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Linear Algebra: Assignment - 4

1. Given V is a vector space of all polynomials in x of degree $x \le 2$ over $\mathbb{R}$.

Basis is given as $B = \{x^2, x, 1\}$

The linear transformation, T is given as:

$T(x^2) = x + m$, $T(x) = (m-1)x$, $T(1) = x^2 + m$.

$$T(x^2) = x + m$$
$$= 0x^2 + 1 \cdot x + 1 \cdot m$$
$$= x^2(0) + x(1) + 1(m).$$
$$= \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ m \end{bmatrix}$$

Similarly, $T(x) = (m-1)x$
$$= 0(x^2) + x(m-1) + 1(0)$$
$$= \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 0 \\ m-1 \\ 0 \end{bmatrix}$$

Also, $T(1) = x^2 + m$
$$= 1(x^2) + x(0) + 1(m)$$
$$= \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ m \end{bmatrix}$$

Hence, the required transformation matrix for $B = \{x^2, x, 1\}$ is
$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix}$$

b) Kernel : If $T : U \to W$ is a linear transformation, then kernel(T) is the set of all $x \in U$, which are mapped on 0, the additive identity of W, by T.

i.e. $\text{Kernel}(T) = \{x \in U \mid T(x) = 0 \in W\}$

So, we need to find $\{x_1, x_2, x_3\}$ such that

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_3 \\ x_1 + x_2(m-1) \\ x_1 m + x_3 m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_3 = 0, \quad -(1)$

$x_1 m + x_3 m = 0$

$\Rightarrow \quad x_1 . m = 0. \quad -(2)$

So either $m = 0$ or $x_1 = 0$.

Case 1: If $m = 0$

Then, $x_1 + x_2(m-1) = 0 \quad -(3)$

$\Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$.

$\therefore$ $k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is present in kernel where $k \in \mathbb{R}$.

i.e, all polynomials of the type $kx^2 + kx$, $k \in \mathbb{R}$ belong to the kernel space.

Case 2: When $m \neq 0$, $x_1 = 0$

Then, $x_1 + x_2(m-1) = 0$ (3rd equation)

$\Rightarrow 0 + x_2(m-1) = 0$

Then $x_2(m-1) = 0$

$\Rightarrow x_2 = 0$ or $m-1=0$ (i.e. $m=1$).

Case 2a. If $x_2 = 0$, then $\{x_1, x_2, x_3\} = \{0, 0, 0\}$.

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is present in kernel space.

Case 2b: If $m = 1$, then we have $\{x_1 = 0, x_2 = k, x_3 = 0\}$ for $k \in \mathbb{R}$.

So, kernel space will contain $k \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

i.e, all polynomials of the type kx belong to the kernel space where $k \in \mathbb{R}$.

So, kernel space for:

$m = 0$ is $\left\{ k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ for $k \in \mathbb{R}$

$m = 1$ is $\left\{ k \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ for $k \in \mathbb{R}$

and $\left\{ \cancel{k} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ otherwise

(c) Image : If $T: U \to V$ is a linear combination, then set of all images of all elements of $x \in U$ is called the range/image of T.

$$Img(T) = T(U) = \{ y \in W \mid y = T(x) \text{ for } x \in U \}$$

We found kernel spaces for three cases: $m=0$, $m=1$ & otherwise. Finding image for them:

Case 1 :

$m = 0$, then $T = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

We can see that v_1 and v_2 are dependent linearly as $v_1 = -v_2$.

So, we have only v_1 and v_3 as independent vectors.

$\therefore$ Range of T when $m=0$ is the span of vectors $\{(0,1,0), (1,0,0)\}$, i.e. polynomials of the form $ax^2 + bx$.

Case 2 :

$m = 1$, then $T = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$

Here, v_1 and v_3 are independent.

$\therefore$ Range of T when $m=1$ is the span of vectors $\{(0,1,1), (1,0,1)\}$

Case 3: $m \neq 0, m \neq 1$

then $T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & m-1 & 0 \\ m & 0 & m \end{bmatrix}$
$\quad\quad\quad\quad v_1 \quad v_2 \quad v_3$

Here, v_1, v_2, v_3 are all independent.
So, range of T will be span of vectors $\{(0,1,m), (0,m-1,0), (1,0,m)\}$

2. $T(x,y,z) = (x+2y-z,\ 2x+3y+z,\ 4x+7y-z)$

The linear-transformation matrix, T can be written as

$$T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 4 & 7 & -1 \end{bmatrix}$$

Let us find the kernel of T. We have to find $\{a, b, c\}$ such that

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 4 & 7 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let us use row transformations to simplify the equations.

$$T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 4 & 7 & -1 \end{bmatrix}.$$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 4R_1.$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & -1 & 3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + 2R_2$

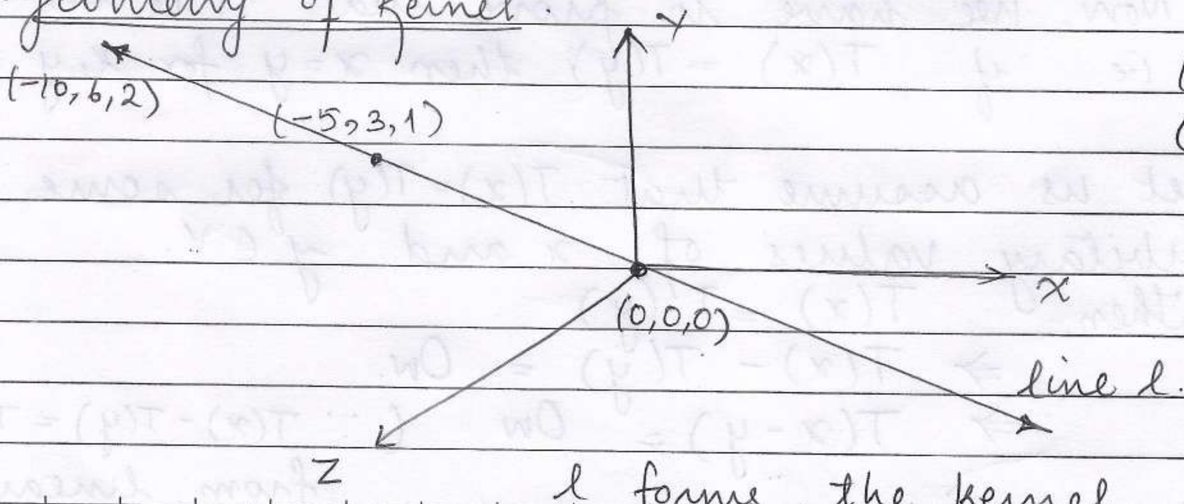
$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{Rank} = 2.$$

We get $-b+3c = 0 \Rightarrow b = +3c$

$a + 5c = 0. \Rightarrow a = -5c.$

$\therefore$ The kernel is $\left\{ k \begin{bmatrix} -5 \\ +3 \\ 1 \end{bmatrix} \right\}$

<u>Geometry of kernel</u>:



(0,0,0)
(-5,3,1)
(-10,6,2)

l forms the kernel.

Finding the range of T.

$$T = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 4 & 7 & -1 \end{bmatrix} \end{array}$$

We have already seen while finding kernel that rank = 2. and that vector v_3 is a free column. So, only v_1 and v_2 are independent.

$\therefore$ Basis of range is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \right\}$

~~Now, both of these vectors will form lines in xyz~~ space. So, the combination of these ~~vectors~~ will form a 2D plane in 3D space.

3. To prove: $T: V \to W$ is one-to-one iff its kernel has only the null vector and T will preserve linear independence.

Proof: Let Kernel$(T) = \{0_V\}$

Now, we have to prove that T is one-to-one i.e. if $T(x) = T(y)$ then $x = y$ for $x, y \in V$.

Let us assume that $T(x) = T(y)$ for some arbitary values of x and $y \in V$.

Then, $T(x) = T(y)$

$\Rightarrow T(x) - T(y) = 0_W$.

$\Rightarrow T(x-y) = 0_W$ ($\because T(x) - T(y) = T(x-y)$ from linearity).

So, we can say that $(x-y) \in$ Kernel(T).
But since kernel(T) only contains O_V,
then $x - y = O_V$.
$\Rightarrow \boxed{x = y}$

Hence, we have proved that if kernel has only the null vector, then $T: V \to W$ is one-to-one. (1)

Now, let us assume that $T: V \to W$ is one-to-one. Let us take any arbitrary value $x \in$ Kernel(T).
Then $T(x) = O_W$ (By the definition of kernel)
$= T(O_V)$
Then since T is a one-to-one function, we know that $T(x) = T(y) \Rightarrow x = y$.
So, $T(x) = T(O_V) \Rightarrow \boxed{x = O_V}$

Hence, we have proved that if $T: V \to W$ is one-to-one, then kernel will only have the null vector, O_V. — (2)

By (1) and (2), we have proved that a linear transformation $T: V \to W$ between two vector spaces is one-to-one <u>iff</u> its kernel has only the null vector.

Now, let us prove that T will preserve linear independence.

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots v_m\}$ be a linearly independent set in V.
Then we have to prove that $U = \{T(v_1), T(v_2) \ldots T(v_m)\}$ will be linearly independent set in W.
So, we have to prove that

$$a_1 T(v_1) + a_2 T(v_2) + \ldots a_m T(v_m) = 0_W.$$

only when $a_1 = a_2 = \ldots = a_m = 0$.

$a_1 T(v_1) + a_2 T(v_2) + \ldots a_m T(v_m) = 0_W$
$\Rightarrow T(a_1 v_1) + T(a_2 v_2) + \ldots T(a_m v_m) = 0_W$ ($\because$ T is linear
$\Rightarrow T(a_1 v_1 + a_2 v_2 + \ldots a_m v_m) = 0_W$ transformation,
$\Rightarrow a_1 v_1 + a_2 v_2 + \ldots a_m v_m = 0_V$. $cT(v) = T(cv)$
We can write this as and $T(v_1) + T(v_2)$
$T: V \to W$ is one one. $= T(v_1 + v_2)$)
$a_1 v_1 + a_2 v_2 + \ldots a_m v_m = 0$, (as S is linearly
then $a_1 = a_2 = \ldots a_m = 0$. independent)

Thus, $a_1 T(v_1) + a_2 T(v_2) + \ldots + a_m T(v_m) = 0_W$
only when $a_1 = a_2 = \ldots = a_m = 0$.
Thus, $U = \{T(v_1), T(v_2), \ldots T(v_m)\}$ is linearly independent iff $S = \{v_1, v_2, \ldots v_m\}$ is linearly independent. Thus, T preserves linear independence if T is one-one.

Hence, proved

4. $A(V,W)$ is set of all linear maps from V to W. where V and W are two vector spaces.
To prove: $A(V,W)$ forms a vector space.
We know that every linear transformation, T preserves:
i) In this content, addition of vectors will mean:
$T_1 + T_2$, i.e. $\cancel{T_1(x+y)} = (T_1 + T_2)x = T_1(x) + T_2(x)$
where $x \in V$.
and $T_1(x), T_2(x) \in W$.

ii) Scalar multiplication will mean:
$(cT)(x) = c \cdot T(x)$ where $x \in V$ and $T(x) \in W$
and $c \in \mathbb{R}$.

Let us see if the set $A(V,W)$ follows the 10 properties of vector space.

V1. For any two vectors $\bar{\alpha}, \bar{\beta} \in V$, $\bar{\alpha} + \bar{\beta} \in V$.
Let, $T_1 \in T_2 \in \cancel{A}$ Then $T_1(x)$ and $T_2(x) \in W$ for $x \in V$
Now, $(T_1 + T_2)(x) = T_1(x) + T_2(x) \in W$. ($\because T_1 \in W, T_2 \in W$)
So, if $T_1, T_2 \in A$ then $T_1 + T_2 \in A$.
Thus, A is closed under addition.

V2. For any two vectors $\bar{\alpha}, \bar{\beta} \in V$, $\bar{\alpha} + \bar{\beta} = \bar{\beta} + \bar{\alpha}$
Let $T_1, T_2 \in A$.
Then, $(T_1 + T_2)(x) = T_1(x) + T_2(x)$ where $x \in V$.
$= T_2(x) + T_1(x)$
$= (T_2 + T_1)x$

$\therefore T_1 + T_2 = T_2 + T_1$.
So, addition is commutative under A.

V3. For any three vectors $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in V$, $\bar{\alpha}+(\bar{\beta}+\bar{\gamma}) = (\bar{\alpha}+\bar{\beta})+\bar{\gamma}$

Let $T_1, T_2, T_3 \in A$ and $x \in V$.

Then, $((T_1+T_2)+T_3)x = (T_1+T_2)x + T_3(x)$

$$= (T_1(x)+T_2(x)) + T_3(x)$$
$$= T_1(x)+(T_2(x)+T_3(x))$$
$$= (T_1+(T_2+T_3))x$$

So, $((T_1+T_2)+T_3)x = (T_1+(T_2+T_3))x$

So, associativity property holds true under addition

V4. There exists a unique vector $\phi \in A$ such that $\bar{\alpha}+\bar{\phi} = \bar{\alpha} = \bar{\phi}+\bar{\alpha}$

Let there be a linear transformation O_T that maps every element in V to O_W.

i.e. $O_T(x) = O_W$

Then, if $T_1 \in A$,

$$(T_1+O_T)x = T_1(x)+O_T(x)$$
$$= T_1(x)+O_W$$
$$= T_1(x)$$
$$= O_W + T_1(x)$$
$$= T_1(x).$$

Hence, there exists an additivity identity, $O_T \in A$ which satisfies the property $T_1+O_T = T_1 = O_T+T_1$ for $T_1 \in A$.

V5. For any vector $\bar{\alpha} \in V$, $\exists$ a unique vector $-\bar{\alpha} \in V$ such that $\bar{\alpha}+(-\bar{\alpha}) = (-\bar{\alpha})+\bar{\alpha} = \phi$.

If $T_1(x) \in A$ then the inverse of that $-T_1 \in A$ from which we can get back x, $x \in V$.
$\therefore$ Additive inverse exists for every $T_1 \in A$.

V6. For any element $a \in F$ and any element $\bar{\alpha} \in V$, $a\bar{\alpha} \in V$.
Let $T_1 \in A$ and $a \in F$ and $x \in V$
Then $(aT_1)(x) = aT_1(x) \in A$.
$\therefore$ A is closed under scalar multiplication.

V7. For any element $a \in F$ and any vector $\bar{\beta}, \bar{\alpha} \in V$,
$a(\bar{\alpha} + \bar{\beta}) = a\bar{\alpha} + a\bar{\beta}$.
Let $a \in F$ and $T_1, T_2 \in A$. $x \in V$.
Then $a((T_1 + T_2)(x)) = a(T_1(x) + T_2(x))$
$= aT_1(x) + aT_2(x)$
$= aT_1 + aT_2$.
Hence, distributivity is preserved.

V8. For any two scalars $a, b \in F$ and any vector $\bar{\alpha} \in V$, $(ab)\bar{\alpha} = a(b\bar{\alpha})$
Let $a, b \in F$ and $T_1 \in A$ and $x \in V$
Then, $(ab)(T_1(x)) = a(bT_1(x))$
So, scalar multiplication is associative.

V9. For any two scalars $a, b \in F$ and any ~~two~~ vector ~~s~~ $\bar{\alpha} \in V$,
$(a+b)\bar{\alpha} = a\bar{\alpha} + b\bar{\alpha}$
Let $a, b \in F$ and $T_1 \in V$
Then $(a+b)T_1(x) = aT_1(x) + b(T_1(x))$
So, V9 is satisfied.

V10. For unit scalar $1 \in F$ and any vector $\bar{x} \in V$, $1\bar{x} = \bar{x}$.
If there will exist a transformation $T_1 \in A$ such that if $1 \in F$,
then $1 \cdot T_1(x) = T_1(x)$.
$\therefore$ A has an identity $1 \in F$ which gives back the same transformation. Thus, there is a multiplicative identity in A.

Since all 10 properties of vector space are satisfied, $A(V,W)$ forms a vector space.

5. V is a vector space such that $V = M \oplus N$.

(A) To prove: $\forall v \in V$, $\exists m \in M$ and $\exists n \in N$ such that $v = m + n$.

Definition of direct sum: Suppose $U_1, U_2, \ldots U_m$ are subspaces of V.
The sum $U_1 \oplus U_2 \oplus \ldots \oplus U_m$ is called the direct sum if each element of $U_1 + U_2 + \ldots U_m$ can be written in only one way as a sum $u_1 + u_2 + \ldots + u_m$ where each u_i is in U_i

Let M and N be subspaces of V.
Given that $V = M \oplus N$.
So, by the definition of direct sum, if we take arbitrary $m \in M$ and $n \in N$, there will exist a unique combination for $(m+n) \in M + N \in V$
So, every element in V can be written as a

5. unique combination of $(m_i + n_i)$.
where $m_i \in M$ and $n_i \in N$.
Hence, $\forall v \in V$, $\exists m \in M$ and $\exists n \in N$ such that $v = m + n$.

5.(B) $P: V \to V$ is a projection of V along M onto N as $P(v) = n$.

(a) To prove: P is linear

Proof: Let $v_1, v_2 \in V$. Since $V = M \oplus N$, there will exist m_1, n_1 such that $v_1 = m_1 + n_1$ and m_2, n_2 such that $v_2 = m_2 + n_2$ where $m_i \in M$ and $n_i \in N$.
Let $a \in \mathbb{F}$. Then,

$$P(av_1 + v_2) = P(a(m_1+n_1) + (m_2+n_2))$$
$$= P(am_1 + an_1 + m_2 + n_2)$$
$$= P((am_1+m_2) + (an_1+n_2))$$
$$= an_1 + n_2 \quad (\because P(v) = n)$$
$$= a(P(v_1)) + P(v_2)$$

$\therefore P(av_1 + v_2) = aP(v_1) + P(v_2)$
Thus, P is linear.

(b) To prove: P is idempotent

Proof: A map P is said to be idempotent if $P^2 = P$. Let v be arbitary element in V.
Then, $P^2(v) = P(P(v))$
$= P(n)$ ($\because V = M \oplus N$, then
$= n$ $\quad v = m + n$ / $m \in M, n \in N$.
$= P(v)$ $\quad n = 0 + n$ / $0 \in M$, $n \in N$
Thus, $P^2(V) = P(V) \Rightarrow P$ is idempotent.)

(c) <u>To prove</u>: range$(P) = N$.
Since $V = M \oplus N$, and $P(v) = n$,
Let $M = \{v : P(v) = 0\}$ and
$N = \{P(v) : v \in N\}$.
Then, $v = I_v . v$
$= I_v . v + P(v) - P(v)$
$= (I_v - P)v - P(v)$
Since $P(v) \in N$, range$(P) = N$.

(d) <u>To prove</u>: Kernel$(P) = M$.
Kernel(P) means all the elements $v \in V$ that map to zero,
i.e. Kernel$(P) = \{v \in V : P(v) = 0\}$
Now since $P(v) = n$,
Kernel$(P) = \{v \in V : n = 0\}$
Since $V = M \oplus N$,
$v = m + n$
$= m + 0 \quad (\because n = 0)$
Thus, Kernel$(P) = \{m + 0 ; m \in M\}$
$= M$.
Thus, Kernel$(P) = M$.

(c) <u>To prove</u>: $I - P$ is the projection of V along N onto M
$(I-P)^2 = (I-P)(I-P) = (I-P) \quad (\because (I-P)$ is idempotent$)$
$(I-P)$ is a projection on M along N. Since $(I-P)$ is linear,
by definition of projection
$(I-P)v = I(v) - P(v)$
$= v - n = m. \quad (\because P(v) = n)$
Hence, proved.

6. Given that A is an $n \times n$ matrix whose all entries are blanks.
Alice wins if $|A| \neq 0$
Bob wins if $|A| \neq 0$.

Alice will win if the vectors in A are linearly independent. For Bob, to win the game, the vectors must be linearly dependent.
So, for each turn, Bob must put a multiple of the value that is put by Alice.
eg. If Alice puts 'a' in position (1,1), then Bob must put 'ka' where $k \in \mathbb{R}$ in any of the positions $(1, c)$ where $c \leq n$.

Now, the strategy for Alice is to keep putting the values that are not multiples of the value in previous vector.

* Let us consider a 1×1 matrix (i.e. $n = 1$)
Case 1 If Alice will start, she will put a non-zero number and win.
Case 2 If Bob will start, he will put zero and win the game.
So for a 1×1 matrix, the person who starts will win the game.

* Let us consider a 2×2 matrix (i.e. $n = 2$)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Case 1 If Alice starts, she will put some value, say '1' in any of the positions. Let her put '1' in 'a'.

$$A = \begin{bmatrix} 1 & b \\ c & d \end{bmatrix}$$

Bob will win only if $|A| = 0$. So, in order to make the vectors (columns) of matrix dependent, he will put $b = k$ (some multiple of 1)

$$A = \begin{bmatrix} 1 & k \\ c & d \end{bmatrix}$$

Now, Alice can put any number in positions c and d. Let us say she puts '2' in d.

$$A = \begin{bmatrix} 1 & k \\ c & 2 \end{bmatrix}$$

In order to make vectors dependent, Bob will put $(2/k)$ in place of c.

$$\text{So, } A = \begin{bmatrix} 1 & k \\ 2/k & 2 \end{bmatrix}$$

Since $|A| = 0$, Bob wins.

Case 2 Let us say Bob starts and puts 1 in position a.

$$A = \begin{bmatrix} 1 & b \\ c & d \end{bmatrix}$$

Now, Alice can put any number in b, c, d. Let Alice put '2' in position c.

$$\text{Then } A = \begin{bmatrix} '1' & b \\ '2' & d \end{bmatrix}$$

Now, whatever Bob puts for making $|A| = 0$ can be broken by Alice.
So, if Bob puts '3' in position 'b'

$$A = \begin{bmatrix} 1 & 3 \\ 2 & d \end{bmatrix}$$

Alice can put any number except '6' in d to win.

$$\text{So, } A = \begin{bmatrix} 1 & 2 \\ 2 & (x \neq 6) \end{bmatrix}$$

So, we can say that if n is even, then the person who starts will not win the game if they play according to the stratengy mentioned.
If n is odd, person who starts will win the game if they choose the stratergy as mentioned above.

So,

n is even $\Rightarrow$ person who starts will lose
n is ~~odd~~ odd $\Rightarrow$ person who starts will win.
~~else if n is odd~~ $\Rightarrow$

Starts \ n	Even	Odd
Alice	Bob	Alice
Bob	Alice	Bob

win matrix.

7. (a) To prove: A linear transformation between vector spaces has a left inverse iff it is ~~surjective~~ injective (one-to-one)

- Let V and W be vector spaces over a field $\mathbb{F}$. A linear transformation $T: V \to W$ is called an <u>injection</u> if $T(x) = T(y) \Rightarrow x = y \ \forall \ x, y \in V$.

- The linear transformation $T: V \to W$ is said to have a left inverse, $T_e: W \to V$ if
$T_e \circ T = I_V$ where I_V is identity operator in V.

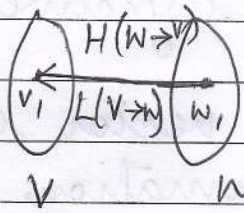
* <u>Part 1</u>: Let the left inverse of T, T_e exist.
<u>To prove</u>: T is injective.

<u>Proof</u>: Let $v_1, v_2 \in V$ such that
$$T(v_1) = T(v_2).$$
Then, $$T_e(T(v_1)) = T_e(T(v_2))$$
$$\Rightarrow (T_e \circ T)(v_1) = (T_e \circ T)(v_2) \quad (\because T_e \cdot T = I_V).$$
$$\Rightarrow v_1 = v_2.$$
Hence, T is injective. —(1)

* <u>Part 2</u>: Let T be injective
<u>To prove</u>: Left inverse of T exists.

<u>Proof</u>: Let $v_1 \in V$ such that $T(v_1) = w_1 \in W$.
Let $H: W \to V$ such that $H(w_1) = v_1$.
Since T is injective, H maps w to a unique $v \in V$.

Then, $(H \circ T) : V \to V$.
such that $(H \circ T) v_1 = H(T(v_1)) = H(w_1) = v_1$



So, $(H \circ T) = I_V$.
Thus, we can say that H is the left inverse, T_l of T. — (2)

From (1) and (2), we have proved that a linear transformation between vector spaces has a left inverse iff it is injective.

7(b) To prove: A linear transformation between vector spaces has a right inverse iff it is surjective (onto)

- A linear transformation $T: V \to W$ is said to be surjective if every vector in W can be the output of T.

- The linear transformation, T is said to have a right inverse, $T_R : W \to V$ if $T \circ T_R = I_W$, where I_W is the identity operator in W.

Part 1 Let the right inverse of $T: V \to W$, i.e. $T_R: W \to V$ exist i.e. $(T \circ T_R) = I_W$.

To prove: T is surjective.

Proof:

Let w_1 be an arbitary element such that $w_1 \in W$. To show that T is surjective, we need to find an element $v_1 \in V$ such that $T(v_1) = w_1$.

$$\begin{aligned} w_1 &= I_W . w_1 \\ &= (T \circ T_R) w_1 \\ &= T(T_R(w_1)) \\ &= T(v_1) \end{aligned}$$

$\therefore$ T is surjective. — (1)

* Part 2: Let T be surjective.

To prove: T is right invertible

Let us define $H: W \to V$ as $H(w_1) = v_1$.

when $T(v_1) = w_1$.

As T is surjective, an inverse map from W to V will always give an element in V.

Then, for an arbitary element $w_1 \in W$,

$$\begin{aligned} (T \circ H) w_1 &= T(H(w_1)) \\ &= T(v_1) \\ &= w_1 = I_W \end{aligned}$$

This means that H is the right inverse of T. So, T_R will exist if T is surjective (2)

From (1) and (2), we have proved that a linear transformation has a right inverse iff it is surjective.