PMAT 22213

Problem sheet 04
Answer sheet

01 (a) $\frac{dy}{dx} + \frac{(2x+1)}{x} y = e^{-2x}$ ——— (1)

Here $P(x) = \frac{2x+1}{x}$ & $Q(x) = e^{-2x}$

Integrating factor; $I = e^{\int P(x)dx} = e^{\int \frac{2x+1}{x} dx}$

$= e^{\int 2\,dx + \int 1/x\, dx}$

$= e^{2x + \ln|x|}$

$= e^{2x} \cdot e^{\ln|x|}$

$= xe^{2x}$

(1) × I $\Rightarrow$ $xe^{2x}\frac{dy}{dx} + \cancel{x}e^{2x} \cdot \frac{(2x+1)}{\cancel{x}} y = xe^{2x} \cdot e^{-2x}$

$\frac{d}{dx}[xe^{2x}y] = x$

$xe^{2x}y = \int x\, dx$

$xe^{2x}y = \frac{x^2}{2} + c$; c is an arbitrary constant.

$y = \left(\frac{x}{2} + \frac{c}{x}\right)e^{-2x}$ //

(b) $(x^2+1)\frac{dy}{dx} + 4xy = x$, $y(2) = 1$

$\frac{dy}{dx} + \frac{4x}{(x^2+1)} y = \frac{x}{x^2+1}$ ——— (1)

Here $P(x) = \frac{4x}{x^2+1}$ & $Q(x) = \frac{x}{x^2+1}$

Integrating factor, $I = e^{\int P(x)\,dx} = e^{\int \frac{4x}{x^2+1}\,dx}$

$$I = e^{2\ln|x^2+1|} = (x^2+1)^2$$

$$① \times I \Rightarrow (x^2+1)^2 \frac{dy}{dx} + \frac{4x}{\cancel{x^2+1}}(x^2+1)^{\cancel{2}} y = \frac{x}{\cancel{(x^2+1)}} \cdot (x^2+1)^{\cancel{2}}$$

$$\frac{d}{dx}\left[(x^2+1)^2 y\right] = x(x^2+1)$$

$$(x^2+1)^2 y = \int (x^3+x)\,dx$$

$$(x^2+1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + C$$; C is an arbitrary constant.

From the given initial condition, $[y(2)=1] \Rightarrow [x=2, y=1]$

$$(2^2+1)^2(1) = \frac{2^4}{4} + \frac{2^2}{2} + C$$

$$25 = 4+2+C$$

$$\therefore C = 19$$

$\therefore$ Particular solution,

$$y(x^2+1)^2 = \frac{x^4}{4} + \frac{x^2}{2} + 19 \quad //$$

(c) $\frac{dy}{dx} + \frac{3y}{x} = 6x^2$ ——— ①

Here $P(x) = \frac{3}{x}$ & $Q(x) = 6x^2$

Integrating factor, $I = e^{\int P(x)\,dx} = e^{\int 3/x\,dx} = e^{3\ln|x|}$

$$I = x^3$$

$$① \times I \Rightarrow x^3 \frac{dy}{dx} + 3x^2 y = 6x^5$$

$$\frac{d}{dx}(x^3 y) = 6x^5$$

$$x^3 y = 6\int x^5\,dx$$

$$x^3 y = \cancel{6} \cdot \frac{x^6}{\cancel{6}} + C$$

$x^3 y = x^6 + c$; c is an arbitrary constant.

$y = x^3 + cx^{-3}$ //

(d) $\frac{dy}{dx} + 3y = 3x^2 e^{-3x}$ —— ①

Here $P(x) = 3$ & $Q(x) = 3x^2 e^{-3x}$

Integrating factor, $I = e^{\int P(x)dx} = e^{\int 3dx} = e^{3x}$

① $\times I \Rightarrow$ $e^{3x}\frac{dy}{dx} + 3e^{3x}y = 3x^2 \cancel{e^{-3x}} \cdot \cancel{e^{3x}}$

$$\frac{d}{dx}\left[e^{3x}y\right] = 3x^2$$

$$e^{3x}y = \int 3x^2 dx$$

$e^{3x}y = x^3 + c$; c is an arbitrary constant.

$y = x^3 e^{-3x} + ce^{-3x}$ //

(e) $\frac{dy}{dx} + 3x^2 y = x^2$, $y(0) = 2$ —— ①

Let $P(x) = 3x^2$, $Q(x) = x^2$

Integrating factor, $I = e^{\int P(x)dx} = e^{\int 3x^2 dx}$

$$I = e^{x^3}$$

① $\times I \Rightarrow$ $e^{x^3}\frac{dy}{dx} + 3x^2 e^{x^3} y = e^{x^3} x^2$

$$\frac{d}{dx}\left[e^{x^3}y\right] = e^{x^3}x^2$$

$$e^{x^3}y = \int \underbrace{x^2 e^{x^3} dx}_{5} \quad \text{—— ②}$$

Let $x^3 = t$, Then $\frac{dt}{dx} = 3x^2$

$$dt/3 = x^2 dx$$

$$S = \int x^2 e^{x^3} dx = \int e^t \frac{dt}{3} = \frac{e^t}{3} + C = \frac{e^{x^3}}{3} + C$$

from ② $\Rightarrow$ $e^{x^3} y = \frac{e^{x^3}}{3} + C$

$$y = \frac{1}{3} + Ce^{-x^3};$$ where C is an arbitrary constant.

02 (a) $\frac{dy}{dx} + y = xy^3$

$y^3 \div \Rightarrow \quad y^{-3}\frac{dy}{dx} + y^{-2} = x$ —①

Let $v = y^{-2}$

$$\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx} \quad \text{—②}$$

① $\times (-2) \Rightarrow \quad -2y^{-3}\frac{dy}{dx} - 2y^{-2} = -2x$

Then from ② $\Rightarrow$

$$\frac{dv}{dx} - 2v = -2x \quad \text{—③}$$

Now, this is first order linear Differential eqⁿ.

Let $P(x) = -2$, $Q(x) = -2x$

$\therefore$ Integrating factor, $I = e^{\int P(x)dx} = e^{\int -2dx}$

$$I = e^{-2x}$$

③ $\times I \Rightarrow \quad e^{-2x}\frac{dv}{dx} - 2e^{-2x}v = -2e^{-2x}x$

$$\frac{d}{dx}\left[e^{-2x}v\right] = -2e^{-2x}x$$

$$e^{-2x}v = -2\int xe^{-2x}dx$$

$$= -2\left[\int x \frac{d\left(e^{-2x}/2\right)}{dx} dx\right]$$

$$= -2\left\{-\frac{xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\frac{dx}{dx}dx\right\}$$

$$e^{-2x}v = \left[xe^{-2x} + \frac{e^{-2x}}{2}\right] + C$$

Now Substitute $v = y^{-2}$;

$$e^{-2x} y^{-2} = x e^{-2x} + \frac{e^{-2x}}{2} + C$$

$$y^{-2} = x + \frac{1}{2} + Ce^{2x}$$

$$\therefore y^{2} = \frac{1}{x + 1/2 + Ce^{2x}}$$

(b) $x \frac{dy}{dx} + y = -2x^6 y^4$

$x \div \Rightarrow \frac{dy}{dx} + \frac{y}{x} = -2x^5 y^4$

$y^4 \div \Rightarrow y^{-4} \frac{dy}{dx} + \frac{y^{-3}}{x} = -2x^5$

$(-3) \times \Rightarrow -3y^{-4} \frac{dy}{dx} - \frac{3}{x} y^{-3} = 6x^5$ ——— ①

Let $v = y^{-3}$, Then $\frac{dv}{dx} = -3y^{-4} \frac{dy}{dx}$

Substituting into ① $\Rightarrow$

$$\frac{dv}{dx} - \frac{3}{x} v = 6x^5 \quad \text{—— ②}$$

Here $P(x) = -\frac{3}{x}$, $Q(x) = 6x^5$

Then integrating factor, $I = e^{\int -3/x \, dx}$

$$= e^{-3\ln|x|}$$

$$I = x^{-3}$$

② $\times I \Rightarrow x^{-3} \frac{dv}{dx} - 3x^{-4} v = 6x^2$

$$\frac{d}{dx}\left[x^{-3} v\right] = 6x^2$$

$$x^{-3} v = 6 \int x^2 \, dx$$

$$x^{-3} v = 2x^3 + C$$

Now Substitute v as y^{-3};

$$\frac{1}{x^3y^3} = 2x^3 + C \;; \quad c \text{ is an arbitrary constant}$$

(c) $x^2\frac{dy}{dx} + xy = \frac{y^3}{x}$, $y(1) = 1$

$x^2 \div \Rightarrow \quad \frac{dy}{dx} + x^{-1}y = \frac{y^3}{x^3}$

$y^3 \div \Rightarrow \quad y^{-3}\frac{dy}{dx} + x^{-1}y^{-3}y = \frac{1}{x^3}$

$(-2)\times \Rightarrow \quad -2y^{-3}\frac{dy}{dx} - 2x^{-1}y^{-2} = \frac{-2}{x^3}$ —— ①

Let $v = y^{-2}$, $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$.

from ① $\Rightarrow$

$$\frac{dv}{dx} - 2x^{-1}v = -\frac{2}{x^3} \quad \text{—— ②}$$

Here, $P(x) = -\frac{2}{x}$ & $Q(x) = -\frac{2}{x^3}$

Integrating factor $I = e^{\int -2/x \, dx}$

$I = e^{-2\ln|x|} = x^{-2} = \frac{1}{x^2}$

② $\times I \Rightarrow \quad \frac{1}{x^2}\frac{dv}{dx} - \frac{2}{x^3}v = -\frac{2}{x^5}$

$$\frac{d}{dx}\left[\frac{v}{x^2}\right] = \frac{-2}{x^5}$$

$$\frac{v}{x^2} = -2\int \frac{1}{x^5}dx$$

$$\frac{v}{x^2} = \frac{(-2)x^{-4}}{(-4)} + C \;; \quad c \text{ is an arbitrary constant.}$$

$v = y^{-2} \Rightarrow \quad \frac{1}{y^2x^2} = \frac{1}{2x^4} + C$

when $y(1)=1 \Rightarrow$

$$\frac{1}{1\cdot 1} = \frac{1}{2} + c$$

$$c = \frac{1}{2}$$

$\therefore$ particular Solution,

$$\frac{1}{y^2x^2} = \frac{1}{2x^4} + \frac{1}{2}$$ //

(d) $\frac{dx}{dt} + tx^3 + \frac{x}{t} = 0$

$$\frac{dx}{dt} + \frac{x}{t} = -tx^3$$

$x^3 \div \Rightarrow$ $\quad x^{-3}\frac{dx}{dt} + \frac{x^{-2}}{t} = -t$

$(-2)\times \Rightarrow$ $\quad -2x^{-3}\frac{dx}{dt} \quad \frac{-2x^{-2}}{t} = 2t$ —— (1)

Let $v = x^{-2}$, then $\frac{dv}{dt} = -2x^{-3}\frac{dx}{dt}$

from (1) $\Rightarrow$ $\quad \frac{dv}{dt} - \frac{2v}{t} = 2t$ —— (2)

Here $P(t) = -\frac{2}{t}$ & $Q(t) = 2t$

Integrating factor, $I = e^{\int -2/t \, dt} = e^{-2\ln|t|} = t^{-2}$

(2) $\times I \Rightarrow$ $\quad t^{-2}\frac{dv}{dt} - 2t^{-3}v = 2t^{-1}$

$$\frac{d}{dt}\left[t^{-2}v\right] = 2t^{-1}$$

$$t^{-2}v = 2\int \frac{1}{t}\,dt$$

$t^{-2}v = 2\ln|t| + c$; c is an arbitrary constant.

$v = x^{-2} \Rightarrow$ $\quad \frac{1}{x^2t^2} = 2\ln|t| + c$ //

(e) $\frac{dy}{dx} + y^3x + y = 0$

$\frac{dy}{dx} + y = -xy^3$

$y^3 \div \Rightarrow \quad y^{-3}\frac{dy}{dx} + y^{-2} = -x$

$(-2)\times \Rightarrow \quad -2y^{-3}\frac{dy}{dx} - 2y^{-2} = 2x \quad$ —— ①

Let $v = y^{-2}$, then $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$

From ① $\Rightarrow$

$\frac{dv}{dx} - 2v = 2x \quad$ —— ②

Let $P(x) = -2$ & $Q(x) = 2x$

Integrating factor, $I = e^{\int -2dx} = e^{-2x}$

$I \times ② \Rightarrow \quad e^{-2x}\frac{dv}{dx} - 2e^{-2x}v = 2xe^{-2x}$

$\frac{d}{dx}\left[e^{-2x}v\right] = 2xe^{-2x}$

$e^{-2x}v = 2\underbrace{\int xe^{-2x}dx}_{S}$

$e^{-2x}v = 2S \quad$ —— ③

$S = \int xe^{-2x}dx = \int x\frac{d}{dx}\left(\frac{e^{-2x}}{-2}\right)dx$

$= \frac{xe^{-2x}}{-2} + \frac{1}{2}\int\left(e^{-2x}\frac{dx}{dx}\right)dx$

$= \frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{4} + c$; c is an arbitra-ry constant

from ③ $\Rightarrow$

$e^{-2x}v = -xe^{-2x} - \frac{e^{-2x}}{2} + c'$; $[c' = 2c]$

$\frac{1}{y^2} = -x - \frac{1}{2} + c'$ //

03 (a) $y^2 dx + 2xy\,dy = 0$ $[M dx + N dy = 0]$

Let $M = y^2$, $N = 2xy$

$\frac{\partial M}{\partial y} = M_y = 2y$ & $\frac{\partial N}{\partial x} = N_x = 2y$

$M_y = N_x$

$\therefore$ Given D.E is exact.

Solution, $\int\limits_{y\ \text{constant}} M\,dx + \int(\text{terms in } N \text{ which are independent of } x)$ c

$\int y^2 dx + \int 0\,dy = c$

$xy^2 + c_1 = c$ (where c, c_1 are constants)

$\therefore$ general soln is : $xy^2 = c_0$ $[c_0 = c - c_1]$

(b) $\cos(x+y)\,dx + (3y^2 + 2y + \cos(x+y))\,dy = 0$

Let $M = \cos(x+y)$, $N = 3y^2 + 2y + \cos(x+y)$

$\frac{\partial M}{\partial y} = M_y = -\sin(x+y)$ $\frac{\partial N}{\partial x} = N_x = -\sin(x+y)$

Since $M_y = N_x$, given D.E is exact.

$\therefore$ general soln :

$\int \cos(x+y)\,dx + \int(3y^2 + 2y)\,dy = c$

$\sin(x+y) + y^3 + y^2 = c$; where c is an arbitrary constant.

(c) $(2x\cos y + 3x^2y)dx + (x^3 - x^2\sin y - y)dy = 0$; $y(0) = 2$

Let $M = 2x\cos y + 3x^2y$, $N = x^3 - x^2\sin y - y$

$M_y = -2x\sin x + 3x^2$, $N_x = 3x^2 - 2x\sin y$

$$M_y = N_x$$

$\therefore$ Given D.E is exact.

$\therefore$ General sol^n is:

$$\int(2x\cos y + 3x^2y)\,dx + \int -y\,dy = c$$

$x^2\cos y + x^3y - \frac{y^2}{2} = c$; c is an arbitrary constant.

when $y(0) = 2$,

$0 + 0 - 2 = c$

$c = -2$

$\therefore$ Particular sol^n is:

$$2x^2\cos y + 2x^3y - y^2 + 4 = 0 \quad //$$

(d) $4\cos 2u\,du - e^{-5v}dv = 0$, $v(0) = -6$

Let $M = 4\cos 2u$, $N = -e^{-5v}$

$M_v = 0$, $N_u = 0$

Since $M_v = N_u$, given D.E is exact.

$\therefore$ General sol^n is:

$$\int 4\cos 2u\,du + \int -e^{-5v}dv = c$$

$$2\sin 2u + \frac{1}{5}e^{-5v} = c$$

when $v(0) = -6$

$2\sin 0 + \frac{1}{5}e^{30} = c$

$\therefore c = \frac{1}{5}e^{30}$

$\therefore$ Particular sol^n is: $2\sin 2u + \frac{1}{5}e^{-5v} = \frac{1}{5}e^{30}$

$$10\sin 2u + e^{-5v} = e^{30}$$

(e) $(x+y)^2 dx - (y^2 - 2xy - x^2) dy = 0$

Let $M = (x+y)^2$, $N = -y^2 + 2xy + x^2$

$\frac{\partial M}{\partial y} = M_y = 2(x+y)$, $\frac{\partial N}{\partial x} = N_x = 2y + 2x = 2(x+y)$

Since $M_y = N_x$, given D.E is exact.

$\therefore$ General Sol^n is:

$$\int (x+y)^2 dx + \int -y^2 dy = c$$

$$\frac{(x+y)^3}{3} - \frac{y^3}{3} = c$$; where c is an arbitrary constant.

$$(x+y)^3 - y^3 = c_1 \;;\; [3c = c_1]$$

(04) (a) $y dx - 3x dy = 0$ —— ①

Let $M = y$ and $N = -3x$

Then $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = -3$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, D.E is not exact.

Here $Mx + Ny = xy - 3xy = -2xy \neq 0$

$\therefore$ Integrating factor $I = \frac{1}{Mx + My} = \frac{1}{-2xy}$

① $\times$ I $\Rightarrow$ $\frac{-1}{2x} dx + \frac{3}{2y} dy = 0$

Now $M' = -\frac{1}{2x}$ and $N' = \frac{3}{2y}$

$\frac{\partial M'}{\partial y} = 0$ and $\frac{\partial N'}{\partial x} = 0$

So, D.E is exact.

$\therefore$ General sol^n is:

$$\int -\frac{1}{2x}\,dx + \int \frac{3}{2y}\,dy = \ln|c|$$

$$-\frac{1}{2}\ln|x| + \frac{3}{2}\ln|y| = \ln|c|$$

$$x^{-1/2}\,y^{3/2} = c$$

$$\frac{y^3}{x} = c_0 \quad [\text{where } c_0 = c^2]$$

$$y^3 = c_0 x$$

$$y = (c_0 x)^{1/3} \;//$$

(b) $xy\,dx + (1+x^2)\,dy = 0$ —— ①

Let $M = xy$ and $N = 1+x^2$

$\frac{\partial M}{\partial y} = x$ and $\frac{\partial N}{\partial x} = 2x$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, D.E is not exact.

$$\frac{-\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}}{M} = \frac{-x+2x}{xy} = \frac{\cancel{x}}{\cancel{x}y} = \frac{1}{y} = g(y)$$

$\therefore$ Integrating factor $I = e^{\int g(y)\,dy} = e^{\int 1/y\,dy}$

$= e^{\ln|y|} = y$

① $\times I \Rightarrow y^2x\,dx + y(1+x^2)\,dy = 0$

Now $M' = y^2 x$ and $N' = y(1+x^2)$

Then $\frac{\partial M'}{\partial y} = 2xy$, $\frac{\partial N'}{\partial x} = 2xy$

Now, D.E is exact.

So, General sol^n is: $\int y^2 x\,dx + \int y\,dy = c$

$$\frac{x^2y^2}{2} + \frac{y^2}{2} = c$$

$$(x^2+1)\,y^2 = c' \quad [c' = 2c] \;//$$

(c) $(e^{x+y} + ye^{y})dx + (xe^{y} - 1)\, dy = 0$, $y(0) = -1$ ①

Let $M = e^{x+y} + ye^{y}$ and $N = xe^{y} - 1$

$$\frac{\partial M}{\partial y} = e^{x+y} + ye^{y} + e^{y} \quad \text{and} \quad N_x = e^{y}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, D.E is not exact.

$$\frac{\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x}}{M} = \frac{\cancel{e^{y}} - (e^{x+y} + ye^{y} + \cancel{e^{y}})}{e^{x+y} + ye^{y}} = -1 = g(y)$$

∴ Integrating factor $I = e^{\int g(y)\,dy} = e^{\int (-1)\,dy} = e^{-y}$

① × I ⇒ $(e^{x} + y)\,dx + (x - e^{-y})\,dy = 0$

Now, $M' = y + e^{x}$, $N' = x - e^{-y}$

$$\frac{\partial M'}{\partial y} = 1 \quad \text{and} \quad \frac{\partial N'}{\partial x} = 1$$

So, exact.

∴ General solⁿ:

$$\int (e^{x} + y)\,dx + \int -e^{-y}\,dy = c$$

$$e^{x} + xy + e^{-y} = c$$

when $y(0) = -1$ ⇒

$$e^{0} + 0 + e^{(1)} = c$$

$$c = e + 1$$

∴ Particular solⁿ:

$$e^{x} + xy + e^{-y} = e + 1$$

(d) $(2x - y^2)\,dx + xy\,dy = 0$ —— ①

Here $M = 2x - y^2$, $N = xy$

$$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = y$$

So, D.E is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2y - y}{xy} = \frac{-3}{x} = f(x)$$

$\therefore$ Integrating factor $I = e^{\int f(x)\,dx} = e^{\int -3/x\,dx}$

$$= e^{\ln|x^{-3}|} = x^{-3}$$

① × I $\Rightarrow$ $(2x^{-2} - y^2x^{-3})\,dx + x^{-2}y\,dy = 0$

Let $M' = 2x^{-2} - y^2x^{-3}$ & $N' = x^{-2}y$

$$\frac{\partial M'}{\partial y} = \frac{-2y}{x^3} \quad \& \quad \frac{\partial N'}{\partial x} = \frac{-2y}{x^3}$$

$\therefore$ Now, D.E is exact.

$\therefore$ General solⁿ is:

$$\int (2x^{-2} - y^2x^{-3})\,dx + \int 0\,dy = c$$

$$\frac{-2}{x} + \frac{y^2}{2x^2} = c$$; where c is an arbitrary constant.

(e) $(3x^2y + 2xy + y^3)\,dx + (x^2+y^2)\,dy = 0$ —— ①

Let $M = 3x^2y + 2xy + y^3$ & $N = x^2+y^2$

$$\frac{\partial M}{\partial y} = 3x^2 + 2x + 3y^2 \quad \& \quad \frac{\partial N}{\partial x} = 2x$$

$\therefore$ D.E is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3x^2 + \cancel{2x} + 3y^2 - \cancel{2x}}{x^2+y^2} = \frac{3\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} = 3$$

$= g(x)$

$\therefore$ Integrating factor $I = e^{\int g(x)dx} = e^{3x}$

① × I.F $\Rightarrow$ $e^{3x}(3x^2y + 2xy + y^3)\,dx + e^{3x}(x^2+y^2)\,dy = 0$

Now $\frac{\partial M'}{\partial y} = e^{3x}(3x^2 + 2x + 3y^2)$

$$\frac{\partial N'}{\partial x} = e^{3x}(2x) + (x^2+y^2)\,3e^{3x}$$

$$= e^{3x}(3x^2 + 3y^2 + 2x)$$

$\therefore$ D.E is exact.

$\therefore$ General soln: $\int e^{3x}(3x^2y + 2xy + y^3)\,dx + \underbrace{\int 0\,dy}_{c_1} = C$

$3y\int x^2e^{3x}dx + 2y\int xe^{3x}dx + y^3\int e^{3x}dx = c_0$ $[c_0 = C - c_1]$ —— (*)

Let $\int x^2e^{3x}dx = \int x^2 \frac{d}{dx}\left(\frac{e^{3x}}{3}\right)dx$

$$= x^2\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}(2x)\,dx$$

$$= \frac{x^2e^{3x}}{3} - \frac{2}{3}\int xe^{3x}dx \quad \text{—— ②}$$

$$\int xe^{3x}\,dx = \int x\,\frac{d\left(\frac{e^{3x}}{3}\right)}{dx}\,dx = \frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}.(1)\,dx$$

$$= \frac{xe^{3x}}{3} - \frac{e^{3x}}{9}$$

$$① \Rightarrow \int x^2 e^{3x}\,dx = \frac{x^2e^{3x}}{3} - \frac{2}{3}\left[\frac{xe^{3x}}{3} - \frac{e^{3x}}{9}\right]$$

$$= \frac{e^{3x}}{27}\left[9x^2 - 18x + 2\right]$$

Substituting values into ⊛ $\Rightarrow$

$$\cancel{3}y \cdot \frac{e^{3x}}{\cancel{27}\,9}(9x^2 - 18x + 2) + 2y \cdot e^{3x}\left(x - \tfrac{1}{9}\right) + \frac{y^3e^{3x}}{3} = c_0$$

$$ye^{3x}(9x^2 - 18x + 2) + 2ye^{3x}(9x - 1) + 3y^3e^{3x} = c'$$

$$ye^{3x}\left[9x^2 \cancel{-18x} \cancel{+2} + \cancel{18x} \cancel{-2} + 3y^2\right] = c'$$

$$3ye^{3x}(3x^2 + y^2) = c'$$

$$ye^{3x}(3x^2 + y^2) = c'' \quad // \quad \left[\text{where } c'' = \frac{c'}{3}\right].$$

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5 (a) $y = c\sin x$ —— ①

Differentiate w.r.t x;

$$\frac{dy}{dx} = c\cos x$$

From ① $\Rightarrow$ $c = \frac{y}{\sin x}$

$\therefore \frac{dy}{dx} = y\cot x$ ← This is the slope of the original family.

There $-\frac{dx}{dy}$ is the slope of the orthogonal trajectories.

$$-\frac{dx}{dy} = y\cot x$$

$$\frac{dx}{dy} = -y\cot x$$

$$\int \tan x\, dx = \int -y\, dy$$

$$-\ln|\cos x| = -\frac{y^2}{2} + \ln|\;|$$

$$\ln|\cos x| + \ln|c| = \frac{y^2}{2}$$

$$\ln|c(\cos x)| = \frac{y^2}{2}$$

$y^2 = 2\ln|c(\cos x)|$ // ← This is the differential eq^{n} of the family of orthogonal trajectories.

(b) $x = ce^{-y^2}$ —— ①

$$\frac{dx}{dy} = ce^{-y^2}.(-2y)$$

from ① $\Rightarrow$

$$\frac{dx}{dy} = -2xy$$

Replace $\frac{dx}{dy}$ by $-\frac{dy}{dx}$.

$$-\frac{dy}{dx} = -2xy$$

$$\int \frac{dy}{y} = \int 2x\,dx$$

$$\ln|y| = x^2 + \ln|c|$$

$$y = ce^{x^2} \quad // \quad \leftarrow$$ This is the differential eq^n of the family of orthogonal trajectories.

(c) $\frac{x^2}{2} + y^2 = c$

$$x + 2y\frac{dy}{dx} = 0$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$$x + 2y\left(-\frac{dx}{dy}\right) = 0$$

$$x - 2y\frac{dx}{dy} = 0$$

$$\int 2\frac{dx}{x} = \int \frac{dy}{y}$$

$$2\ln|x| = \ln|y| + \ln|c|$$

$$\ln|x^2| = \ln|yc|$$

$$x^2 = yc \quad //$$

(d) $xy = c^2$

$$x\frac{dy}{dx} + y = 0$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$$-x\frac{dx}{dy} + y = 0$$

$$x\frac{dx}{dy} = y$$

$$\int x\,dx = \int y\,dy$$

$$x^2 = y^2 + C$$

(e) $y^2 = 4Cx$ —— ①

$$2y\frac{dy}{dx} = 4C$$

from ① $\Rightarrow$ $\cancel{2y}\frac{dy}{dx} = \frac{y^{\cancel{2}}}{x}$

Replace $\frac{dy}{dx}$ by $\left(-\frac{dx}{dy}\right)$.

$$2\left(-\frac{dx}{dy}\right) = \frac{y}{x}$$

$$-2x\,dx = y\,dy$$

$$-x^2 = \frac{y^2}{2} + C$$

$$-2x^2 = y^2 + C$$

$$y^2 + 2x^2 = C_1$$