Solution Manuals

Date

1.1

1. a) $a_{ii} = a_{11} + a_{12} + a_{33} = 1+4+1 = 6$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

$= 1+1+1+0+16+4+0+1+1 = 25$ (scalar)

$a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{bmatrix}$ (matrix)

$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ (vector)

$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$

$= 1+0+2+0+0+0+0+0+4 = 7$ (scalar)

$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$ (matrix)

$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 1+0+4 = 5$ (Scalar)

b) $a_{ii} = a_{11} + a_{22} + a_{33} = 1+2+2 = 5$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

Date

$= 1+4+0+0+4+1+0+16+4 = 30$ (scalar)

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 2 \\ 0 & 8 & 4 \\ 0 & 16 & 8 \end{bmatrix} \text{(matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix} \text{ vector}$$

$= 4+4+0+0+2+1+0+4+2 = 17$ (scalar)

$$b_i b_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \text{(matrix)}$$

$b_i b_i = b_1b_1 + b_2b_2 + b_3b_3 = 4+1+1 = 6$ (scalar)

c) $a_{ii} = a_{11} + a_{22} + a_{33} = 1+0+4 = 5$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

$= 1+1+1+1+0+4+0+1+16 = 25$ (scalar)

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 7 \\ 1 & 3 & 9 \\ 1 & 4 & 18 \end{bmatrix} \text{(matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{(vector)}$$

$= 1+1+0+1+0+0+0+0+0 = 3$ (scalar)

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Date

$$b_i \cdot b_j = \begin{bmatrix} b_1 b_1 & b_1 b_2 & b_1 b_3 \\ b_2 b_1 & b_2 b_2 & b_2 b_3 \\ b_3 b_1 & b_3 b_2 & b_3 b_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (matrix)}$$

$$b_i b_i = b_1 b_1 + b_2 b_2 + b_3 b_3 = 1 + 1 + 0 = 2 \text{ (scalar)}$$

2)

a) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions

b) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}$$

Clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions.

c) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Clearly $a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate conditions

Date

3) $a_{ij} b_{ij} = -a_{ji} b_{ji} = -a_{ij} b_{ij} \Rightarrow 2a_{ij} b_{ij} = 0 \Rightarrow a_{ij} b_{ij} = 0$

From exercise 1-2(a): $a_{(ij)} a_{[ij]} = \frac{1}{4} tr\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right) = 0$

From Exercise 1-2(b): $a_{(ij)} a_{[ij]} = \frac{1}{4} tr\left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}^T\right) = 0$

From exercise 1-2(c): $a_{(ij)} a_{[ij]} = 1/4\, tr\left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right) = 0$

1)

$\delta_{ij} a_j = \delta_{i1} a_1 + \delta_{i2} + \delta_{i3} a_3 = \begin{bmatrix} \delta_{11} a_1 + \delta_{12} a_2 + \delta_{13} a_3 \\ \delta_{21} a_1 + \delta_{22} a_2 + \delta_{23} a_3 \\ \delta_{31} a_1 + \delta_{32} a_2 + \delta_{33} a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_i$

$\delta_{ij} a_{jk} = \Big[\delta_{11} a_{11} + \delta_{12} a_{21} + \delta_{13} a_{31} \quad \delta_{11} a_{12} + \delta_{12} a_{22} + \delta_{13} a_{32} \quad \delta_{11} a_{13} + \delta_{12} a_{23} + \delta_{13} a_{33}$

$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ij}$

5)

$$\det(a_{ij}) = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} = \varepsilon_{123} a_{11} a_{22} a_{33} + \varepsilon_{231} a_{12} a_{23} a_{31} + \varepsilon_{312} a_{13} a_{21} a_{32}$$

$$+ \varepsilon_{321} a_{13} a_{22} a_{31} + \varepsilon_{132} a_{11} a_{23} a_{32} + \varepsilon_{213} a_{12} a_{21} a_{33}$$

$$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33}$$

$$= a_{11}(a_{22} a_{33} - a_{23} a_{32}) - a_{12}(a_{21} a_{33} - a_{23} a_{31}) + a_{13}(a_{21} a_{32} - a_{22} a_{31})$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

6)

$45°$ rotation about x_1-axis $\Rightarrow Q_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

From Exercise 1-1(a): $b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$

$a'_{ij} = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & 4 & -1 \\ 0 & -2 & 1 \end{bmatrix}$

From exercise 1-1(b): $b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \\ 0 \end{bmatrix}$

$a'_{ij} = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$

Date

$$= \begin{bmatrix} 1 & \sqrt{2} & -\sqrt{2} \\ 0 & 4.5 & -1.5 \\ 0 & 1.5 & -0.5 \end{bmatrix}$$

From Exercise 1-1(c): $b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2}/2 & 3.5 & 2.5 \\ -\sqrt{2}/2 & 1.5 & 0.5 \end{bmatrix}$$

7)

$$Q_{ij} = \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) \\ \cos(x_2', x_1) & \cos(x_2', x_2) \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos(90^\circ - \theta) \\ \cos(90^\circ + \theta) & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$b_i' = Q_{ij} b_j = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 \cos\theta + b_2 \sin\theta \\ -b_1 \sin\theta + b_2 \cos\theta \end{bmatrix}^T$$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$$

$$\begin{bmatrix} a_{11}\cos^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\sin^2\theta & a_{12}\cos^2\theta - (a_{11}-a_{22}) \\ a_{21}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{12}\sin^2\theta & a_{11}\sin^2\theta - (a_{12}-a_{21}) \end{bmatrix}$$

$$\begin{matrix} \sin\theta\cos\theta - a_{21}\sin^2\theta \\ \sin\theta\cos\theta + a_{22}\cos^2\theta \end{matrix}\Bigg]$$

8)

$$a'\delta'_{ij} = Q_{ip}Q_{jq}\,a\delta_{pq} = a\,Q_{ip}Q_{jp} = a\delta_{ij}$$

9)

$$a'\delta'_{ij}\delta'_{kl} + \beta'\delta'_{ik}\delta'_{jk} = Q_{im}Q_{jn}Q_{kp}Q_{lq}\left(a\delta_{mn}\delta_{pq} + \beta\delta_{mp}\delta_{nq} + \gamma\delta_{mq}\delta_{np}\right)$$

$$= a\,Q_{im}Q_{jm}Q_{kp}Q_{lp} + \beta\,Q_{lm}Q_{jn}Q_{km}Q_{ln} + \gamma\,Q_{im}Q_{jn}Q_{kn}Q_{lm}$$

$$= a\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{j}$$

10)

$$C_{ijkl} = a\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} = a\delta_{ij}\delta_{kl} + \beta(\delta_{ij}\delta_{kl}$$

$$+\beta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$= a\delta_{kl}\delta_{ij} + \beta(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}) = C_{klij}$$

11)

If $a = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$$I_a = a_{ii} = \lambda_1 + \lambda_2 + \lambda_3$$

$$II_a = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} + \begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

$$III_a = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{vmatrix} = \lambda_1\lambda_2\lambda_3$$

Date

12) $a - a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow I_a = -1 \ , II_a = -2 \ , III_a = 0$

Roots $\Rightarrow \lambda_1 = -2, \ \lambda_2 = 0, \ \lambda_3 = 1$

$\lambda_1 = -2$ Case.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0 \Rightarrow \begin{array}{l} n_1^{(1)} + n_2^{(1)} = 0 \\ n_3^{(1)} = 0 \\ n_1^{(1)2} + n_2^{(2)} + n_3^{(1)^2} = 1 \end{array} \Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2, \ n^{(1)} = \pm(\sqrt{2}/2)(-1,1,0)$$

$\lambda_2 = 0$ Case:

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} -n_1^{(2)} + n_2^{(2)} = 0 \\ n_3^{(2)} = 0 \\ n_1^{(2)^2} + n_2^{(2)^2} + n_3^{(2)^2} = 1 \end{array} \Rightarrow n_1 = n_2 = \pm\sqrt{2}/2 = n^{(2)} = \pm(\sqrt{2}/2)(1,1,0)$$

$\lambda_3 = 1$ Case:

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} -2n_1^{(3)} + n_2^{(3)} = 0 \\ n_1^{(3)} - 2n_2^{(3)} = 0 \\ n_1^{(3)^2} + n_2^{(3)^2} + n_3^{(3)^2} = 1 \end{array} \Rightarrow n_1 = n_2 = 0, \ n_3^{(3)} = 1 = n^{(3)} = \pm(0,0,1)$$

The rotation matrix is given by

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \text{ and}$$

Date

$$a'_{ij} = \ell_{ip}\ell_{jp}a_{pq} = \frac{1}{2}\begin{bmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 2/\sqrt{2}\end{bmatrix}\begin{bmatrix}-1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 2/\sqrt{2}\end{bmatrix}^T = \begin{bmatrix}-2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

12 (b)

$$a_{ij} = \begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 0\\ 0 & 0 & 0\end{bmatrix} \Rightarrow I_a = -4,\ II_a = 3,\ III_a = 0$$

Roots $\Rightarrow \lambda_1 = -3,\ \lambda_2 = -1,\ \lambda_3 = 0$

$\lambda_1 = -3$ Case:

$$\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 3\end{bmatrix}\begin{bmatrix}n_1^{(1)}\\ n_2^{(1)}\\ n_3^{(1)}\end{bmatrix} = 0 \Rightarrow \begin{matrix}n_1^{(1)} + n_2^{(1)} = 0\\ n_3^{(1)} = 0\\ n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1\end{matrix} \Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2,\ n^{(1)} = \pm(\sqrt{2}/2)(-1,1,0)$$

$\lambda_2 = -1$ Case:

$$\begin{bmatrix}-1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}n_1\\ n_2\\ n_3\end{bmatrix} = 0 \Rightarrow \begin{matrix}-n_1^{(2)} + n_2^{(2)} = 0\\ n_3^{(2)} = 0\\ n_1^{(2)2} + n_2^{(2)2} + n_3^{(2)2} = 1\end{matrix} \Rightarrow n_1 = n_2 \pm\sqrt{2}/2 \Rightarrow n^{(2)} = \pm(\sqrt{2}/2)(1,1,0)$$

$\lambda_3 = 0$ Case:

$$\begin{bmatrix}-2 & 1 & 0\\ 1 & -2 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}n_1\\ n_2\\ n_3\end{bmatrix} = 0 \Rightarrow \begin{matrix}-2n_1^{(3)} + n_2^{(3)} = 0\\ n_1^{(3)} - 2n_2^{(3)} = 0\\ n_1^{(3)2} + n_2^{(3)2} + n_3^{(3)2} = 1\end{matrix} \Rightarrow n_1 = n_2 = 0;$$

$n_3^{(3)} = 1 \Rightarrow n^{(3)} = \pm(0,0,1)$

Date

The rotation matrix is given by (

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$ and

$$a'_{ij} = Q_{ip}Q_{jq}a_{pq} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

12 (c)

$$a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow I_a = -2, \; II_a = 0, \; III_a = 0$$

Roots $\Rightarrow \lambda = -2, \quad \lambda_2 = \lambda_3 = 0$

$\lambda_1 = -2$ Case

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0 \Rightarrow \begin{matrix} n_1^{(1)} + n_2^{(1)} = 0 \\ n_3^{(1)} = 0 \\ n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1 \end{matrix} \Rightarrow n_1^{(1)} = -n_2^{(1)}$$

$= \pm\sqrt{2}/2, \; n^{(1)} = \pm\sqrt{2}/2$

$(-1, 1, 0)$

$\lambda_2 = \lambda_3 = 0$ Case:

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} -n_1 + n_2 = 0 \\ n_1^2 + n_2^{(1)2} + n_3^{(1)2} = 1 \end{matrix} \Rightarrow n_1 = n_2, \; n_3^2$$

$= 1 - 2n_1^2 \Rightarrow n = \pm(k, k, \sqrt{1-2k^2})$

Date

For arbitary k and thus directions are not uniquely determined. For convenience we may choose

$k = \sqrt{2}/2$ and 0 to get $n^{(2)} = \pm \sqrt{2}/2\,(1,1,0)$ and
$n^{3} = \pm (0,0,1)$

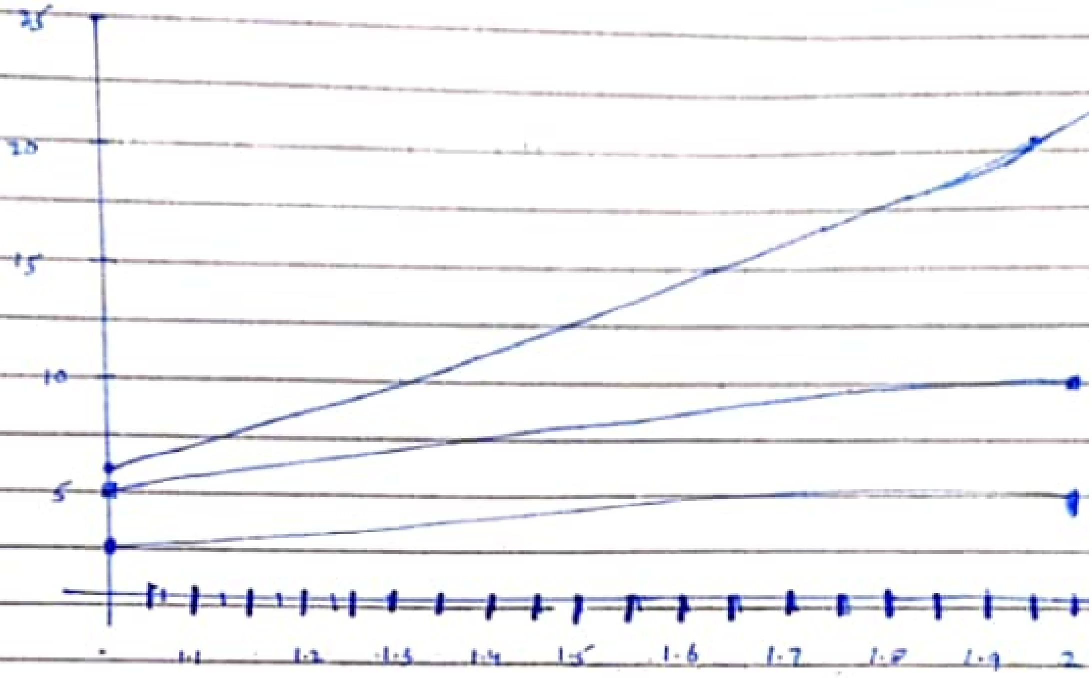
The rotation matrix is given by

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \text{ and}$$

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = 1/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Question 1.13 :



Q 1-14:

a) ~~$u = x_1 e_1 + x_1 x_2 e_2 + 2x_1 x_2 x_3 e_3$~~

$$\nabla \cdot u = u_{1,1} + u_{2,2} + u_{3,3} = 1 + x_1 + 2x_1 x_2$$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1 & x_1 x_2 & 2x_1 x_2 x_3 \end{vmatrix} = 2x_1 x_3 e_1 - 2x_2 x_3 e_2 + x_2 e_3$$

$$\nabla^2 u = 0e_1 + 0e_2 + 0e_3 = 0$$

$$\nabla u = \begin{bmatrix} 1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 2x_2 x_3 & 2x_1 x_3 & 2x_1 x_2 \end{bmatrix}, \quad tr(\nabla u) = 1 + x_1 + 2x_1 x_2$$

$$\nabla^2 U = 0e_1 + 0e_2 + 0e_3 = 0$$

$$\nabla U = \begin{bmatrix} 1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 2x_2x_3 & 2x_1x_3 & 2x_1x_2 \end{bmatrix}, \quad tr(\nabla U) = 1 + x_1 + 2x_1x_2$$

(b)

$$U = x_1^2 e_1 + 2x_1x_2 e_2 + x_3^3 e_3$$

$$\nabla . U = U_{1,1} + U_{2,2} + U_{3,3} = 2x_1 + 2x_1 + 3x_3^2$$

$$\nabla^2 U = 2e_1 + 0e_2 + 6x_3 e_3 = 0$$

$$\nabla U = \begin{bmatrix} 2x_1 & 0 & 0 \\ 2x_2 & 2x_1 & 0 \\ 0 & 0 & 3x_3^2 \end{bmatrix}, \quad tr(\nabla . U) = 4x_1 + 3x_3^2$$

(c)

$$U = x_2^2 e_1 + 2x_2x_3 e_2 + 4x_1^2 e_3$$

$$\nabla . U = U_{1,1} + U_{2,2} + U_{3,3} = 0 + 2x_3 + 0$$

$$\nabla \times U = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_2^2 & 2x_2x_3 & 4x_1^2 \end{vmatrix} = -2x_2 e_1 - 8x_1 e_2 - 2x_2 e_3$$

$$\nabla^2 U = 2e_1 + 0e_2 + 8e_3 = 0$$

$$\nabla U = \begin{bmatrix} 0 & 2x_2 & 0 \\ 0 & 2x_3 & 2x_2 \\ 8x_1 & 0 & 0 \end{bmatrix}, \quad tr(\nabla U) = 3x_3$$

Question 1.15 :

$$a_i = -\frac{1}{2}\varepsilon_{ijk}\, a_{jk}$$

$$\varepsilon_{imn}\, a_i = -\frac{1}{2}\varepsilon_{ijk}\,\varepsilon_{imn}\, a_{jk} = -\frac{1}{2}\begin{vmatrix}\delta_{ii} & \delta_{im} & \delta_{in}\\ \delta_{ji} & \delta_{jm} & \delta_{jn}\\ \delta_{ki} & \delta_{km} & \delta_{kn}\end{vmatrix} a_{jk}$$

$$= -\frac{1}{2}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) a_{jk}$$

$$= -\frac{1}{2}\left(a_{mn} - a_{nm}\right) = -\frac{1}{2}\left(a_{mn} + a_{mn}\right) = -a_{mn}$$

$$a_{jk} = -\varepsilon_{ijk}\, a_i$$

⟵ x ⟶

Question 1.16 :

(a)

$$\nabla(\phi\psi) = (\phi\psi)_{,k} + \phi_{,k}\psi = \nabla\phi\psi + \phi\nabla\psi$$

$$\nabla^2(\phi\psi) = (\phi\psi)_{,kk} = (\phi\psi_{,k} + \phi_{,k}\psi)_{,k}$$

$$= \phi\psi_{,kk} + \phi_{,k}\psi_{,k} + \phi_{,k}\psi_{,k} + \phi_{,kk}\psi$$

$$= \phi_{,kk}\psi + \phi\psi_{,kk} + 2\phi_{,k}\psi_{,k}$$

$$= (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\nabla\phi\cdot\nabla\psi$$

$$\nabla\cdot(\phi u) = (\phi u_k)_{,k} = \phi u_{k,k} + \phi_{,k} u_k = \nabla\phi\cdot u + \phi(\nabla\cdot u)$$

(b)

$$\nabla\times(\phi u) = \varepsilon_{ijk}(\phi u_k)_{,j} = \varepsilon_{ijk}(\phi u_{k,j} + \phi_{,j} u_k)$$

$$= \varepsilon_{ijk}\phi_{,j} u_k + \phi\varepsilon_{ijk} u_k = \nabla\phi\times u + \phi(\nabla\times u)$$

$$\nabla\cdot(u\times v) = (\varepsilon_{ijk} u_j v_k)_{,i} = \varepsilon_{ijk}(u_j v_{k,i} + u_{j,i} v_k)$$

$$= v_k\varepsilon_{ijk} u_{j,i} = v\cdot(\nabla\times u) - u\cdot(\nabla\times v)$$

$$\nabla\times\nabla\phi = \varepsilon_{ijk}(\phi_{,k})_{,j} = \varepsilon_{ijk}\phi_{,kj} = 0$$ ∵ Because of symmetry & antisymmetry in jk

$$\nabla\cdot\nabla\phi = (\phi_{,k})_{,k} = \phi_{,kk} = \nabla^2\phi$$

(c)

$$\nabla\cdot(\nabla\times u) = (\varepsilon_{ijk} u_{k,j})_{,i} = \varepsilon_{ijk} u_{k,ji} = 0$$

$$\nabla\times(\nabla\times u) = \varepsilon_{mni}(\varepsilon_{ijk} u_{k,j})_{,n} = \varepsilon_{imn}\varepsilon_{ijk} u_{k,jn}$$

$$= (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}) u_{k,jn} = u_{n,nm} - u_{m,nn}$$

$$= \nabla(\nabla\cdot u) - \nabla^2 u$$

$$u\times(\nabla\times u) = \varepsilon_{ijk} u_j(\varepsilon_{kmn} u_{n,m}) = \varepsilon_{kij}\varepsilon_{kmn} u_j u_{n,m}$$

$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) u_j u_{n,m} = u_n u_{n,i} - u_m u_{i,m}$$

$$= \frac{1}{2}\nabla(u\cdot u) - u\cdot\nabla u$$

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Question 1.17

Cylindrical Coordinates : $\xi^1 = r$, $\xi^2 = \theta$, $\xi^3 = z$

$$(ds)^2 = (dr)^2 + (r\,d\theta)^2 + (dz)^2 \Rightarrow h_1 = 1,\ h_2 = r,\ h_3 = 1$$

$$\hat{e}_r = \cos\theta\, e_1 + \sin\theta\, e_2,\quad \hat{e}_\theta = -\sin\theta\, e_1 + \cos\theta\, e_2,\quad \hat{e}_z = e_3$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta,\quad \frac{\partial \hat{e}_r}{\partial r} = \frac{\partial \hat{e}_z}{\partial r} = \frac{\partial \hat{e}_z}{\partial \theta} = \frac{\partial \hat{e}_z}{\partial z} = 0$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\nabla f = \hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \hat{e}_z \frac{\partial f}{\partial z}$$

$$\nabla \cdot U = \frac{1}{r} \frac{\partial}{\partial r}(r u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla \times U = \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right)\hat{e}_z$$

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Question 18:

Spherical coordinates : $\xi^1 = R,\ \xi^2 = \phi,\ \xi^3 = \theta$

$x^1 = \xi^1 \sin\xi^2 \cos\xi^3,\quad x^2 = \xi^1 \sin\xi^2 \sin\xi^3$

$x^3 = \xi^1 \cos\xi^2$

<u>Scale factors</u> :

$$(h_1)^2 = \frac{\partial x^K}{\partial \xi^1}\frac{\partial x^K}{\partial \xi^1} = (\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + \cos^2\phi$$

$$= 1 \quad \Rightarrow \boxed{h_1 = 1}$$

$$(h_2)^2 = \frac{\partial x^K}{\partial \xi^2}\frac{\partial x^K}{\partial \xi^2} = R^2 \Rightarrow h_2 = R$$

$$(h_3)^2 = \frac{\partial x^K}{\partial \xi^3}\frac{\partial x^K}{\partial \xi^3} = R^2\sin\phi \Rightarrow h_3 = R\sin\phi$$

<u>Unit vectors</u>

$\vec{e}_R = \cos\theta\sin\phi\, e_1 + \sin\theta\sin\phi\, e_2 + \cos\phi\, e_3$

$\hat{e}_\phi = \cos\theta\cos\phi\, e_1 + \sin\theta\cos\phi\, e_2 - \sin\phi\, e_3$

$\vec{e}_\theta = -\sin\theta\, e_1 + \cos\theta\, e_2$

$$\frac{\partial \hat{e}_R}{\partial R} = 0 \quad , \quad \frac{\partial \hat{e}_R}{\partial \phi} = \hat{e}_\phi \quad , \quad \frac{\partial \hat{e}_R}{\partial \theta} = \sin\phi\, \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\phi}{\partial R} = 0 \quad , \quad \frac{\partial \hat{e}_\phi}{\partial \phi} = -\hat{e}_r \quad , \quad \frac{\partial \hat{e}_\phi}{\partial \theta} = \cos\phi\, \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\theta}{\partial R} = 0 \quad , \quad \frac{\partial \hat{e}_\theta}{\partial \phi} = 0 \quad , \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\cos\phi\, \hat{e}_\phi$$

$\underline{f}$

$$\nabla = \hat{e}_R \frac{\partial}{\partial R} + \hat{e}_\phi \frac{1}{R} \frac{\partial}{\partial \phi} + \hat{e}_\theta \frac{1}{R\sin\phi} \frac{\partial}{\partial \theta}$$

$$\nabla f = \hat{e}_R \frac{\partial f}{\partial R} + \hat{e}_\phi \frac{1}{R} \frac{\partial f}{\partial \phi} + \hat{e}_z \frac{1}{R\sin\phi} \frac{\partial f}{\partial \theta}$$

$$\nabla \cdot U = \frac{1}{R^2 \sin\phi} \frac{\partial}{\partial R}\left(R^2 \sin\phi\, U_R\right) + \frac{1}{R^2 \sin\phi} \frac{\partial}{\partial \phi}\left(R\sin\phi\, U_\phi\right)$$

$$+ \frac{1}{R^2 \sin\phi} \frac{\partial}{\partial \theta}\left(R\, U_\theta\right)$$

$$= \frac{1}{R^2} \frac{\partial}{\partial R}\left(R^2 U_R\right) + \frac{1}{R\sin\phi} \frac{\partial}{\partial \phi}\left(\sin\phi\, U_\phi\right) + \frac{1}{R\sin\phi} \frac{\partial}{\partial \theta}($$

$$\nabla^2 f = \frac{1}{R^2 \sin\phi}\frac{\partial}{\partial R}\left(R^2 \sin\phi \frac{\partial f}{\partial R}\right) + \frac{1}{R^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right)$$

$$+ \frac{1}{R^2 \sin\phi}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\phi}\frac{\partial f}{\partial \theta}\right)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial f}{\partial R}\right) + \frac{1}{R^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right)$$

$$+ \frac{1}{R^2 \sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$$

$$\nabla \times U = \left(\frac{1}{R^2 \sin\phi}\left[\frac{\partial}{\partial \phi}(R \sin\phi\, U_\theta) - \frac{\partial}{\partial \theta}(R U_\phi)\right]\right)\hat{e}_R$$

$$+ \left(\frac{1}{R \sin\phi}\left[\frac{\partial}{\partial \theta}(U_R) - \frac{\partial}{\partial R}(R \sin\phi\, U_\theta)\right]\right)\hat{e}_\phi$$

$$+ \left(\frac{1}{R}\frac{\partial}{\partial R}\left[(R U_\phi - \frac{\partial}{\partial \phi}(U_R)\right)\hat{e}_\theta$$

$$= \left[\frac{1}{R \sin\phi}\left(\frac{\partial}{\partial \phi}(\sin\phi U_\theta) - \frac{\partial U_\phi}{\partial \theta}\right)\right]\hat{e}_R + \left[\frac{1}{R \sin\phi}\frac{\partial U_R}{\partial \theta} -\right.$$

$$\left.\frac{1}{R}\frac{\partial}{\partial R}(R U_\theta)\right]\hat{e}_\phi + \left[\frac{1}{R}\left(\frac{\partial}{\partial R}(R U_\phi) - \frac{\partial U_R}{}\right)\right]\hat{e}$$

$$\nabla^2 f = \frac{1}{R^2 \sin\phi}\frac{\partial}{\partial R}\left(R^2 \sin\phi \frac{\partial f}{\partial R}\right) + \frac{1}{R^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right)$$

$$+ \frac{1}{R^2 \sin\phi}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\phi}\frac{\partial f}{\partial \theta}\right)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial f}{\partial R}\right) + \frac{1}{R^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right)$$

$$+ \frac{1}{R^2 \sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$$

$$\nabla \times U = \left(\frac{1}{R^2 \sin\phi}\left[\frac{\partial}{\partial \phi}(R \sin\phi\, U_\theta) - \frac{\partial}{\partial \theta}(R U_\phi)\right]\right)\vec{e}_R$$

$$+ \left(\frac{1}{R \sin\phi}\left[\frac{\partial}{\partial \theta}(U_R) - \frac{\partial}{\partial R}(R \sin\phi\, U_\theta)\right]\right)\vec{e}_\phi$$

$$+ \left(\frac{1}{R}\frac{\partial}{\partial R}\left[(R U_\phi - \frac{\partial}{\partial \phi}(U_R)\right)\vec{e}_\theta$$

$$= \left[\frac{1}{R \sin\phi}\left(\frac{\partial}{\partial \phi}(\sin\phi\, U_\theta) - \frac{\partial U_\phi}{\partial \theta}\right)\right]\vec{e}_R + \left[\frac{1}{R \sin\phi}\frac{\partial U_R}{\partial \theta} -\right.$$

$$\left.\frac{1}{R}\frac{\partial}{\partial R}(R U_\theta)\right]\vec{e}_\phi + \left[\frac{1}{R}\left(\frac{\partial}{\partial R}(R U_\phi) - \frac{\partial U_R}{\partial \phi}\right)\right]\vec{e}_\theta$$