INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY
HYDERABAD

COMPLEX INNER PRODUCT SPACES & FORMULATION OF QUANTUM THEORY

COURSE TITLE - LINEAR ALGEBRA

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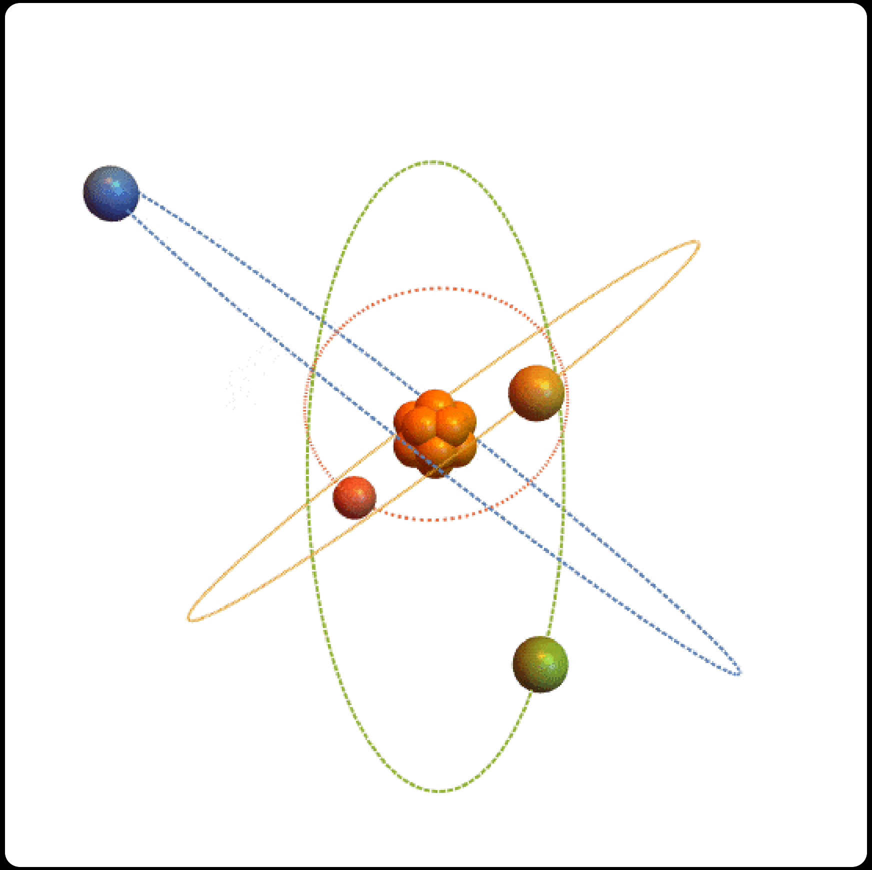


INTRODUCTION

Vector

A vector is a column of numbers (any numbers, even complex). The amount of numbers is referred to as the dimension of the vector.

A n-dimension vector can be represented as ,

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Matrices

- A matrix is a box of numbers (real or complex).
- Since a vector is also a matrix that has only one column, we can think of matrices as functions on vectors. For ex:- an $m \times n$ matrix maps an n-dimensional vector to an m-dimensional vector.

Inner product

- Mathematical operation between two vectors, $\vec{v}$ and $\vec{w}$ of the same dimension that returns a scalar number
- Denoted by $\vec{v} \cdot \vec{w}$

$$\boxed{\vec{v} \bullet \vec{w} = \sum_{j=1}^{n} \overline{v_j} w_j}$$

Example

$$\vec{v} = \begin{bmatrix} i \\ 2+i \end{bmatrix}, \quad \text{and} \quad \vec{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

then:

$$\vec{v} \bullet \vec{w} = (-i) \cdot 2 + (2-i) \cdot (-1)$$

Basis

- A basis is a finite set of vectors that can be used to describe any other vectors of the same dimension.

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
$$= v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The set:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Dirac's Notation

- It is also known as Bra-ket notation.
- Basis in kets format.

$$|1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix},\ |2\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}\ ,\ \ldots\ ,\ |i\rangle = \begin{bmatrix}0\\\ldots\\i\end{bmatrix}$$

- Any given vector can be written in kets format will be written as,

$$\overrightarrow{v} = \sum_i v_i|i\rangle = x|1\rangle + y|2\rangle + z|3\rangle$$

VECTOR SPACE

- The collection of all the complex vectors of a given dimension with vector addition and scalar multiplication, is called a vector space.

LINEARITY

- A linear function, or linear map, or linear operator, f is a function that satisfies:
- 1. $f(x+y) = f(x) + f(y)$, for any input x and y
- 2. $f(cx) = c\, f(x)$ for any input x and any scalar c

INNER PRODUCT SPACE

- The inner product, also known as the dot product or the scalar product, of two vectors and is a mathematical operation between two vectors of the same dimension that returns a scalar number. It is denoted $\vec{v}$ $\vec{w}$.

Explicitly, for vectors:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

the inner product of $\vec{v}$ and $\vec{w}$ gives:

$$\vec{v} \bullet \vec{w} = \bar{v_1}\, w_1 + \bar{v_2}\, w_2 + \bar{v_2}\, w_2 + \ldots + \bar{v_n}\, w_n$$

COMPLEX INNER PRODUCT SPACES

- By an inner product on a complex vector space we mean a device of assigning to each pair of vectors x and y a complex number denoted by such that the following conditions are satisfied:

(C1) $\langle \mathbf{x}, \mathbf{y} \rangle \geq 0$, and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

(C2) $\langle \mathbf{y}, \mathbf{x} \rangle = \overline{\langle \mathbf{x}, \mathbf{y} \rangle}$.

(C3) The inner product is a "*sesquelinear* map", i.e.

$$\langle a_1\mathbf{x}_1 + a_2\mathbf{x}_2,\ \mathbf{y} \rangle = a_1\langle \mathbf{x}_1, \mathbf{y} \rangle + a_2\langle \mathbf{x}_2, \mathbf{y} \rangle$$

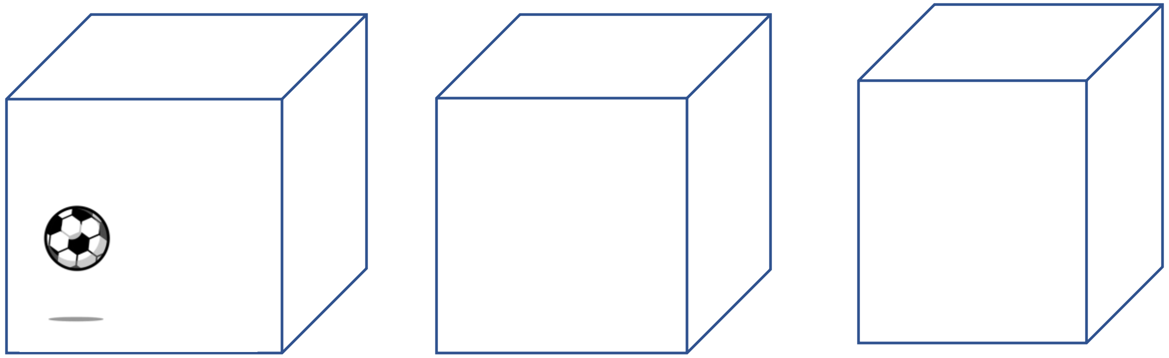
$$\langle \mathbf{x},\ b_1\mathbf{y}_1 + b_2\mathbf{y}_2 \rangle = \overline{b}_1\langle \mathbf{x}, \mathbf{y}_1 \rangle + \overline{b}_2\langle \mathbf{x}, \mathbf{y}_2 \rangle.$$

POSTULATE 1

- A **state** is a complete description of a physical system. In quantum mechanics, a state is a **vector** in **Hilbert** space.
  - Quantum state: The collection of all relevant physical properties of a quantum system (e.g., position, momentum, spin, polarization) is known as the state of the system.
    - Exclusive states: Two states are said to be exclusive if the fact of being in one of the states with certainty implies that there are no chances of being in any of the other states.

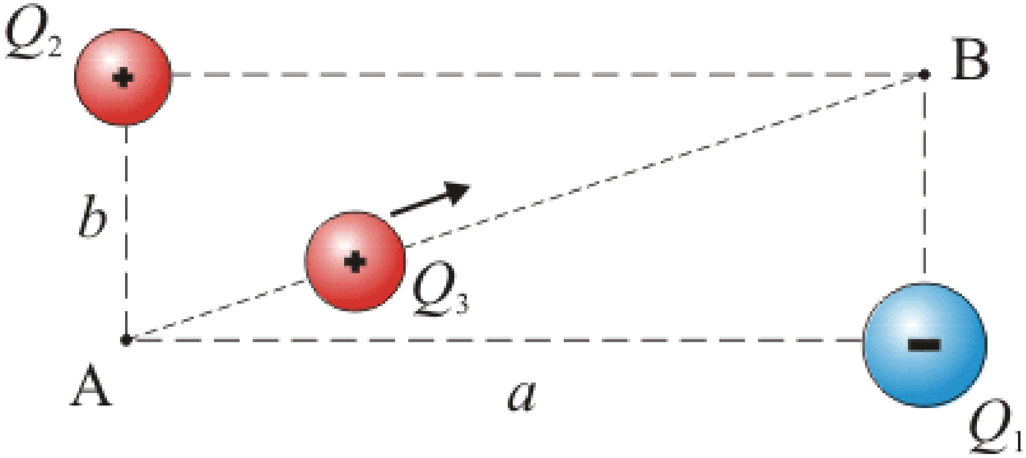
QUANTUM STATES

Exclusive state: 1 particle, 3 boxes



$$|1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad |3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Non-exclusive state: Moving particle



http://physicstasks.eu/

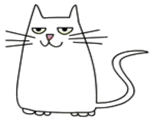
Exclusivity = Orthogonal vectors?

Energy of an electron as qubit

Ground state (lowest energy): Quantum-0
Excited state(higher energy): Quantum-1
Since the ground and excited states are mutually exclusive, we could represent:

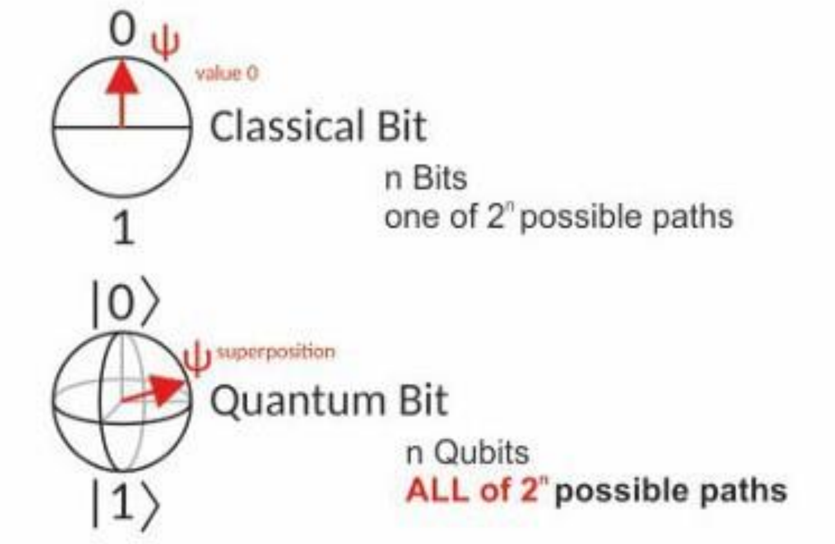
$$\text{ground state} \leftrightarrow |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{excited state} \leftrightarrow |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Quantum superposition principle



If a quantum system can be in the state-0 and can also be in state-1, then quantum mechanics allows the system to be in any arbitrary state.

$$|\psi\rangle = a|0\rangle + b|1\rangle$$
$$= \begin{bmatrix} a \\ b \end{bmatrix}$$



We say that $|\psi\rangle$ is in a superposition of $|0\rangle$ and $|1\rangle$ with probability amplitudes *a* and *b*.

POSTULATE 2

<u>Born's Rule:</u>

Suppose we have a quantum state $|\psi i\rangle$ & an orthonormal basis $\{|a_1\rangle, \ldots, |a_n\rangle\}$. Then we can measure $|\psi\rangle$ with respect to this orthonormal basis.The probability of measuring the state $|a_i\rangle$

$P(a_i)$, is given by:

$P(a_i) = |\langle a_i|\psi\rangle|^2$

The wave function of a quantum mechanical system can be represented as a super-position of basis states in a n-dimensional vector space.

$$|\psi\rangle = \sum_k c_k \mid \Phi_k\rangle$$

where k=1,2,...n

Expectation value of A in state is:

$$\langle A\rangle = \frac{\langle\psi|A|\psi\rangle}{\langle\psi|\psi\rangle}$$

Hence we get,

$$A \mid \psi\rangle = \sum_k c_k a_k \mid \Phi_k\rangle$$

$$A \mid \Phi_k\rangle = a_k \mid \Phi_k\,\rangle$$

where : a_k eigenvalue and k=1..n

$$A \mid \psi\rangle = \sum_k c_k a_k \mid \Phi_k\rangle$$

$$\langle\psi \mid = \sum_l c_l^* \langle\Phi_l|$$

$$\left\langle\Phi_l \mid \Phi_k\right\rangle = \delta_{lk}$$

As it is orthonormal basis

$$\begin{aligned}
\langle\psi \mid A \mid \psi\rangle &= \sum_l \sum_k c_l^* c_k a_k \langle\phi_l \mid \phi_k\rangle \\
&= \sum_k c_k^* c_k a_k \\
&= \sum_k \mid c_k \mid^2 a_k
\end{aligned}$$

$$\langle A \rangle = \frac{\sum_k |c_k|^2 a_k}{\langle \psi | \psi \rangle}$$

If ψ is normalized

$$\langle A \rangle = \sum_k \mid c_k \mid^2 a_k$$

Then,then measurement outcome would be one of the eigen value a_k of with probability $|c_k|^2$

Wave Collapse:

After the measurement is performed the original state collapses in the measured state,

i.e., we're left with one of the states $|a_1\rangle$,..., $|a_n\rangle$

POSTULATE 3

"Quantum operations are represented by unitary operators on the Hilbert space"

Quantum operation: A quantum operation transforms a quantum state to another quantum state, therefore, we must have it so that the norm of the vector is preserved.

NORM: The norm (or length) of a n-dimensional vector $\vec{x}$, denoted $||\vec{x}||$ is given by:

$$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{N} |x_i|^2\right)^{1/2} = \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2}$$

POSTULATE 3

U*nitary operator* *: It* is a bounded linear operator $U : H \rightarrow H$ on a Hilbert space H for which the following hold:

1. U is surjective

2. U preserves the inner product of the Hilbert space, H.

In other words, for all vectors x and y in H we have :

$\langle U_x U_y \rangle_H = \langle x,y \rangle_H$

POSTULATE 3 : OBSERVATION

If one implements several quantum operations one after the other,

say $U_1, U_2, \ldots U_m$ then the matrix representation of the combined quantum operation U_{tot}, is given by:

$$U_{tot} = U_m \ldots U_2 U_1$$

So, the order of the multiplication goes from right to left in chronological order.

Composite Quantum Systems

We are accustomed from classical physics to the fact that composite systems can be decomposed into their subsystems and that conversely, individual systems can be combined to give overall composite systems.

Postulate 4:

"The Hilbert space of a composite system is given by the Kronecker Product (also known as the tensor product) of the separate, individual Hilbert spaces."

Joint quantum systems must follow the following properties:

1. Dimensions: We should be able to see a composite system made of two qubits (2 dimensions each) as a single quantum system with 4 dimensions.

$$|00\rangle_4 = |0\rangle_2 \otimes |0\rangle_2 \longleftrightarrow \text{qubit 1 in } |0\rangle_2 \text{ and qubit 2 in } |0\rangle_2$$
$$|01\rangle_4 = |0\rangle_2 \otimes |1\rangle_2 \longleftrightarrow \text{qubit 1 in } |0\rangle_2 \text{ and qubit 2 in } |1\rangle_2$$
$$|10\rangle_4 = |1\rangle_2 \otimes |0\rangle_2 \longleftrightarrow \text{qubit 1 in } |1\rangle_2 \text{ and qubit 2 in } |0\rangle_2$$
$$|11\rangle_4 = |1\rangle_2 \otimes |1\rangle_2 \longleftrightarrow \text{qubit 1 in } |1\rangle_2 \text{ and qubit 2 in } |1\rangle_2$$

$$|00\rangle_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad |01\rangle_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad |10\rangle_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad |11\rangle_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

2. Measurement Probabilities: The joint system can be seen as a single, larger system of higher dimension and **Born's rule** still applies. Therefore, measuring the probability of qubit 1 in ϕ_1 and the probability of qubit 2 in ϕ_2 in i.e., measuring the joint system in Φ will be given by

$$[\quad |\Phi\rangle_4 = |\phi_1\phi_2\rangle_4 = |\phi_1\rangle_2 \otimes |\phi_2\rangle_2 \quad]$$

$$[\quad |\Psi\rangle_4 = |\psi_1\psi_2\rangle_4 = |\psi_1\rangle_2 \otimes |\psi_2\rangle_2. \quad]$$

$$\begin{aligned} P(\Phi) &= |\langle\Phi|_4|\Psi\rangle_4|^2 \\ &= |\langle\phi_1\phi_2|_4|\psi_1\psi_2\rangle_4|^2 \\ &= \left|\left[\langle\phi_1|_2 \otimes \langle\phi_2|_2\right]\left[|\phi_1\rangle_2 \otimes |\phi_2\rangle_2\right]\right|^2 \end{aligned}$$

$$P(\Phi) = P(\phi_1)P(\phi_2)$$

$$\boxed{\left|\left[\langle\phi_1|_2 \otimes \langle\phi_2|_2\right]\left[|\phi_1\rangle_2 \otimes |\phi_2\rangle_2\right]\right|^2 = |\langle\phi_1|\psi_1\rangle|^2 |\langle\phi_2|\psi_2\rangle|^2}$$

3. Joint Quantum Operations: If U_1 is a unitary operator on qubit 1 and U_2 is a unitary operator on qubit 2, then the joint operation $U_1 \otimes U_2$ must have the property that:

$$\begin{aligned}[U_1 \otimes U_2]|\psi_1 \psi_2\rangle_4 &= [U_1 \otimes U_2]\,[|\psi_1\rangle_2 \otimes |\psi_2\rangle_2] \\ &= [U_1|\psi_1\rangle_2] \otimes [U_2|\psi_2\rangle_2]\end{aligned}$$

The resulting joint state after applying the joint quantum operation, must be equal to the joint state of the individual state after the individual operations, respectively.

Kronecker Product

- Suppose we have two vectors:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

The Kronecker product is defined as:

$$\vec{v} \otimes \vec{w} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \otimes \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ v_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} v_1 w_1 \\ v_1 w_2 \\ v_2 w_1 \\ v_2 w_2 \end{bmatrix}$$

Kronecker Product for matrices:

- If A is an m × n matrix and B is a p × q matrix, then the Kronecker product A ⨂ B is defined as the pm × qn block matrix:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1q} & \cdots & \cdots & a_{1n}b_{11} & a_{1n}b_{12} & \cdots & a_{1n}b_{1q} \\ a_{11}b_{21} & a_{11}b_{22} & \cdots & a_{11}b_{2q} & \cdots & \cdots & a_{1n}b_{21} & a_{1n}b_{22} & \cdots & a_{1n}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{11}b_{p1} & a_{11}b_{p2} & \cdots & a_{11}b_{pq} & \cdots & \cdots & a_{1n}b_{p1} & a_{1n}b_{p2} & \cdots & a_{1n}b_{pq} \\ \vdots & \vdots & & \vdots & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \ddots & \vdots & \vdots & & \vdots \\ a_{m1}b_{11} & a_{m1}b_{12} & \cdots & a_{m1}b_{1q} & \cdots & \cdots & a_{mn}b_{11} & a_{mn}b_{12} & \cdots & a_{mn}b_{1q} \\ a_{m1}b_{21} & a_{m1}b_{22} & \cdots & a_{m1}b_{2q} & \cdots & \cdots & a_{mn}b_{21} & a_{mn}b_{22} & \cdots & a_{mn}b_{2q} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{p1} & a_{m1}b_{p2} & \cdots & a_{m1}b_{pq} & \cdots & \cdots & a_{mn}b_{p1} & a_{mn}b_{p2} & \cdots & a_{mn}b_{pq} \end{bmatrix}.$$

Multiple qubits: If we have two qubits with individual states $|\psi\rangle$ and $|\phi\rangle$, their joint quantum state $|\Psi\rangle$ is given by:

$|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle$

where $\otimes$ represents the Kronecker product.

POSTULATE 5

- Physical observables are represented by the eigenvalues of a Hermitian operator on the Hilbert space.
- The fifth postulate of quantum mechanics says that for a given physically meaningful quantity with multiple possible values, there is a Hermitian operator that associates with the possible value of the measurements.
- The observable is physical quantity that can be measured. For example energy, momentum, spin etc can be regarded as observables.

Hermitian Operator

- An operator Is a tool that extract useful information from quantum state

$$\hat{A} f(x) = k * f(x)$$

Hermitian Operator is an operator whose conjugate transpose is equal to itself.

$$M^\dagger = M$$

Eigen values of Hermitian Operator are always real.

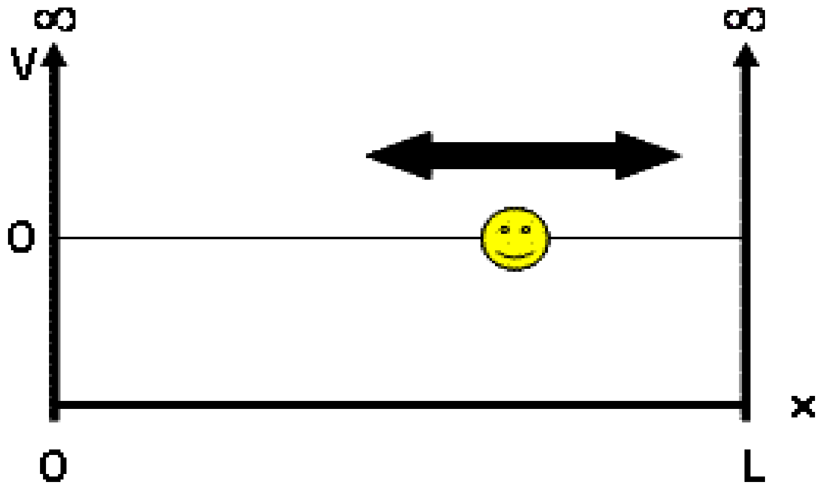
Quantum mechanical theory:

- The information about quantum system is contained within a mathematical entity called the wavefunction(ψ)
- Each physical property or observable, of interest has a corresponding operator which operates on the wavefunction.
- If the wavefunction is an eigenfunction of that operator, then its eigenvalue is the value of that observable.
- A prime example of this is the Schrödinger equation:

$$\hat{H}\,\psi = E\,\psi$$

Application

- **Problem of Particle in the box:**



Equation

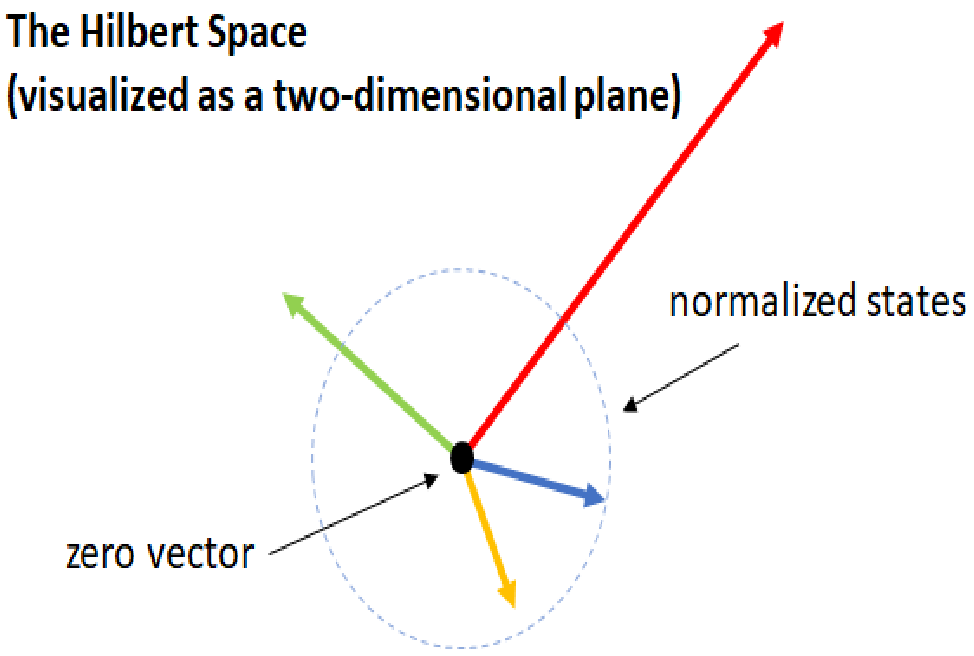
$$(\frac{\hbar^2}{2m})*(\frac{d^2\psi(x)}{dx^2})+V(x)*\psi(x)=E\psi(x)$$

$$En=\frac{n^2*h^2}{8mL^2}$$

Here by operating the Hermitian operator on the wave function, we get Energy as eigen value which is an observable of the particle.

Application Of Postulates :-

- Two State System



Let's consider the normalized ket pointing to the right is $|0\rangle$ and the one pointing up is $|1\rangle$.

The following state can be represented as $|\psi_{\frac{\pi}{2}}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$.

In general we can have states that could look like $|\psi_{\varphi}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\varphi}|1\rangle\right)$,
where φ is called the relative phase.

Suppose I wanted to measure whether the state is in state $|1\rangle$, which corresponds to the observable $\hat{Q}_1 = |1\rangle\langle 1|$. Then the possible results are “yes” (1) and “no” (0).

Suppose we measure $\hat{Q}_1$ for the state $|\psi_{\varphi}\rangle$. The probability of measuring 1 for this observable is

$$P(\lambda = 1) = |\hat{P}_{1,\lambda=1}|\psi_{\varphi}\rangle|^2 = ||1\rangle\langle 1|\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\varphi}|1\rangle\right)|^2 = |\frac{1}{\sqrt{2}}\left(|1\rangle\langle 1|0\rangle + e^{i\varphi}|1\rangle\langle 1|1\rangle\right)|^2 = |\frac{1}{\sqrt{2}}\left(|1\rangle \cdot 0 + e^{i\varphi}|1\rangle \cdot 1\right)|^2 = \frac{1}{2}$$

after which the state would collapse to state $|1\rangle$.

Please note that the answer doesn't depend on the relative phase.

we can instead consider the observable corresponding to whether or not the state is in state $|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$. This corresponds to the operator $\hat{Q}_2 = \frac{1}{2}\left(|0\rangle + |1\rangle\right)\left(\langle 0| + \langle 1|\right)$.

Now the projection operators we need for this observable are $\hat{P}_{2,\lambda=1} = \frac{1}{2}\left(|0\rangle + |1\rangle\right)\left(\langle 0| + \langle 1|\right)$ and $\hat{P}_{2,\lambda=0} = \frac{1}{2}\left(|0\rangle - |1\rangle\right)\left(\langle 0| - \langle 1|\right)$. The probability of measuring that the state is in $|0\rangle + |1\rangle$ is

$$P(\lambda = 1) = |\hat{P}_{2,\lambda=1}|\psi_\varphi\rangle|^2 = |\frac{1}{2}\left(|0\rangle + |1\rangle\right)\left(\langle 0| + \langle 1|\right)\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\varphi}|1\rangle\right)|^2 = |\frac{1}{2\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right)\left(|0\rangle + e^{i\varphi}|1\rangle\right)|^2 = |\frac{1}{2\sqrt{2}}\left(|0\rangle\langle 0|0\rangle + |0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle + |1\rangle\langle 1|0\rangle + e^{i\varphi}|0\rangle\langle 0|1\rangle + e^{i\varphi}|0\rangle\langle 1|1\rangle + e^{i\varphi}|1\rangle\langle 0|1\rangle + e^{i\varphi}|1\rangle\langle 1|1\rangle\right)|^2 = |\frac{1}{2\sqrt{2}}\left((1 + e^{i\varphi})|0\rangle + (1 + e^{i\varphi})|1\rangle\right)|^2 = \frac{1}{2} + \frac{1}{2}\cos\varphi$$

after which the state would collapse to state $|\psi_0\rangle$.

While a bit more nuanced, in quantum mechanics, a similar thing goes on with position and momentum. If you measure the position over and over again, you will get the same answer each time. However, if you measure the momentum, the state will collapse to an eigenstate of the momentum operator, and, if you measured the position again, it wouldn't be in the exact place anymore. In fact, if you measured the momentum perfectly, it turns out that you lose *all* position information. This underlies **Heisenberg's uncertainty principle** between position and momentum, which says that the product of the uncertainties for position and momentum must be at least a nonzero constant .

THANK YOU!

- FROM TEAM 1