SIMPLE CURRENTS, COSET CONSTRUCTION FOR THE N=2 MINIMAL MODELS AND GEPNER MODELS

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Department of Physics
ETH Zurich

by
Yiming Alan Li

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Abstract

We show how the existence of primary fields having unique fusion with any other field in a rational conformal field theory implies other non-diagonal modular invariant partition functions by applying a generalized orbifold construction. After a brief account on the coset construction of N=2 superconformal minimal models, we indicate how a supersymmetric compactification of the heterotic string on a Calabi-Yau manifold can be described by a tensor product of these $\mathcal{N} = 2$ minimal models. These theories are exactly solvable and the theory of simple currents allows us to determine a modular invariant partition in an elegant way.

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1 Introduction

The study of two-dimensional conformal field theory finds application in both statistical mechanics, particularly in critical phenomena in two-dimensional second-order phase transitions, and in string theories. A central objective is to determine a modular invariant partition function which arises when studying these conformal field theories on higher-genus Riemann surfaces. The Verlinde formula, one of the deepest results in 2D CFT, states a relationships between the modular properties of the theory and the underlying fusion algebra of the primary fields. Many rational conformal field theories have primary fields with unique fusion rules with all other primary fields. Just this simple property gives rise to a new discrete symmetry in the theory and is enough to state the existence of modular invariant partition functions other than the diagonal-invariant and to construct them following an orbifold-like procedure, meaning performing GSO-like projections and then summing over twisted sectors.
This technique finds applications in space-time supersymmetric string compactifications, which have received a considerable amount of attention in the last forty years. In compactifications from ten-dimensional to four-dimensional space-time, the internal sector of the theory is usually described on Calabi-Yau manifolds, which are known to preserve supersymmetry in four dimensions. Solving the non-linear sigma model on these highly curved backgrounds can get quite involved, however, more evidence has been collected that these highly complex geometries can be solved exactly by a tensor product of $\mathcal{N} = 2$ superconformal minimal models. In particular, the tensor product CFT has the same discrete symmetry group as certain Calabi-Yau manifolds defined as hypersurfaces in complex projective space. These CFTs describing the worldsheet of string propagation on the supersymmetric internal sector are known as Gepner models. The simple currents found in Gepner models have the feature that they generate space-time supersymmetry and it is therefore possible to write down a space-time supersymmetric, modular invariant partition function and to analyze its spectrum. Thus, they are fully suited for realistic phenomenology.
This paper is organized as follows; in sect. 2 we present all the necessary tools of the theory of simple currents for Gepner models. For a more careful analysis and an actual algebraic proof of modular invariance of the newly constructed partition functions, we recommend the papers in the references. Sect. 3 introduces the $\mathcal{N} = 2$ superconformal minimal models via a coset construction of current algebras. The spectral flow operator is identified to be one of the simple currents in the theory. Sect. 4 presents Gepner models and space-time supersymmetric string compactifications. Two projections, namely worldsheet supersymmetry and actual space-time supersymmetry, can be achieved elegantly by carefully designed simple supercurrents. The last result will be an explicit expression for a partition function of a heterotic string in supersymmetric compactified space-time.

2 Simple currents

In the study of modular invariant partition functions for a conformal field theory defined on a torus it is useful to define the characters of the representations on the Verma module generated by a primary field ϕ_i as

$$\chi_i(\tau) = Tr(e^{2\pi i\tau(L_0 - c/24)}) \tag{1}$$

with τ being the modular parameter of the torus. L_0 is the zero-Laurent mode of the underlying Virasoro algebra and c is the central charge of the conformal field theory. The trace is taken over all non-null states created by the generators of the algebra (e.g. Virasoro algebra) acting on a ground state. The characters transform under modular T and modular S transformations as follows:

$$\begin{aligned} \chi_i(\tau+1) &= T_{ij}\chi_j \quad \text{(modular transformation T)} \\ \chi_i(-\frac{1}{\tau}) &= S_{ij}\chi_j \quad \text{(modular transformation S)} \end{aligned} \tag{2}$$

The matrices T and S must form a representation of the modular group, i.e. $S^2 = (ST)^3 = C$, where C is the charge conjugation matrix.

In general terms, a partition function of a conformal field theory can be expressed in terms of the characters of the chiral algebra as

$$Z(\tau, \bar{\tau}) = \sum_{i,j} \chi_i(\tau) M_{ij} \bar{\chi}_j(\bar{\tau}) \tag{3}$$

where the indices i, j run over the primary fields associated with the A and $\bar{A}$ chiral algebra respectively. Here M is a matrix with non-negative integer entries and it describes how the characters of the left and right moving sectors mix with each other. For the vacuum representation to be unique, the matrix M is subject to an additional constraint which is $M_{00} = 1$. Modular invariance of the partition function requires the T and S matrix in (44) to fulfill the following conditions:

$$[S.M] = [T, M] = 0 \tag{4}$$

One easy solution to this is the diagonal invariant $M_{ij} = \delta_{ij}$ which corresponds to the mixing of representations of the same type in the left and right moving sectors. The problem can now be formulated as follows: Find all M which satisfy above equation.

This is a non-trivial problem in CFT and one can easily convince themselves that for larger rational conformal field theories, (i.e. more primary fields in the theory), one can not find all solutions merely through brute force. One method of finding new, non-diagonal modular invariant partition functions is through the use of simple currents. Although this method unfortunately does not complete the classifications of modular invariants of for example Kač-Moody algebras, it still has the advantage of not being restricted to the algebra itself.

Consider the fusion algebra of a rational conformal field theory:

$$[\phi_i] \times [\phi_j] = \sum_k N_{ij}^k \ [\phi_k] \tag{5}$$

Suppose there exists a primary field J with the following property:

$$J \times \phi = \phi' \tag{6}$$

with only one term appearing on the right-hand side for any primary field ϕ in the theory. This primary field J is called a *simple current*. Just this single property allows us to make a number of observations. The conjugate J^c is defined by the fusion rule $J \times J^c = 1$ Consider now the fusion of J with itself: $J \times J = J^2$. $J^2 = J$ is not possible since applying the fusion of J^c on both sides would result in $J = 1$ which is not true in the non-trivial case. Therefore, $J^2 = 1$ or J^2 is another new primary field different from J. Due to the associativity of the fusion algebra it follows that J^2 is also a simple current. One can now act with J on J^2 and show that the result is either the identity, or a new simple current J^3. Finally, because rational conformal field theories have a finite number of primary fields, we conclude that the action of J on itself generates an orbit consisting of a finite amount of different simple currents, i.e. $\exists N \in \mathbb{Z}$ such that

$$J^N = 1 \tag{7}$$

Similarly, the action of J divides the set of primary fields into orbits. The fusion rules for the set of fields J^n, $n = 0, 1, ..., N-1$ generate a group isomorphic to Z_N, i.e. it is discrete and abelian. The conserved charge associated to this discrete symmetry is the monodromy charge which is best understood when looking at the operator product expansion between the simple current J and a primary field ϕ

$$J(z)\phi(w) \propto \frac{1}{(z-w)^{h_J + h_\phi - h_{J\times\phi}}}(J \times \phi)(w) \tag{8}$$

The monodromy charge Q is defined as

$$Q = h_J + h_\phi - h_{J\times\phi} \mod 1 \tag{9}$$

Moving J(z) around $\phi(w)$ counterclockwise, or in mathematical terms, sending (z-w) to $e^{2\pi i}(z-w)$, the fusion of J and ϕ acquires a factor $e^{-2\pi i Q(\phi)}$. Therefore, the monodromy charge can be seen as the phase that is being picked up due to the monodromy of the current J on the primary field ϕ and due to the unique fusion of J with any field, this phase is unique for each field.

To see now that this charge is indeed conserved, consider the general operator product between two fields in the theory:

$$\phi_i(z)\phi_j(w) = \sum_k C_{ijk}(z-w)^{h_k - h_i - h_j}\phi_k(w) + \text{descendants} \tag{10}$$

Taking the monodromy of J with respect to both sides and using the fact that J has unique monodromy with all fields, we can conclude that $Q(\phi_i) + Q(\phi_j) = Q(\phi_k)$ for all fields ϕ_k that appear on the right-hand side of the equation (10). This can also be illustrated like in Figure 1. With the rules of complex analysis, we can deform the contour of the operator product between ϕ_i and ϕ_j as depicted on the right-hand side.

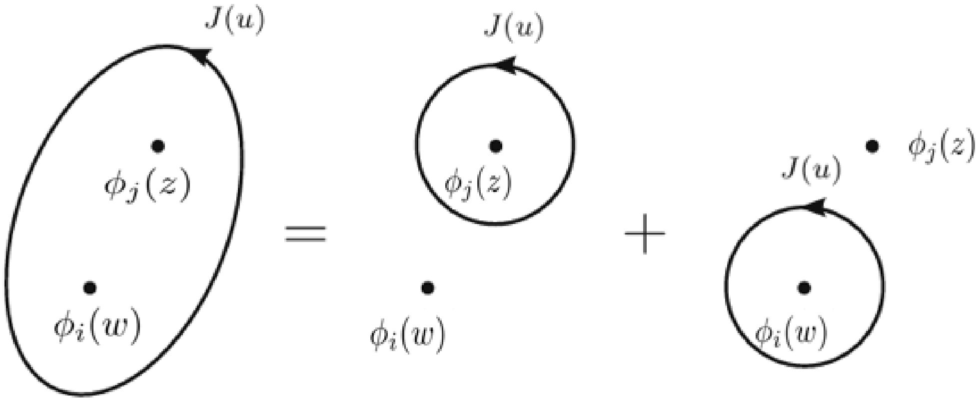

Figure 1: Monodromy charge is conserved in OPE
left: monodromy of OPE between ϕ_i and ϕ_j
right: monodromy of ϕ_i and ϕ_j

Since the monodromy charge is conserved in all operator products, it is conserved in all correlation functions, and hence corresponds to a symmetry of the theory.
The action of J on itself does in general not exhaust all the simple currents in the theory. But one can easily convince oneself that the set of all simple currents generates a discrete group of the following form

$$Z_{N_1} \times ... \times Z_{N_k} \tag{11}$$

called the center.
One can now construct new modular invariants by using this Z_N-symmetry through orbifold arguments. First, using the formalism developed in ref. [3], we introduce a twist field $\mathscr{F}(z,\bar{z}) = J(z)J^c(\bar{z})$. Every primary field $\Phi_{ij}(z)$, which consists of the highest weight representation with label "i" in the holomorphic algebra and label "j" in the anti-holomorphic one, has a well defined monodromy with respect to this twist field. The phase that is being picked up when carried around the twist field is $e^{2\pi i(Q(\Phi_i)-Q^c(\Phi_j))}$. The generator of this symmetry (the orbifold twist) is defined as $\mathscr{L} = Q_L - Q_R^c = Q_L + Q_R$. For the case when J(z) has integer conformal spin, we can also instead define the chiral twist field as $\mathscr{F}(z) = J(z)$. To construct now the modular invariant, one starts with any modular invariants, e.g. the diagonal invariant, constructs the untwisted factor by projecting out all states which are not invariant under $\mathscr{L}$, and adds all twisted sectors by acting with $\mathscr{F}$ on the untwisted sector.
In the case for a simple current with conformal spin, we have

$$Z_J(\tau,\bar{\tau}) = \frac{1}{\mathcal{N}} \sum_i \sum_{\substack{\alpha=0 \\ Q(\Phi_i)=0}}^{l_i-1} \sum_{\beta=0}^{N-1} \chi_{\alpha+\beta,i}(\tau)\bar{\chi}_{\alpha,i}(\bar{\tau}) \tag{12}$$

where the normalisation constant $\mathcal{N}$ is fixed by the requirement that the vacuum should only appear once in $Z_J(\tau,\bar{\tau})$. Here, the index i denotes the different orbits given by the action of J. The index α is the position of the primary field within the orbit i. The index β runs over the order of J.

One can show for the case of a simple current with integer conformal spin, that the monodromy charge stays constant within an orbit. In Figure 1, this is also clearly seen by setting $\phi_j(z) = J(z)$. Therefore, whole orbits are projected out and their characters will not appear in the partition function (12). For the general case where we use instead the twist field $\mathscr{F}(z,\bar{z}) = J(z)J^c(\bar{z})$, all the characters from the left- and right-moving sector mix with each other. Both cases describe the D_{2n+2}- and D_{2n+1}-series in the A-D-E classification of all $\widehat{su}(2)_k$ modular invariant partition functions respectively.

3 Coset Construction for the N=2 Minimal Models

Let us remember first the $\mathcal{N}$=2 superconformal algebra consisting of the Laurent modes L_m of the energy-momentum tensor, its superpartners $G_r^{\pm}$ and the modes j_n of a U(1) current.

$$\begin{aligned}
[L_m, L_n] &= (m-n)\ L_{m+n} + \frac{c}{12}\ (m^3 - m)\ \delta_{m+n,0}, \\
[L_m, j_n] &= -n\ j_{m+n} \\
[L_m, G_r^{\pm}] &= \left(\frac{m}{2} - r\right)\ G_{m+r}^{\pm}, \\
[j_m, j_n] &= \frac{c}{3}\ m\ \delta_{m+n,0}, \\
[j_m, G_r^{\pm}] &= \pm G_{m+r}^{\pm}, \\
\{G_r^+, G_s^-\} &= 2\ L_{r+s} + (r-s)\ j_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}, \\
\{G_r^+, G_s^+\} &= \{G_r^-, G_s^-\} = 0
\end{aligned} \tag{13}$$

For half-integer moding of $G_r^{\pm}$, this algebra is known as the Neveu-Schwarz algebra while for integer moded $G_r^{\pm}$, it is called the Ramond algebra.
The Cartan subalgebra is generated not only by L_0, like it is for example the case in $\mathcal{N}$=1 supersymmetry, but also by j_0. Hence, states are labeled both by their eigenvalue with respect to L_0 and j_0.

$$L_0 \left|h, q\right\rangle = h \left|h, q\right\rangle, \qquad j_0 \left|h, q\right\rangle = q \left|h, q\right\rangle \tag{14}$$

Many rational conformal field theories can be defined from Kač-Moody algebras using a coset construction. For the $\mathcal{N} = 2$ minimal models, we consider the following coset:

$$\frac{\widehat{su}(2)_k \times \widehat{u}(1)_2}{\widehat{u}(1)_{k+2}} \tag{15}$$

The central charge of this CFT is calculated via the central charges of the individual theories. For the $su(2)_k$ Kač-Moody algebra at level k we have

$$c_{su(2)_k} = \frac{3k}{k+2} \quad \text{for k} \geq 1 \tag{16}$$

For the boson theory we have $c = 1$. Therefore, for $0 < c < 3$, we have

$$c = c_{\widehat{su}(2)_k} + c_{\widehat{u}(1)_2} - c_{\widehat{u}(1)_{k+2}} = \frac{3k}{k+2} + 1 - 1 = \frac{3k}{k+2} \quad \text{for k} \geq 1 \tag{17}$$

A highest weight representation of the tensor product theory $\widehat{su}(2)_k \times \widehat{u}(1)_{k+2}$ decomposes into a direct sum of tensor products of highest weight representations of $u(1)_{k+2}$ and the coset:

$$(\lambda_{\widehat{su}(2)_k}) \otimes (\lambda_{\widehat{u}(1)_2}) = \bigoplus_{\lambda_{\widehat{u}(1)_{k+2}}} (\lambda_{\widehat{u}(1)_{k+2}}) \otimes (\lambda_{\widehat{su}(2)_k \times \widehat{u}(1)_2 / \widehat{u}(1)_{k+2}}) \tag{18}$$

Characters under a tensor product multiply.

$$\chi_l^{\widehat{su}(2)_k}(\tau)\chi_s^{\widehat{u}(1)_2}(\tau) = \sum_{m=-k-1}^{k+2} \chi_m^{\widehat{u}(1)_{k+2}}(\tau)\chi_{m,s}^l(\tau) \tag{19}$$

where $\chi_{m,s}^l$ are called branching functions. They carry three labels l,m,s from each individual theory in the coset respectively.
The conformal dimension of the highest weight representations can be calculated using the decomposition (18):

$$h_{m,s}^l = h_l^{\widehat{su}(2)_k} + h_s^{\widehat{(u)}(1)_2} - h_m^{\widehat{u}(1)_{k+2}} = \frac{l(l+2)}{4(k+2)} + \frac{s^2}{4\cdot 2} - \frac{m^2}{4(k+2)} \tag{20}$$

On the other hand, for the U(1) charge we have

$$q_{m,s} = -\frac{m}{k+2} + \frac{s}{2} \tag{21}$$

The constraint on the integers l,m and s take the following form:

$$0 \leq l \leq k, \qquad -(k+1) \leq m \leq k+2, \qquad s = \begin{cases} 0,2 & \text{Neveu-Schwarz sector} \\ \pm 1 & \text{Ramond sector} \end{cases}$$

m is defined modulo 2(k+2) and s is defined modulo 4. Given these restrictions on (l,m,s), one finds a $\mathbb{Z}_2$ identification of fields

$$\phi_{m,s}^l \sim \phi_{m+k+2,s+2}^{k-l} \tag{23}$$

The behaviour of the branching functions under modular S transformation can be inferred from (19). We find

$$S_{l,l'}^{\widehat{su}(2)_k} S_{s,s'}^{\widehat{u}(1)_2} = S_{m,m'}^{\widehat{u}(1)_{k+2}} S_{(l,m,s),(l,m',s')}^{\mathcal{N}=2} \tag{24}$$

and therefore

$$S_{(l,m,s),(l,m',s')}^{\mathcal{N}=2} = (S_{m,m'}^{\widehat{u}(1)_{k+2}})^{-1} S_{l,l'}^{\widehat{su}(2)_k} S_{s,s'}^{\widehat{u}(1)_2} \tag{25}$$

Applying now the Verlinde formula, we can group the S-matrix entries corresponding to the same theory together to recover the fusion rules of the $\mathcal{N}=2$ superconformal unitary models to be of the following form:

$$[\phi_{m_1,s_1}^{l_1}] \times [\phi_{m_2,s_2}^{l_2}] = \sum_{l_3,m_3,s_3} \underbrace{N_{l_1,l_2}^{l_3}}_{\widehat{su}(2)_k} \underbrace{\delta_{m_1+m_2-m_3,0}^{(k+2)}}_{\widehat{u}(1)_{k+2}} \underbrace{\delta_{s_1+s_2-s_3,0}^{(2)}}_{\widehat{u}(1)_2} [\phi_{m_3,s_3}^{l_3}] \tag{26}$$

For the fusion coefficients for $\widehat{su}(2)_k$ one finds:

$$N_{l_1,l_2}^{l_3} = \begin{cases} 1 & \text{if } |l_1 - l_2| \leq l_3 \leq \min(l_1 + l_2, 2k - l_1 - l_2) \text{ and } l_1 + l_2 + l_3 = 0 \mod 2 \\ 0 & \text{otherwise} \end{cases}$$

For the fusion rules for $\widehat{u}(1)_k$ one finds:

$$N_{a,b}^{c} = \delta^{(2k)}(a + b - c) = \begin{cases} 1 & \text{if } a + b - c = 0 \mod 2\text{k} \\ 0 & \text{otherwise} \end{cases}$$

By comparing now with the definition of a simple current, we see that all fields $\phi_{m,s}^{0}$ with $m + s \in 2\mathbb{Z}$ are simple currents. One can show that they form a discrete subgroup

$$\begin{cases} \mathbb{Z}_{4k+8} & \text{for k odd} \\ \mathbb{Z}_{2k+4} \times \mathbb{Z}_2 & \text{for k even} \end{cases}$$

In particular, the fields

$$\phi_{-1,-1}^{0} \text{ and } \phi_{0,2}^{0} \qquad \text{are simple currents} \tag{30}$$

The field $\phi_{-1,-1}^{0}$ is the spectral flow operator mapping the Neveu-Schwarz vacuum to the Ramond sector. It generates $\mathcal{N} = 2$ supersymmetry. We will use this simple current to construct a space-time supersymmetric worldsheet.

4 Gepner models

4.1 Connection to Calabi-Yau manifolds

Calabi-Yau manifolds are often defined as hypersurfaces on a complex projective space. Consider the manifold M_k defined as a hypersurface,

$$V(X_i) = X_1^{k+2} + X_2^{k+2} + \ \dots \ + X_{k+2}^{k+2} = 0 \tag{31}$$

in complex projective space $\mathbb{C}P^{k+1}$, i.e. modulo the identification of fields $X_i \equiv \lambda X_i$ where λ is a complex number. The manifold M_3 would correspond to the Quintic Calabi-Yau manifold defined by $\sum_{i=1}^{5} x_i^5 = 0$ in $\mathbb{C}P^4$.
The discrete symmetry group of the manifold M_k is as follows. Each X_i can be multiplied by a complex phase which is a k + 2 root of unity. We have k + 2 of these, therefore we have a Z_{k+2}^{k+2} symmetry on the surface. Furthermore, we can factor out phases of k + 2 roots of unity, since we are in complex projective space and the overall phase is irrelevant. There exists also a permutation symmetry between the variables X_i. Overall, the symmetry group of the manifold M_k is

$$\frac{S_{k+2} \times Z_{k+2}^{k+2}}{Z_{k+2}} \tag{32}$$

Consider next the tensor product theory consisting of $\mathcal{N} = 2$ minimal models at level k

$$\underbrace{(\mathcal{N} = 2)_k \ \otimes \ \dots \ \otimes \ (\mathcal{N} = 2)_k}_{k+2} \tag{33}$$

Each minimal model has a Z_{k+2} symmetry given by the orbit of the spectral flow operator. There exist k+2 minimal models in the theory. The charge of a field is given by $\frac{q+\bar{q}}{2}$ mod k+2. Since the discrete symmetry is embedded in the U(1) charge, and the total U(1) charge is an odd integer, as we will see later, the action of the diagonal subgroup of the product CFT is actually trivial, which allows us to quotient out a Z_{k+2} symmetry. Additionally, one can permute any of the minimal models. The overall symmetry group is

$$S_{k+2} \times Z_{k+2}^{k+2}/Z_{k+2} \tag{34}$$

This is the same symmetry group found in the Calabi-Yau manifold.

4.2 Gepner's construction

The worldsheet of a space-time supersymmetric string theory requires a central charge of c=15 in order to cancel the anomaly, i.e. it requires ten space-time dimensions. In superstring compactifications to four space-time dimensions (c=6), the internal sector is described by a six-dimensional Calabi-Yau manifold (c=9). Gepner's idea was to describe this internal sector as a tensor product of $\mathcal{N}=2$ minimal models.

$$(\mathcal{N}=2)_{c=9} = \bigoplus_{i=1}^{r} (\mathcal{N}=2)^{Vir}_{c_i} \quad \text{with} \quad \sum_{i=1}^{r} c_i = \sum_{i=1}^{r} \frac{3k_i}{k_i+2} = 9 \tag{35}$$

There exist many solutions to this equation. For example, one can choose r=5 with $k_i = 3$, which would correspond to the quintic Calabi-Yau manifold defined by the hypersurface $x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0$.
The external sector (4D space-time) in the light-cone gauge, can be described by a theory of two free bosons X_1, X_2 and a theory of two free fermions $\widehat{so}(2)_1$. Once can show that the characters of the $\widehat{so}(2)_1$ theory take the following form:

$$\begin{aligned}
\chi_O^{(0,0)}(\tau) &= \frac{\Theta_{0,2}}{\eta(\tau)} \quad && (h,q) = (0,0) \\
\chi_V^{(\frac{1}{2},1)}(\tau) &= \frac{\Theta_{2,2}}{\eta(\tau)} \quad && (h,q) = \left(\tfrac{1}{2},1\right) \\
\chi_S^{(\frac{1}{8},+\frac{1}{2})}(\tau) &= \frac{\Theta_{1,2}}{\eta(\tau)} \quad && (h,q) = \left(\tfrac{1}{8},+\tfrac{1}{2}\right) \\
\chi_C^{(\frac{1}{8},-\frac{1}{2})}(\tau) &= \frac{\Theta_{-1,2}}{\eta(\tau)} \quad && (h,q) = \left(\tfrac{1}{8},-\tfrac{1}{2}\right)
\end{aligned} \tag{36}$$

where the superscripts on χ denote the conformal weight h and the charge q with respect to the U(1) current. Employing the modular properties of the Θ-function and the Dedekind η-function, the modular S-matrix can be computed to be

$$S^{\widehat{so}(2)_1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & +i \\ 1 & -1 & +i & -i \end{pmatrix}$$

where this matrix is understood as acting on the vector $\chi = (\chi_O, \chi_V, \chi_S, \chi_C)^T$. Using again the Verlinde formula, we can determine the fusion rules of $\widehat{so}(2)_1$ to be of the following form

$$\begin{aligned}
[V]\times[V] &= [O], \quad & [V]\times[S] &= [C], \quad & [V]\times[C] &= [S] \\
[S]\times[S] &= [V], \quad & [S]\times[C] &= [O], \quad & [C]\times[C] &= [V]
\end{aligned} \tag{38}$$

We can now write down the combined CFT for the worldsheet. Except the two free bosons, the CFT has the following tensor product structure:

$$\bigotimes_{i=1}^{r} (\mathcal{N}=2)^{Vir}_{c_i} \otimes \widehat{so}(2)_1 \tag{39}$$

The highest weight representations of this CFT are again tensor products of the highest representations of the individual theories which we will denote using their labels as

$$\bigotimes_{i=1}^{r} (l_i, m_i, s_i) \otimes (s_0) \tag{40}$$

where s_0 labels the O,V,S,C- representation.
The energy-momentum tensor and the U(1) current of the entire CFT is given by

$$\begin{aligned} T(z) &= \sum_{i=1}^{r} T_i(z) + T_{\widehat{so}(2)_1}(z) \\ j(z) &= \sum_{i=1}^{r} j_i(z) + N(\Psi_\alpha \bar{\Psi}_\alpha)(z) \end{aligned} \tag{41}$$

where T_i and j_i are the energy-momentum tensor and the U(1) current for each $\mathcal{N}=2$ minimal model, and the complex fermion Ψ_α realises the $\widehat{so}(2)_1$ current algebra.

4.3 Worldsheet supersymmetry projection

One problem for our tensor product CFT is the fact that there is no clear definition of the Neveu-Schwarz and Ramond sectors. We want states of the following form $(NS)_{\text{int.}} \otimes (R)_{\text{ext.}}$ or $(R)_{\text{int.}} \otimes (S)_{\text{ext.}}$ to be removed. Let us consider the following set of simple currents given by

$$J_i = \underbrace{(0,0,0) \otimes \ \dots \ \otimes \underbrace{(0,0,2)}_{i\text{-th pos.}} \otimes \ \dots \ \otimes (0,0,0)}_{r} \otimes (V) \qquad i = 1, \dots, r \tag{42}$$

Calculating the conformal weight of this supercurrent, we find

$$h_{J_i} = (r-1) \cdot h^0_{0,0} + h^0_{0,2} + h_V = 0 + \frac{1}{2} + \frac{1}{2} = 1 \tag{43}$$

Let us now look at the action of these supercurrents on a general state in the theory. From the fusion rules of the $\mathcal{N}=2$ minimal models and the fusion rules of the $\widehat{so}(2)_1$ current algebra we have

$$\begin{aligned} &[\phi^0_{0,0}] \times [\phi^l_{m,s}] = [\phi^l_{m,s}], \quad [\phi^0_{0,2}] \times [\phi^l_{m,s}] = [\phi^l_{m,s+2}] \quad \text{(internal sector)} \\ [V] \times [O] = [V], \quad [V] \times [V] = [O], \quad &[V] \times [S] = [C], \quad [V] \times [C] = [S] \quad \text{(external sector)} \end{aligned} \tag{44}$$

The action of J on a general state (40) is

$$J \times \bigotimes_{i=1}^{r} (l_i, m_i, s_i) \otimes (s_0) = (l_0, m_0, s_0) \otimes \ \dots \ \otimes \underbrace{(l_i, m_i, s_i + 2)}_{i\text{-th pos.}} \otimes \ \dots \ \otimes (l_r, m_r, s_r) \otimes (s_0 + 2) \tag{45}$$

From the fusion rules one can see that states in the NS and R sectors are mapped to NS and R sectors respectively in both the internal and the external sector.
We can now compute the monodromy charge for a general state:

$$
\begin{aligned}
Q\big(\bigotimes_{i=1}^{r}(l_i,m_i,s_i)\otimes(s_0)\big) =& h_J + h_{\bigotimes_{i=1}^{r}(l_i,m_i,s_i)\otimes(s_0)} - h_{J\times\bigotimes_{i=1}^{r}(l_i,m_i,s_i)\otimes(s_0)} \mod 1 \\
=&1+\sum_i\Big(\frac{l_i(l_i+2)-m_i^2}{4(k_i+2)}+\frac{s_i^2}{8}\Big)+\frac{s_0^2}{8} \\
&-\Big(\sum_i\Big(\frac{l_i(l_i+2)-m_i^2}{4(k_i+2)}+\frac{s_i^2}{8}\Big)+\frac{4s_i+4}{8}+\frac{(s_0+2)^2}{8}\Big) \mod 1 \\
=&-\frac{1}{2}(s_0+s_i) \mod 1 \\
=&\begin{cases} 0 & \text{if } s_0+s_i\in 2\mathbb{Z}, \\ \frac{1}{2} & \text{else.} \end{cases}
\end{aligned}
$$

(46)

Applying the projection Q=0, we are removing all mixed states and keeping only states that are in the NS sector in both the $\mathcal{N}=2$ factor and the $\widehat{so}(2)_1$ theory or in the R sector in both theories. This gives us a clear distinction between those two sectors.

4.4 Space-time supersymmetry projection

Another problem with our tensor product CFT is the fact that there are not the same number of states in the Neveu-Schwarz as in the Ramond sector which leads to an unequal amount of bosons and fermions and therefore violating space-time supersymmetry. What we need is a one-to-one map between the Neveu-Schwarz and the Ramond sectors. For $\mathcal{N}=2$ minimal models, we have such a map available, namely the spectral flow operator. From the above discussion in sect. 3 we know that the spectral flow operator is also a simple current of the minimal models. We can now perform a simple current projection such that we achieve space-time supersymmetry.
Consider the supercurrent acting on the internal sector:

$$
J_{sf} = \underbrace{(0,-1,-1)\otimes\ \dots\ \otimes(0,-1,-1)}_{r} \tag{47}
$$

The conformal weight of this supercurrent is

$$
h_{J_{sf}} = \sum_i (h^0_{1,1})_i = \sum_i\Big(\frac{-1}{4(k_i+2)}+\frac{1}{8}\Big) \tag{48}
$$

The action of the supercurrent on a general state in the internal sector is found as

$$
J_{sf}\times\bigotimes_{i=1}^{r}(l_i,m_i,s_i) = \bigotimes_{i=1}^{r}(l_i,m_i-1,s_i-1) \tag{49}
$$

The monodromy charge for this simple current now reads:

$$
\begin{aligned}
Q\Big(\bigotimes_{i=1}^{r}(l_i,m_i,s_i)\Big) =& h_{J_{sf}} + h_{\bigotimes_{i=1}^{r}(l_i,m_i,s_i)} - h_{\bigotimes_{i=1}^{r}(l_i,m_i+1,s_i+1)} \mod 1 \\
=& \sum_i \Big(\frac{-1}{4(k_i+2)} + \frac{1}{8}\Big) + \sum_i \Big(\frac{-2m_i+1}{4(k_i+2)} + \frac{2s_i-1}{8}\Big) \quad \mod 1 \\
=& \sum_i \Big(\frac{-m_i}{2(k_i+2)} + \frac{s_i}{4}\Big) \quad \mod 1 \\
=& \frac{q_{\bigotimes_{i=1}^{r}(l_i,m_i,s_i)}}{2} \quad \mod 1
\end{aligned} \tag{50}
$$

where we have used Eq. (21). Applying the projection Q=0, we are keeping only states with even U(1) charge. If we include the external sector as well, with the aim of constructing a full modular invariant partition function, this condition on the U(1) charge gets modified to $q \in 2\mathbb{Z}+1$. This generalized projection to ensure space-time supersymmetry is known as a Gliozzi-Scherk-Olive (GSO) projection.

4.5 Space-time supersymmetric modular invariant partition functions

After having studied the necessary projections for space-time supersymmetry, we are now in a position to construct a modular invariant partition function. Quite generally, it has the form

$$
Z = \frac{1}{\mathcal{N}} \sum_{i,j} \chi_i(\tau)\ M_{ij}(J_{sf}) \prod_{s=1}^{r} M_{ij}(J_s)\ \bar{\chi}_j(\bar{\tau}) \tag{51}
$$

where the matrices M(J) encode the information about which holomorphic characters couple to which anti-holomorphic ones due to the extension by the simple current J.
Let us now introduce a compact notation for the characters

$$
\chi_{\vec{\lambda}}^{\vec{l}}(\tau) := \prod_{i=1}^{r} \chi_{m_i,s_i}^{l_i}(\tau) \cdot \chi_{s_0}^{\widehat{so}(2)_1}(\tau) \quad \text{with } \vec{\lambda} = (s_0, m_1, ..., m_r, s_1, ..., s_r) \text{ and } \vec{l} = (l_1, ..., l_r) \tag{52}
$$

The charge vector for the simple currents J_i from Eq. (42) is

$$
\beta_i = (2, 0, ..., 0, 0, ..., \underset{\substack{\uparrow \\ \text{i-th pos.}}}{2}, ..., 0). \tag{53}
$$

One can define a scalar product between two charge vectors, which is given by

$$
\vec{\lambda} \cdot \vec{\mu} = \frac{s_0 s_0'}{4} + \frac{1}{2} \sum_{i=1}^{r} \Big(-\frac{m_i m_i'}{k_i+2} + \frac{s_i s_i'}{2} \Big) \tag{54}
$$

As we have seen before, states which are neither purely in the Neveu-Schwarz nor purely in the Ramond sector will be removed with respect to the simple current projection Eq. (42). This can be expressed as

$$\vec{\beta}_i \cdot \vec{\lambda} \in \mathbb{Z} \tag{55}$$

The charge vector for the simple current J_{sf} from Eq. (47) reads

$$\vec{\beta}_0 = (1, 1, ..., 1, 1, ..., 1). \tag{56}$$

The projection onto all states with odd U(1) charge can be formulated as

$$\vec{\beta}_0 \cdot \vec{\lambda} \in \frac{1}{2} + \mathbb{Z} \tag{57}$$

We can now write down the simple current extended, space-time supersymmetric, modular invariant partition function with the results from sect. 2 as

$$Z(\tau, \bar{\tau}) = \frac{1}{\mathcal{N}} \sum_{\vec{l}, \vec{\lambda}} \sum_{\substack{\upsilon_i = 0,1 \\ \vec{\beta}_i \cdot \vec{\lambda} \in \mathcal{Z} \\ \vec{\beta}_0 \cdot \vec{\lambda} \in \frac{1}{2} + \mathcal{Z}}} \sum_{\upsilon_0 = 0}^{L-1} \chi^{\vec{l}}_{\vec{\lambda}}(\tau) \bar{\chi}^{\vec{l}}_{\vec{\lambda} + \sum_{i=1}^{r} \upsilon_i \beta_i + \upsilon_0 \beta_0}(\bar{\tau}) (-1)^{\upsilon_0} \tag{58}$$

where $\mathcal{N}$ is again an overall normalisation constant, L is the order of J_{sf} and the $(-1)^{\upsilon_0}$ is needed to ensure the right spin-statistics.

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