ x, y \in F$.
From the table 2, we can see that $xy = yx$.
Hence, this property is proved.

6. Multiplication (Boolean AND) is associative.
$$x.(y.z) = (x.y).z$$
for all x, y, z in F.
We know this is true for Boolean AND.

7. There exists a unique non-zero element 1 in F such that $x.1 = 1.x = x$, for every x in F.
In Boolean AND, this element is "1".

8. To each non-zero x in F there exists a unique element x^{-1} in F such that $x.x^{-1} = 1$. For Boolean AND, $x^{-1} = 1$.

9. Multiplication distributes over addition i.e. $x(y \oplus z) = xy \oplus xz \;\forall\; x, y, z \in F$.
This is true using Boolean Algebra.

LHS

x	y	z	$y \oplus z$	$x(y \oplus z)$
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	0	0

RHS

x	y	z	xy	xz	$xy \oplus xz$
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	0	0
0	1	1	0	0	0
1	0	0	0	0	0
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	1	0

Since the truth table values are same, we proved that $x(y \oplus z) = xy \oplus xz$.

10. Along with (1-9) properties, the trivial property of closure under addition and multiplication is satisfied as the operations are boolean algebra operations under binary numbers $\{0,1\}$.
Obviously, the result of the operations will lie in $\{0,1\}$.

Hence, $\oplus$ and $\cdot$ is closed under F.

Since all the properties of field are satisfied, we can say that the set $F = \{0,1\}$ is a field under addition (Boolean XOR) and multiplication (Boolean AND)

Q2. Which of the following pairs of sets V and F form a valid vector space? Prove all your claims. (Addition and multiplication operations are defined as usual arithmetic on real numbers. Note that you have to check if V is a vector space over F.)

(a) $V = \mathbb{R}$ and $F = \mathbb{N}$

In Qn(1), we have specified in property (3), a field must satisfy the property that:
For each x in F there must correspond a unique element ($-x$) in F such that
$x + 0 = x$.

$F = \mathbb{N}$ does not satisfy this property as there does not exist any zero element in $\mathbb{N}$ like this.
So, $F = \mathbb{N}$ is not a field.
Thus $V = \mathbb{R}$ and $F = \mathbb{N}$ cannot form a valid vector space.

(b) $V = Q$ and $F = \mathbb{R}$.
Step 1: Proving $\mathbb{R}$ is a field.
In Qn 1. we have seen properties for a set to be a field.
The set $\mathbb{R}$ satisfies the properties in Qn 1.
So, $\mathbb{R}$ is a field.

Step 2: Checking if $V = Q$ and $F = \mathbb{R}$ forms a vector space or not.

Let us check if $V = Q$ over $F = R$ follows the following properties of vector space:

V1. For any two vectors $\bar{\alpha}, \bar{\beta} \in \mathbb{Q}$ $\bar{\alpha} + \bar{\beta} \in \mathbb{Q}$.
We can see that additive closure is maintained for Q.

V2. For any two vectors $\bar{\alpha}, \bar{\beta} \in Q$, $\bar{\alpha} + \bar{\beta} = \bar{\beta} + \bar{\alpha}$
This is true as rational numbers are commutative under addition.

V3. For any three vectors $\bar{\alpha}, \beta, \bar{\gamma} \in Q$, $\bar{\alpha} + (\bar{\beta} + \bar{\gamma}) = (\bar{\alpha} + \bar{\beta}) + \bar{\gamma}$

This property is satisfied as addition is associative under Q.

V4. There exists a unique vector $\phi \in V$ such that $\bar{\alpha} + \bar{\phi} = \bar{\alpha} = \bar{\phi} + \bar{\alpha}$.
Here, the vector is O_V or the zero vector.
Hence, this property is satisfied.

V5. For any vector $\bar{\alpha} \in Q$, $\exists$ a unique vector $-\bar{\alpha} \in V$ such that $\bar{\alpha} + (-\bar{\alpha}) = (-\bar{\alpha}) + \bar{\alpha} = \phi$
The additive inverse of a vector in Q is the negative vector that also belongs to Q.
Hence, this property is satisfied.

V6. For any element $a \in \mathbb{R}$ and any element $\bar{\alpha} \in Q$, $a\bar{\alpha} \in Q$.
Now, this property is not satisfied.
Let us take an example to prove this.
Let $\bar{\alpha} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$, Let $a = \sqrt{2}\cancel{+2i}$ ($\sqrt{2}$ is irrational)

$$a\bar{\alpha} = (\sqrt{2}\cancel{+2i})\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} \cancel{1+i} \\ \cancel{2+2i} \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2} \end{bmatrix} \notin Q_V$$

Now, $a\bar{\alpha} \notin Q$.
So, Q and C are not closed under scalar multiplication. Thus, we can say that $V = Q$ and $F = R$ DOES NOT form a vector space.

(c) $V = \mathbb{R}$ and $F = \mathbb{Q}$.

Step 1: proving that $\mathbb{Q}$ is a field.
$F = \mathbb{Q}$ satisfies all the 10 properties that we have seen in Qn 1 for a set to be a field. Thus, $\mathbb{Q}$ is a field.

Step 2: Proving that $V = \mathbb{R}$ forms a vector space over $F = \mathbb{Q}$.

Let us see if the properties of vector space are followed or not:

V1. For any two vectors $\bar{\alpha}, \bar{\beta} \in \mathbb{R}$, $\bar{\alpha} + \bar{\beta} \in \mathbb{R}$.
This is true since addition is closed under $\mathbb{R}$.

V2. For any two vectors $\bar{\alpha}, \bar{\beta} \in \mathbb{R}$, $\bar{\alpha} + \bar{\beta} = \bar{\beta} + \bar{\alpha}$.
This is true since addition is commutative.

V3. For any three vectors $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in \mathbb{R}$,
$(\bar{\alpha} + \bar{\beta}) + \bar{\gamma} = \bar{\alpha} + (\bar{\beta} + \bar{\gamma})$.
This is true as addition is associative.

V4. There exists a unique vector $\phi \in \mathbb{R}$ such that $\bar{\alpha} + \phi = \bar{\alpha} = \bar{\phi} + \bar{\alpha}$
This is true as vector $\bar{O}_V$ is the zero vector in $\mathbb{R}$ that is the additive identity of any vector $\bar{\alpha} \in \mathbb{R}$.

V5. For any vector $\bar{\alpha} \in \mathbb{R}$, $\exists$ an unique vector $-\bar{\alpha} \in \mathbb{R}$ such that $\bar{\alpha} + (-\bar{\alpha}) = (-\bar{\alpha}) + \bar{\alpha} = \phi$
This is true as for every element of form $(a+bi)$, $(-a-bi)$ will form its additive inverse in the vector which will result in zero vector.

V6. For any element $a \in \mathbb{Q}$ and any element $\bar{\alpha} \in \mathbb{R}$, $a\bar{\alpha} \in \mathbb{Q}$.
This is true as when a rational number is multiplied by a real number, the result will lie in $\mathbb{R}$.
Thus, $\mathbb{R}$ is closed under scalar multiplication with $\mathbb{Q}$.

V7. For any element $a \in \mathbb{Q}$ and any vector $\bar{\beta}, \bar{\alpha} \in \mathbb{R}$, we have $a(\bar{\alpha} + \bar{\beta}) = a\bar{\alpha} + a\bar{\beta}$.
This is true as multiplication is distributive over addition.

V9. For any two scalars $a, b \in \mathbb{Q}$ and any vector $\bar{\alpha} \in \mathbb{R}$ $\quad (ab)\bar{\alpha} = a(b\bar{\alpha})$
This is true as both LHS and RHS will have the final result as every element in vector $\bar{\alpha}$ being multiplied by (ab).

V8. For any two scalars, $a, b \in \mathbb{Q}$ and any vector $\bar{\alpha} \in \mathbb{R}$, $(a+b)\bar{\alpha} = a\bar{\alpha} + b\bar{\alpha}$.
We know this is true because multiplication is distributive over addition.

V10. For any scalar $1 \in Q$ and any vector $\bar{\alpha} \in \mathbb{R}$,
$1 \cdot \bar{\alpha} = \bar{\alpha}$.
This is true as 1, i.e. a unit scalar $\in Q$ when multiplied by any element in $\mathbb{R}$ will give the element itself.

Since all the 10 properties are satisfied, we can say that $V = \mathbb{R}$ is a vector space under the scalar $F = Q$.

(d) $V = \mathbb{R}$ and $F = C$

Step 1: Proving that $F = C$ is a field.
We have seen the properties a set should satisfy in order to be a field in Qn 1.
We can see that $F = C$ follows all those 10 properties.
Thus $F = C$ is a field under $(+, \cdot)$ operations.

Step 2: Checking if $V = \mathbb{R}$ is a vector space under $F = C$.

We have seen all the 10 properties V should satisfy to be a vector space in Qn 2.(c).
Let us see property V6.

Let $\bar{\alpha} = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$ and $a = 2 + 2i$

$$a \cdot \bar{\alpha} = (2+2i)\begin{bmatrix} 2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 4+4i \\ 1+i \end{bmatrix} \notin \mathbb{R}^2.$$

Here, for element $a \in C$ and $\bar{\alpha} \in \mathbb{R}$, $a\bar{\alpha} \notin \mathbb{R}$.

Thus, $\mathbb{R}$ is NOT closed under scalar multiplication with C (complex number set).

Thus, $V = \mathbb{R}$ DOES NOT form a vector space over $F = C$ as closure under scalar multiplication property (V6) is not satisfied.

Qn 3. Prove that a field F is a vector space over itself. Also show that the direct sums of a field F will form a vector space V over F. Do you see a pattern here related to the previous question?

Ans Given that F is a field. Let A be an arbitrary field

To prove: $V = A$ is a vector space under $F = A$ ~~iff~~ if A is a field.

<u>Proof</u>: Let us consider the ten properties of a vector space.

V1. For any two vectors $\bar{\alpha}, \bar{\beta} \in A$, $\bar{\alpha} + \bar{\beta} \in A$. Since A is a field, addition will be ~~commutative~~ closed in A. Hence, this property will hold true.

V2. For any two vectors, $\bar{\alpha}, \bar{\beta} \in A$, $\bar{\alpha}+\bar{\beta} = \bar{\beta}+\bar{\alpha}$
This is true as A is a field and addition is commutative in a field, A.

V3. For any three vectors, $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in A$, $(\bar{\alpha}+\bar{\beta})+\bar{\gamma} = \bar{\alpha}+(\bar{\beta}+\bar{\gamma})$. This is true as A is a field and addition is associative in a field.

V4. There exists a unique vector $\phi \in A$ such that $\bar{\alpha}+\phi = \bar{\alpha} = \bar{\phi}+\bar{\alpha}$
Since A is a field, there will exist an additive identity element in A. So, this property holds true.

V5. For any vector $\bar{\alpha} \in A$, $\exists$ an unique vector $-\bar{\alpha} \in A$ such that $\bar{\alpha}+(-\bar{\alpha}) = (-\bar{\alpha})+\bar{\alpha} = \phi$.
Since A is a field, each element $a \in A$ will have a unique additive inverse. So this property holds true for vectors $\bar{\alpha} \in A$.

V6. For any element $a \in A$ and any element vector $\bar{\alpha} \in A$, $a\bar{\alpha} \in A$.
Since A is a field, it will be closed under multiplication. So, when we multiply an element in the vector $\bar{\alpha} \in A$ with element $a \in A$, we will get the elements that belong to A only.
So, this property holds true.

V7. For any element $a \in A$ and any vector $\bar{\alpha}, \bar{\beta} \in A$ we have $a(\bar{\alpha}+\bar{\beta}) = a\bar{\alpha} + a\bar{\beta}$. Since A is a field, multiplication is distributive over addition. So, this property holds true.

V8. For any two scalars, $a, b \in A$ and any vector $\bar{\alpha} \in A$, $(a+b)\bar{\alpha} = a\bar{\alpha} + b\bar{\alpha}$. This is again true as multiplication is distributive over addition in field A.

V9. For any two scalars, $a, b \in A$ and any vector $\bar{\alpha} \in A$,
$(ab)\bar{\alpha} = a(b\bar{\alpha})$.
This is true as multiplication is associative in field A.

V10. For any scalar $1 \in A$ and any vector $\bar{\alpha} \in A$,
$1 \cdot \bar{\alpha} = \bar{\alpha}$.
This is true as A will have a unique multiplicative inverse for each element in A as it is a field.

Thus, we have seen that all the properties of a field A can be extended to prove the vector A under field A will form a valid vector space.

Hence, proved.

To prove: Direct sums of a field F will form a vector space V over F.

→ In abstract algebra, the direct sum is a construction which combines several modules into a new, larger module.

eg. If $\mathbb{R}$ is the set of real numbers, then

$$\mathbb{R} \oplus \mathbb{R} = \mathbb{R}^2$$
$$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} = \mathbb{R}^3$$
$$\vdots$$
$$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \ldots n \text{ times} = \mathbb{R}^n.$$

where $\oplus$ represents direct sum of components

Similarly, according to the question,
$F_1 \oplus F_2$ consists of pairs (a, b) where $a \in F$ and $b \in F$.

$F_1 \oplus F_2 \oplus \ldots \oplus F_n = F^n$ consists of n tuples $(a_1, a_2, \ldots a_n)$ where $a_1 \in F, a_2 \in F, \ldots a_n \in F$.

In Q2 (c), we have seen the 10 properties that a set must follow to be a valid vector space.

Let us prove the ten properties:

V1. For any two vectors $\bar{\alpha}, \bar{\beta} \in V$, $\bar{\alpha} + \bar{\beta} \in V$.

Let $\bar{\alpha} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\bar{\beta} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$$\bar{\alpha}+\bar{\beta} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_n+b_n \end{bmatrix}$$

Since $a_i \in F$ and $b_i \in F$, then $a_i+b_i \in F$, as addition is closed under a field.

Hence, $\bar{\alpha}+\bar{\beta} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix} \in F^n$.

So, this property is satisfied.

V2. For any two vectors, $\bar{\alpha}, \bar{\beta} \in V$, $\bar{\alpha}+\bar{\beta} = \bar{\beta}+\bar{\alpha}$

$$\bar{\alpha}+\bar{\beta} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix} = \begin{bmatrix} b_1+a_1 \\ b_2+a_2 \\ \vdots \\ b_n+a_n \end{bmatrix} = \bar{\beta}+\bar{\alpha}$$

This is possible as $a_i \in F$ and $b_i \in F$ and addition is commutative in a field.

$\therefore a_i+b_i = b_i+a_i$

So, this property is satisfied.

V3. For any three vectors, $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in V$,

$\bar{\alpha}+(\bar{\beta}+\bar{\gamma}) = (\bar{\alpha}+\bar{\beta})+\bar{\gamma}$.

~~Since~~ Let $a_i \in F$ and $b_i \in F$ and $c_i \in F$ and since addition is associative in a field, we get:

$$a_i+(b_i+c_i) = (a_i+b_i)+c_i$$

$$\bar{\alpha} + (\bar{\beta} + \bar{\gamma}) = \begin{bmatrix} a_1 + (b_1 + c_1) \\ a_2 + (b_2 + c_2) \\ \vdots \\ a_n + (b_n + c_n) \end{bmatrix} = \begin{bmatrix} (a_1 + b_1) + c_1 \\ (a_2 + b_2) + c_2 \\ \vdots \\ (a_n + b_n) + c_n \end{bmatrix} = (\bar{\alpha} + \bar{\beta}) + \bar{\gamma}$$

So, we have proved this property.

V4 There exists a unique element in V $\phi \in V$ such that $\bar{\alpha} + \phi = \bar{\alpha} = \phi + \bar{\alpha}$

Since F is a field, there will exist a unique additive identity element $e = 0 \in F$.

So, we can write $\phi = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ such that

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

So, we get a $\phi \in V$ such that $\bar{\alpha} + \phi = \bar{\alpha} = \phi + \bar{\alpha}$.

V5. For any vector $\bar{\alpha} \in V$, $\exists$ a unique vector $-\bar{\alpha} \in V$ such that $\bar{\alpha} + (-\bar{\alpha}) = (-\bar{\alpha}) + \bar{\alpha} = \phi$

Since F is a field, each element $a_i \in F$ will have an additive inverse $-a_i$.

$\therefore$ For vector $\bar{\alpha} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, $-\bar{\alpha} = \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{bmatrix}$

such that $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{bmatrix} = \phi$. So, V5 is proved to be true.

V6. For any element $a \in F$ and any element $\bar{\alpha} \in V$, $a\bar{\alpha} \in V$.

Let $a_i \in F$ and $c \in F$. According to closure property of multiplication, $a_i c \in F$.

So, if $\bar{\alpha} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$,

$$c\bar{\alpha} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix} \in F^n \text{ i.e } (V).$$

So, V6 is satisfied.

V7. For any element $a \in F$ and any vector $\bar{\alpha}, \bar{\beta} \in V$, we have

$$a(\bar{\alpha} + \bar{\beta}) = a\bar{\alpha} + a\bar{\beta}$$

In fields, multiplication is distributive over addition.

If $a_i \in F$, $b_i \in F$ and $c_i \in F$ then

$$a_i(b_i + c_i) = a_i b_i + a_i c_i$$

So, if $\bar{\alpha} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in F^n$ and $\bar{\beta} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in F^n$

and $c \in F$; then

$$c(\bar{\alpha}+\bar{\beta}) = c\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix} = \begin{bmatrix} ca_1+cb_1 \\ ca_2+cb_2 \\ \vdots \\ ca_n+cb_n \end{bmatrix} = c\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + c\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= c\bar{\alpha} + c\bar{\beta}$$

$$\therefore c(\bar{\alpha}+\bar{\beta}) = c\bar{\alpha} + c\bar{\beta}$$

So, this property V7 is satisfied.

V8. For any two scalars, $a, b \in F$ and any vector $\bar{\alpha} \in V$, $(a+b)\bar{\alpha} = a\bar{\alpha} + b\bar{\alpha}$

We know that multiplication is distributive over addition in fields.

So, if $a_i \in F$, $b_i \in F$ and $c_i \in F$
then $(a_i+b_i)c_i = a_ic_i + b_ic_i$

So, if $\bar{\alpha} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in F^n$ ~~then~~ and $b, c \in F$

$$(b+c)\bar{\alpha} = (b+c)\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ba_1 \\ ba_2 \\ \vdots \\ ba_n \end{bmatrix} + \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix} = b\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + c\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$= b\bar{\alpha} + c\bar{\alpha}$$

$$\therefore (b+c)\bar{\alpha} = b\bar{\alpha} + c\bar{\alpha}$$

So, we proved V8 is true.

V9. For any two scalars $a, b \in F$ and any vector $\bar{\alpha} \in V$, $(ab)\bar{\alpha} = a(b\bar{\alpha})$

We know that multiplication is associative in fields, i.e.

if $a, b, c \in F$ then

$$(ab)c = a(bc).$$

So, if $\bar{\alpha} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in F^n$ and $b, c \in F$, then

$$(bc)\bar{\alpha} = (bc)\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = b\begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix} = b(c\bar{\alpha})$$

$\therefore (bc)\bar{\alpha} = b(c\bar{\alpha})$

So, V9 is proved to be true.

V10. For unit ~~vector~~ scalar $1 \in F$ and any vector $\bar{\alpha} \in V$, $1\bar{\alpha} = \bar{\alpha}$. Here 1 is the multiplicative identity in F. So, if $a_i \in F$, then

$$1 . a_i = a_i$$

If $\bar{\alpha} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in F^n$ and $1 \in F$

then $$1\bar{\alpha} = 1 . \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 1.a_1 \\ 1.a_2 \\ \vdots \\ 1.a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \bar{\alpha}.$$

So, we have proved that V10 is true.

Hence, direct sums of field F will form a vector space $\mathbb{F}^n$ which will ~~form~~ be a vector space V over $\mathbb{F}$. This is true as we have proved all 10 properties that a vector space should satisfy.

Pattern found

I have seen that if we consider a ~~vector space~~ set V ~~over~~ & field F, then V will form a vector space over field F if F is a component of V.

→ eg. If $V = C$ and $F = \mathbb{R}$, V will form a vector space over field $\mathbb{R}$ as $\mathbb{R}$ is a subcomponent of C.

All the properties of field $\mathbb{R}$ can be extended to prove that $V = C$ is a vector space, since $\mathbb{R}$ is a subset of C.

→ On the other hand $V = \mathbb{R}$ on $F = C$ WILL NOT form a vector space since C is not contained in $\mathbb{R}$. This will lead to scalar multiplication property not being followed.

In Q2 (b). $V = \mathbb{Q}$ and $F = \mathbb{R}$.
Again, $\mathbb{R}$ is NOT a submodule of $\mathbb{Q}$. So, $V = \mathbb{Q}$ will not form a vector space over $F = \mathbb{R}$.

In Q2 (c), $V = \mathbb{R}$ and $F = \mathbb{Q}$
Here, $\mathbb{Q}$ is a submodule or subset of $\mathbb{R}$ set. So, $V = \mathbb{R}$ will form a vector space over $F = \mathbb{Q}$.

In Q2 (d), $V = \mathbb{R}$ and $F = \mathbb{C}$
Here, $\mathbb{C}$ is the larger set and is not contained in $\mathbb{R}$. So, $V = \mathbb{R}$ will NOT form a vector space over $F = \mathbb{C}$.

This is true because ~~$V = \mathbb{R}$~~ a field $\mathbb{F}$ will form a vector space over itself (as proved in the 1st part of this question).

So, if we just consider the superset of the field, that is also a field, then all the properties of the vector space can be extended to the larger vector set and the larger vector set will be a vector space over field $\mathbb{F}$ also, since scalar multiplication will become closed.
This is the pattern I found in Qn 2.

Q4. Let V be the set of pairs (x,y) of real numbers and let F be the field of real numbers.

Define

$$(x,y) + (x_1,y_1) = (x+x_1, 0)$$
$$c(x,y) = (cx, 0)$$

Is V, with these operations a vector space? Prove or disprove.

Ans. We have seen the properties V set must follow for it to be a vector space.

Let us consider the properties V4 and V5.

V4. Existence of an additive identity
There exists a unique vector $\phi \in V$ such that $\bar{\alpha} + \bar{\phi} = \bar{\alpha} = \bar{\phi} + \bar{\alpha}$
Let $\bar{\alpha} = (2,3) \in V$.
There does not exist any $\bar{\phi} = (x_1, y_1)$ such that
$$(2,3) + (x_1,y_1) = (2,3)$$
as the second element of the result pair will always be 0.
i.e. $(2,3) + (x,y_1) = (2+x, 0) \neq (2,3)$.
Now since there is no additive identity there will be no additive inverse. So, V5 is also not satisfied.

Thus, V is not a vector space. Hence, proved.

Q5. Which of the following sets of vectors $\alpha = (a_1, \ldots a_n)$ in $\mathbb{R}^n$ are subspaces of R^n $(n \geq 3)$?

(a) All α such that $a_1 \geq 0$.

We know that
A non-empty subset W of V is a subspace of V iff:
(i) W is closed w.r.t. addition
(ii) W is closed under scalar multiplication
i.e $\bar{\alpha} \in W, c \in F \Rightarrow c\bar{\alpha} \in W$

Now, consider the set of all α such that $a_1 \geq 0$. Let that be W.
Let $\beta = (2, a_2, a_3, \ldots a_n)$
$\beta \subset W$ as $2 \geq 0$.
Now, let us consider an element $c = -2$ from $\mathbb{R}$.
Let us check property (ii)
$c \cdot \beta = -2(2, a_2, a_3, \ldots a_n)$
$= (-4, -2a_2, -2a_3, \ldots -2a_n) \notin W.$

So, W is NOT closed under scalar multiplication.

Thus, the subset of W, i.e. all α such that $a_1 \geq 0$ is NOT a subspace of R^n $(n \geq 3)$.

(b) All α such that $a_1 + 3a_2 = a_3$.

Let us consider properties a subset W of V must follow for W to be a subspace of V.
Here, W is all sets $\alpha = (a_1, \dots a_n)$ where
$a_1 + 3a_2 = a_3$.

i) W should be closed under addition.
Consider any two arbitary ~~pairs~~ elements of W:

$$\beta = (a_1, a_2, a_3, \dots a_n)$$
$$\gamma = (b_1, b_2, b_3, \dots, b_n)$$

Now, $\beta + \gamma$
$= (a_1+b_1, a_2+b_2, a_3+b_3, \dots a_n+b_n)$

Given $a_3 = a_1 + 3a_2$
and $b_3 = b_1 + 3b_2$

$$\therefore a_3 + b_3 = a_1 + 3a_2 + b_1 + 3b_2$$
$$= (a_1+b_1) + 3(a_2+b_2). \in W.$$

So, the result of the sum is in W.
i.e. if $\beta \in W$ and $\gamma \in W$, then $\beta + \gamma \in W$.
Thus, W is closed w.r.t. addition.

ii) W should be closed w.r.t. scalar multiplication.
Consider $x \in \mathbb{R}$.
and $\bar{\beta} = (a_1, a_2, a_3, \dots a_n)$
i.e. $\bar{\beta} = (a_1, a_2, a_1 + 3a_2, \dots a_n)$

$$x \cdot \bar{\beta} = (xa_1, xa_2, xa_1 + 3xa_2, \dots a_n)$$
$$= (xa_1, xa_2, x(a_1+3a_2), \dots a_n) \in W.$$

Thus, we have proved that if $\bar{\alpha} \in W$, $c \in F$

then $c\bar{\alpha} \in W$.
Hence, W is closed w.r.t. scalar multiplication
Since both properties are satisfied,
the set ~~that~~ of all α such that $a_1 + 3a_2 = a_3$
is a subspace of R^n.

c) All α such that $a_2 = a_1^2$.
Let W be the subset given in question.
Consider two elements $\beta \in W$ and $\gamma \in W$:

$$\beta = (-1, 1, a_3, \ldots a_n)$$
and $$\gamma = (1, 1, a_3, \ldots a_n)$$

$$\beta + \gamma = (0, 2, 2a_3, 2a_4, \ldots 2a_n) \notin W$$
as $2 \neq 0^2$.
So, W is NOT closed w.r.t. addition.
Thus, W is NOT a subspace of R^n.

d) All α such that $a_1 a_2 = 0$.
Let W be the subset of V that has elements of α such that $a_1 a_2 = 0$.
Consider two elements $\beta \in W$ and $\gamma \in W$:

$$\beta = (2, 0, a_3, a_4, \ldots a_n)$$
$$\gamma = (0, 2, a_3, a_4, \ldots a_n)$$
$$\beta + \gamma = (2, 2, 2a_3, 2a_4, \ldots 2a_n) \notin W$$
as $2 \cdot 2 \neq 0$
Thus, W is NOT closed w.r.t. addition.
Thus, W is NOT a subspace of R^n.

e) All α such that a_2 is rational.
Consider W to consist of elements α such that a_2 is rational.
Now, consider the property (ii):
W should be closed w.r.t. scalar multiplication.

Consider an element $\beta \in W$ such that:

$$\beta = (a_1, 2, a_3, \ldots a_n)$$

Here '2' is a rational number. So, $\beta \in W$.
Now, consider an irrational element that belongs to $\mathbb{R}$. Let $c = \sqrt{2}$.

$$c.\beta = \sqrt{2}(a_1, 2, a_3, \ldots a_n)$$
$$= (\sqrt{2}a_1, 2\sqrt{2}, \sqrt{2}a_3, \ldots \sqrt{2}a_n) \notin W$$

as $2\sqrt{2}$ is not rational.
$\therefore$ W is NOT closed w.r.t. scalar multiplication.
Thus, W is NOT a subspace of $\mathbb{R}^n$.