Iterative methods: Fixed Point Iteration Method

Numerical Methods

October 1, 2021

Lecture Notes

Fixed Point Iteration Method

Algorithm

To find a root of a function given as $f(x) = 0$,

1. Given an equation $f(x) = 0$
2. Convert $f(x) = 0$ into the form $x = g(x)$
3. Let the initial guess be x_0
4. Repeat
$$x_{i+1} = g(x_i)$$
for a maximum iterations or if $|x_{i+1} = g(x_i)| < \epsilon$ where ϵ is some tolerance limit.

Conditions for Convergence

- If $g(x)$ and $g'(x)$ are continuous on a interval about their root.
- If $|g'(x)| < 1$ for all x in a interval, then the fixed point iterative process $x_{i+1} = g(x_i), i = 0, 1, 2, \ldots,$ will converge to the root $x = s$ for any initial approximations x_0 belonging to the interval.

Example:

Find the root of the expression $f(x) = x^3 + 4x^2 - 3x - 12 = 0$.
Converting given equation in the form of $x = g(x)$,

$$x = \frac{x^3 + 4x^2 - 12}{3} = g_1(x)$$
$$or, x = \sqrt{\frac{3x + 12 - x^3}{4}} = g_2(x)$$
$$or, x = \sqrt[3]{3x + 12 - 4x^2} = g_3(x)$$

Applying convergence test $|g'(x)| < 1$ on all above equation at $x = 1$

$$g_1'(x) = \frac{3x^2 + 8x}{3}$$
$$|g_1'(1)| = 3.667 \not< 1$$
and,
$$g_2'(x) = \frac{3 - 3x^2}{4\sqrt{-x^3 + 3x + 12}}$$
$$|g_2'(1)| = 0 < 1$$
and,
$$g_3'(x) = \frac{3 - 8x}{3(-4x^2 + 3x + 12)^{2/3}}$$
$$|g_3'(1)| = 0.3369 < 1$$

Here, $g_1(x)$ is divergent because $|g_1(x)| > 1$, and
$g_3(x)$ is divergent because $g_3(x)$ and $g_3'(x)$ are discontinuous around 1.

Thus, taking $x = g_(x) = \sqrt{\frac{3x+12-x^3}{4}}$ for iteration

Caution:*This function is continuous in the interval of* $[-2.9, 2.7]$*. Hence, the starting value should only be taken from this interval.*

i	x_{i+1}	$g(x_i)$
0	1	1.870828693
1	1.870828693	1.663173597
2	1.663173597	1.759896241
3	1.759896241	1.719656716
4	1.719656716	1.737352026
5	1.737352026	1.729743226
6	1.729743226	1.733047737
7	1.733047737	1.731618698
8	1.731618698	1.732237836
9	1.732237836	1.731969807
10	1.731969807	1.732085879
11	1.732085879	1.732035621
12	1.732035621	1.732057384
13	1.732057384	1.73204796

Table 1: Solution of$f(x) = x^3 + 4x^2 - 3x - 12$

From the table, the root of given equation is 1.73204.

Practice Exercises

- Find the root of the following expressions using Fixed Point Iteration method:
  1. $\cos(x) - x * e^x = 0$
  2. $f(x) = \sin(x) - 4e^{-2x}$
  3. $4\ln(x) + 0.2x^2$
  4. $x^5 + 6x^3 - 4x + 10$
  5. $x^4 - x - 10 = 0$