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Linear Algebra - Assignment 5

1. Given matrix H is $n \times n$ Hermitian matrix.
To prove: Eigen values of H are real.

Proof:
Hermitian matrices are the matrices that equal their conjugate transpose.
i.e,
A hermitian matrix A; if $A^\theta = \overline{A}^T$ has entries $(A^H)_{ij} = \overline{A}_{ji}$
then $\boxed{A = A^\theta}$

Let H be the $n \times n$ Hermitian matrix and let λ be the characteristic root of H with characteristic vector x. Then $H^\theta = H$.

Then, $Hx = \lambda x$ — (1)

Taking conjugate transpose on both sides, we get

$$(Hx)^\theta = (\lambda x)^\theta$$

So, $x^\theta H^\theta = x^\theta . \overline{\lambda}$ $\quad (\because (AB)^\theta = B^\theta . A^\theta)$

$\Rightarrow x^\theta H^\theta = \overline{\lambda} x^\theta$

$\Rightarrow x^\theta H = \overline{\lambda} x^\theta$ $\quad (\because H^\theta = H)$

Postmultiplying both sides by x, we get

$$x^\theta H x = \overline{\lambda} x^\theta x$$

$\Rightarrow x^\theta \lambda x = \overline{\lambda} x^\theta x$

$\Rightarrow \lambda x^\theta x = \overline{\lambda} x^\theta x$

$\Rightarrow (\lambda - \overline{\lambda}) x^\theta x = 0$

Here, if x will be a zero vector, then the definition of Eigen vectors is lost. We do not consider the 0 vector as Eigen vector since $A0 = \lambda 0$ for every scalar λ. Then, the

associated Eigen value would be undefined.
So, we have

$$x^{\theta}.x \neq 0.$$

This means that $\lambda - \bar{\lambda} = 0$
$\Rightarrow \lambda = \bar{\lambda}$

If conjugate of a complex number $z = a+ib$, $\bar{z} = a-ib$, then if $z = \bar{z}$
$\Rightarrow a+ib = a-ib$
$\Rightarrow b = 0$

So, there is no imaginary part for λ.
This means that Eigen values of H are only real.

Hence, proved.

5. Given A is a diagonalizable $n \times n$ matrix
To prove: $\det(\exp(A)) = \exp(\text{trace}(A))$

Proof

A square matrix A is called diagonalizable if it is similar to a diagonal matrix.
i.e. if there exists an invertible matrix, P and a diagonal matrix D such that

$$P^{-1}.A.P = D$$

or $$A = PDP^{-1}.$$

such ~~that~~ P, D are not unique

The exponential function of a square matrix, $A_{n \times n}$ is given by:

$$\exp(A) = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \ldots \quad -(i)$$

where I is the identity matrix of size $n \times n$.

Now, substituting (PDP^{-1}) in place of A, we get

$$A^2 = (PDP^{-1})^2$$
$$= (PDP^{-1})(PDP^{-1})$$
$$= PD(P^{-1}P)DP^{-1}$$
$$= PD\,I_{n\times n}\,DP^{-1}$$
$$= PDDP^{-1}$$
$$= PD^2P^{-1}.$$

$$A^3 = A^2 . A$$
$$= PD^2P^{-1}(PDP^{-1})$$
$$= PD^2(P^{-1}P)DP^{-1}$$
$$= PD^3P^{-1} \quad (\because P^{-1}P = I_{n\times n}).$$

Basically,

$$A^i = PD^iP^{-1}$$

$\therefore$ Equation (1) becomes:

$$\exp(A) = P.P^{-1} + PDP^{-1} + \frac{1}{2!}(PD^2P^{-1}) + \frac{1}{3!}(PD^3P^{-1}) + \ldots$$

$$(\because I = PP^{-1})$$

$$= P\left[I + D + \frac{1}{2!}D^2 + \frac{1}{3!}(D^3) + \frac{1}{4!}(D^4) + \ldots\right]P^{-1} \quad -(2)$$

Now, $\exp(D) = I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \ldots$

$\therefore$ Equation (2) can be written as

$$\exp(A) = P.\exp(D).P^{-1}.$$

Taking determinant on both sides, we get

$$\det(\exp(A)) = \det(P\,\exp(D).\,P^{-1})$$
$$= \det(P).\det(\exp(D).P^{-1}) \quad (\because |AB| = |A|.|B|)$$
$$= \det(P).\det(P^{-1}).\det(\exp(D))$$
$$= \det(P.P^{-1}).\det(\exp(D))$$
$$= \det(I).\det(\exp(D))$$
$$= 1.\det(\exp(D))$$
$$= \det(\exp(D)). \quad -③$$

Here, D is a diagonal matrix.

Let $D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$

Then exp(D) will also be a diagonal matrix

as $\exp(D) = \begin{bmatrix} \exp(\lambda_1) & 0 & 0 & 0 \\ 0 & \exp(\lambda_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \exp(\lambda_3) \end{bmatrix}$

Since determinant(D) is product of diagonal values,

Then, $\det(\exp(D)) = \exp(\lambda_1).\exp(\lambda_2)\ldots\exp(\lambda_n)$
$$= \exp(\lambda_1 + \lambda_2 + \lambda_3 + \ldots \lambda_n)$$

$$\therefore \det(\exp(D)) = \exp(\text{trace}(D)) \quad -④$$

($\because$ trace(D) is sum of diagonal values).

Now, from ③ and ④, we have

$\det(\exp(A)) = \exp(\text{trace}(D))$ — ⑤

Since trace of a matrix is the sum of its eigen values, we can say

$\text{trace}(A) = \text{trace}(D)$ — ⑥

From ⑤ and ⑥,

$$\boxed{\det(\exp(A)) = \exp(\text{trace}(A))}$$

Hence, proved.

A. ~~(a)~~ Given $A_{n\times n}$ is a complex matrix.

(a) To prove: trace (A) is the sum of its eigen values.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & & a_{nn} \end{bmatrix}$

and let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the eigen values of A.

Here $\text{trace}(A) = a_{11} + a_{22} + \cdots a_{nn}$.

The eigen values are roots of the characteristic polynomial $p(\lambda) = |A - \lambda I_{n\times n}|$

$$p(\lambda) = |A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ a_{31} & \dots & \dots & \dots \\ a_{n1} & a_{n2} & & a_{nn}-\lambda \end{vmatrix}$$

Evaluating this determinant will give us the characteristic polynomial.
This can be written as:

$$p(\lambda) = a\lambda^n + b\lambda^{n-1} + \dots + c$$

(Let a, b be coefficients of λ^n and λ^{n-1} respectively and let 'c' be some constant).

$$a = \begin{cases} \cancel{1} & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

$$\therefore \boxed{a = (-1)^n}$$

Let us see what 'b' is
When we multiply the diagonal elements we get $(a_{11}-\lambda)(a_{22}-\lambda).(a_{33}-\lambda)..(a_{nn}-\lambda)$.

Multiplying these terms, we will get the coefficient of λ^{n-1} as

$$\boxed{b = \begin{cases} -(a_{11}+a_{22}+a_{33}+..a_{nn}) & \text{if } n \text{ is even} \\ (a_{11}+a_{22}+a_{33}+..a_{nn}) & \text{if } n \text{ is odd} \end{cases}}$$

To get c, let us take $\lambda = 0$. Then,

$$P(0) = |A| = c.$$

$$\therefore \boxed{c = |A|}$$

Hence, the characteristic polynomial

$$p(\lambda) = (-1)^n\lambda^n + (-1)^{n-1}(a_{11}+a_{22}+a_{33}+\ldots a_{nn})\lambda^{n-1} \ldots + |A|$$
$$= (\lambda_1-\lambda)(\lambda_2-\lambda)\ldots(\lambda_n-\lambda) \quad — (1)$$

where $\lambda_1, \lambda_2, \ldots \lambda_n$ are roots (eigen values) of the characteristic equation.

Let us compare the coefficients of λ^{n-1} in equation (1).

We know that for a polynomial of degree n, the sum of the roots is equal to the coefficient of λ^{n-1}.

So,
$$\boxed{(-1)^{n-1}(\lambda_1+\lambda_2+\lambda_3+\ldots\lambda_n) = (-1)^{n-1}a_{11}+a_{22}+a_{33}+\ldots a_{nn}}$$

$$\therefore \boxed{\lambda_1+\lambda_2+\ldots\lambda_n = trace(A)}.$$

Thus, the sum of Eigen values of $A_{n\times n}$ is equal to trace (A).

Hence, proved.

(B) To prove: Determinant (A) is the product of eigen values.

We know from proof (A) that

$$p(\lambda) = (-1)^n\lambda^n + (-1)^{n-1}(a_{11}+a_{22}+\ldots a_{nn})\lambda^{n-1}+\ldots+|A|$$
$$= (\lambda_1-\lambda)(\lambda_2-\lambda)\ldots(\lambda_n-\lambda).$$

We get the constant (|A|) when we multiply the roots of the polynomial equation. (i.e. product of roots = free coefficient)

$$\therefore \boxed{|A| = \lambda_1 . \lambda_2 . \lambda_3 . . . \lambda_n}$$

Determinant (A) is the product of eigen values. Hence, proved.

6. Given points in a 2D plane:

$$\left\{ \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$$

i. A covarience matrix is a square matrix giving the covarience between each pair of elements of a given random vector.

We know that

$$\text{covarience}(x, y) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$

Here

n	1	2	3
x	4	-2	-1
y	4	-3	-1

$$\text{Mean}(x) = \bar{x} = \frac{4-2-1}{3} = 1/3.$$

$\text{Mean}(Y) = \overline{Y} = \frac{4-3-1}{3} = 0.$

Now, to fill up the covarience matrix, we need to find cov(x,y), cov(y,x), cov(x,x) and cov(y,y).

$$cov(x,y) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$

Here, $n = 3$.

x_i	y_i	$x_i - \bar{x} = x_i - 1/3$	$y_i - \bar{y} = y_i - 0$	$(x_i - \bar{x})(y_i - \bar{y})$
4	4	$11/3$	4	$44/3$
-2	-3	$-7/3$	-3	$\cancel{21/3}$ 7
-1	-1	$-4/3$	-1	$4/3$
		$\cancel{\Sigma x_i - \bar{x}}$ 0	$\cancel{\Sigma y_i - \bar{y}} = 0$	$\Sigma = 23$

$cov(x,y) = \frac{1}{3}(23) = 23/3.$

Now,

$$cov(y,y) = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2$$

y_i	$y_i - \bar{y} = y_i - 0$	$(y_i - \bar{y})^2$
4	4	16
3	3	9
-1	-1	1
		$\Sigma = 26$

$cov(y,y) = \frac{1}{3} \times 26 = 26/3.$

$cov(y,x) = cov(x,y) = 23/3.$

$$cov(x,x) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$x_i - \bar{x}$	$(x_i - \bar{x})^2$
$11/3$	$121/9$
$-7/3$	$49/9$
$-4/3$	$16/9$

$\Sigma = 62/3.$

$$\therefore cov(x,x) = \frac{1}{3} \times \frac{62}{3} = 62/9.$$

So, we get the covarience matrix as:

$$\begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y. \end{array} & \begin{bmatrix} 62/9 & 23/3 \\ 23/3 & 26/3 \end{bmatrix} \end{array}$$

Let this covarience matrix be A.

$$A = \begin{bmatrix} 62/9 & 23/3 \\ 23/3 & 26/3 \end{bmatrix}$$

We find the eigen values of A by

$$|A - \lambda I| = 0.$$

$$\Rightarrow \begin{vmatrix} 62/9-\lambda & 23/3 \\ 23/3 & 26/3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{62}{9}-\lambda\right)\left(\frac{26}{3}-\lambda\right) - \left(\frac{23}{3}\right)^2 = 0$$

$$\Rightarrow \left(\frac{62}{9}-\lambda\right)\left(\frac{26}{3}-\lambda\right) - \frac{529}{9} = 0$$

$$\Rightarrow \left(\frac{62-9\lambda}{\cancel{9}}\right)\left(\frac{26-3\lambda}{3}\right) - \frac{529}{\cancel{9}} = 0$$

$$\Rightarrow \frac{(62-9\lambda)(26-3\lambda)}{3} = 529$$

$$\Rightarrow 1612 - 186\lambda - 234\lambda + 27\lambda^2 = 1587$$

$$\Rightarrow 27\lambda^2 - 420\lambda + 25 = 0.$$

The roots of the above equation are:

$$\lambda_1 = \frac{420+\sqrt{420^2-4\times27\times25}}{2\times27}, \quad \lambda_2 = \frac{420-\sqrt{420^2-4(27)(25)}}{2\times27}$$

$$\therefore \lambda_1 = 15.4958, \quad \lambda_2 = 0.0597$$

$$\lambda_1 = (70+5\sqrt{193})/9, \quad \lambda_2 = (70-5\sqrt{193})/9.$$

Finding Eigen vector for λ_1

$$A-\lambda_1 I = \begin{bmatrix} (-8+5\sqrt{193})/9 & 23/3 \\ 23/3 & (8-5\sqrt{193})/9 \end{bmatrix}$$

Let eigen vector be x_1. Then

$$Ax_1 = \lambda x_1$$

$$\Rightarrow (A-\lambda I)x_1 = 0$$

$$\begin{bmatrix} (-8-5\sqrt{193})/9 & 23/3 \\ 23/3 & (8-5\sqrt{193})/9 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving by Gaussian elimination, we get:

$$R_1 \rightarrow R_1 \times \frac{9}{-8-5\sqrt{193}}.$$

$$\left[\begin{array}{cc|c} 1 & \frac{-5\sqrt{193}+8}{69} & 0 \\ \frac{23}{3} & \frac{8-5\sqrt{193}}{9} & 0 \end{array}\right]$$

$$R_2 \rightarrow R_2 - 23/3\, R_1$$

$$\left[\begin{array}{cc|c} 1 & \frac{-5\sqrt{193}+8}{69} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

We get $a_1 + \left(\frac{-5\sqrt{193}+8}{69}\right) a_2 = 0$

$$\therefore \quad a_1 = \left(\frac{5\sqrt{193}-8}{69}\right) a_2.$$

$$\therefore \quad x_1 = \begin{bmatrix} (5\sqrt{193}-8)/69 \\ 1 \end{bmatrix} = a \begin{bmatrix} 0.89 \\ 1 \end{bmatrix}$$

$a \in R$

Finding Eigen vector for $\lambda_2 = \frac{70-5\sqrt{193}}{9}$.

$$(A-\lambda_2 I)X_2 = \begin{bmatrix} (-8+5\sqrt{193})/9 & 23/3 \\ 23/3 & (8+5\sqrt{193})/9 \end{bmatrix} X_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $X_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$R_1 \rightarrow R_1 \times \frac{9}{-8+5\sqrt{193}}$.

$$\left[\begin{array}{cc|c} 1 & (5\sqrt{193}+8)/69 & 0 \\ 23/3 & (5\sqrt{193}+8)/9 & 0 \end{array}\right]$$

$R_2 \rightarrow R_2 - 23/3\, R_1$.

$$\left[\begin{array}{cc|c} 1 & 5\sqrt{193}+8/69 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So, we get $b_1 + \frac{5\sqrt{193}+8}{69} b_2 = 0$

$\therefore b_1 = \frac{-5\sqrt{193}-8}{69} b_2$.

$$\therefore X_2 = \begin{bmatrix} \frac{-8-5\sqrt{193}}{69} \\ 1 \end{bmatrix}$$

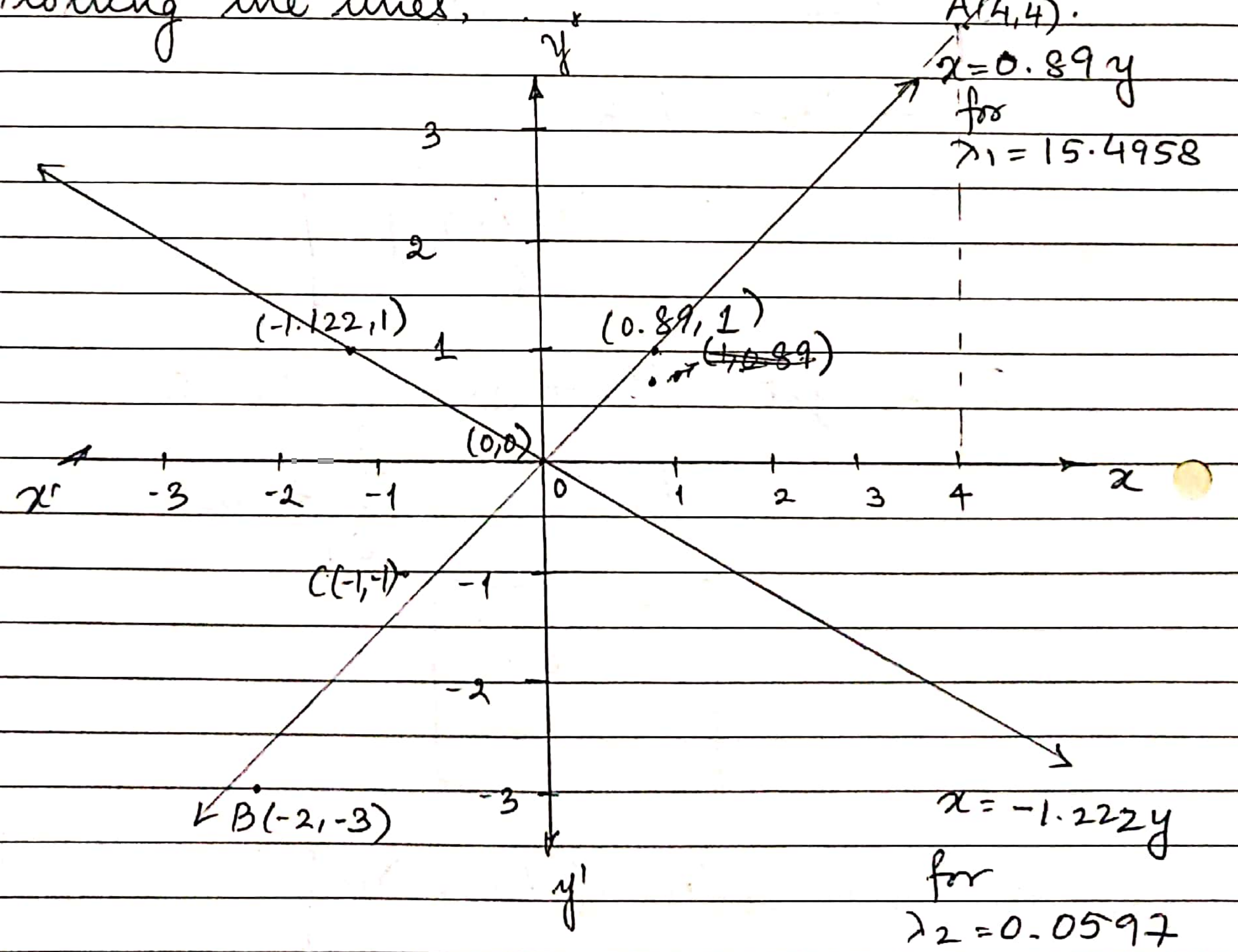
So, eigen vectors of A are $X_1 = \begin{bmatrix} (-8+5\sqrt{193})/69 \\ 1 \end{bmatrix}$

and $X_2 = \begin{bmatrix} (-8-5\sqrt{193})/69 \\ 1 \end{bmatrix} = b \begin{bmatrix} -1.122 \\ 1 \end{bmatrix}$, $b \in R$.

Plotting the points for:

Eigen value.	Line
$\lambda_1 = 15.4958$	$x = 0.89y$
$\lambda_2 = 0.0597$	$x = -1.122y$.

Plotting the lines,



Eigen vector for $\lambda_1 = 15.4958$ is aligned in a direction such that a line can best fit through pts.

8. Given $A = \begin{bmatrix} 15 & 0 & 6 \\ 0 & 15 & 3 \\ 6 & 3 & 27 \end{bmatrix}$

A is symmetric.

(a) Form $A = UDU'$

The spectrum of a matrix A is the set of Eigen values of A.

The spectral theorem for symmetric ~~matrices~~, $A_{n\times n}$ has following properties:

i) A has 'n' real eigen values (counting multiplicities)

ii) Dimension of the eigen space for λ = multiplicity of λ as root in the characteristic equation.

iii) Eigen spaces are mutually orthogonal

iv) A is orthogonally diagonalizable.

Now, let A be a symmetric matrix with orthogonal diagonalization

$$A = UDU'$$

with $U = [\vec{u}_1 \ldots \vec{u}_n]$.

Spectral Decomposition: Let

$$A = UDU' = UDU^T = [\vec{u}_1, \ldots \vec{u}_n] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \\ \vec{u}_n^T \end{bmatrix}$$

Here $\lambda_1, \lambda_2, \ldots \lambda_n$ are Eigen values of A.

$$A = [\lambda_1 \vec{u_1} \ldots \lambda_n \vec{u_n}] \begin{bmatrix} \vec{u_1}^T \\ \vdots \\ \vec{u_n}^T \end{bmatrix} = \lambda_1 \vec{u_1}\vec{u_1}^T + \ldots + \lambda_n \vec{u_n}\vec{u_n}^T.$$

Let us find Eigen values of A.

$$A = \begin{bmatrix} 15 & 0 & 6 \\ 0 & 15 & 3 \\ 6 & 3 & 27 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 15-\lambda & 0 & 6 \\ 0 & 15-\lambda & 3 \\ 6 & 3 & 27-\lambda \end{vmatrix} = 0$$

$\Rightarrow (15-\lambda)[(15-\lambda)(27-\lambda)] + 6(-6(15-\lambda)) = 0.$

$\Rightarrow (15-\lambda)(\lambda^2 - 42\lambda + 405) + 6(-90 + 6\lambda) = 0$

$\Rightarrow -\lambda^3 + 42\lambda^2 - 405\lambda + 15\lambda^2 - 630\lambda + 6075$
$- 540 + 36\lambda = 0$

$\Rightarrow -\lambda^3 + 57\lambda^2 - 990\lambda + 5400 = 0$

Solving this, we get

$\lambda_1 = 30$

$\lambda_2 = 15$

$\lambda_3 = 12.$

Let us find Eigen vectors for λ_1, λ_2 and λ_3

- Finding eigen vector for $\lambda_I = 30$

$$(A-\lambda_1 I)X_1 = \begin{bmatrix} -15 & 0 & 6 \\ 0 & -15 & 3 \\ 6 & 3 & -3 \end{bmatrix} x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $x_1 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} -15 & 0 & 6 & 0 \\ 0 & -15 & 3 & 0 \\ 6 & 3 & -3 & 0 \end{array}\right]$$

$R_1 \to R_1/-15$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2/5 & 0 \\ 0 & -15 & 3 & 0 \\ 6 & 3 & -3 & 0 \end{array}\right]$$

$R_3 \to R_3 - 6R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2/5 & 0 \\ 0 & -15 & 3 & 0 \\ 0 & 3 & -3/5 & 0 \end{array}\right]$$

$R_2 \to -R_2/15$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2/5 & 0 \\ 0 & 1 & -1/5 & 0 \\ 0 & 3 & -3/5 & 0 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2/5 & 0 \\ 0 & 1 & -1/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then we have $a_1 = 2/5\, a_3$

$a_2 = 1/5\, a_3$.

$\therefore\ x_1 = a \begin{bmatrix} 2/5 \\ 1/5 \\ 1 \end{bmatrix}, \quad a \in R.$

- Finding Eigen vector for $\lambda_2 = 15$

$$(A - \lambda_2 I)X_2 = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 3 \\ 6 & 3 & 12 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 6 & 0 \\ 0 & 0 & 3 & 0 \\ 6 & 3 & 12 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 6 & 3 & 12 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right]$$

$R_1 \rightarrow R_1/6$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right]$$

$R_3 \rightarrow R_3/6$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$R_3 \rightarrow R_3 - 3R_2$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \to R_1 - 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then, $b_1 = -1/2\, b_2$

$b_3 = 0.$

So, $X_2 = b \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \; b \in \mathbb{R}.$

- Finding Eigen vector for $\lambda_3 = 12.$

$$(A - \lambda_3 I)X_3 = \begin{bmatrix} 3 & 0 & 6 \\ 0 & 3 & 3 \\ 6 & 3 & 15 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 0 & 6 & 0 \\ 0 & 3 & 3 & 0 \\ 6 & 3 & 15 & 0 \end{array}\right]$$

$R_1 \to R_1/3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 6 & 3 & 15 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - 6R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right]$$

$R_2 \to R_2/3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c_1 + 2c_3 = 0 \Rightarrow c_1 = -2c_3$

$c_2 + c_3 = 0 \qquad c_2 = -c_3.$

$$x_3 = c\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \quad c \in \mathbb{R}.$$

Now, D is the diagonal matrix with diagonal elements as $\lambda_1 = 30$, $\lambda_2 = 15$, $\lambda_3 = 12$.

$$D = \begin{bmatrix} 30 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \frac{x_2}{\|x_2\|} & \frac{x_3}{\|x_3\|} \end{bmatrix}$$

$$\frac{x_1}{\|x_1\|} \quad U = \begin{bmatrix} 2/\sqrt{30} & -1/\sqrt{5} & -2/\sqrt{6} \\ 1/\sqrt{30} & 2/\sqrt{5} & -1/\sqrt{6} \\ 5/\sqrt{30} & 0 & 1/\sqrt{6} \end{bmatrix}$$

$$U' = U^{-1} = U^T = \begin{bmatrix} 2/\sqrt{30} & 1/\sqrt{30} & 5/\sqrt{30} \\ -1/\sqrt{5} & 2/\sqrt{5} & 0 \\ -2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$A = UDU'$ form is

$$\underbrace{\begin{bmatrix} 15 & 0 & 6 \\ 0 & 15 & 3 \\ 6 & 3 & 27 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2/\sqrt{30} & -1/\sqrt{5} & -2/\sqrt{6} \\ 1/\sqrt{30} & 2/\sqrt{5} & -1/\sqrt{6} \\ 5/\sqrt{30} & 0 & 1/\sqrt{6} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 30 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 12 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 2/\sqrt{30} & 1/\sqrt{30} & 5/\sqrt{30} \\ -1/\sqrt{5} & 2/\sqrt{5} & 0 \\ -2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}}_{U'}$$

(6) Form $A = \sum_{i=0}^{rank} \lambda_i u_i u_i^T$

where u_i is the ith Eigen vector (normalized) of the matrix.

$$\|\vec{x}_1\| = \sqrt{(2/5)^2 + (1/5)^2 + (1)^2} = \sqrt{30}/5$$

$$\|\vec{x}_2\| = \sqrt{(-1/2)^2 + (1)^2} = \sqrt{5}/2$$

$$\|\vec{x}_3\| = \sqrt{(-2)^2 + (-1)^2 + 1^2} = \sqrt{6}.$$

$$u_1 = \frac{5}{\sqrt{30}} \begin{bmatrix} 2/5 \\ 1/5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{30} \\ 1/\sqrt{30} \\ 5/\sqrt{30} \end{bmatrix} \quad \left(\because u_i = \frac{x_i}{\|\vec{x}_i\|}\right)$$

$$u_2 = \frac{2}{\sqrt{5}} \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}$$

$$u_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

u_1, u_2, u_3 are the orthogonal sets.

$$\therefore A = 30 \begin{bmatrix} 2/\sqrt{30} \\ 1/\sqrt{30} \\ 5/\sqrt{30} \end{bmatrix} \begin{bmatrix} 2/\sqrt{30} & 1/\sqrt{30} & 5/\sqrt{30} \end{bmatrix} + 15 \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} & 0 \end{bmatrix}$$
$$+ 12 \begin{bmatrix} -2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} -2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

7. Given $A = \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a) To prove: $R^3 \to R : (x, y) \to x^T A y$ gives a scalar product.

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Then $x^T A y$

$$= [x_1 \quad x_2 \quad x_3]_{1\times3} \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}_{3\times1}$$

$$= \left[x_1 + x_2/2 \quad x_1/2 + x_2 \quad x_3\right] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= \left[x_1 y_1 + x_2 y_1/2 + x_1 y_2/2 + x_2 y_2 + x_3 y_3\right]_{1\times1}$$

The result is a scalar product as we can see.

Hence $R^2 \to R : (x, y) \to x^T A y$ gives a scalar product.

(b) $\alpha : R^3 \to R$ such that $\alpha : (x_1, x_2, x_3) \to x_1 + x_2$
$v_1 = (1,1,1)$ $v_2 = (2,2,0)$ $v_3 = (1,0,0)$
are basis of R^3.

To find: orthonormal basis e_1, e_2, e_3 of $\mathbb{R}^3$
with $e_1 \in \text{span}(v_1)$, $e_2 \in (\text{kern}(\alpha))$

given e_1 is an orthonormal basis $\in \text{span}(v_1)$
since it is orthonormal,
let $e_1 \in E_1$.
such that $\vec{e_1} = \dfrac{\vec{v_1}}{\|\vec{v_1}\|} = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now, $\alpha : (x_1, x_2, x_3) \to x_1 + x_2$
The transformation matrix for α is $[1 \;\; 1 \;\; 0]$.

Then $\text{kern}(\alpha)$ is given by x such that
$$[1 \;\; 1 \;\; 0] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$
i.e. $c_1 + c_2 = 0$

The columns in the matrix that do not contain pivot elements can be taken by arbitrary values.

So, $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} c_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} c_3$.

$\therefore \text{kern}(\alpha) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ $e_2 \in \text{kern}(\alpha)$.

2. Given $A, B \in \mathbb{C}^{n\times n}$ such that $AB = BA$.
λ is an Eigen value of A.
V_λ is the subspace of all eigenvectors corresponding to the eigen value.

(a) To prove: $\exists v \in V_\lambda$ such that v is eigen vector of B.

Let us consider that v_1 ~~is~~ ~~an~~ be eigen vectors of A with eigen value λ.
Then let basis of V_λ be $(v_1, v_2, \dots v_k)$
Now, $Av_1 = \lambda v_1$.
Premultiplying ~~B~~ with B, we get

$$BAv_1 = B\lambda v_1$$
$$\Rightarrow A(Bv_1) = \lambda(Bv_1)$$

So, (Bv_1) is an eigen vector of A.
i.e. $Bv_1 \in V_\lambda$.
Since $v_1, v_2, \dots v_k$ are basis of V_λ,
$$Bv_1 = a_1v_1 + a_2v_2 + \dots a_kv_k$$
Now, $v_1, v_2, \dots v_k$ are linearly independent as they form the basis of V_λ.
Then
$$Bv_1 = a_1v_1 \quad \text{and} \quad a_2 = a_3 = \dots = a_k = 0$$
Since $Bv_1 = a_1v_1$, v_1 is eigen vector of B.
a_1 need not be equal to λ.
So, Eigen values need not be same.

(b) Spectrum of A is non-degenerate.
To prove: There exists a basis such that A and B are simultaneously diagonal in that basis.

Spectrum of A is non-degenerate so, A has 'n' distinct Eigen values.

Let A have an eigen value λ and v be the corresponding eigen vector

Then, $Av = \lambda v$.

Pre multiplying by B,

$$BAv = B\lambda v$$

$$\Rightarrow ABv = \lambda Bv \quad (\because AB = BA)$$

We proved that (Bv) is an eigen vector of A with eigen value λ.

As A has 'n' distinct eigen values, their multiplicity will become 1. So, all eigen spaces of A will be one-dimensional.

Then $Bv = \mu v$ for $\mu \in \mathbb{R}$.

$\therefore$ v is an eigen vector of B for eigen value μ.

This means that eigen vectors of A are also eigen vectors of B. — (1)

Since A has n distinct eigen values, A will be diagonalizable. — 2

From 1 and 2, A and B ~~can~~ be simultaneously diagonal.

So, there will exist a basis such that A and B are simultaneously diagonal in that basis.

3. Given A is a matrix such that $\lambda_i < 1$ where λ_i is its eigen value.

To prove: $\sum_{k=0}^{\infty} A^k = (I-A)^{-1}$

Proof:

Let $S = I + A + A^2 + \ldots A^k$. — (1)

Postmultiplying the above equation with A,

we get

$SA = IA + A^2 + A^3 + \ldots A^{k+1}$

$\Rightarrow SA = A + A^2 + A^3 + \ldots A^{k+1}$ — (2)

Subtracting equation (2) from (1) we get,

$S - SA = I - A^{k+1}$

$\Rightarrow S(I-A) = I - A^{k+1}$

Now, $\lim_{k\to\infty} S(I-A) = \lim_{k\to\infty} I - A^{k+1}$

$\Rightarrow S(I-A) = I - \lim_{k\to\infty} A^{k+1}$ — (3)

Now since $|A^{k+1}| \leq |A|^{k+1}$ and $|A| < 1$,

we have $\lim_{k\to\infty} A^{k+1} = 0$ — (4)

Substituting this value in (3), we have

$S(I-A) = I$ — (5)

This means that S is left inverse of $I-A$. — (5)

Let us ~~pre~~ multiply (1) with ~~A~~,

$$AS = A + A^2 + \dots + A^{k+1} \quad -(6)$$

Subtracting equation (6) from (1), we get

$$S - AS = I - A^{k+1}$$

$$S(I-A) = I \qquad (\because \text{ from } (4))$$

$\therefore$ S is right inverse of $I-A$! — (7)

From (5) and (7), S becomes both the left inverse and right inverse of $(I-A)$.

$$\therefore \quad S = \sum_{k=0}^{\infty} A^k = (I-A)^{-1}.$$

$$\therefore \quad \sum_{k=0}^{\infty} A^k = (I-A)^{-1}$$

Hence, proved.