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Linear Algebra Assignment 3.

1(a) given system of linear equations is :

$$x + 3y + 5z = 14$$
$$2x - y - 3z = 3$$
$$4x + 5y - z = 7$$

Step 1: Writing the augmented matrix of the system of linear equations, we get:

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right]$$

Step 2: For Gaussian elimination, we use elementary row operations to reduce the augmented matrix to a row Echelon form.

So,
$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 4 & 5 & -1 & 7 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 - 4R_1} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & -7 & -21 & -49 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow -R_2/7} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & 1 & 13/7 & 25/7 \\ 0 & -7 & -21 & -49 \end{array}\right]$$

Applying $R_3 \rightarrow R_3 + 7R_2$, the matrix becomes :

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & 1 & 13/7 & 25/7 \\ 0 & 0 & -8 & -24 \end{array}\right]$$

Applying $R_3 \leftarrow -(R_3/8)$, we get

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & 1 & 13/7 & 25/7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This is a row echelon form.
Corresponding system of linear equations:

$$x + 3y + 5z = 14$$
$$y + (13/7)z = 25/7$$
$$z = 3$$

Step 3: Applying back subtitution to solve the equation, we get:

$$z = 3$$
$$y + 39/7 = 25/7 \Rightarrow y = -2.$$
$$x + (-6) + 15 = 14 \Rightarrow \cancel{z}\, x = 5$$
$$\therefore \boxed{x = 5,\ y = -2,\ z = 3.}$$

$\therefore \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$ is the solution for the given system of linear equations.

1.(b) Given system of linear equations is:

$$y + z = 4$$
$$3x + 6y - 3z = 3$$
$$-2x - 3y + 7z = 10$$

<u>Step 1</u>: Writing the augmented matrix of the system of linear equations, we get:

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 4 \\ 3 & 6 & -3 & 3 \\ -2 & -3 & 7 & 10 \end{array}\right]$$

<u>Step 2</u>: For Gauss Jordan elimination, we use elementary row operations to reduce to augmented matrix to a reduced row Echelon form.

So,
$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 4 \\ 3 & 6 & -3 & 3 \\ -2 & -3 & 7 & 10 \end{array}\right]$$

Applying $R_2 \leftrightarrow R_1$

$$\left[\begin{array}{ccc|c} 3 & 6 & -3 & 3 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{array}\right]$$

Applying $R_1 \rightarrow R_1/3$,

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{array}\right]$$

Applying $R_2 \rightarrow R_2 + 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right]$$

Applying $R_3 \rightarrow R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

Applying $R_1 \rightarrow R_1 - 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -7 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

Applying $R_3 \rightarrow R_3/4$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -7 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Applying $R_2 \rightarrow R_2 - R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Applying $R_1 \rightarrow R_1 + 3R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Step 3 : Now, the equations are directly of the form :

$$x = -1 , y = 2 , z = 2$$

$\therefore \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ is the solution for the given system of linear equations.

Q2. Given $V = \mathbb{R}^{\infty}$. Let sequences given be S.

1. Checking if absolutely summable sequences, $((x_i)$ such that $\sum_{i=1}^{\infty} |x_i| < \infty$, (i.e. the sum is finite) is a subspace of V or not.

Let us check the three conditions :

(i) The zero vector belongs to S as zero vector is absolutely summable.

$$A = \{0, 0, 0, \dots\} \in S \quad \text{as} \quad \sum_{i=1}^{\infty} 0 = 0$$

So, this condition is verified.

(ii) Closure under addition.

Let $A = \{a_1, a_2, a_3, \dots\} \in S$, i.e. $\sum_{i=1}^{\infty} |a_i| < \infty$.

and $B = \{b_1, b_2, \dots\} \in S$,

i.e. $\sum_{i=1}^{\infty} |b_i| < \infty$.

Now, $C = A + B = \{(a_1+b_1), (a_2+b_2), \dots\}$

Let $c_i = a_i + b_i$ where $c_i \in C$

We know that $|a+b| \le |a| + |b|$. $\forall a,b \in \mathbb{R}$.
Hence, if $a_i \in A$ and $b_i \in B$ are finite, then obviously $|a_i + b_i|$ will also be finite.

Taking sum of elements in $(A+B)$, we get.

$$\sum_{i=1}^{\infty} |c_i| \le |a_1 + b_1| + |a_2 + b_2| + \dots$$
$$\le |a_1| + |b_1| + |a_2| + |b_2| + \dots$$
$$\le (|a_1| + |a_2| + |a_3| + \dots) + (|b_1| + |b_2| + \dots)$$
$$\le \sum_{i=1}^{\infty} |a_i| + \sum_{i=1}^{\infty} |b_i|$$
$$< \infty. \quad \left(\text{as } \sum_{i=1}^{\infty} a_i < \infty \text{ and } \sum_{i=1}^{\infty} b_i < \infty\right)$$

$\therefore$ If $A \in S$ & $B \in S$ then $(A+B) \in S$.
So, the second condition is satisfied.

iii) closure under scalar multiplication

Let $A = \{a_1, a_2, \dots\} \in S$. So $\sum_{i=1}^{\infty} |a_i| < \infty$.

Let $c \in \mathbb{R}$ be an arbitary element.
$B = c.A = \{ca_1, ca_2, \dots\}$
Let $ca_i = b_i \in B$.
We know that $|a * b| \le |a| * |b|$ $\forall a,b \in \mathbb{R}$.

$$\text{So, } \sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} c.a_i = c.\sum_{i=1}^{\infty} a_i$$
$$= c(|a_1| + |a_2| + |a_3| + \dots)$$

since $\sum_{i=1}^{\infty} a_i < \infty$, $c\sum_{i=1}^{\infty} a_i < \infty$.

$\therefore$ If $A \in S$, then $c.A \in S$ where $c \in \mathbb{R}$.

Since all the three conditions are satisfied, absolute summable sequences form a subspace of V over $\mathbb{R}$.

2. Bounded sequences ((x_i) such that $\exists M > 0$ such that $|x_i| \leq M \quad \forall i$)
Let the sequence be S
Checking three conditions :

i) The zero vector belongs to S as:
$$A = \{0, 0, 0, \ldots\}$$
If we take $M > 0$, then $0 \leq M$.
eg. Let $M = 1$.
Then $0 \leq 1$
$\therefore \quad |a_i| \leq 1. \quad \forall \; a_i \in A.$
So, this condition is satisfied.

ii) closure under addition.
Let $A = \{a_1, a_2, a_3, \ldots\} \in S$, so, $|a_i| \leq M_a$ — (1)
$B = \{b_1, b_2, b_3, \ldots\} \in S$, So, $|b_i| \leq M_b$ — (2)

Let $C = A + B$
$= \{(a_1+b_1), (a_2+b_2), (a_3+b_3), \ldots\}$
Let $a_i + b_i = c_i$
Then, $|c_i| = |a_i + b_i|$
$\leq |a_i| + |b_i|$
$\leq M_A + M_B$ ($\because$ from (1) and (2))
$\therefore$ Let $M_A + M_B = M_C$, then
$|c_i| \leq M_C$. where $M_C \in \mathbb{R}$ and
as $|M_A| > 0$ & $|M_B| > 0$ then $M_C > 0$.

$\therefore \forall c_i$ in C, $|c_i| < M_C$.
So, closure under addition property is satisfied.

(iii) Closure under scalar multiplication.
Let $A = \{a_1, a_2, \ldots\} \in S$, i.e. $|a_i| < M$ $\forall a_i \in A$
Let $B = c.A = \{ca_1, ca_2, \ldots\}$ where $c \in \mathbb{R}$
Let $ca_i = b_i$

$$|b_i| = ca_i$$
$$\leq |c||a_i|$$
$$\leq M_C \text{ where } M_C = |c||a_i|$$

So, if $A \in S$ and $c \in \mathbb{R}$, then $c.A \in S$.
So, closure under scalar multiplication property is satisfied.

Hence, bounded sequences form a subspace of V over $\mathbb{R}$.

3. Arithmetic sequences ((x_i) such that $x_i = a + di$ for some fixed a and d)
Let S be this sequence

i) Zero vector $\in S$ as
$A = \{0, 0, 0 \ldots\}$
Here, $a = d = 0$.
So, condition holds true.

ii) Closure under addition.
Let $A = \{a_1, a_2, \ldots\} \in S$.
For some x and $y \in \mathbb{R}$, we can write

$A = \{x, x+y, x+2y, x+3y, \dots\}$.
Also, let $B = \{b_1, b_2, \dots\} \in S$
For some p and $q \in \mathbb{R}$, we can say
$B = \{p, p+q, p+2q, \dots\}$

Now, let $C = A+B$
$= \{(x+p), (x+y+p+q) + (x+2y+p+2q) + \dots\}$
$= \{(x+p), (x+p+(y+q)), (x+p+2(y+q)) + \dots\}$
This C vector also forms an arithmetic sequence with first element as $(x+p)$ and common difference as $(y+q)$.
$\therefore$ If $A \in S$ and $B \in S$, then $(A+B) \in S$.
Since closure under addition is satisfied, this property holds true.

iii) closure under scalar multiplication
Let $A = \{a_1, a_2, \dots\} \in S$
$A = \{x, x+y, x+2y, \dots\}$ for some x and $y \in$
If $c \in \mathbb{R}$, then
$cA = \{cx, cx+cy, cx+2cy, \dots\}$
We can see that the sequence (cA) also forms an arithmetic sequence with first element as 'cx' and common difference as cy.
So, if $A \in S$, then $cA \in S$ for $c \in \mathbb{R}$.
So, this condition holds true.

Since all conditions are satisfied, arithmetic sequences form of subspace of V over $\mathbb{R}$.

4. Geometric sequences ((x_i) such that $x_i = ar^i$ for some fixed a and r).

Let us check the three conditions:

i) closure under addition:

Consider $A = \{1, 1, 1, \dots\} \in S$

This is a G.P. with $a=1$ and $r=1$

Let $B = \{1, 1/2, 1/4, 1/8, \dots\} \in S$.

This is a G.P. with $a=1$ and $r = 1/2$.

Let $C = A+B$

$$C = \{(1+1), (1+1/2), (1+1/4), \dots\}$$

$$= \{2, 3/2, 5/4, \dots\}$$

$$\frac{c_2}{c_1} = \frac{3/2}{2} = 3/4$$

$$\frac{c_3}{c_2} = \frac{5/\cancel{4}2}{3/\cancel{2}} = 5/6$$

There is no common ratio. So, if $A \in S$ and $B \in S$ then $A+B \notin S$.

So, geometric sequences are NOT closed under addition.

Hence, they do not form a subspace of V over $\mathbb{R}$.

3. To prove: Set of all continuous real valued functions on the domain $[0,1] \subset \mathbb{R}$ denoted by $\mathbb{R}[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$ forms a vector space over $\mathbb{R}$.

Let the set of all continuous real valued function on $[0,1]$ be denoted as S. Let us see if S satisfies the 10 properties for being a vector space over field $\mathbb{R}$.

V1. For any two vectors, $\bar{\alpha}, \bar{\beta} \in V$, $\bar{\alpha} + \bar{\beta} \in V$.
Let $f(x)$ and $g(x)$ be two functions in S.
If $f(x) \in S$, $g(x) \in S$ then $(f(x) + g(x)) \in S$ as sum of two continuous functions is continuous, i.e. closure under addition holds.

V2. For any two vectors $\bar{\alpha}, \bar{\beta} \in V$, $\bar{\alpha} + \bar{\beta} = \bar{\beta} + \bar{\alpha}$
Let $\bar{\alpha} = f(x)$ and $\bar{\beta} = g(x)$, then
$\bar{\alpha} + \bar{\beta} = (f+g)x = f(x) + g(x) = g(x) + f(x) = (g+f)x = \bar{\beta} + \bar{\alpha}$
So, V2 is true i.e. commutativity under addition holds

V3. For any three vectors $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in S$, $\bar{\alpha} + (\bar{\beta} + \bar{\gamma}) = (\bar{\alpha} + \bar{\beta}) + \bar{\gamma}$
Let $\bar{\alpha} = f(x)$, $\bar{\beta} = g(x)$, $\bar{\gamma} = h(x)$. Then,

$$
\begin{aligned}
\bar{\alpha} + (\bar{\beta} + \bar{\gamma}) = (f + (g+h))x &= f(x) + (g+h)x \\
&= f(x) + (g(x) + h(x)) \\
&= (f(x) + g(x)) + h(x) \\
&= (f+g)x + h(x) \\
&= ((f+g) + h)(x) \\
&= (\bar{\alpha} + \bar{\beta}) + \bar{\gamma}
\end{aligned}
$$

So, V3 is satisfied. So, addition is associative.

V4. There exists an unique vector $\phi \in V$ such that $\bar{\alpha} + \bar{\phi} = \bar{\alpha} = \bar{\phi} + \bar{\alpha}$.

Let $z(x)$ be a zero function that maps every x to 0. $z(x)$ is a continuous function.

$\therefore z(x) \in S$.

For every $f(x) \in S$, $f(x) + z(x) = f(x) = z(x) + f(x)$.

So, there is a presence of identity element in S.

V5. For any vector $\bar{\alpha} \in V$, $\exists$ an unique vector $-\bar{\alpha} \in V$ such that $\bar{\alpha} + (-\bar{\alpha}) = (-\bar{\alpha}) + \bar{\alpha} = \bar{\phi}$

For every $f(x) \in S$, there will exist $-f(x) \in S$ which is also continuous such that:

$$f(x) + (-f(x)) = 0$$

Here 0 can be $z(x)$.

So, additive inverse exists.

V6. For any element $a \in \mathbb{F}$ and any element $\bar{\alpha} \in V$, $a\bar{\alpha} \in V$.

If $f(x) \in S$ then $c.f(x) \in S$ where $c \in \mathbb{R}$ as a continuous function when multiplied by a scalar remains continuous.

So, scalar multiplication is closed.

V7. For any element $a \in \mathbb{F}$ and any vector $\bar{\beta}, \bar{\alpha} \in V$, we have $a(\bar{\alpha} + \bar{\beta}) = a\bar{\alpha} + a\bar{\beta}$

Let $\bar{\alpha} = f(x)$ and $\bar{\beta} \in g(x)$ such that $f(x) \in S$ & $g(x) \in S$ and let $c \in \mathbb{R}$

Then, $c.((f+g)x) = c(f(x) + g(x)) = cf(x) + c.g(x)$
$= c\bar{\alpha} + c\bar{\beta}$.

Hence, scalar multiplication is distributive over addition.

V8. For any two scalars $a, b \in \mathbb{F}$ and any vector $\bar{\alpha} \in V$, $(a+b)\bar{\alpha} = a\bar{\alpha} + b\bar{\alpha}$

Let $a, b \in \mathbb{R}$ and $\bar{\alpha} = f(x) \in S$.

Then, $(a+b)\bar{\alpha} = (a+b)f(x)$

$$= af(x) + bf(x)$$

$$= a\bar{\alpha} + b\bar{\alpha}$$

Hence, V8 holds true.

V9. For any two scalars, $a, b \in \mathbb{F}$ and any vector $\bar{\alpha} \in V$, $(ab)\bar{\alpha} = a(b\bar{\alpha})$

Let $a, b \in \mathbb{R}$ and $\bar{\alpha} = f(x) \in S$

Then, $(ab)\bar{\alpha} = (ab)f(x)$

$$= a(bf(x))$$

$$= a(b\bar{\alpha}).$$

So, associativity over scalar multiplication holds true.

V10. For unit scalar $1 \in \mathbb{F}$ and any vector $\bar{\alpha} \in V$, $1.\bar{\alpha} = \bar{\alpha}$

$1 \in \mathbb{R}$ is the unit scalar. Let $\bar{\alpha} = f(x) \in S$.

Then, $1.\bar{\alpha} = 1.f(x) = f(x) = \bar{\alpha}$

So, identity exists for scalar multiplication also.

Since all the 10 properties of vector space are satisfied for S over R, it forms a vector space V over R.

Hence, proved.

4. Given f, g, h are three vectors as $f(x) = x$, $g(x) = e^{x}$ and $h(x) = e^{-x}$, $x \in [0, 1]$, from $\mathbb{R}[0,1]$.

To prove: f, g, h are linearly independent

A list $v_1, v_2, \ldots v_m$ of vectors in V is called linearly independent if the only choice of $a_1, a_2, \ldots a_m \in \mathbb{F}$ that makes $a_1 v_1 + \ldots + a_m v_m = 0$ is $a_1 = a_2 = \ldots a_m = 0$.

Let us assume that f, g, h are linearly dependent. Then, there will exist some a, b, c not all zero such that

$$a f(x) + b g(x) + c h(x) = 0$$

$$\Rightarrow \quad ax + be^{x} + ce^{-x} = 0 \quad \text{for } x \in [0,1] \quad -(1)$$

Let us calculate values of a, b, c by putting different values of x in equation (1).

Putting $x = 0$,

$$a \cdot 0 + b + c = 0$$

$$\Rightarrow \boxed{b = -c} \quad -(2)$$

Putting $x = 1$ in equation (1),

$$\boxed{a + be + \frac{c}{e} = 0} \quad -(3)$$

Putting $x = 1/2$ in equation (1),

$$\boxed{\frac{a}{2} + be^{1/2} + ce^{-1/2} = 0} \quad -(3)$$

Let us substitute c as (-b) in equation 2 & 3

$$a + be - \frac{b}{e} = 0 \quad , \quad -(5)$$

$$\frac{a}{2} + be^{1/2} - be^{-1/2} = 0 \quad -(4)$$

Multiplying 4 by 2,

$$a + 2be^{1/2} - be^{-1/2} = 0 \quad -(6)$$

Subtracting 5 from 6, we get

$$2be^{1/2} - be - be^{-1/2} + \frac{b}{e} = 0$$

$$\Rightarrow b \underbrace{\left[2e^{1/2} - e - e^{-1/2} + \frac{1}{e}\right]}_{\text{constant, } k} = 0$$

$$\therefore \quad bk = 0 \Rightarrow \boxed{b = 0.}$$

From equation 1, $c = -b \Rightarrow \boxed{c = 0}$

From equation 3, $a + 0.e + 0(1/e) = 0$

$$\Rightarrow \boxed{a = 0}$$

So, the only option for $ax + be^{x} + ce^{-x} = 0$ is that all $a = b = c = 0$.

Hence, f, g, h are linearly independent vectors. Hence, proved.