NAME: SAI SIRISHA NADIMINTI
ROLL NO: 2020201044

Q1. Given that S is a linearly independent subset of a vector space V and β is a vector in V that is not in the subspace spanned by S.

To prove: set obtained by adjoing β to S is linearly independent.

Proof:

A list $v_1, v_2, \ldots v_m$ of vectors in V is called linearly independent if the only choice of $a_1, a_2, \ldots a_m \in \mathbb{F}$ that makes
$$a_1v_1 + a_2v_2 + \ldots + a_mv_m = 0$$
is $a_1 = a_2 = a_3 = \ldots = a_m = 0$.

Let the set $S = \{s_1, s_2, \ldots s_m\}$. As S is linearly independent set, $a_1, a_2, \ldots a_m \in \mathbb{F}$ makes
$$a_1s_1 + a_2s_2 + \ldots a_ms_m = 0$$
only when $a_1 = a_2 = a_3 = \ldots a_m = 0$

Given β is not present in the subspace spanned by S, it means that no linear combination of vectors in S can form β,
i.e. $a_1s_1 + a_2s_2 + \ldots + a_ms_m \neq \beta$ for $a_i \in F$. —(1)

To prove $A = [\beta | S]$ is linearly independent, we have to prove that
$$a_0\beta + a_1s_1 + a_2s_2 + \ldots a_ms_m = 0 \quad a_i \in \mathbb{F}$$
only when all $a_1, a_2, \ldots a_{m+1} = 0$.

Let us assume that A is a set of linearly dependent vectors, i.e for $a_i \in \mathbb{F}$ not all 0,

$$a_0\beta + a_1 s_1 + a_2 s_2 + \dots + a_m s_m = 0 \quad \cancel{a_i \in \mathbb{F}}$$

$$\Rightarrow a_0\beta = -(a_1 s_1 + a_2 s_2 + \dots + a_m s_m)$$

$$\Rightarrow \beta = \left(-\frac{a_1}{a_0}\right) s_1 + \left(-\frac{a_2}{a_0}\right) s_2 + \left(-\frac{a_3}{a_0}\right) s_3 + \dots + \left(-\frac{a_m}{a_0}\right) s_m.$$

It means that there exists a linear combination of vectors in S that produce β. This statement contradicts equation (1), that says that β is NOT in the subspace spanned by S.

So, our assumption that A is linearly dependent is false.
Thus, the set obtained by adjoining β to S is linearly independent.

Q2. Given that S is a subspace of a vector space V.
To prove: Span(S) = S.

__Definition of span__ : Set of all combinations of a list of vectors $v_1, v_2, \dots v_m$ in V is called the span of $v_1, v_2, \dots v_m$ such that

$$\text{Span}(v_1, v_2, \dots v_m) = \{a_1 v_1 + a_2 v_2 + \dots a_m v_m\}$$

where $a_1, a_2, \dots a_m \in \mathbb{F}$.

→ Since S is a subspace of V, it will be a vector space and will follow the 10 properties of vector space.

So, there will exist a multiplicative identity in $\mathbb{F}$ ($1 \in \mathbb{F}$). If there is an arbitary element $x \in S$, then

$$1.x = x \in span(S).$$

So, $\forall\ x \in S$, $x \in span(S)$

$$\therefore S \subseteq span(S) \quad -(1)$$

Now, consider an element $x \in span(S)$.
It means that there will be a combination of $(\alpha_1, \alpha_2, .. \alpha_n) \in S$ and $(a_1, a_2, .. a_n) \in \mathbb{F}$ such that $v = a_1\alpha_1 + a_2\alpha_2 + .. a_n\alpha_n \in S$.

$$\therefore span(S) \subseteq S \quad -(2).$$

From (1) and (2), we can say that $span(S) = S$.
Hence, proved.

Q3.
(a) <u>Definition of null space:</u>
For $T \in \mathcal{L}(V, W)$, the null space of T, denoted by null T, is the subset of V consisting of those vectors that T maps to 0:

$$\text{null}\ T = \{v \in V : Tv = 0\}., \quad \text{null}\ T \in V.$$

So, for a given matrix, A:
X such that $AX = 0$
forms a null space of A.
If A consists of linearly independent vectors, then the null space will be the null vector only.
To get X, we will convert A to row-reduced form.

FOR EDUCATIONAL USE

3(a) $A = \begin{bmatrix} 12 & 4 & 4 \\ 5 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{R_1 \to R_1/12} \begin{bmatrix} 1 & 1/3 & 1/3 \\ 5 & 4 & 5 \\ 2 & 3 & 4 \end{bmatrix}$

$\downarrow R_2 \to R_2 - 5R_1$, $R_3 \to R_3 - 2R_1$.

$\begin{bmatrix} 1 & 1/3 & 1/3 \\ 0 & 7/3 & 10/3 \\ 0 & 0 & 0 \end{bmatrix} \xleftarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1/3 & 1/3 \\ 0 & 7/3 & 10/3 \\ 0 & 7/3 & 10/3 \end{bmatrix}$

Here rank(A) = 2.

So, there will be one arbitary value in

we get

$x_1 + x_2/3 + x_3/3 = 0$ — (1)

and $7x_2/3 + 10x_3/3 = 0$ — (2)

Let $x_3 = t$

then from (2), we will get $x_2 = -10/7\, t$

from (1), if we substitute x_2 and x_3, we get

$x_1 = 1/7\, t$.

So $X = \begin{bmatrix} 1/7\, t \\ -10/7\, t \\ t \end{bmatrix} = \left\{ \begin{bmatrix} 1/7 \\ -10/7 \\ 1 \end{bmatrix} t \right\}$ is the null space of A.

3(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 6R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -7 & -14 \\ 0 & -2 & -5 \end{bmatrix}$

$\downarrow R_2 \to R_2 / -7$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \xleftarrow{R_3 \to R_3 + 2R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & -5 \end{bmatrix}$

So, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\Rightarrow$ Rank(A) = 3.

So, $x_3 = 0$,

$x_2 + 2x_3 = 0 \Rightarrow x_2 = 0$

$x_1 + 2x_2 + 3x_3 = 0 \Rightarrow x_1 = 0$.

So, null space of A is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

3(c). $A = \begin{bmatrix} 12 & 3 & 9 \\ 20 & 5 & 15 \\ 16 & 4 & 12 \end{bmatrix} \xrightarrow{R_1 = R_1/12} \begin{bmatrix} 1 & 1/4 & 3/4 \\ 20 & 5 & 15 \\ 16 & 4 & 12 \end{bmatrix}$

$\downarrow \begin{array}{l} R_2 \to R_2 - 20R_1 \\ R_3 \to R_3 - 16R_1 \end{array}$

$\begin{bmatrix} 1 & 1/4 & 3/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ Rank(A) = 1.

So, we can take $x_2 = a$ and $x_3 = b$

then $x_1 + \frac{x_2}{4} + \frac{3x_3}{4} = 0$

$\Rightarrow x_1 + \frac{a}{4} + \frac{3b}{4} = 0$

$\Rightarrow x_1 = \frac{-a-3b}{4}$

$\therefore X = \begin{bmatrix} (-a-3b)/4 \\ a \\ b \end{bmatrix}$

$\therefore$ Null space is $\left\{ \begin{bmatrix} -1/4 \\ 1 \\ 0 \end{bmatrix} a + \begin{bmatrix} -3/4 \\ 0 \\ 1 \end{bmatrix} b \right\}$

Q4. Given that W is a subspace of V with a basis $\{\alpha_i : i \in [m]\}$, $\beta \in V \setminus W$.
To show: set $\{\alpha_i + \beta : i \in [m]\}$ spans an m-dimensional subspace of V.

→ Basis: A list $v_1, .. v_n$ of vectors V is a basis of V iff every $v \in V$ can be written uniquely in the form
$$v = a_1 v_1 + a_2 v_2 + .. + a_n v_n, \quad a_1, a_2, .. a_n \in \mathbb{F}.$$

Given W has a basis $\{\alpha_i : i \in [m]\}$. So, W is a vector space where every $w \in W$ can be written as
$$w = c_1 \alpha_1 + c_2 \alpha_2 + .. c_m \alpha_m, \quad c_1, c_2, .. c_m \in \mathbb{F}$$
$\alpha_1, \alpha_2, .. \alpha_m$ are linearly independent.
Dimension of a finite-dimensional vector space is the length of the basis of the vector space.
So, here Dimension of $W = m$.

Given that $\beta \in V \setminus W$, i.e. $\beta \in V$ but $\beta \notin W$.
Let $A = \{\alpha_i + \beta : i \in [m]\}$
$A = \{\alpha_1 + \beta, \alpha_2 + \beta, \ldots, \alpha_m + \beta\}$

Now, we have to prove that A spans an m-dimensional subspace of V. Let that subspace of V be called S, i.e.
$$\text{span}(A) = S$$
i.e. $\text{span}(\alpha_1 + \beta, \alpha_2 + \beta, \alpha_3 + \beta, .. \alpha_m + \beta) = S.$

LHS $\quad span(A)$

$= span(\alpha_1+\beta, \alpha_2+\beta, \alpha_3+\beta, \ldots \alpha_m+\beta)$

$= c_1(\alpha_1+\beta) + c_2(\alpha_2+\beta) + \ldots c_m(\alpha_m+\beta)$ $c_i \in \mathbb{F}$.

Clearly, the dimension of this set of vectors is 'm'. — (1)

$span(A) = c_1\alpha_1 + c_1\beta + c_2\alpha_2 + c_2\beta + \ldots c_m\alpha_m + c_m\beta$

$= c_1\alpha_1 + c_2\alpha_2 + \ldots c_m\alpha_m + c_1\beta + c_2\beta + \ldots c_m\beta$

$S = W + c_1\beta + c_2\beta + \ldots c_m\beta$, $c_i \in \mathbb{F}$.

i.e. $\forall w \in W$, $w + \sum c_i\beta \in S$, $w + C_i\beta \in S$; $\sum c_i = C_i$

To prove S is a subspace of V, we have to prove that:

i) S is closed w.r.t. addition, i.e. if $s_1, s_2 \in S$ then $s_1 + s_2 \in S$.

ii) S is closed w.r.t scalar multiplication, i.e. $s_1 \in S$, $c \in F \Rightarrow c \cdot s_1 \in S$.

i) Closure under addition

Let $s_1 = w_1 + C_1\beta \in S$, $w_1 \in W$.

$S_2 = W_2 + C_2\beta \in S$, $w_2 \in W$.

Then $s_1 + s_2 = (w_1 + w_2) + (C_1 + C_2)\beta$

Now, we already know that W is a vector space, and vector spaces are closed under addition. So, if $w_1 \in W$ and $w_2 \in W$, then $w_1 + w_2 \in W$.

Also, $c_i \in F$ and field is also closed under addition. Hence, $C_1, C_2 \in F$ and $C_1 + C_2 \in F$.

Thus, $w_1 + w_2 \in W$, $C_1 + C_2 \in \mathbb{F}$

$\Rightarrow s_1 + s_2 \in S$.

Thus, we can say that S is closed under addition.

ii) Closure under scalar multiplication

Let $s \in S$ such that $s = w + c\beta$.

where $w \in W$ and $c \in F$ (proved in (i))

Let $b \in F$.

$b.s = b.w + b.c.\beta$

Since $w \in W$ and $b \in F$ and vector space, W is closed under scalar multiplication, $bw \in W$.

Similarly, $b \in F$ and $c \in F$, so, $b.c \in F$ as field is closed under multiplication.

So, since $b.w \in W$ and $b.c \in F$,

$b.s \in S$.

So, S is closed under scalar multiplication.

Thus, S is a subspace of V. — (2)

From (1), (2), we have proved that set A $\{\alpha_i + \beta : i \in [m]\}$ spans an m-dimensional subspace of V.

Q5. F is a finite field with p^n elements (p is prime). and V is a k-dimensional vector space over F.

1. The number of linear transformations $T: V \to V$. ~~Here, V is a~~ Every linear transformation can be represented by a $k \times k$ matrix where k is the dimension of vector space.

Each element in that matrix takes the value from field, $\mathbb{F}$. that has p^n values.

So, number of $k\times k$ matrices possible canbe found out as:

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & k \end{array} \\ \begin{bmatrix} p^n\text{ways} & p^n\text{ways} & \cdots & \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ & & & p^n\text{ways} \end{bmatrix}_{k\times k} \end{array}$$

$\therefore$ No. of matrices $= p^n \times p^n \times \ldots$ ($k\times k$) times
$= (p^n)^{k^2}$ matrices.

$\therefore$ No. of transformations $= (p^n)^{k^2}$ transformations.

Thus, the number of linear transformations $T: V\to V$ are $(p^n)^{k^2}$

2. The number of invertible linear transformations $T: V\to V$.

The number of Invertible linear transformations can be found out by finding number of invertible $k\times k$ matrices.

To find invertible (non-singular) $k\times k$ matrices, all the columns must be linearly independent.

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \cdots & k \end{array} \\ \begin{bmatrix} \; & & & & \; \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \end{array}$$

non-zero
$(p^n)^k - 1$ ways $\quad (p^n)^k - p^n$

1st column of matrix (k-sized column) must be comprised of non-zeroes. elements in $\mathbb{F}$.

$\therefore$ No. of ways to form 1st column $= (p^n)^k - 1$.

2nd column of matrix must be linearly independent from 1st column. Since there are p^n vectors in the span of 1st column, we have

no. of ways to form second column $= (p^n)^k - p^n$

Now, let v_1, v_2 be first two column vectors. Then the set of vectors in the span of v_1, v_2 is of the form $\{c_1 v_1 + c_2 v_2 \mid c_1, c_2 \in \mathbb{F}\}$. This set is of size $(p^n)^2$ as we have p^n choices for c_1 and p^n choices for c_2

$\therefore$ No. of ways to form ~~second~~ third column
$$= (p^n)^k - (p^n)^2$$

Similarly,

no. of ways to form forth column $= (p^n)^k - (p^n)^3$

We must continue this process for all 'k' colums to get

the number of invertible matrices
$$= ((p^n)^k - 1) \cdot ((p^n)^k - p^n) \cdot ((p^n)^k - (p^n)^2) \cdot ((p^n)^k - (p^n)^3) \cdots ((p^n)^k - (p^n)^{k-1})$$
$$= \prod_{j=0}^{k-1} \left((p^n)^k - (p^n)^j \right) = \prod_{j=0}^{k-1} \left[(p^n)^k - (p^n)^j \right]$$

This is the required number of invertible linear transformations $T: V \to V$.